MATH 257/316 ASSIGNMENT 3 SOLUTIONS

Q1. #1 $Ly = -2(x-1)^2(x^2-2)^3 y'' + 2(x^2-1)y' - 4xy = 0$

$\therefore$ $x=1$ AND $x = \pm\sqrt{2}$ ARE SP

$x=1$: $\lim_{x\to 1} \frac{\cancel{(x-1)}\{2\cancel{(x-1)}(x+1)\}}{-2\cancel{(x-1)^2}(x^2-2)^3} = +2 = p_0 < \infty$, $\lim_{x\to 1} \frac{\cancel{(x-1)^2}\{-4x\}}{-2\cancel{(x-1)^2}(x^2-2)^3} = -2 = q_0 < \infty$ RSP

$\frac{L_0 x^r}{x^r} = r(r-1) + 2r - 2 = r^2 + r - 2 = (r+2)(r-1) = 0 \Rightarrow r = -2 \quad r = 1$

$\therefore$ EXPONENTS AT THE SINGULARITY ARE $r = 1$ & $r = -2$ $\{x^1 \,\&\, x^{-2}\}$

$x = +\sqrt{2}$ $\lim_{x\to\sqrt{2}} \frac{\cancel{(x-\sqrt{2})}\{2(x^2-1)\}}{-2(x-\sqrt{2})^{\cancel{3}\,2}(x+\sqrt{2})^3(x-1)^2} \to \infty$ $\therefore$ $x = \sqrt{2}$ IS AN IRREGULAR SINGULAR POINT

$x = -\sqrt{2}$ SIMILARLY IS AN IRREGULAR SINGULAR POINT.

#2: $Ly = 2\tan^2 x\, y'' - \sin x\, y' - y = 0$

$\sec^2 x = 1 + \tan^2 x$

$\tan x \sec^2 x = \tan x + \tan^3 x$

$\tan x = 0$ WHEN $x = 0; \pm\pi, \ldots = k\pi$

$x = k\pi$: $\lim_{x\to k\pi} \frac{(x-k\pi)(-\sin x)}{2\sin^2 x \sec^2 x} \overset{0/0}{=} \lim_{x\to k\pi} \frac{(-1)}{2\sec x + 4\tan^2 x} = \frac{(-1)^{k+1}}{2} = p_0 < \infty$

$\lim_{x\to k\pi} \frac{(x-k\pi)^2\{-1\}}{2\tan^2 x} \overset{0/0}{=} \lim_{x\to k\pi} \frac{-2(x-k\pi)}{4\tan x \sec^2 x} =$

$\overset{0/0}{=} \lim_{x\to k\pi} \frac{-2}{4\{\sec^2 x + 3\tan^2 x \sec^2 x\}} = -\frac{1}{2} = q_0 < \infty$ RSP

$\therefore$ $L_0 y = x^2 y'' + \frac{(-1)^k}{2} x y' + \frac{1}{2} y \Rightarrow r(r-1) + \frac{(-1)^{k+1}}{2} r + \frac{1}{2} = 0$

$2r^2 + [(-1)^{k+1} - 2]r + 1 = 0$

$k = 0, \pm 2, \ldots$ $2r^2 - 3r + 1 = (2r-1)(r-1) = 0$ $r = 1/2$ $r = 1$ EXPONENTS

$k = \pm 1, \pm 3, \ldots$ $2r^2 - r + 1 = (2r+1)(r-1) = 0$ $r = -1/2$ $r = 1$ EXPONENTS

Q2 $Ly = 4x^2y'' + 4x(x+1)y' - y = 0$

$\lim_{x\to 0} x \dfrac{4x(x+1)}{4x^2} = 1 = p_0 < \infty$ $\lim_{x\to 0} x^2 \dfrac{(-1)}{4x^2} = -\dfrac{1}{4} = q_0 < \infty$ $x=0$ RSP

$L_0 y = x^2y'' + xy' - \frac{1}{4}y = 0$ INDICIAL EQ.: $r(r-1) + r - \frac{1}{4} = 0 \Rightarrow r = \pm\frac{1}{2}$

FROBENIUS EXPANSION: $y = \sum_{n=0}^{\infty} c_n x^{n+r}$, $y' = \sum_{n=0}^{\infty} c_n(n+r)x^{n+r-1}$, $y'' = \sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r-2}$

