1. Consider the differential equation

$$Ly = 6x^2y'' + 5xy' - (1+x)y = 0 \tag{1}$$

(a) Classify the points $0 \le x < \infty$ as ordinary points, regular singular points, or irregular singular points.

(b) Find two values of r such that there are solutions of the form $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$.

(c) Use the series expansion in (b) to determine two independent solutions of (1). You only need to calculate the first three non-zero terms in each case.

[20 marks]

a) $0 < x < \infty$ ARE ORDINARY POINTS & $x=0$ IS A SINGULAR POINT

$\lim_{x\to 0} \frac{x(5x)}{6x^2} = \frac{5}{6} = p_0$ $\quad \lim_{x\to 0} \frac{x^2\{-(1+x)\}}{6x^2} = -\frac{1}{6} = q_0$ $\quad |p_0| \& |q_0| < \infty \Rightarrow x=0$ IS A REGULAR SINGULAR PT.

b) INDICIAL EQ $\quad r(r-1) + \frac{5}{6}r - \frac{1}{6} = 0 \quad 6r^2 - r - 1 = (3r+1)(2r-1) = 0 \quad r = -\frac{1}{3}, \frac{1}{2}$

c) $y = \sum_{n=0}^{\infty} a_n x^{n+r}$, $\quad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$, $\quad y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$

$$Ly = \overbrace{6x^2y''}^{} \; \overbrace{+5xy'}^{} \; \overbrace{-y}^{} \; \overbrace{-xy}^{} = 0$$

$$= \sum_{n=0}^{\infty} 6(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty} 5(n+r)a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

$\boxed{m=n}$ $\quad \boxed{m=n}$ $\quad \boxed{m=n}$ $\quad \boxed{m=n+1,\ n=m-1,\ n=0 \Rightarrow m=1}$

$$= \sum_{m=0}^{\infty} [(m+r)\{6(m+r-1)+5\} - 1] a_m x^{m+r} - \sum_{m=1}^{\infty} a_{m-1} x^{m+r} = 0$$

$$= \{r(6r-1) - 1\} a_0 x^r + \sum_{m=1}^{\infty} \{[(m+r)\{6(m+r)-1\} - 1] a_m - a_{m-1}\} x^{m+r} = 0$$

$x^r]$ $\quad 6r^2 - r - 1 = (3r+1)(2r-1) = 0 \quad r = -\frac{1}{3}, \frac{1}{2}$ (AS ABOVE)

$x^{m+r}, m \geq 1]$ $\quad a_m = \dfrac{a_{m-1}}{(m+r)\{6(m+r)-1\} - 1}$

$\underline{r = -\frac{1}{3}}$: $\quad a_m = \dfrac{a_{m-1}}{(m-\frac{1}{3})(6m-3) - 1} = \dfrac{a_{m-1}}{(3m-1)(2m-1) - 1} = \dfrac{a_{m-1}}{6m^2 - 5m + \cancel{1} - \cancel{1}} = \dfrac{a_{m-1}}{m(6m-5)}$

$a_1 = \dfrac{a_0}{1.1} = a_0 \qquad a_2 = \dfrac{a_1}{2.7} = \dfrac{a_0}{14}$

$y_1(x) = a_0 x^{-1/3}\left[1 + x + \frac{x^2}{14} + \cdots\right]$

$r = \frac{1}{2}$: $\quad a_m = \dfrac{a_{m-1}}{(m+\frac{1}{2})(6m+2) - 1} = \dfrac{a_{m-1}}{(2m+1)(3m+1) - 1} = \dfrac{a_{m-1}}{6m^2 + 5m + \cancel{1} - \cancel{1}} = \dfrac{a_{m-1}}{m(6m+5)}$

$a_1 = \dfrac{a_0}{1.11} = \dfrac{a_0}{11} \qquad a_2 = \dfrac{a_1}{2.17} = \dfrac{a_0}{22.17}$

$y_2(x) = a_0 x^{1/2}\left[1 + \frac{x}{11} + \frac{x^2}{22.17} + \cdots\right]$

2. Consider the following diffusion initial-boundary value problem

$$\begin{aligned} u_t &= u_{xx},\ 0 < x < \pi/2,\ t > 0 \\ u_x(0,t) &= 0 = u(\pi/2,t) \\ u(x,0) &= x \end{aligned} \tag{2}$$

(a) Determine the solution to (2) by separation of variables.

