

M406 MIDTERM SOLUTIONS

1/3

1. LET  $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

SINCE THE GAUSS-HERMITE RULE MUST INTEGRATE THIS FUNCTION EXACTLY

$$I = \int_{-\infty}^{\infty} f(a_0 + a_1 x + a_2 x^2 + a_3 x^3) e^{-x^2} dx = a_0 \sqrt{\pi} + a_2 \sqrt{\pi}/2 \quad (\text{USING THE HINT})$$

THE QUADRATURE RULE YIELDS

$$I = W_1 \{ f(-\xi_2) + f(\xi_2) \} \quad \text{USING THE SYMMETRY}$$

$$\begin{aligned} &= W_1 \{ a_0 - a_1 \xi_2 + a_2 \xi_2^2 - a_3 \xi_2^3 + a_0 + a_1 \xi_2 + a_2 \xi_2^2 + a_3 \xi_2^3 \} \\ &= 2W_1 a_0 + 2W_1 a_2 \xi_2^2 \\ &= a_0 \sqrt{\pi} + a_2 \sqrt{\pi}/2. \end{aligned}$$

EQUATING COEFFICIENTS OF  $a_0$  &  $a_2$ , WHICH ARE ARBITRARILY, WE HAVE  $2W_1 = \sqrt{\pi}$  AND  $\xi_2^2 = 1/2$   $\xi_2 = \pm \frac{1}{\sqrt{2}}$   $W_1 = \sqrt{\pi}/2$ .

2.  $Lu = u'' + \mu^2 u = f$  } (1)  
 $u'(0) = 0 = u'(1)$

$$(a) (V, Lu) = \int_0^1 V \{ u'' + \mu^2 u \} dx,$$

$$= V u' \Big|_0^1 - V' u \Big|_0^1 + \int_0^1 u (V'' + \mu^2 V) dx = V'(1)u(1) - V'(0)u(0) + \int_0^1 u V'' dx$$

SINCE  $u(0)$  AND  $u(1)$  ARE UNKNOWN WE REQUIRE  $V'(1) = 0 = V'(0)$

SO THAT THE OPERATOR  $L + \partial C$  (1) IS ESSENTIALLY SELF ADJOINT.

THE GREEN'S FUNCTION MUST SATISFY

$$L_S G(S, X) = \delta(S-X)$$

$$G(0, X) = 0 = G(1, X)$$

FOR THERE TO BE A SOLUTION WE REQUIRE  $\mu \neq n\pi$ ,  $n \in \mathbb{Z}$ . COMBINING SOLUTIONS TO THE HOMOGENEOUS EQUATION  $L_S V = 0$  WE ASSUME

$$G(S, X) = \begin{cases} A \cos \mu S & S < X \\ B \cos \mu(1-S) & S > X \end{cases} \quad G_S = \begin{cases} -A \mu \sin \mu S & S < X \\ B \mu \sin \mu(1-S) & S > X \end{cases}$$

CONTINUITY:  $G(X, X) = A \cos \mu X = B \cos \mu(1-X)$

JUMP:  $\int_{X-E}^{X+E} G_S + \mu^2 G dx = G_S \Big|_{X-E}^{X+E} + \int_{X-E}^{X+E} \mu^2 G dx = 1$

$$\therefore B \mu \sin \mu(1-X) + A \mu \sin \mu X = 1$$

$$\begin{bmatrix} -\cos \mu X & \cos \mu(1-X) \\ \mu \sin \mu X & \mu \sin \mu(1-X) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow A = -\cos \mu(1-X)/w(x)$$

$$B = -\cos \mu X / w(x)$$

WHERE  $w(x) = -\mu [\sin \mu(1-X) \cos \mu X + \cos \mu(1-X) \sin \mu X] = -\mu \sin \mu$

$$\therefore G(s, x) = \frac{1}{\mu \sin \mu x} \begin{cases} \cos \mu(1-x) \cos \mu s & s < x \\ \cos \mu x \cos \mu(1-s) & s > x. \end{cases}$$

$$u(x) = \int_0^x G(s, x) f(s) ds$$

(b) IF  $\mu$  IS AN EIGENVALUE OF THE HOMOGENEOUS PROBLEM

$$\begin{aligned} Lu = u'' + \mu^2 u = 0 & \quad \left. \begin{array}{l} \mu = n\pi \quad n=0, 1, 2, \dots \\ u'(0) = 0 = u'(1) \end{array} \right\} \\ u_n(x) &= \cos(n\pi x) \end{aligned}$$