$Ly = 4x^2y'' + 4x^2y' + 4xy' - y$

$= \sum_{n=0}^{\infty} 4c_n(n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} 4c_n(n+r)x^{n+r+1} + \sum_{n=0}^{\infty} 4c_n(n+r)x^{n+r} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0$

$\boxed{n=m}$ $n+1 \to n$ $\boxed{n+1=m}$ $n = m-1$ $n=0 \Rightarrow m=1$ $\boxed{n=m}$ $\boxed{n=m}$

$\therefore Ly = \sum_{m=0}^{\infty} c_m[4(m+r)(m+r-1) + 4(m+r) - 1]x^{m+r} + \sum_{m=1}^{\infty} 4c_{m-1}(m+r-1)x^{m+r} = 0$

$= c_0[4r(r-1) + 4r - 1]x^r$

$+ \sum_{m=1}^{\infty} \{c_m[4(m+r)\{(m+r-1)+1\} - 1] + 4c_{m-1}(m+r-1)\} x^{m+r} = 0$

$x^r >$ $4r^2 - 1 = 0$ $r = \pm 1/2$

$x^{m+r} >$ $c_m[4(m+r)^2 - 1] + 4c_{m-1}(m+r-1) = 0$ $m \geq 1$.

RECURSION $c_m = \dfrac{-4c_{m-1}(m+r-1)}{4(m+r)^2 - 1}$

WE ONLY CONSIDER THE LARGER ROOT $r = +1/2$

FOR $r = 1/2$: $c_m = \dfrac{-4c_{m-1}(m - 1/2)}{4(m+1/2)^2 - 1} = \dfrac{-4c_{m-1}(m-1/2)}{4m(m+1)} = \dfrac{-c_{m-1}(m-1/2)}{m(m+1)}$

$c_1 \overset{m=1}{=} -\dfrac{c_0(1/2)}{1.2} = -\dfrac{c_0}{4}$ $c_2 \overset{m=2}{=} \dfrac{-c_1(\cancel{3}/2)}{2.\cancel{3}} = +\dfrac{c_0}{16}$

$\therefore y_{1/2}(x) = c_0 x^{1/2}\left[1 - \dfrac{x}{4} + \dfrac{x^2}{16} - \dots\right]$

Q3 $Ly = 8x^2y'' - 2xy' + 3(x+1)y = 0$

$\lim_{x\to 0} x\,\frac{(-2x)}{8x^2} = -\frac{1}{4} = p_0 < \infty \qquad \lim_{x\to 0} x^2\,\frac{3(x+1)}{8x^2} = \frac{3}{8} = q_0 < \infty \quad \therefore$ RSP

INDICIAL EQ: $r(r-1) - \frac{r}{4} + \frac{3}{8} = 0 \Rightarrow 8r^2 - 10r + 3 = (4r-3)(2r-1) = 0 \quad r = 3/4,\ 1/2.$

LET $y = \sum_{n=0}^{\infty} c_n x^{n+r}$, $y' = \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1}$, $y'' = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2}$.

$$Ly = 8x^2y'' - 2xy' + 3xy + 3y = 0$$

$$= \sum_{n=0}^{\infty} c_n 8(n+r)(n+r-1)x^{n+r} - \sum_{n=0}^{\infty} 2c_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} c_n 3x^{n+r+1} + \sum_{n=0}^{\infty} 3c_n x^{n+r} = 0$$

$\boxed{n=m}$ $\boxed{n=m}$ $\boxed{n+1=m}$ $\boxed{n=m-1}$ $n=0 \Rightarrow m=1$ $\boxed{n=m}$

$$Ly = \sum_{m=0}^{\infty} c_m\{8(m+r)(m+r-1) - 2(m+r) + 3\}x^{m+r} + \sum_{m=1}^{\infty} 3c_{m-1}x^{m+r} = 0$$

$$= c_0\{8r^2 - 10r + 3\}x^r + \sum_{m=1}^{\infty}\{c_m[(m+r)\{8(m+r)-10\}+3] + 3c_{m-1}\}x^{m+r} = 0$$

$x^r >$ $\quad 8r^2 - 10r + 3 = (4r-3)(2r-1) = 0 \quad r = 3/4$ & $r = 1/2$

$x^{m+r} >$ $\quad c_m[(m+r)\{8(m+r)-10\}+3] + 3c_{m-1} = 0$

$r = 1/2$: $\quad c_m = \frac{-3c_{m-1}}{(m+1/2)\{8m+4-10\}+3} = \frac{-3c_{m-1}}{2m(4m-1)}$

