

Rescaling the spatial Lambda-Fleming-Viot process and convergence to super-Brownian motion*

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Abstract

We show that a space-time rescaling of the spatial Lambda-Fleming-Viot process of Barton and Etheridge converges to super-Brownian motion. This can be viewed as an extension of a result of Chetwynd-Diggle and Etheridge [5]. In that work the scaled impact factors (which govern the event based dynamics) vanish in the limit, here we drop that requirement. The analysis is particularly interesting in the biologically relevant two-dimensional case.

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1 Introduction

Our purpose in this paper is to extend a result in [5] which shows that certain suitably rescaled spatial Lambda-Fleming-Viot (SLFV) processes converge weakly to super-Brownian motion (SBM). Our extension is analogous to that of allowing nearest neighbour interactions in interacting particle models, as opposed to taking long range limits, and is particularly delicate in the critical two-dimensional case. SBM is a well known measure-valued diffusion, introduced in [26] and [9], for which there is an extensive research literature (e.g., for reviews see [10], [13] and [22]). SLFV processes were introduced more recently, in [12], to serve as models for the evolution of allele frequencies in populations distributed across spatial continua. An analytic construction was given in [2], along with a discussion of the biological significance of the model. A more probabilistic construction was given in [25], one which gives a very useful connection between SLFV processes and their duals. Following [5], we consider here a neutral two-type version of the general SLFV model, taking “space” to be \mathbb{R}^d . Informally, our process (constructed below) is a Markov process $(\mu_t)_{t \geq 0}$ where for each $x \in \mathbb{R}^d$,

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$\mu_t(x)$ is a probability distribution on the type space $\{0, 1\}$, with the interpretation that $\int_B \mu_t(x)(\{i\})dx$ represents the proportion of the population of type i in a region $B \subset \mathbb{R}^d$ at time t . We will consider an extension of the fixed radius case from [5] (Theorem 2.6 of that reference) and not the interesting variable radius case, also discussed there in Theorem 2.7, in which stable branching arises in the limit.

SBM arises as the limit under Brownian space-time rescaling of a range of critical spatially interacting models in mathematical physics and biology *above the critical dimension* including critical oriented percolation [17], critical lattice trees [18], the critical contact process [16], and the voter model [6]; it is believed to be the scaling of critical ordinary percolation in the same regime. The only scaling limit of the above which has been verified *at the critical dimension* is the voter model [6] where the critical dimension is two. In this case the simple nature of the dual process, a coalescing random walk, allows one to carry out the required explicit calculations. Now our challenge is to use the related but more complex dual of the Barton-Etheridge model to carry through the analysis. It is understood here that we are not taking “long-range” limits (e.g. as was done for the contact process in [11]) which will weaken the interaction and make the analysis considerably easier. In our setting this means not letting the impact factor (described below) approach zero in the rescaling.

We start a rigorous description of the model by recalling the definition of the fixed radius SLFV process given in [5]. Let $r > 0$ be the “interaction radius”, let $\rho \in [0, 1]$ be the “impact factor,” and let Π be a Poisson point process on $\mathbb{R}^d \otimes (0, \infty)$ with intensity $dx \otimes dt$. We suppose the distribution of types in the population changes over time according to “reproduction events” determined by Π . Given μ_{t-} , if $(x, t) \in \Pi$, choose an independent point z uniformly at random from the Euclidean ball

$$B_r(x) = \{y : |y - x| \leq r\},$$

and (independently) a type α according to the distribution $\mu_{t-}(z)$, and then set

$$\mu_t(y) = (1 - \rho)\mu_{t-}(y) + \rho\delta_\alpha \quad \forall y \in B_r(x).$$

We keep $\mu_t(y) = \mu_{t-}(y)$ for $y \notin B_r(x)$. Writing $\mu_t(x)$ in the form $w_t(x)\delta_1 + (1 - w_t(x))\delta_0$, we can reformulate the above dynamics more conveniently in terms of w_t as follows. Starting from a Borel $w_0 : \mathbb{R}^d \rightarrow [0, 1]$ with compact support, for $(x, t) \in \Pi$, choose an independent parental location z uniformly at random from $B_r(x)$, independent of everything, and then:

- (i) with probability $w_{t-}(z)$ put $w_t(y) = (1 - \rho)w_{t-}(y) + \rho$ for all $y \in B_r(x)$,
- (ii) with probability $1 - w_{t-}(z)$ put $w_t(y) = (1 - \rho)w_{t-}(y)$ for all $y \in B_r(x)$, (1.1)
- (iii) for all $y \notin B_r(x)$ keep $w_t(y) = w_{t-}(y)$.