FOR A SOLUTION TO EXIST AT ALL IF  $\mu = n\pi$

WE REQUIRE THAT  $\int_0^1 f(x) \cos(n\pi x) dx = 0$ ,

FOR THE SPECIAL CASE  $n=0$  WE REQUIRE  $\int_0^1 f(x) dx = 0$ .

$$(c) \text{ IF } \mu = 0 \quad \left. \begin{array}{l} u'' = 0 \\ u'(0) = 0 = u'(1) \end{array} \right\} \quad \phi = 1 \text{ IS A NONTRIVIAL SOLUTION}$$

WE LOOK FOR A SOLUTION TO THE PROBLEM

$$\int_0^s \tilde{G}(s, x) = S(s-x) + C\phi(x) \quad \phi(x) = 1$$

$$\tilde{G}_S(0, x) = 0 = \tilde{G}_S(1, x)$$

WHERE  $C$  IS CHOSEN SO THAT  $\int_0^1 \{S(s-x) + C\phi\} \phi dx = 0 \Rightarrow C = -1$

$$L_S \tilde{G} = \tilde{G}_{SS} = S(s-x) - 1$$

$$\tilde{G}_S = H(s-x) - S + A$$

$$\tilde{G} = (s-x)H(s-x) - S^2/2 + AS + B$$

$$\tilde{G}_S(0, x) = A = 0 \quad \tilde{G}_S(1) = 1 - 1 = 0$$

$$\tilde{G}(s, x) = B \cdot 1 + (s-x)H(s-x) - S^2/2$$

$$(d) \int_0^1 \{u'' + \mu^2 u\} v dx = 0$$

$$u'v' - \int_0^1 u'v - \mu^2 uv dx = 0$$

$$u'(1)v(1) - u'(0)v(0) - \int_0^1 u'v - \mu^2 uv dx = 0.$$

$\therefore$  FIND  $u \in H' = \{u : \int_0^1 (u')^2 dx < \infty\}$  such that

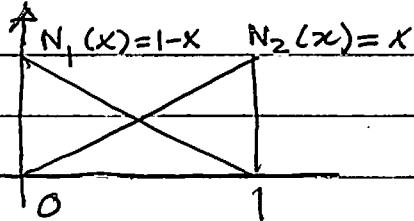
$$a(u, v) = \int_0^1 u'v' dx = \mu^2 \int_0^1 uv dx \quad \forall v \in H'$$

$$\text{LET } u^h(x) = u_1 N_1(x) + u_2 N_2(x)$$

$$= u_1(1-x) + u_2 x$$

$$N_1'(x) = -1 \quad N_2'(x) = 1 \quad N_i'(x) = (-1)^i$$

$$\text{LET } v_j^h \in \{N_1, N_2\}$$



THEN

$$a(u^h, N_i^h) = \sum_{j=1}^2 u_j \int_0^1 N_i' N_j' dx = \mu^2 \sum_{j=1}^2 u_j \int_0^1 N_i N_j dx$$

$$\text{OR } K u = \mu^2 M u$$

$$K_{ij} = \int_0^1 (-1)^i (-1)^j dx = (-1)^{i+j} \Rightarrow K = \begin{bmatrix} +1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$M_{ij} = \int_0^1 N_i N_j dx$$

$$M_{11} = \int_0^1 (1-x)^2 dx = \int_0^1 1-2x+x^2 dx = x - x^2 + \frac{x^3}{3} \Big|_0^1 = \frac{1}{3}$$

$$M_{21} = M_{12} = \int_0^1 (1-x)x dx = \int_0^1 x - x^2 dx = \frac{x^2}{2} - \frac{x^3}{3} \Big|_0^1 = \frac{3-2}{6} = \frac{1}{6}$$

$$M_{22} = \int_0^1 x^2 dx = \frac{x^3}{3} \Big|_0^1 = \frac{1}{3}$$

$\therefore$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{\mu^2}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

FOR A NONTRIVIAL SOLUTION TO

$$A u = \begin{bmatrix} 1 - 2\mu^2/6 & -1/\mu^2/6 \\ -1/\mu^2/6 & 1 - 2\mu^2/6 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0 \quad \text{WE REQUIRE } \det(A) = 0$$

$$\therefore \det(A) = (1 - \mu^2/3)^2 - (1 + \mu^2/6) = -2\mu^2/3 + \mu^4/9 - 1 - 2\mu^2/6 - \mu^4/36$$

$$\therefore -\mu^2(1 - \mu^2/12) = 0 \Rightarrow \mu^2 = 0 \text{ OR } \mu^2 = 12$$

WHICH ARE APPROXIMATIONS TO THE FIRST TWO EIGENVALUES  $\mu_1 = 0$  &  $\mu_2 = 11$