1. Consider the differential equation

$$Ly = 2x^2y'' + xy' - (1+x)y = 0 \tag{1}$$

(a) Classify the points $0 \le x < \infty$ as ordinary points, regular singular points, or irregular singular points.

(b) Find two values of r such that there are solutions of the form $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$.

(c) Use the series expansion in (b) to determine two independent solutions of (1). You only need to calculate the first three non-zero terms in each case.

[20 marks]

(a) $0<x<\infty$ ARE ALL ORDINARY POINTS.

$x=0$ IS A SINGULAR POINT

TO FURTHER CLASSIFY $x=0$ CONSIDER: $\lim_{x\to 0} x\dfrac{x}{2x^2} = \dfrac{1}{2} = p_0$ $\lim_{x\to 0} \dfrac{x^2\cdot-(1+x)}{2x^2} = -\dfrac{1}{2} = q_0$

SINCE $|p_0|$ & $|q_0| < \infty$ $x=0$ IS A REGULAR SINGULAR POINT.

b) THE INDICIAL EQ IS: $0 = r(r-1) + p_0 r + q_0 = r^2 - \frac{1}{2}r - \frac{1}{2} \Rightarrow 2r^2 - r - 1 = (2r+1)(r-1) = 0$

THUS $r = -1/2$ AND $r = 1$

c) LET $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ $y' = \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1}$ $y'' = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2}$

$$0 = Ly = 2x^2y'' + xy' - y - xy$$

$$= \sum_{n=0}^{\infty} 2a_n(n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+1}$$

$m=n$, $m=n$, $m=n$, $m=n+1$, $n=m-1$, $n=0 \Rightarrow m=1$

$$= \sum_{m=0}^{\infty} \{(m+r)[2(m+r-1)+1]-1\} a_m x^{m+r} - \sum_{m=1}^{\infty} a_{m-1} x^{m+r}$$

$$= \{r[(2r-2)+1]-1\} a_0 x^r + \sum_{m=1}^{\infty} \Big(\{(m+r)[2(m+r)-1]-1\} a_m - a_{m-1}\Big) x^{m+r} = 0$$

SINCE THE x^{m+r} ARE LINEARLY INDEPENDENT FUNCTIONS THE COEFFICIENT OF EACH POWER VANISHES

$x^r]$ $2r^2 - r - 1 = (2r+1)(r-1) = 0 \Rightarrow r = -1/2$ OR $r = 1$ AS ABOVE.

$x^{m+r}]_{m\ge 1}$ $a_m = \dfrac{a_{m-1}}{(m+r)[2(m+r)-1]-1}$ RECURSION RELATION FOR a_m.

$r = -\frac{1}{2}$: $a_m = \dfrac{a_{m-1}}{(m-\frac{1}{2})2(m-1)-1} = \dfrac{a_{m-1}}{(2m-1)(m-1)-1} = \dfrac{a_m}{2m^2-3m+\cancel{1}-\cancel{1}} = \dfrac{a_m}{m(2m-3)}$

$\therefore$ $a_1 = \dfrac{a_0}{1(2-3)} = -a_0$ $a_2 = \dfrac{a_1}{2(4-3)} = \dfrac{-a_0}{2}$ ...

$$y_1(x) = x^{-1/2}\left[1 - x - \frac{x^2}{2} - \cdots\right]$$

$r = 1$: $a_m = \dfrac{a_{m-1}}{(m+1)(2m+1)-1} = \dfrac{a_{m-1}}{2m^2+3m+\cancel{1}-\cancel{1}} = \dfrac{a_{m-1}}{m(2m+3)}$

$\therefore$ $a_1 = \dfrac{a_0}{5}$, $a_2 = \dfrac{a_1}{2\cdot 7} = \dfrac{a_0}{70}$

$$\therefore\ y_2(x) = x^1\left[1 + \frac{x}{5} + \frac{x^2}{70} + \cdots\right]$$

2. Consider the following diffusion initial-boundary value problem

$$\begin{aligned} u_t &= \alpha^2 u_{xx},\ 0 < x < L,\ t > 0 \\ u(0,t) &= 0 = u_x(L,t) \\ u(x,0) &= 1 \end{aligned} \tag{2}$$

(a) Determine the solution to (2) by separation of variables.