As noted in Section 3 of [5], this description gives a well-defined $w_t : \mathbb{R}^d \rightarrow [0, 1]$ which has compact support at all times. (See [25] for more details on the construction.) It will be useful to regard w_t as the measure $w_t(x)dx$, and for bounded Borel $\phi : \mathbb{R}^d \rightarrow [0, \infty)$, write

$$w_t(\phi) = \int_{\mathbb{R}^d} \phi(x) w_t(x)dx. \tag{1.2}$$

Closely associated with the process w_t is a dual process of coalescing “lineages”. If we sample a finite number of spatial locations $\{x_i\}$ at time T , it is easy to see that the values $w_T(x_i)$ can be determined from w_0 by using Π to trace the lineages backward in time. Since Π run backwards is still a Poisson process, we may define a version of the lineages process starting at backwards time 0 from a finite number of locations $\{x_i\}$ as

follows. If $(x, t) \in \Pi$, mark each lineage in $B_r(x)$ independently with probability ρ , and choose a point z uniformly at random from $B_r(x)$. If at least one of the lineages in $B_r(x)$ is marked, all marked lineages in $B_r(x)$ coalesce and the resulting lineage is moved to z . If no lineage is marked, no lineage moves. Lineages outside of $B_r(x)$ are not affected. In this paper it will suffice to consider only the one and two-lineage systems, so we will ignore the higher lineage systems which are more complex to analyze.

We now give a more precise description of these Markov jump processes, using the language of “particles” instead of lineages. Let $|\Gamma|$ be the Lebesgue measure of $\Gamma \subset \mathbb{R}^d$. Let U, U^1, U^2 be independent random variables uniformly distributed on $B_r = B_r(0)$, and let \bar{U} have the law of $U^1 + U^2$, i.e., \bar{U} has density

$$P(\bar{U} \in dz) = \frac{|B_r(0) \cap B_r(z)|}{|B_r(0)|^2} dz := h_{\bar{U}}(z) dz. \tag{1.3}$$

We let $\bar{\sigma}^2 1_{d \times d}$ denote the covariance matrix of \bar{U} , so that if $x = (x_1, \dots, x_d)$, then

$$\bar{\sigma}^2 = \frac{2}{|B_r|} \int_{B_r} (x_1)^2 dx. \tag{1.4}$$

We will use this notation throughout, along with η_t for the single particle dual and $\xi_t = (\xi_t^1, \xi_t^2)$ for the two particle dual.

(a) *The single-particle dual η_t .* If we start with a single particle at x , it is easy to see that η_t is the random walk on \mathbb{R}^d starting at x which makes jumps at rate $\rho|B_r|$ with jump distribution given in (1.3). We write $P_{\{x\}}$ for the underlying law of η .

(b) *The two-particle dual (ξ_t^1, ξ_t^2) .* If we start with two particles, one at x_1 and the other at $x_2 \neq x_1$, (ξ_t^1, ξ_t^2) is the Markov jump process starting at (x_1, x_2) , and with law $P_{\{x_1, x_2\}}$, which makes transitions

$$(y_1, y_2) \rightarrow \begin{cases} (y + \bar{U}, y + \bar{U}) & \text{at rate } \rho|B_r| \text{ if } y_1 = y_2 = y \\ (y_1 + \bar{U}, y_2) & \text{at rate } \rho(|B_r| - \rho|B_r(y_1) \cap B_r(y_2)|) \text{ if } y_1 \neq y_2 \\ (y_1, y_2 + \bar{U}) & \text{at rate } \rho(|B_r| - \rho|B_r(y_1) \cap B_r(y_2)|) \text{ if } y_1 \neq y_2 \\ (U + U_{y_1, y_2}, U + U_{y_1, y_2}) & \text{at rate } \rho^2|B_r(y_1) \cap B_r(y_2)| \text{ if } y_1 \neq y_2, \end{cases} \tag{1.5}$$