[14 marks]

(b) Briefly describe how you would use the method of finite differences to obtain an approximate solution this boundary value problem that is accurate to $O(\Delta x^2, \Delta t)$ terms. Use the notation $u_n^k \simeq u(x_n, t_k)$ to represent the nodal values on the finite difference mesh. Explain how you propose to approximate the boundary condition $u_x(0,t) = 0$ with $O(\Delta x^2)$ accuracy.

Hint: Taylor's expansion may prove useful: $f(x+\Delta x) = f(x) + \frac{f'(x)}{1!}\Delta x + \frac{f''(x)}{2!}\Delta x^2 + O(\Delta x^3)$. $(*)$

[6 marks]

[total 20 marks]

a) LET $u(x,t) = X(x)T(t)$ AND SUBSTITUTE INTO THE HEAT EQ

$$X\dot{T} = X''T$$

$\div XT]$ $\quad \dfrac{\dot{T}}{T(t)} = \dfrac{X''}{X(x)} = \lambda = -\mu^2$ CONSTANT

$T]$ $\quad \dot{T} = -\mu^2 T \Rightarrow T(t) = Ce^{-\mu^2 t}$

$X]$ $\quad \left.\begin{array}{l} X'' + \mu^2 X = 0 \\ X'(0) = 0 = X(0) \end{array}\right\}$ $\mu_n = \dfrac{(2n+1)\pi}{2(\pi/2)} = (2n+1)$ $n = 0, 1, \ldots$ $\quad X_n = \cos(\mu_n x)$ $n = 0, 1, \ldots$

$$u(x,t) = \sum_{n=0}^{\infty} a_n e^{-\mu_n^2 t} \cos(2n+1)x$$

$$x = u(x,0) = \sum_{n=0}^{\infty} a_n 1 \cos(2n+1)x$$

$$a_n = \frac{2}{\pi/2}\int_0^{\pi/2} x\cos(2n+1)x\,dx = \frac{4}{\pi}\left[x\frac{\sin(2n+1)x}{(2n+1)}\Big|_0^{\pi/2} - \frac{1}{(2n+1)}\int_0^{\pi/2} 1.\sin(2n+1)x\,dx\right]$$

$$= \frac{4}{\pi}\left[\frac{\pi}{2}\frac{(-1)^n}{(2n+1)} + \frac{\cos(2n+1)x}{(2n+1)^2}\Big|_0^{\pi/2}\right] = \frac{2(-1)^n}{(2n+1)} - \frac{4}{\pi}\frac{1}{(2n+1)^2}$$

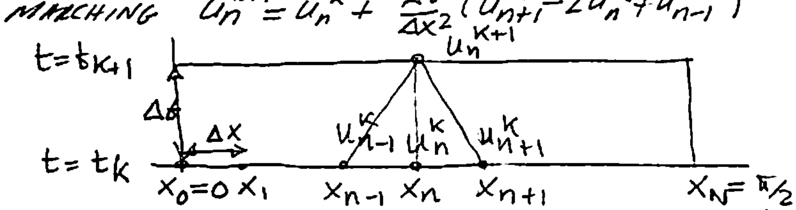
$$\therefore\ u(x,t) = \sum_{n=0}^{\infty}\left\{\frac{2(-1)^n}{(2n+1)} - \frac{4}{\pi}\frac{1}{(2n+1)^2}\right\} e^{-(2n+1)^2 t}\cos(2n+1)x.$$

b) FROM $(*)$ $\quad f'(x) = \dfrac{f(x+\Delta x) - f(x)}{\Delta x} + O(\Delta x) \Rightarrow \dfrac{\partial u}{\partial t}(x_n, t_k) = \dfrac{u_n^{k+1} - u_n^k}{\Delta t} + O(\Delta t)$

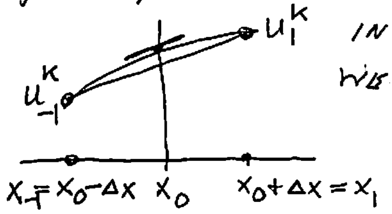
ALSO FROM $(*)$ $\quad f''(x) = \dfrac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{\Delta x^2} + O(\Delta x^2) \Rightarrow \dfrac{\partial^2 u}{\partial x^2}(x_n, t_k) = \dfrac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{\Delta x^2} + O(\Delta x^2)$