$c_1 = \frac{-3c_0}{2\cdot 3} = -\frac{c_0}{2} \qquad c_2 = \frac{-3c_1}{4\cdot 7} = \frac{+3c_0}{56}$

$\therefore\ y_{1/2}(x) = c_0 x^{1/2}[1 - x/2 + 3x^2/56 - \cdots]$

$r = 3/4$: $\quad c_m = \frac{-3c_{m-1}}{(m+3/4)[8m+6-10]+3} = \frac{-3c_{m-1}}{2m(4m+1)}$

$c_1 = \frac{-3c_0}{2.5} \qquad c_2 = \frac{-3c_1}{4.9} = \frac{+\cancel{9}c_0}{2.5.4.\cancel{9}} = \frac{c_0}{40}$

$y_{3/4}(x) = c_0 x^{3/4}\left[1 - \frac{3x}{10} + \frac{x^2}{40} - \cdots\right]$

Q4 $Ly = (1-x^2)y'' - xy' + \alpha y = 0$

(a) $y = \sum_{n=0}^{\infty} c_n x^n \quad y' = \sum_{n=1}^{\infty} c_n n x^{n-1} \quad y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$

$$Ly = y'' - x^2 y'' - xy' + \alpha y = 0$$

$$= \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} - \sum_{n=2}^{\infty} c_n n(n-1) x^n - \sum_{n=1}^{\infty} c_n n x^n + \alpha \sum_{n=0}^{\infty} c_n x^n$$

$n-2 = m \quad n = m+2 \qquad n = m \qquad n = m \qquad n = m$

$n = 2 \Rightarrow m = 0$

$$0 = Ly = \sum_{m=0}^{\infty} c_{m+2}(m+2)(m+1)x^m - \sum_{m=2}^{\infty} c_m m(m-1)x^m - \sum_{m=1}^{\infty} c_m m x^m + \alpha \sum_{m=0}^{\infty} c_m x^m$$

$$= c_2 \cdot 2 \cdot 1 x^0 + c_3\, 3 \cdot 2 x^1 \cdots - c_1 \cdot 1 x^1 + \alpha c_0 x^0 + \alpha c_1 x^1$$

$$+ \sum_{m=2}^{\infty} \left[c_{m+2}(m+2)(m+1) - c_m\{m(m-1) + m - \alpha\} \right] x^m$$

$$= (2c_2 + \alpha c_0)x^0 + (6c_3 + (\alpha - 1)c_1)x^1 + \sum_{m=2}^{\infty} \{c_{m+2}(m+2)(m+1) - c_m(m^2 - \alpha)\} x^m$$

$x^0 >$ $\quad c_2 = -\frac{\alpha}{2} c_0$

$x^1 >$ $\quad c_3 = (1-\alpha)c_1/6$

$x^m \; m \geq 2 >$ $\quad c_{m+2} = \dfrac{c_m(m^2 - \alpha)}{(m+2)(m+1)}$

$$c_4 = \frac{c_2(4-\alpha)}{4 \cdot 3} = -\frac{c_0(4-\alpha)\alpha}{4 \cdot 3 \cdot 2}$$

$$y_0(x) = c_0\left[1 - \frac{\alpha x^2}{2} - \frac{(4-\alpha)\alpha}{24} x^4 + \cdots\right]$$

$$c_5 = \frac{c_3(9-\alpha)}{5 \cdot 4} = c_1 \frac{(1-\alpha)(9-\alpha)}{5!}$$

$$\therefore y_1(x) = c_1\left[x + \frac{(1-\alpha)}{6} x^3 + \frac{(1-\alpha)(9-\alpha)}{5!} x^5 + \cdots\right]$$

WHAT HAPPENS IF $\alpha \in \mathbb{Z}$:

$\alpha = 0 \quad y_0(x) = c_0 \cdot 1 \quad y_1(x) = c_1[x + x^3/6 + \cdots]$

$\alpha = 1 \quad y_0(x) = c_0\left[1 - \frac{x^2}{2} - \frac{x^4}{8} - \cdots\right] \quad y_1(x) = c_1 x$

$\alpha = 4 \quad y_0(x) = c_0[1 - 2x^2] \quad y_1(x) = c_1\left[x - \frac{x^3}{2} - \cdots\right]$