[14 marks]

(b) Briefly describe how you would use the method of finite differences to obtain an approximate solution this boundary value problem that is accurate to $O(\Delta x^2, \Delta t)$ terms. Use the notation $u_n^k \simeq u(x_n, t_k)$ to represent the nodal values on the finite difference mesh. Explain how you propose to approximate the boundary condition $u_x(L,t) = 0$ with $O(\Delta x^2)$ accuracy.

Hint: Taylor's expansion may prove useful: $f(x+\Delta x) = f(x) + \frac{f'(x)}{1!}\Delta x + \frac{f''(x)}{2!}\Delta x^2 + O(\Delta x^3)$.

[6 marks]

[total 20 marks]

a) LET $u(x,t) = X(x)T(t) \Rightarrow \dfrac{\dot{T}(t)}{\alpha^2 T(t)} = \dfrac{X''(x)}{X(x)} = -\lambda^2$

T] $\dot{T} = -\alpha^2\lambda^2 T \Rightarrow T(t) = Ce^{-\alpha^2\lambda^2 t}$

X] $\lambda \neq 0$: $X'' + \lambda^2 X = 0 \Rightarrow X(x) = A\cos\lambda x + B\sin\lambda x \quad X' = -A\lambda\sin\lambda x + B\lambda\cos\lambda x$

$X(0) = 0 = X'(L) \quad X(0) = A = 0 \quad X'(L) = B\lambda\cos\lambda L = 0 \Rightarrow \lambda_n L = \dfrac{(2n+1)\pi}{2} \quad n = 0, 1, \dots$

$\therefore \lambda_n \ (2n+1)\pi/2L \quad X_n(x) = \sin\lambda_n x.$

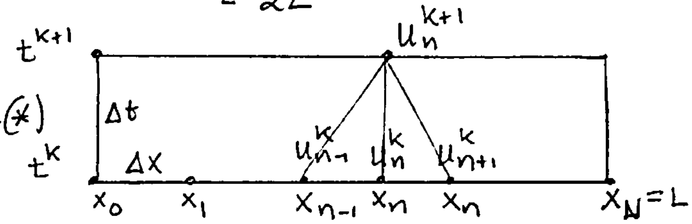
$\lambda = 0$: $X'' = 0 \Rightarrow X' = A \quad X = Ax + B \quad X(0) = B = 0 \quad X'(L) = A = 0$ ONLY TRIVIAL SOLN.

$u(x,t) = \sum_{n=0}^{\infty} b_n e^{-\alpha^2\lambda_n^2 t}\sin(\lambda_n x)$

IC: $1 = u(x,0) = \sum_{n=0}^{\infty} b_n \sin(\lambda_n x)$ WHERE $b_n = \frac{2}{L}\int_0^L 1 \sin\left(\frac{(2n+1)\pi x}{2L}\right)dx$

$\therefore b_n = \frac{2}{L}\left[\dfrac{-\cos\left(\frac{(2n+1)\pi x}{2L}\right)}{\lambda_n}\right]_0^L = \dfrac{2}{\lambda_n L}\cdot 1 = \dfrac{2}{\left(\frac{(2n+1)\pi}{2L}\right)L} = \dfrac{4}{\pi(2n+1)}$

$\therefore u(x,t) = \frac{4}{\pi}\sum_{n=0}^{\infty} \dfrac{e^{-\alpha^2\lambda_n^2 t}}{(2n+1)}\sin(\lambda_n x)$ WHERE $\lambda_n = \left[\frac{(2n+1)\pi}{2L}\right]$



b) LET $u_n^k \simeq u(x_n, t_k)$

$u_{n\pm1} = u_n \pm \Delta x u_n' + \frac{\Delta x^2}{2}u_n'' \pm \frac{\Delta x^3}{6}u_n^{(3)} + \frac{\Delta x^4}{24}u_n^{(4)} + \dots \ (*)$