where U_{y_1, y_2} is an independent random variable, uniformly distributed over $B_r(y_1) \cap B_r(y_2)$. For $y_1 \neq y_2$, the total jump rate at (y_1, y_2) , $y_1 \neq y_2$, is $2\rho|B_r| - \rho^2|B_r(y_1) \cap B_r(y_2)|$. To see the above rates consider, for example, the second transition from (y_1, y_2) to $(y_1 + \bar{U}, y_2)$ for $y_1 \neq y_2$ where (y_1, y_2) is the current site of our two-particle dual. The next jump in the first coordinate can only occur at a point $(x, t) \in \Pi$ with $x \in B_r(y_1)$ so let (x, t) be the next such point. At (x, t) such a jump (affecting the first coordinate but not the second) can occur in one of two ways: if x lands in $B_r(y_1) \setminus (B_r(y_1) \cap B_r(y_2))$ and the particle ξ^1 at y_1 is marked, or if x lands in $B_r(y_1) \cap B_r(y_2)$ and the particle at y_1 is marked and the particle at y_2 is not. The total rate in t is obtained by integrating out x and so is

$$\rho(|B_r(y_1)| - |B_r(y_1) \cap B_r(y_2)|) + \rho(1 - \rho)|B_r(y_1) \cap B_r(y_2)| = \rho|B_r| - \rho^2|B_r(y_1) \cap B_r(y_2)|.$$

In either of the above scenarios the particle at y_1 will jump to z , a uniformly selected site in $B_r(x)$. Given y_1 , x will be uniformly distributed on $B_r(y_1)$ and so $x - y_1$ will be uniform on B_r . Clearly given (y_1, x) , $z - x$ is uniformly distributed over B_r and so $(x - y_1, z - x)$ is a pair of independent uniforms on B_r . Therefore the jump in ξ^1 at time t is $z - y_1 = (z - x) + (x - y_1)$ and so has law \bar{U} as claimed. The other transitions are similar to analyze.

The coalescence time for the two-particle dual starting at (x_1, x_2) is

$$\tau = \inf\{t \geq 0 : \xi_t^1 = \xi_t^2\}. \tag{1.6}$$

Although (ξ_t^1, ξ_t^2) is Markov, the individual coordinates ξ_t^1, ξ_t^2 are not (i.e., ξ^1 is not Markov with respect to the filtration $\sigma(\xi_s^1, 0 \leq s \leq t)_{t \geq 0}$). However, when $B_r(\xi_t^1) \cap B_r(\xi_t^2) = \emptyset$, both coordinates move independently according to the single particle dynamics, while for $t > \tau$, the coalesced coordinates move together according to the single particle dynamics. It is also clear from (1.5) that the two-particle dual is translation invariant, that is,

$$P_{\{x_1+x, x_2+x\}}((\xi^1, \xi^2) \in \cdot) = P_{\{x_1, x_2\}}((x + \xi^1, x + \xi^2) \in \cdot) \quad \forall x, x_1, x_2 \in \mathbb{R}^d. \tag{1.7}$$

The two special cases of the general duality equation in Proposition 2.5 of [5] that we need are the following. For all $t \geq 0$, $\psi_1 \in C(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ and $\psi_2 \in C(\mathbb{R}^d \times \mathbb{R}^d) \cap L^1(\mathbb{R}^d \times \mathbb{R}^d)$,

$$E_{w_0}[w_t(\psi_1)] = \int_{\mathbb{R}^d} \psi_1(x) E_{\{x\}}[w_0(\eta_t)] dx, \text{ and} \tag{1.8}$$

$$\begin{aligned} E_{w_0} \left[\int_{\mathbb{R}^d \times \mathbb{R}^d} \psi_2(x_1, x_2) w_t(x_1) w_t(x_2) dx_1 dx_2 \right] \\ = \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi_2(x_1, x_2) E_{\{x_1, x_2\}} [w_0(\xi_t^1) 1_{\{\tau \leq t\}} + w_0(\xi_t^1) w_0(\xi_t^2) 1_{\{\tau > t\}}] dx_1 dx_2. \end{aligned} \tag{1.9}$$

By standard approximation arguments, these equations then hold for all Borel ψ_1, ψ_2 which are either nonnegative or integrable (on one side or the other). In particular, letting $\mathbf{1}$ denote the constant function 1 on \mathbb{R}^d , we have

$$E_{w_0}[w_t(\mathbf{1})] = \int_{\mathbb{R}^d} E_{\{x\}}[w_0(\eta_t)] dx = w_0(\mathbf{1}). \tag{1.10}$$