THUS SUBSTITUTING INTO THE PDE: $\quad \dfrac{u_n^{k+1} - u_n^k}{\Delta t} = \dfrac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{\Delta x^2} + O(\Delta x^2, \Delta t)$

OR FOR TIME MARCHING $\quad u_n^{k+1} = u_n^k + \dfrac{\Delta t}{\Delta x^2}(u_{n+1}^k - 2u_n^k + u_{n-1}^k)$



...ING $(*)$ WITH $f(x - \Delta x)$ AND SUBTRACTING WE OBTAIN $\quad f'(x) = \dfrac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} + O(\Delta x^2)$



INTRODUCING A FALSE MESHPOINT AT $x_0 - \Delta x$.

WE HAVE $0 = \dfrac{\partial u}{\partial x}(0,t) = \dfrac{u_1^k - u_{-1}^k}{2\Delta x} + O(\Delta x^2)$

FROM WHICH IT FOLLOWS THAT $\boxed{u_{-1}^k = u_1^k}$

3. Solve the following inhomogeneous initial boundary value problem for the wave equation:

$$(1)\qquad u_{tt} = c^2u_{xx} + e^{-t}\sin(3x) + 1,\ 0 < x < \frac{\pi}{2},\ t > 0$$

$$u(0,t) = t^2/2 \text{ and } u_x(\frac{\pi}{2},t) = t,\ \ t > 0$$

$$u(x,0) = 0,\quad u_t(x,0) = \sin(5x) + x,\ \ 0 < x < \frac{\pi}{2}$$

[total 20 marks]

FIND A FUNCTION $W(x,t) = A(t)x + B(t)$ THAT SATISFIES THE NONZERO BC

$t^2/2 = W(0,t) = B(t) \quad t = W_x(\frac{\pi}{2},t) = A(t) \Rightarrow \boxed{W(x,t) = xt + t^2/2} \Rightarrow W_t = x+t$

$W_{tt} = 1$

NOW LET $u(x,t) = W(x,t) + V(x,t)$ THEN SUBSTITUTE INTO (1)

$(W_{tt} + V_{tt}) = (\cancel{1} + V_{tt}) = c^2(\cancel{W_{xx}} + V_{xx}) + e^{-t}\sin(3x) + \cancel{1} \Rightarrow V_{tt} = c^2V_{xx} + e^{-t}\sin(3x)$

BC: $\cancel{t^2/2} = u(0,t) = W(0,t) + V(0,t) = \cancel{t^2/2} + V(0,t) \Rightarrow V(0,t) = 0$

$\cancel{t} = u_x(\frac{\pi}{2},t) = W_x(\pi/2,t) + V_x(\pi/2,t) = \cancel{t} + V_x(\pi/2,t) \Rightarrow V_x(\frac{\pi}{2},t) = 0$ } BC (2)

IC: $0 = u(x,0) = W(x,0) + V(x,0) = 0 + V(x,0) \Rightarrow V(x,0) = 0$

$\sin 5x + \cancel{x} = u_t(x,0) = W_t(x,0) + V_t(x,0) = \cancel{x} + V_t(x,0) \Rightarrow V_t(x,0) = \sin 5x$

SINCE THE BVP(2) HAS HOMOG BC THE ASSOCIATED EIGENFUNCTIONS ARE $\mu_n = (2n+1)$, $X_n = \sin\mu_n x$

NOW USE THE EIGENFUNCTION EXPANSION: $n = 0,1,2,\ldots$

$$e^{-t}\sin(3x) = \sum_{n=0}^{\infty} S_n(t)\sin\mu_n x \Rightarrow S_n(t) = \delta_{n1}e^{-t}$$

AND ASSUME $V(x,t) = \sum_{n=0}^{\infty} V_n(t)\sin\mu_n x \quad V_{tt} = \sum_{n=0}^{\infty}\ddot{V}_n(t)\sin\mu_n x \quad V_{xx} = \sum_{n=0}^{\infty} V_n[-\mu_n^2\sin\mu_n x]$

$$\therefore 0 = V_{tt} - c^2V_{xx} - e^{-t}\sin(3x) = \sum_{n=0}^{\infty}\left\{\frac{d^2V_n}{dt^2} + \mu_n^2c^2V_n - e^{-t}\delta_{n1}\right\}\sin(\mu_n x)$$