ADD ⊕ & ⊖: $\dfrac{u_{n+1} - 2u_n + u_{n-1}}{\Delta x^2} = u_n'' + O(\Delta x^2)$

LET $x_n = n\Delta x \quad n = 0, \dots, N \quad \Delta x = L/N.$

$u_n^{k+1} = u_n^k + \Delta t\dot{u}_n^k + \frac{\Delta t^2}{2}\ddot{u}_n^k + \dots$

$\therefore \dfrac{u_n^{k+1} - u_n^k}{\Delta t} = \dot{u}_n^k + O(\Delta t).$

THUS SUBSTITUTING INTO THE PDE YIELDS: $\dfrac{u_n^{k+1} - u_n^k}{\Delta t} = \alpha^2\left(\dfrac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{\Delta x^2}\right) + O(\Delta x^2, \Delta t)$

INTRODUCING A FICTICIOUS MESH POINT $x_{N+1} = x_N + \Delta x$ AT WHICH WE ASSUME $u \simeq u_{N+1}$

SUBTRACTING ⊕ & ⊖ TERMS IN (*) YIELDS $\dfrac{u_{N+1}^k - u_{N-1}^k}{2\Delta x} = u_N' + O(\Delta x^2)$

SINCE $u_x(L,t) = 0$ THIS REDUCES TO $u_{N+1}^k = u_{N-1}^k$

3. The motion of a string on an elastic foundation with a stiffness γ satisfies the following initial-boundary value problem:

$$u_{tt} = u_{xx} - \gamma u,\ 0 < x < 1,\ t > 0 \quad (3)$$
$$u(0,t) = u(1,t) = 0 \quad (4)$$
$$u(x,0) = 0,\ u_t(x,0) = g(x)$$

(a) Solve (3) subject to the boundary conditions (4) using separation of variables.

[15 marks]

(b) For $\gamma = 7\pi^2$ and $g(x) = \sin 3\pi x$, sketch the solution at $t = 3/8$.

[5 marks]
[total 20 marks]

a) LET $u(x,t) = X(x)T(t) \Rightarrow X\ddot{T} = X''T - \gamma XT$

$\div XT]$ $\quad \frac{\ddot{T}}{T} + \gamma = \frac{X''}{X} = -\lambda^2$

$T]$ $\quad \ddot{T} + (\gamma+\lambda^2)T = 0 \Rightarrow T(t) = A\cos\mu t + B\sin\mu t$ WHERE $\mu = \sqrt{\gamma+\lambda^2}$

$X]$ $\quad \left.\begin{array}{l} X'' + \lambda^2 X = 0 \\ X(0) = 0 = X(1) \end{array}\right\} \Rightarrow X = A\cos\lambda x + B\sin\lambda x$

$X(0) = A = 0 \quad X(1) = B\sin\lambda = 0 \Rightarrow \lambda_n = n\pi \quad n = 1,2,\ldots;\ X_n = \sin(n\pi x)$

$\therefore u(x,t) = \sum_{n=1}^{\infty} [A_n\cos\mu_n t + B_n\sin\mu_n t]\sin(\lambda_n x)$

$0 = u(x,0) = \sum_{n=1}^{\infty} A_n \sin(\lambda_n x) \Rightarrow A_n = 0 \ \forall n.$

$u_t(x,t) = \sum_{n=1}^{\infty} B_n\mu_n\cos\mu_n t\,\sin(\lambda_n x)$

$g(x) = u_t(x,0) = \sum_{n=1}^{\infty} B_n\mu_n\sin\lambda_n x \Rightarrow \mu_n B_n = 2\int_0^1 g(x)\sin(n\pi x)\,dx.$

