Math 257/316 (Supplementary Exercises 1- not to be handed in)

Problem 1: (Sample Exercises): Find the general solution of the following equations:

a. $6y'' + y' - 2y = 0$, Exact solution is: $y(x) = C_1 e^{-\frac{2}{3}x} + C_2 e^{\frac{1}{2}x}$

b. $y^{(4)} - 3y'' - 4y = 0$, Exact solution is: $y(x) = C_1 \cos x + C_2 \sin x + C_3 e^{2x} + C_4 e^{-2x}$

c $y'' - 3y' + 2y = 4e^x$, Exact solution is: $y(x) = -4xe^x - 4e^x + C_1 e^x + C_2 e^{2x}$

d. $x^2 y'' - 2xy' - 4y = 0$, Exact solution is: $y(x) = \frac{C_1}{x} + C_2 x^4$

e.$9x^2 y'' + 3xy' + y = 0$, Exact solution is: $y(x) = C_1 \sqrt[3]{x} + C_2 \sqrt[3]{x} \ln x$

f. $x^2 y'' + 3xy' + 2y = 0$, $y(1) = 1$, $y'(1) = -2$

$$x^2 y'' + 3xy' + 2y = 0$$
$$y(1) = 1$$
$$y'(1) = -2$$
, Exact solution is: $y(x) = \frac{1}{x} \cos(\ln x) - \frac{1}{x} \sin(\ln x)$

Problem 2: (**Sample Exercise**): Consider the following first order linear ODE:

$$(1 - x^2) y' - 2xy = 0 \tag{1}$$

The exact solution is: $y(x) = \frac{C_1}{(x-1)(x+1)}$

$$\frac{1}{1 - x^2}$$

$= 1 + x^2 + x^4 + O\left(x^6\right)$

a. Solve this differential equation using an integrating factor.

b. Expand the above solution in a Taylor series about the point $x_0 = 0$. For what values of x does this series fail to converge?

c. Now assume a power series solution of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Substitute this series into the differential equation (1) and obtain a recursion for the coefficients a_n. Use this recursion to determine the series representation of solution. Compare this result to the series obtained in part b above. Is there any relationship between the points of divergence of the series and the coefficient $(1 - x^2)$ of the derivative in (1)?

PROBLEM 4: $Ly = (1-x^2)y' - 2xy = [(1-x^2)y]' = 0$

(a) $(1-x^2)y = c$

$$y = \frac{c}{1-x^2}$$

(b) FROM THE GEOMETRIC SERIES $1 + r + r^2 + \cdots = \frac{1}{1-r}$ $|r| < 1$

SUBSTITUTE $r = x^2$ INTO THE GEOMETRIC SERIES

$$\therefore \frac{1}{1-x^2} = 1 + x^2 + x^4 + \cdots$$

$$y(x) = c(1 + x^2 + x^4 + \cdots)$$

THE SERIES FAILS TO CONVERGE IF $|x| \geq 1$.

(c) $Ly = (1-x^2)y' - 2xy = 0$

LET $y(x) = \sum_{n=0}^{\infty} a_n x^n$, $xy = \sum_{n=0}^{\infty} a_n x^{n+1}$, $y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$, $x^2 y' = \sum_{n=1}^{\infty} a_n n x^{n+1}$

THEN $Ly = \sum_{n=1}^{\infty} a_n n x^{n-1} - \sum_{n=1}^{\infty} a_n n x^{n+1} - 2\sum_{n=0}^{\infty} a_n x^{n+1} = 0$

$$= \sum_{n=1}^{\infty} a_n n x^{n-1} - \sum_{n=1}^{\infty} a_n (n+2) x^{n+1} - 2a_0 x^1 = 0$$

LET US CHANGE THE INDEX IN THE FIRST SUM FROM $n-1 \Rightarrow n+1$

LET $n-1 = m+1$ $\quad n = m+2$ $\quad n=1 \Rightarrow m=-1$

$$\therefore \sum_{n=1}^{\infty} a_n n x^{n-1} = \sum_{m=-1}^{\infty} a_{m+2}(m+2)x^{m+1} = a_1 x^0 + 2a_2 x + \sum_{m=1}^{\infty} a_{m+2}(m+2)x^{m+1}$$

NOW SINCE m IS A DUMMY INDEX WE CAN REPLACE m BY n IN THIS SUM.

$$\therefore Ly = a_1 x^0 + 2a_2 x^1 + \sum_{n=1}^{\infty} a_{n+2}(n+2)x^{n+1} - \sum_{n=1}^{\infty} a_n (n+2) x^{n+1} - 2a_0 x^1 = 0$$

$$\therefore Ly = a_1 x^0 + 2(a_2 - a_0)x^1 + \sum_{n=1}^{\infty} \{a_{n+2}(n+2) - a_n(n+2)\} x^{n+1} = 0. \quad (*)$$

NOW SINCE THE POWERS OF x: $1, x, x^2, \ldots$ ARE LINEARLY INDEPENDENT AND THE SUM OF POWERS $(*)$ ABOVE IS ZERO THE ONLY POSSIBILITY IS THAT THE COEFFICIENTS ARE ALL ZERO

$x^0 > a_1 = 0$

$x^1 > a_2 = a_0$

$x^{n+1} > a_{n+2} = a_n \quad n \geq 1$

$\therefore 0 = a_1 = a_3 = a_5 = \cdots$ ALL VANISH

AND $a_0 = a_2 = a_4 = \ldots$

THUS RETURNING TO THE ORIGINAL SERIES

$$y(x) = a_0 \cdot 1 + a_2 x^2 + a_4 x^4 + \cdots$$

NOTE THAT THE SERIES DIVERGES FOR $|x| \geq 1$, WE NOTE THAT THE COEFFICIENT OF THE HIGHEST DERIVATIVE y' VANISHES AT THE POINT $|x| = 1$ AT WHICH THE SERIES DIVERGENCE STARTS.