SINCE THE $\sin\mu_n x$ ARE L.I. FUNCTIONS $\{\} = 0$: $\frac{d^2V_n}{dt^2} + \mu_n^2c^2V_n = e^{-t}\delta_{n1}$

THE SOL'N TO THE HOMOG EQ IS: $V_n^H = A_n\cos(\mu_n ct) + B_n\sin(\mu_n ct)$

FOR A PARTICULAR SOL'N $V_n^P = De^{-t} \Rightarrow \ddot{V}_n^P + \mu_n^2c^2V_n^P = D(1+\mu_n^2c^2)e^{-t} = e^{-t}\delta_{n1} \Rightarrow D = \frac{\delta_{n1}}{1+\mu_n^2c^2}$

USING $V_n^H + V_n^P$ TO GET THE GENERAL SOL'N WE OBTAIN

$$V(x,t) = \sum_{n=0}^{\infty}\left\{A_n\cos\mu_n ct + B_n\sin\mu_n ct + \frac{\delta_{n1}e^{-t}}{1+\mu_n^2c^2}\right\}\sin(2n+1)x$$

IMPOSE IC: $0 = V(x,0) = \sum_{n=0}^{\infty}\left(A_n + \frac{\delta_{n1}}{1+\mu_n^2c^2}\right)\sin(2n+1)x \Rightarrow A_n = -\frac{\delta_{n1}}{1+\mu_n^2c^2}$

$$V_t = \sum_{n=0}^{\infty}\left[\mu_n c[-A_n\sin\mu_n ct + B_n\cos\mu_n ct] - \frac{\delta_{n1}e^{-t}}{1+\mu_n^2c^2}\right]\sin(2n+1)x$$

$$\sin 5x = \sum_{n=0}^{\infty}\left(\mu_n cB_n - \frac{\delta_{n1}}{1+\mu_n^2c^2}\right)\sin(2n+1)x \Rightarrow B_n = \frac{\delta_{n2}}{\mu_n c} + \frac{\delta_{n1}}{\mu_n c[1+\mu_n^2c^2]}$$

$\therefore u(x,t) = W(x,t) + V(x,t)$

$$= xt + t^2/2 + \sum_{n=0}^{\infty}\left\{\frac{-\delta_{n1}}{(1+\mu_n^2c^2)}\cos\mu_n ct + \left(\frac{\delta_{n2}}{\mu_n c} + \frac{\delta_{n1}}{\mu_n c[1+\mu_n^2c^2]}\right)\sin\mu_n ct + \frac{\delta_{n1}e^{-t}}{1+\mu_n^2c^2}\right\}\sin(2n+1)x$$

$$= xt + \frac{x^2}{2} - \frac{\cos 3ct}{(1+9c^2)}\sin 3x + \frac{\sin 5ct\sin 5x}{5c} + \frac{\sin 3ct\sin 3x}{3c(1+9c^2)}$$

$$+\left(\frac{e^{-t}}{1+9c^2}\right)\sin 3x$$

4. Consider the eigenvalue problem

$$x^2y'' + xy' + \lambda y = 0 \qquad (1)$$
$$y(1) = 0 = y'(2)$$

(a) Reduce this problem to the form of a Sturm-Liouville eigenvalue problem. Determine the eigenvalues and corresponding eigenfunctions. [8 marks]

(b) Use the eigenfunctions in (a) to solve the following mixed boundary value problem for Laplace's equation on the quarter-annular region:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 1 < r < 2, \quad 0 < \theta < \pi/2$$
$$u(r,0) = 0 \text{ and } \frac{\partial u(r,\pi/2)}{\partial \theta} = f(r)$$
$$u(1,\theta) = 0 \text{ and } \frac{\partial u(2,\theta)}{\partial r} = 0$$

[12 marks]
[total 20 marks]

a) $W(x) = \dfrac{e^{\int \frac{x}{x^2}dx}}{x^2} = \dfrac{e^{\ln x}}{x^2} = \dfrac{1}{x}$ SO MULTIPLY (1) BY $W(x)$ TO OBTAIN

$\mathcal{L}y = -(xy')' = \lambda \frac{1}{x} y$ WHICH IS IN S-L FORM.