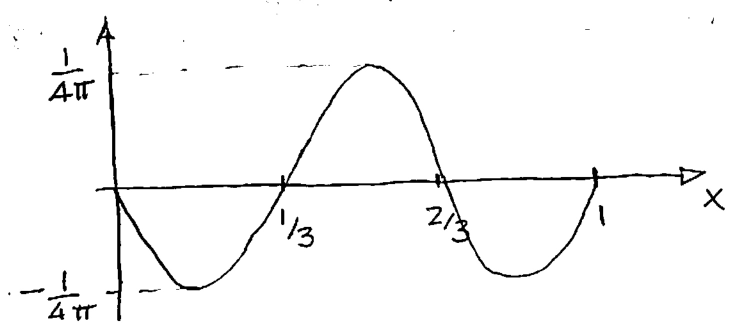
$\therefore u(x,t) = \sum_{n=1}^{\infty} B_n \sin\mu_n t\,\sin(\lambda_n x)$ WHERE $B_n = \frac{2}{\mu_n}\int_0^1 g(x)\sin(n\pi x)\,dx$

b) $\gamma = 7\pi^2 \quad g(x) = \sin 3\pi x \Rightarrow B_n = \frac{2}{\mu_n}\int_0^1 \sin(3\pi x)\sin(n\pi x)\,dx = \frac{2}{\mu_n}\delta_{n3}\frac{1}{2} = \frac{\delta_{n3}}{\mu_3}$

$\therefore u(x,t) = \frac{1}{\mu_3}\sin\mu_3 t\,\sin(3\pi x) \qquad \mu_3 = \sqrt{7\pi^2+9\pi^2} = 4\pi$

$= \frac{1}{4\pi}\sin(3\pi x)\sin(4\pi t)$

$u(x,\frac{3}{8}) = \frac{1}{4\pi}\sin(3\pi x)\sin\left(\frac{3\pi}{2}\right) = -\frac{1}{4\pi}\sin(3\pi x)$ HAS A WAVELENGTH OF 2/3

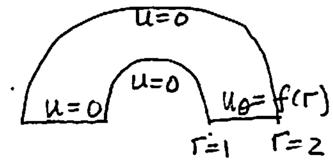


4. Consider the eigenvalue problem

$$x^2y'' + xy' + \lambda y = 0 \qquad (1)$$
$$y(1) = 0 = y(2)$$

(a) Reduce this problem to the form of a Sturm-Liouville eigenvalue problem. Determine the eigenvalues and corresponding eigenfunctions. [8 marks]

(b) Use the eigenfunctions in (a) to solve the following mixed boundary value problem for Laplace's equation on the semi-annular region:



$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 1 < r < 2, \quad 0 < \theta < \pi$$
$$u_\theta(r,0) = f(r) \quad \text{and} \quad u(r,\pi) = 0$$
$$u(1,\theta) = 0 \quad \text{and} \quad u(2,\theta) = 0$$

[12 marks]

[total 20 marks]

a) $F(x) = \frac{e^{\int \frac{x}{x^2}dx}}{x^2} = \frac{1}{x} \Rightarrow -xy'' - y' = -(xy')' = \frac{\lambda}{x}y$ WHICH IS IN SL FORM WITH WEIGHT $\frac{1}{x}$.

SINCE (1) IS A CAUCHY EULER EQ LET $y = x^\gamma \Rightarrow \gamma(\gamma-1)+\gamma+\lambda = \gamma^2+\lambda = 0 \Rightarrow \gamma = \pm i\sqrt{\lambda} = \pm i\mu$

$\lambda \neq 0$: ∴ $y(x) = A\cos\mu\ln x + B\sin\mu\ln x$ IS THE GENERAL SOL'N

BC ⇒ $y(1) = A = 0$ $\quad y(2) = B\sin\mu\ln 2 \Rightarrow \mu_n = \frac{n\pi}{\ln 2}$ $n=1,2,\ldots$ $y_n(x) = \sin\left(\frac{n\pi\ln x}{\ln 2}\right)$

b) SEPARATE VARIABLES $u(r,\theta) = R(r)\Theta(\theta) \Rightarrow \frac{r^2R''+rR'}{R(r)} = -\frac{\Theta''}{\Theta} = -\mu^2$ (SINCE HOMOG. BC IN r)

