M257/316 MIDTERM 2 SECTION 102

Q1. PDE $u_t = u_{xx} + e^{-\beta t}\cos 3x - 2 \qquad 0<x<\pi,\ t>0$

BC $u_x(0,t)=\pi \qquad u_x(\pi,t)=3\pi$

IC $u(x,0)=\cos x + x^2.$

CHOOSE $w(x)=Ax^2+Bx$ WITH $w'(x)=2Ax+B$ MATCH THE NON-ZERO BC

$\pi=u_x(0,t)=B \qquad 3\pi=u_x(\pi,t)=2A\pi+\pi \Rightarrow A=1 \Rightarrow \boxed{w(x)=x^2+\pi x}$

NOW LET $u(x,t)=w(x)+v(x,t)$ AND PLUG INTO THE PDE, BC & IC:

PDE: $u_t = \cancel{w_t}+v_t = w_{xx}+v_{xx}+e^{-\beta t}\cos 3x-2 = 2+v_{xx}+e^{-\beta t}\cos 3x-2 \Rightarrow v_t=v_{xx}+e^{-\beta t}\cos 3x$

$\cancel{\pi} = u_x(0,t)=w_x(0)+v_x(0,t)=\cancel{\pi}+v_x(0,t) \Rightarrow v_x(0,t)=0$ (*)

$3\pi = u_x(\pi,t)=w_x(\pi)+v_x(\pi,t)=3\pi+v_x(\pi,t) \Rightarrow v_x(\pi,t)=0$

$\cos x + \cancel{x^2} = u(x,0)=w(x)+v(x,0)=\cancel{x^2}+\pi x+v(x,0) \Rightarrow v(x,0)=\cos x-\pi x$

THE EIGENVALUES & EIGENFUNCTIONS ARE $\mu_n\in\{0\}\cup\{n\}_{n=1}^{\infty} \quad X_n\in\{1\}\cup\{\cos nx\}$

LET $S(x,t)=e^{-\beta t}\cos 3x = \frac{S_0(t)}{2}+\sum_{n=1}^{\infty} S_n(t)\cos nx \Rightarrow S_n(t)=e^{-\beta t}\delta_{n3}$

LET $v(x,t)=\frac{v_0(t)}{2}1+\sum_{n=1}^{\infty} v_n(t)\cos nx$ AND PLUG INTO THE PDE

$0 = v_t - v_{xx} - e^{-\beta t}\cos 3x = \frac{\dot v_0}{2} + \sum_{n=1}^{\infty}\left\{\dot v_n + n^2 v_n - e^{-\beta t}\delta_{n3}\right\}\cos(nx)$

SINCE $\{1,\cos nx\}$ ARE LINEARLY INDEPENDENT $\boxed{\dot v_0=0 \Rightarrow v_0=C_0}$ & $\boxed{\dot v_n+n^2v_n=e^{-\beta t}\delta_{n3} \quad n\geq 1}$

APPLYING THE INTEGRATING FACTOR $F=e^{n^2t} \Rightarrow \frac{d}{dt}\left[e^{n^2t}v_n\right]=e^{(n^2-\beta)t}\delta_{n3}$

$\therefore\ e^{n^2t}v_n = \frac{e^{(n^2-\beta)t}}{n^2-\beta}\delta_{n3}+C_n \Rightarrow v_n(t)=\delta_{n3}\frac{e^{-\beta t}}{n^2-\beta}+C_ne^{-n^2t}$

$\therefore\ v(x,t)=\frac{C_0}{2}+\sum_{n=1}^{\infty}\left[\frac{\delta_{n3}e^{-\beta t}}{n^2-\beta}+C_ne^{-n^2t}\right]\cos nx$

NOW $\cos x-\pi x = v(x,0)=\frac{C_0}{2}+\sum_{n=1}^{\infty}\left[\frac{\delta_{n3}}{n^2-\beta}+C_n\right]\cos nx.$

$\therefore\ C_0=\frac{2}{\pi}\int_0^{\pi}(-\pi x)dx = -\frac{2x^2}{2}\Big|_0^{\pi} = -\pi^2$

AND $C_n+\frac{\delta_{n3}}{n^2-\beta}-\delta_{n1} = -\frac{2}{\pi}\int_0^{\pi}\pi x\cos nx\,dx = 2\frac{[1-(-1)^n]}{n^2}$ FROM THE GIVEN INTEGRAL