LET $y = x^\gamma \Rightarrow \gamma(\gamma-1) + \gamma + \lambda = \gamma^2 + \lambda = \gamma^2 + \mu^2 = 0 \Rightarrow \gamma = \pm i\mu$ WHERE $\lambda = \mu^2$

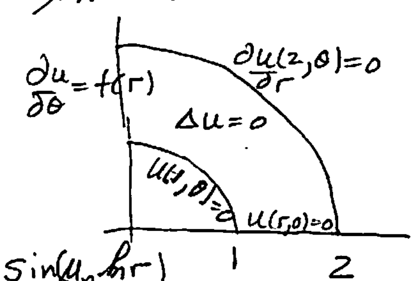
$\therefore y(x) = A\cos(\mu \ln x) + B\sin(\mu \ln x)$ $\quad y' = -A\mu(\sin\mu\ln x)\frac{1}{x} + B\mu(\cos\mu\ln x)\frac{1}{x}$

$0 = y(1) = A$ $\quad 0 = y'(2) = B\frac{\mu}{2}\cos(\mu\ln 2) \Rightarrow \mu_n = \dfrac{(2n+1)\pi}{2\ln 2}$ $n = 0, 1, \ldots$ ARE EIGENVALUES

& $y_n(x) = \sin(\mu_n \ln x)$ ARE THE EIGENFCNS.

b) LET $u(r,\theta) = R(r)\Theta(\theta)$ THEN

$\dfrac{r^2R'' + rR'}{R(r)} = -\dfrac{\Theta''}{\Theta(\theta)} = -\lambda = -\mu^2$ CONSTANT



$\Theta]$ $\left.\begin{array}{l}\Theta'' - \mu^2\Theta = 0 \\ \Theta(0) = 0\end{array}\right\}$ $\Theta = A\cosh\mu\theta + B\sinh\mu\theta$
$0 = \Theta(0) = A \Rightarrow \Theta = B\sinh\mu\theta$.

$R]$ $\left.\begin{array}{l}r^2R'' + rR' + \mu^2 R = 0 \\ R(1) = 0 = R'(2)\end{array}\right\}$ $\Rightarrow \mu_n = \dfrac{(2n+1)\pi}{2\ln 2}$ $n = 0, 1, \ldots$ $\quad R_n = \sin(\mu_n \ln r)$

$\therefore u(r,\theta) = \sum_{n=0}^{\infty} B_n \sinh\mu_n\theta \, \sin(\mu_n \ln r)$

$\dfrac{\partial u}{\partial \theta} = \sum_{n=0}^{\infty} B_n\mu_n\cosh(\mu_n\theta)\sin\mu_n\ln r$

LAST BC: $f(r) = \dfrac{\partial u}{\partial \theta}(r,\pi/2) = \sum_{n=0}^{\infty} \underbrace{B_n\mu_n\cosh(\mu_n\pi/2)}_{d_n}\sin(\mu_n \ln r) = \sum_{n=0}^{\infty} d_n \sin(\mu_n \ln r)$

THEN $\int_1^2 \frac{1}{r} f(r)\sin(\mu_m \ln r)\,dr = \sum_{n=0}^{\infty} d_n \int_1^2 \frac{1}{r}\sin(\mu_m \ln r)\sin(\mu_n \ln r)\,dr$

let $x = \ln r$ $\quad dx = \frac{dr}{r}$ $\quad = \sum_{n=0}^{\infty} d_n \underbrace{\int_0^{\ln 2} \sin(\mu_m x)\sin(\mu_n x)\,dx}_{\delta_{mn}\ln 2/2}$

$= d_m \ln 2/2$.

$\therefore d_m = B_m\mu_m\cosh(\mu_m\pi/2) = \dfrac{2}{\ln 2}\int_1^2 \frac{1}{r} f(r)\sin(\mu_m \ln r)\,dr$

$\therefore B_m = \dfrac{2}{\ln 2\,\mu_m\cosh(\mu_m\pi/2)}\int_1^2 \frac{1}{r} f(r)\sin(\mu_m \ln r)\,dr$

5. Solve the inhomogeneous heat conduction problem with heat loss, a time dependent source, and subject to time dependent boundary conditions:

$$\begin{aligned} u_t &= u_{xx} - u + e^{-t}\sin(x),\ 0 < x < \frac{\pi}{2}, t > 0 \qquad (1) \\ u(0,t) &= 0, \text{ and } \frac{\partial u(\pi/2,t)}{\partial x} = e^{-t} \\ u(x,0) &= x. \end{aligned}$$

[20 marks]

LET $W(x,t) = A(t)x + B(t)$ MATCH THE BC.