R] $r^2R'' + rR' + \mu^2R = 0$, $R(1) = 0 = R(2) \Rightarrow \mu_n = \frac{n\pi}{\ln 2}$ $n=1,2\ldots$ $R_n(r) = \sin(\mu_n\ln x)$

Θ] $\Theta'' - \mu^2\Theta = 0$ $\quad \Theta(\pi) = 0 \Rightarrow \Theta_n(\theta) = B_n\sinh\mu_n(\theta-\pi)$

$$\therefore u(r,\theta) = \sum_{n=1}^{\infty} B_n\sinh\mu_n(\theta-\pi)\sin(\mu_n\ln r)$$

$$u_\theta(r,\theta) = \sum_{n=1}^{\infty} B_n\mu_n\cosh\mu_n(\theta-\pi)\sin(\mu_n\ln r)$$

$$f(r) = u_\theta(r,0) = \sum_{n=1}^{\infty} B_n\mu_n\cosh(\mu_n\pi)\sin(\mu_n\ln r)$$

$$\int_1^2 \frac{1}{r}f(r)\sin(\mu_m\ln r)\,dr = \sum_{n=1}^{\infty} B_n\mu_n\cosh(\mu_n\pi)\int_1^2\frac{1}{r}\sin(\mu_m\ln r)\sin(\mu_n\ln r)\,dr$$

NOW $\int_1^2 \frac{1}{r}\sin(\mu_m\ln r)\sin(\mu_n\ln r)\,dr = \int_0^{\ln 2}\sin\left(\frac{n\pi x}{\ln 2}\right)\sin\left(\frac{m\pi x}{\ln 2}\right)dx = \frac{\ln 2}{2}\delta_{mn}$.

$x = \ln r$ $\quad dx = \frac{dr}{r}$

$r=1 \Rightarrow x=0$ $\quad r=2 \Rightarrow x = \ln 2$

$$\therefore \quad B_m\mu_m\cosh(\mu_m\pi).\ln 2/2 = \int_1^2 \frac{1}{r}f(r)\sin(\mu_m\ln r)\,dr.$$

OR $$B_m = \frac{2}{\mu_m\cosh(\mu_m\pi)\ln 2}\int_1^2\frac{1}{r}f(r)\sin(\mu_m\ln r)\,dr$$

AND $$u(r,\theta) = \sum_{n=1}^{\infty} B_n\sinh\mu_n(\theta-\pi)\sin(\mu_n\ln r)$$

5. **Solve the inhomogeneous heat conduction problem subject to time dependent boundary conditions:**

$$\begin{aligned} u_t &= u_{xx} + xt + 1, \ 0 < x < 1, \ t > 0 \\ u_x(0,t) &= 0, \text{ and } u(1,t) = t \\ u(x,0) &= 0. \end{aligned}$$

[20 marks]

FIND A SIMPLE FUNCTION $W(x,t) = A(t)x + B(t)$ THAT WE CONSTRUCT TO SATISFY THE BC: $W_x = A(t) = 0$ $W(1,t) = B(t) = t$ $\therefore W(x,t) = t$ DOES THE JOB

LET $u(x,t) = W(x,t) + V(x,t)$ THEN

$$u_t = (W+V)_t = \cancel{1} + V_t = u_{xx} + xt + 1 = (\cancel{W}+V)_{xx} + xt + \cancel{1} \Rightarrow \quad V_t = V_{xx} + xt$$

BC: $0 = u_x(0,t) = W_x(0,t) + V_x(0,t) = 0 + V_x(0,t) \quad \Rightarrow \quad V_x(0,t) = 0$

$\cancel{t} = u(1,t) = W(1,t) + V(1,t) = \cancel{t} + V(1,t) \quad \Rightarrow \quad V(1,t) = 0$

IC: $0 = u(x,0) = W(x,0) + V(x,0) = 0 + V(x,0) \quad \Rightarrow \quad V(x,0) = 0.$

SINCE V SATISFIES HOMOGENEOUS BC WE CAN USE AN EXPANSION IN THE EIGENFUNCTIONS OF V.

