SECTION 201 MIDTERM MATH 257/316

Q1 $\quad u_t = u_{xx} + x \qquad 0<x<1;\ t>0 \quad (1)$

$u(0,t) = 1-e^{-t} \qquad u_x(1,t) = t \quad (2)$

$u(x,0) = 0 \quad (3)$

LET $w(x,t) = (1-e^{-t}) + tx \qquad w_x(x,t) = t \qquad w(0,t) = 1-e^{-t} \qquad w_t = x + e^{-t}$

NOW LET $u(x,t) = w(x,t) + v(x,t)$

$$(1) \Rightarrow 0 = u_t - u_{xx} - x = w_t + v_t - \cancel{w_{xx}} - v_{xx} - x = v_t - v_{xx} + (x+e^{-t}) - x = v_t - v_{xx} + e^{-t}$$

$$\therefore \boxed{v_t = v_{xx} - e^{-t}} \quad (4)$$

$$\cancel{1-e^{-t}} = u(0,t) = w(0,t) + v(0,t) = \cancel{1-e^{-t}} + v(0,t) \Rightarrow \boxed{v(0,t) = 0} \quad (5)$$

$$\cancel{t} = u_x(1,t) = w_x(1,t) + v_x(1,t) = \cancel{t} + v_x(1,t) \Rightarrow \boxed{v_x(1,t) = 0} \quad (6)$$

$$0 = u(x,0) = w(x,0) + v(x,0) = 0 + v(x,0) \Rightarrow \boxed{v(x,0) = 0} \quad (7)$$

THE EIGENFUNCTIONS ASSOCIATED WITH THE HOMOG. BC (5)-(6) ARE THOSE OF THE BVP

$$\left.\begin{array}{l} X'' + \lambda^2 X = 0 \\ X(0) = 0 = X'(1) \end{array}\right\} \quad \lambda_n = \left(\frac{(2n+1)\pi}{2}\right) \quad n = 0,1,2 \qquad \text{AND } X_n = \sin(\lambda_n x)$$

EXPAND THE SOURCE ~~IN (4) IN TERMS OF EIGENFUNCTIONS~~ i.e.

$$S(x,t) = -e^{-t} = \sum_{n=0}^{\infty} \hat{S}_n(t)\sin(\lambda_n x) \Rightarrow \hat{S}_n(t) = -2e^{-t}\int_0^1 \sin(\lambda_n x)\,dx = \frac{-2e^{-t}(1-\cos\lambda_n)}{\lambda_n}$$

NOW LET $v(x,t) = \sum_{n=0}^{\infty} \hat{v}_n(t)\sin(\lambda_n x)$ SO THAT WHEN WE PLUG INTO (4):

$$0 = v_t - v_{xx} + e^{-t} = \sum_{n=0}^{\infty}\left\{\frac{d\hat{v}_n}{dt} + \lambda_n^2\hat{v}_n - \hat{S}_n(t)\right\}\sin\lambda_n x \Rightarrow \frac{d\hat{v}_n}{dt} + \lambda_n^2\hat{v}_n = \frac{-2e^{-t}(1-\cos\lambda_n)}{\lambda_n}$$

$$\therefore \frac{d}{dt}\left[e^{\lambda_n^2 t}\hat{v}_n\right] = \frac{2[\cos\lambda_n - 1]}{\lambda_n}e^{[\lambda_n^2-1]t} \Rightarrow e^{\lambda_n^2 t}\hat{v}_n = \frac{2[\cos\lambda_n - 1]}{\lambda_n}\frac{e^{(\lambda_n^2-1)t}}{(\lambda_n^2-1)} + C_n. \quad C_n \text{ ARBITR}$$

$$\therefore \hat{v}_n(t) = \frac{2(\cos\lambda_n - 1)}{\lambda_n(\lambda_n^2-1)}e^{-t} + C_n e^{-\lambda_n^2 t}$$