$\therefore\ C_n=\delta_{n1}-\frac{\delta_{n3}}{n^2-\beta}+2[1-(-1)^n]/n^2.$

$\therefore\ u(x,t)=x^2+\pi x+\left(\frac{-\pi^2}{2}\right)+\sum_{n=1}^{\infty}\left\{\frac{\delta_{n3}e^{-\beta t}}{n^2-\beta}+\left(\delta_{n1}-\frac{\delta_{n3}}{n^2-\beta}+\frac{2[1-(-1)^n]}{n^2}\right)e^{-n^2t}\right\}\cos nx$

$= x^2+\pi x-\frac{\pi^2}{2}+\frac{(e^{-\beta t}-e^{-9t})}{(9-\beta)}\cos 3x+e^{-t}\cos x+2\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^2}e^{-n^2t}\cos nx$

Q2. PDE: $u_{tt} = u_{xx} - \gamma u \qquad 0 < x < \pi,\ t > 0$

BC: $u(0,t) = 0 = u(\pi, t)$

IC: $u(x,0) = \sin x \qquad u_t(x,0) = \sin 2x$

LET $u(x,t) = X(x)T(t) \Rightarrow u_{tt} = X(x)\ddot{T}(t) = X''(x)T(t) - \gamma X(x)T(t) = u_{xx} - \gamma u$

$\div XT$ AND REARRANGE: $\dfrac{\ddot{T}(t)}{T(t)} + \gamma = \dfrac{X''(x)}{X(x)} = \lambda = \text{CONST}$

X] $\lambda \geq 0$ YIELD TRIVIAL SOLUTIONS, SO LET $\lambda = -\mu^2 < 0 \Rightarrow X'' + \mu^2 X = 0 \quad X(0) = 0 = X(\pi)$ FOR WHICH THE EIGENVALUES ARE $\mu_n = n \quad n = 1, 2 \ldots$ AND EIGENFUNCTIONS ARE $X_n = \sin nx$.

T] $\ddot{T} + (\gamma + \mu^2)T = 0 \Rightarrow T(t) = A\cos\nu t + B\sin\nu t$ WHERE $\nu_n = \sqrt{\gamma + \mu_n^2} = \sqrt{\gamma + n^2}$

$$\therefore \quad u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\nu_n t + B_n \sin\nu_n t\right] \sin nx$$

$$u_t(x,t) = \sum_{n=1}^{\infty} \left[-A_n \nu_n \sin\nu_n t + B_n \nu_n \cos\nu_n t\right] \sin nx$$

$$\sin x = u(x,0) = \sum_{n=1}^{\infty} A_n \sin nx \qquad A_n = \delta_{n1}$$

$$\sin 2x = u_t(x,0) = \sum_{n=1}^{\infty} B_n \nu_n \sin nx \qquad B_n \nu_n = \delta_{n2} \qquad B_n = \delta_{n2}/\nu_n.$$

$$\therefore \quad u(x,t) = \sum_{n=1}^{\infty} \left[\delta_{n1} \cos\nu_n t + \frac{\delta_{n2}}{\nu_n} \sin\nu_n t\right] \sin nx$$

$$= \cos\sqrt{\gamma+1}\,t \sin x + \frac{\sin\sqrt{\gamma+4}\,t}{\sqrt{\gamma+4}} \sin 2x$$

IF $\gamma = 0$ $\quad u(x,t) = \cos t \sin x + \dfrac{\sin 2t}{2} \sin 2x$

$$= \frac{1}{2}\left[\sin(x-t) + \sin(x+t)\right] + \frac{1}{2}\left[\cos 2(x-t) - \cos 2(x+t)\right]$$

BY D'ALEMBERT'S SOLUTION (SINCE $\gamma = 0$)

$$u(x,t) = \frac{1}{2}\left[\sin(x-t) + \sin(x+t)\right] + \frac{1}{2}\int_{x-t}^{x+t} \sin 2s\, ds$$

$$= \frac{1}{2}\left[\sin(x-t) + \sin(x+t)\right] + \frac{1}{2}\left[\cos 2(x-t) - \cos 2(x+t)\right] \quad \text{AS ABOVE}$$