$0 = W(0,t) = B(t) \quad W_x = A(t) \quad W_x(\pi/2,t) = A(t) = e^{-t} \quad \boxed{W = xe^{-t}} \quad W_t = -xe^{-t}$

NOW LET $u(x,t) = W(x,t) + V(x,t)$ AND SUBSTITUTE INTO (1)

$u_t = W_t + V_t = -\cancel{x}e^{-t} + V_t = (\cancel{W_{xx}} + V_{xx}) - (\cancel{x}e^{-t} + V) + e^{-t}\sin x \Rightarrow V_t = V_{xx} - V + e^{-t}\sin x$

BC: $0 = u(0,t) = W(0,t) + V(0,t) = 0 + V(0,t) \Rightarrow V(0,t) = 0$

$\cancel{e^{-t}} = u_x(\pi/2,t) = W_x(\pi/2,t) + V_x(\pi/2,t) = \cancel{e^{-t}} + V_x(\pi/2,t) \Rightarrow V_x(\pi/2,t) = 0$

IC: $\cancel{x} = u(x,0) = W(x,0) + V(x,0) = \cancel{x} + V(x,0) \Rightarrow V(x,0) = 0$ (2)

SINCE V SATISFIES HOMOGENEOUS BC ASSOCIATED WITH THE EIGENVALUES & EIGENFCNS

OF $\left.\begin{matrix} X'' + \mu^2 X = 0 \\ X(0) = 0 = X'(\pi/2) \end{matrix}\right\}$ $\mu_n = (2n+1)\ n = 0, 1, 2.\ X_n = \sin(2n+1)x.$

EXPAND THE SOURCE IN TERMS OF THE EIGENFUNCTIONS:

$e^{-t}\sin x = \sum_{n=0}^{\infty} S_n(t)\sin(2n+1)x \Rightarrow S_n(t) = \delta_{n0}e^{-t}$

NOW LET $V(x,t) = \sum_{n=0}^{\infty} V_n(t)\sin\mu_n x \quad V_t = \sum_{n=0} \dot{V}_n \sin\mu_n x \quad V_{xx} = \sum_{n=0} V_n\{-\mu_n^2 \sin\mu_n x\}$

$\therefore 0 = V_t - V_{xx} + V - e^{-t}\sin x = \sum_{n=0}^{\infty}\left\{\frac{dV_n}{dt} + (\mu_n^2+1)V_n - e^{-t}\delta_{n0}\right\}\sin\mu_n x$

$\sin\mu_n x$ L.I $\Rightarrow \frac{dV_n}{dt} + (1+\mu_n^2)V_n = e^{-t}\delta_{n0} \Rightarrow \frac{d}{dt}\left\{e^{+(1+\mu_n^2)t}V_n\right\} = e^{\mu_n^2 t}\delta_{n0}$

$\therefore e^{(1+\mu_n^2)t}V_n = \int_0^t e^{\mu_n^2\tau}\delta_{n0}\,d\tau + C_n = \left(\frac{e^{\mu_n^2 t} - 1}{\mu_n^2}\right)\delta_{n0} + C_n.$

$\therefore V_n(t) = \left(\frac{e^{-t} - e^{-(1+\mu_n^2)t}}{\mu_n^2}\right)\delta_{n0} + C_n e^{-(1+\mu_n^2)t}; \quad V_n(0) = C_n$

NOW $0 = \sum_{n=0}^{\infty} V_n(0)\sin(\mu_n x) \Rightarrow V_n(0) = C_n = 0.$

$\therefore u(x,t) = xe^{-t} + \sum_{n=0}^{\infty}\left(\frac{e^{-t} - e^{-(1+\mu_n^2)t}}{\mu_n^2}\right)\delta_{n0}\sin\mu_n x \qquad \mu_0 = 1.$

$= xe^{-t} + \left(\frac{e^{-t} - e^{-2t}}{1}\right)\sin x$