THUS

$$v(x,t) = \sum_{n=0}^{\infty}\left[\frac{2(\cos\lambda_n - 1)}{\lambda_n(\lambda_n^2-1)}e^{-t} + C_n e^{-\lambda_n^2 t}\right]\sin(\lambda_n x)$$

$$\text{IC }(7) \Rightarrow 0 = v(x,0) = \sum_{n=0}^{\infty}\left\{\frac{2(\cos\lambda_n - 1)}{\lambda_n(\lambda_n^2-1)} + C_n\right\}\sin\lambda_n x \Rightarrow C_n = -\frac{2(\cos\lambda_n - 1)}{\lambda_n(\lambda_n^2-1)}$$

THUS

$$u(x,t) = (1-e^{-t}) + tx + \sum_{n=0}^{\infty}\frac{2(\cos\lambda_n - 1)}{\lambda_n(\lambda_n^2-1)}\left\{e^{-t} - e^{-\lambda_n^2 t}\right\}\sin(\lambda_n x)$$

Q2: $u_{tt} = c^2 u_{xx}$

$u_x(0,t)=0 \quad u(1,t)=0$

$u(x,0)=x$

EIGENFUNCTONS ARE $X_n = \cos\lambda_n x$ & EIGENVECTORS $\lambda_n = \left(\frac{(2n+1)\pi}{2}\right), n=0,1,2..$

LET $u(x,t) = \sum_{n=0}^{\infty} \hat{u}_n(t)\cos(\lambda_n x)$

$$0 = u_{tt} - c^2 u_{xx} = \sum_{n=0}^{\infty}\left\{\frac{d^2\hat{u}_n(t)}{dt^2} + \lambda_n^2 c^2 \hat{u}_n(t)\right\}\cos(\lambda_n x)$$

$$\therefore \frac{d^2\hat{u}_n}{dt^2} + \lambda_n^2 c^2 \hat{u}_n = 0 \Rightarrow \hat{u}_n(t) = A_n\cos(\lambda_n ct) + B_n\sin(\lambda_n ct)$$

$$\therefore u(x,t) = \sum_{n=0}^{\infty}\{A_n\cos(\lambda_n ct) + B_n\sin(\lambda_n ct)\}\cos(\lambda_n x)$$

$$u_t(x,t) = \sum_{n=0}^{\infty}\{-A_n\lambda_n c\sin(\lambda_n ct) + \lambda_n c B_n\cos(\lambda_n ct)\}\cos(\lambda_n x)$$

$$x = u(x,0) = \sum_{n=0}^{\infty} A_n\cos(\lambda_n x) \Rightarrow A_n = 2\int_0^1 x\cos(\lambda_n x)\,dx = 2\left[\frac{x\sin\lambda_n x}{\lambda_n}\right]_0^1 - \frac{1}{\lambda_n}\int_0^1 \sin\lambda_n x\,dx$$

$$A_n = \frac{2\sin(\lambda_n)}{\lambda_n} + \frac{1}{\lambda_n^2}(\cos\lambda_n - 1)$$

$$0 = u_t(x,0) = \sum_{n=0}^{\infty}\lambda_n c B_n\cos(\lambda_n x) \Rightarrow B_n = 0$$

$$\therefore u(x,t) = \sum_{n=0}^{\infty} A_n\cos(\lambda_n ct)\cos(\lambda_n x)$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} A_n\{\cos\lambda_n(x+ct) + \cos\lambda_n(x-ct)\}$$

$$= \frac{1}{2}\{F(x+ct) + F(x-ct)\}$$ WHICH IS IN THE FORM OF D'ALEMBERT'S SOLUTION

WHERE $F(x) = \sum_{n=0}^{\infty} A_n\cos(\lambda_n x)$ $\quad A_n = \frac{2\sin\lambda_n}{\lambda_n} + \frac{1}{\lambda_n^2}(\cancel{\cdots}-1)$

$\cos\lambda_n(x+\Lambda) = \cos\lambda_n x$

$\lambda_n\Lambda = \left(\frac{2n+1}{2}\right)\pi\Lambda = 2\pi$

$\Lambda_n = \frac{4\pi}{2n+1}$

$\Lambda_0 = 4\pi$