ONE OF THE TWO SERIES TERMINATES TO YIELD A POLYNOMIAL WHEN α IS A PERFECT SQUARE $\rightarrow$ THE TCHEBYSHEV POLYNOMIALS

(b) LET $z = x-1$ $\quad \frac{d}{dx} = \frac{d}{dz}$

$$Ly = (1-x^2)y'' - xy' + \alpha y = [-z(2+z)Y'' - [1+z]Y' + \alpha Y] = \mathcal{L}Y = 0$$

$$\lim_{z\to 0} \frac{z\{-(1+z)\}}{-z(2+z)} = +\frac{1}{2} = p_0 \qquad \lim_{z\to 0} z^2 \frac{\alpha}{-z(2+z)} = 0 = q_0 < \infty \quad \text{RSP.}$$

INDICIAL EQ: $r(r-1) + r/2 = 0 \qquad 2r^2 - r = (2r-1)r = 0 \qquad r = 0, \tfrac{1}{2}$.

(c) $\alpha \neq 0, 1$: LET $Y = \sum_{n=0}^{\infty} C_n z^{n+r} \quad Y' = \sum_{n=0}^{\infty} C_n (n+r) z^{n+r-1}, \quad Y'' = \sum_{n=0}^{\infty} C_n (n+r)(n+r-1) z^{n+r-2}$

$$0 = \mathcal{L}Y = -2zY'' \qquad - z^2 Y'' \qquad - Y' \qquad - zY'$$

$$= -\sum_{n=0}^{\infty} 2C_n(n+r)(n+r-1)z^{n+r-1} - \sum_{n=0}^{\infty} C_n(n+r)(n+r-1)z^{n+r} - \sum_{n=0}^{\infty} C_n(n+r)z^{n+r-1} - \sum_{n=0}^{\infty} C_n(n+r)z^{n+r} + \alpha \sum_{n=0} C_n z^{n+r}$$

$n-1=m \quad n=m+1 \qquad n=m \qquad n=m+1 \qquad n=m$

$n=0 \Rightarrow m=-1$

$$0 = \mathcal{L}Y = \sum_{m=-1}^{\infty} C_{m+1}\{-2(m+r+1)(m+r) - (m+r+1)\} X^{m+r} - \sum_{m=0}^{\infty} C_m[(m+r)(m+r-1) + (m+r) - \alpha] X^{m+r}$$

$$= C_0\{-2r(r-1) - r\} X^r - \sum_{m=0}^{\infty}\left[C_{m+1}\{(m+r+1)[2(m+r)+1]\} + C_m[(m+r)^2 - \alpha]\right] X^{m+r}$$

$X^r >$ $\quad 2r^2 - r = (2r-1)r = 0 \quad r = 0, \tfrac{1}{2}$

$X^{m+r} >$ $\quad C_{m+1} = -C_m[(m+r)^2 - \alpha] / (m+r+1)[2(m+r)+1]$

$r = \tfrac{1}{2}$: $\quad C_{m+1} = -C_m[m^2 + m + 1/4 - \alpha] / (m + 3/2)[2(m+1)] = -\dfrac{C_m[m^2 + m + \frac{1}{4} - \alpha]}{(2m+3)(m+1)}$

$$C_1 = -C_0\left(\tfrac{1}{4} - \alpha\right)/3.1 = \frac{C_0}{12}(3\alpha - 1) \qquad C_2 = -\frac{C_1\left(\frac{9}{4} - \alpha\right)}{5.2} = \frac{C_0\left(\frac{9}{4} - \alpha\right)(1 - 3\alpha)}{120}$$

$$\therefore \quad y_{1/2}(x) = X^{1/2} C_0\left[1 + \frac{(3\alpha - 1)X}{12} + \frac{\left(\frac{9}{4} - \alpha\right)(1 - 3\alpha)}{120} X^2 + \cdots\right.$$

$r = 0$: $\quad C_{m+1} = -C_m[m^2 - \alpha] / (m+1)(2m+1)$

$$C_1 = -C_0(-\alpha)/1.1 = C_0\alpha$$

$$C_2 = -C_1(1-\alpha)/2.3 = -C_0\alpha(1-\alpha)/6$$

$$\therefore \quad y_0 = C_0 X^0\left[1 - \alpha X + \frac{(\alpha - 1)}{6}\alpha X^2 + \cdots\right]$$

IF $\alpha = 0$ OR $\alpha = 1$ THEN THE SERIES TERMINATES.