$X'' + \lambda^2 X = 0$ $X'(0) = 0 = X(1) \Rightarrow X = A\cos\lambda x + B\sin\lambda x$ $X' = -A\sin\lambda x + B\lambda\cos\lambda x$

$X'(0) = B\lambda = 0 \Rightarrow B = 0$, $X(1) = A\cos\lambda = 0 \Rightarrow \lambda_n = (2n+1)\pi/2$ $n = 0,1,2..$ $X_n = \cos\lambda_n x$

EXPAND SOURCE, LET $xt = \sum_{n=0}^{\infty} S_n(t)\cos\lambda_n x \Rightarrow S_n(t) = 2\int_0^1 (xt)\cos\lambda_n x\,dx$

$$\therefore S_n(t) = 2t\left[x\frac{\sin\lambda_n x}{\lambda_n}\Big|_{\cancel{0}}^1 - \frac{1}{\lambda_n}\int_0^1 1\sin\lambda_n x\,dx\right] = 2t\left[\frac{1}{\lambda_n}\sin\left[\left(\frac{2n+1}{2}\right)\pi\right] + \frac{\cos\lambda_n x}{\lambda_n^2}\Big|_0^1\right] = 2t\left[\frac{(-1)^n}{\lambda_n} - \frac{1}{\lambda_n^2}\right] = \sigma_n t$$

NOW LET $V(x,t) = \sum_{n=0}^{\infty} V_n(t)\cos\lambda_n x$ $\quad V_t = \sum_{n=1}^{\infty} \dot{V}_n \cos\lambda_n x$ $\quad V_{xx} = \sum_{n=1}^{\infty} (-\lambda_n^2)V_n\cos\lambda_n x$

$$\therefore V_t - V_{xx} - xt = \sum_{n=0}^{\infty}\left\{\frac{dV_n}{dt} + \lambda_n^2 V_n - \sigma_n t\right\}\cos(\lambda_n x) = 0 \Rightarrow \frac{dV_n}{dt} + \lambda_n^2 V_n = \sigma_n t.$$

$$\therefore \frac{d}{dt}\left\{e^{\lambda_n^2 t} V_n\right\} = \sigma_n t e^{\lambda_n^2 t} \Rightarrow e^{\lambda_n^2 t} V_n = \sigma_n \int_0^t \tau e^{\lambda_n^2 \tau}\,d\tau + c_n$$

$$\text{NOW} \quad \int_0^t \tau e^{\lambda_n^2 \tau}\,d\tau = \tau\frac{e^{\lambda_n^2 \tau}}{\lambda_n^2}\Big|_{\cancel{0}}^t - \frac{1}{\lambda_n^2}\int_0^t 1 e^{\lambda_n^2 \tau}\,d\tau = t\frac{e^{\lambda_n^2 t}}{\lambda_n^2} - \frac{1}{\lambda_n^4}\left(e^{\lambda_n^2 t} - 1\right)$$

$$\therefore \quad V_n(t) = \sigma_n\left[\frac{t}{\lambda_n^2} - \frac{1}{\lambda_n^4}\left(1 - e^{-\lambda_n^2 t}\right)\right] + c_n. \qquad V_n(0) = c_n.$$

$$\text{IC: } 0 = V(x,0) = \sum_{n=0}^{\infty} V_n(0)\cos\lambda_n x \Rightarrow V_n(0) = c_n = 0.$$

$$\therefore \quad u(x,t) = t + \sum_{n=0}^{\infty}\sigma_n\left[\frac{t}{\lambda_n^2} - \frac{1}{\lambda_n^4}\left(1 - e^{-\lambda_n^2 t}\right)\right]\cos(\lambda_n x)$$

WHERE $\sigma_n = 2\left[\frac{(-1)^n}{\lambda_n} - \frac{1}{\lambda_n^2}\right]$ AND $\lambda_n = \frac{(2n+1)\pi}{2}$ $\quad n = 0,1,\ldots$