

The University of British Columbia
Final Examination - December 17, 2012
Mathematics 257/316
All Sections

Closed book examination

Time: 2.5 hours

Last Name _____ **First** _____ **Signature** _____

Student Number _____

Special Instructions:

No books, notes, or calculators are allowed.

Rules governing examinations		
<ul style="list-style-type: none">● Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.● Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.● No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.● Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.● Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:<ul style="list-style-type: none">(a) speaking or communicating with other candidates, unless otherwise authorized;(b) purposely exposing written papers to the view of other candidates or imaging devices;(c) purposely viewing the written papers of other candidates;(d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,(e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)—(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).● Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.● Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.● Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).		

1		20
2		20
3		20
4		20
5		20
Total		100

1. Consider the differential equation

$$8x^2y'' + 2xy' + (1 + 2x)y = 0 \quad (1)$$

- (a) Classify the points $0 \leq x < \infty$ as ordinary points, regular singular points, or irregular singular points.
- (b) Find two values of r such that there are solutions of the form $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$.
- (c) Use the series expansion in (b) to determine two independent solutions of (1). You only need to calculate the first three non-zero terms in each case.

[20 marks]

(Question 1 Continued)

(Question 1 Continued)

2. Consider the following initial boundary value problem for the heat equation:

$$\begin{aligned}u_t &= u_{xx} - u, & 0 < x < 1, & \quad t > 0 \\u_x(0, t) &= 0, & u_x(1, t) &= 0 \\u(x, 0) &= x\end{aligned}\tag{2}$$

- (a) Determine the solution to the boundary value problem (2) by separation of variables.

[14 marks]

- (b) Briefly describe how you would use the method of finite differences to obtain an approximate solution this boundary value problem that is accurate to $O(\Delta x^2, \Delta t)$ terms. Use the notation $u_n^k \simeq u(x_n, t_k)$ to represent the nodal values on the finite difference mesh. Explain how you propose to approximate the boundary condition $u_x(0, t) = 1$ with $O(\Delta x^2)$ accuracy.

Hint: Taylor's expansion may prove useful: $f(x + \Delta x) = f(x) + \frac{f'(x)}{1!}\Delta x + \frac{f''(x)}{2!}\Delta x^2 + O(\Delta x^3)$.

[6 marks]

[total 20 marks]

(Question 2 Continued)

(Question 2 Continued)

3. The motion of a string subject to a gravitational load satisfies the following initial-boundary value problem:

$$u_{tt} = c^2 u_{xx} - g, \quad 0 < x < 1, \quad t > 0 \quad (3)$$

$$u(0, t) = u(1, t) = 0 \quad (4)$$

$$u(x, 0) = \sin(\pi x), \quad u_t(x, 0) = 0.$$

Here g is the acceleration due to gravity, which you may assume is constant.

- (a) Determine the static deflection of the string, which is determined by solving (3) in which it is assumed that $u_{tt} = 0$ subject to the boundary conditions (4). [8 marks]
- (b) Use the solution obtained in (a) to reduce the initial-boundary value problem to solving a homogeneous wave equation subject to homogeneous boundary conditions. Now use separation of variables to determine the solution to this boundary value problem and hence the complete solution of the entire initial-boundary value problem.

[12 marks]

HINT: The following integral may be useful:

$$\int_0^1 (x^2 - x) \sin(n\pi x) dx = 2 \frac{\cos n\pi - 1}{n^3 \pi^3}$$

[total 20 marks]

(Question 3 Continued)

(Question 3 Continued)

4. (a) Consider the eigenvalue problem

$$\begin{aligned}r^2 R'' + rR' + \lambda R &= 0 \\ R(1) &= 0 = R'(e^2)\end{aligned}$$

Reduce this problem to the form of a Sturm-Liouville eigenvalue problem. Determine the eigenvalues and corresponding eigenfunctions. [8 marks]

- (b) Use the eigenfunctions in (a) to solve the following mixed boundary value problem for the annular region:

$$\begin{aligned}u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} &= 0, \quad 1 < r < e^2, \quad 0 < \theta < \pi \\ u(r, 0) &= 0 \quad \text{and} \quad u(r, \pi) = f(r) \\ u(1, \theta) &= 0 \quad \text{and} \quad \frac{\partial}{\partial r}u(e^2, \theta) = 0\end{aligned}$$

[12 marks]
[total 20 marks]

(Question 4 Continued)

(Question 4 Continued)

5. Solve the inhomogeneous heat conduction problem:

$$\begin{aligned}u_t &= u_{xx} + xt, \quad 0 < x < 1, \quad t > 0 \\u_x(0, t) &= \frac{t^2}{2}, \quad \text{and } u(1, t) = 0 \\u(x, 0) &= 0.\end{aligned}$$

[20 marks]

(Question 5 Continued)

(Question 5 Continued)

(Additional Page)