Before stating the main fixed radius result of [5], Theorem 2.6, we introduce super-Brownian motion using the martingale problem formulation. If $(X_t)_{t \geq 0}$ is a stochastic process, $(\mathcal{F}_t^X)_{t \geq 0}$ will denote the right-continuous filtration generated by X . Let $\mathcal{M}_F(\mathbb{R}^d)$ denote the space of finite Borel measures on \mathbb{R}^d endowed with the topology of weak convergence, and for $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ let $\mu(\phi) = \int_{\mathbb{R}^d} \phi d\mu$. The space of bounded continuous functions on \mathbb{R}^d is denoted by $C_b(\mathbb{R}^d)$, and $C_0^3(\mathbb{R}^d)$ is the space of continuous functions on \mathbb{R}^d which vanish at infinity and have bounded continuous partials of order 3 and less. Then (see, e.g., Theorem A.1 of [6] for uniqueness, and Theorem II.5.1 and Remark II.5.5 of [22] for existence) Super-Brownian motion with diffusion coefficient σ^2 and branching rate b , denoted $\text{SBM}(X_0, \sigma^2, b)$, is the unique $\mathcal{M}_F(\mathbb{R}^d)$ -valued Markov process $(X_t)_{t \geq 0}$ with continuous paths and initial state X_0 , such that for every $\phi \in C_0^3(\mathbb{R}^d)$,

$$M_t(\phi) = X_t(\phi) - X_0(\phi) - \int_0^t X_s \left(\frac{\sigma^2}{2} \Delta \phi \right) ds \tag{1.11}$$

is a local (\mathcal{F}_t^X) -martingale with predictable quadratic variation process

$$\langle M(\phi) \rangle_t = b \int_0^t X_s(\phi^2) ds. \tag{1.12}$$

Theorem 2.6 in [5] considers a sequence $X_t^N = K w_t^N$ of scaled versions of w_t , defined with sequences $K = K_N$, $M = M_N$, and $J = J_N$. Namely, given w_0^N , w_t^N is constructed in the same way as w_t , but with the following modifications. If $(x, t) \in \Pi$ then there is a reproduction event at $(\frac{x}{M}, \frac{t}{N})$, with impact factor $\frac{b}{J}$ and reproduction region $B_{\frac{r}{M}}(\frac{x}{M})$.

Thus, for w_t^N , time is sped up by N , space is shrunk by M_N , and the impact factor is reduced by $J_N \geq 1$. Denote the rescaled Poisson point process with intensity $M^d N dx \otimes dt$ by

$$\Pi^N = \left\{ \left(\frac{x}{M}, \frac{t}{N} \right) : (x, t) \in \Pi \right\}.$$

If $n_t^N(A) = \#\{(s, x) \in \Pi^N \cap ([0, t] \times A)\}$, let (\mathcal{F}_t^N) be the right-continuous filtration generated by $\{n_s^N(A) : s \leq t, A \text{ a Borel set in } \mathbb{R}^d\}$.

Let $D([0, \infty), \mathcal{M}_F(\mathbb{R}^d))$ denote the space of cadlag $\mathcal{M}_F(\mathbb{R}^d)$ -valued paths equipped with the Skorokhod (J1) topology.

Theorem 1.0 (Theorem 2.6 in [5]). *Suppose that for a compact set $D_0 \subset \mathbb{R}^d$, $\text{supp}(w_0^N) \subset D_0$ for all N , and as elements of $\mathcal{M}_F(\mathbb{R}^d)$, $X_0^N \rightarrow X_0 \in \mathcal{M}_F(\mathbb{R}^d)$ as $N \rightarrow \infty$. In addition, suppose there are constants $C_1, C_2 \in (0, \infty)$ such that, as $N \rightarrow \infty$,*

(1) $M \rightarrow \infty$,

(2) $\frac{N}{JM^2} \rightarrow C_1$,

(3) $\frac{KN}{J^2M^d} \rightarrow C_2$,

(4) $\begin{cases} \frac{M}{J} \rightarrow 0 & \text{if } d = 1, \\ \frac{\log M}{J} \rightarrow 0 & \text{if } d = 2, \\ \frac{1}{J} \rightarrow 0 & \text{if } d \geq 3. \end{cases}$

Then the sequence $(X^N)_{N \geq 1}$ converges weakly in $D([0, \infty), \mathcal{M}_F(\mathbb{R}^d))$ to $SBM(X_0, \sigma^2, b)$ with

$$\sigma^2 = 2C_1\rho \int_{B_r} (x_1)^2 dx \text{ and } b = C_2\rho^2 |B_r|^2.$$

(The constant $C(d)$ in Definition 4.1 in [5] should be $C(d) = \int_{B_1} (x_1)^2 dx$.) As noted in [5], this result is similar in spirit to Theorem 1.1 in [6], which proves convergence to SBM for certain sparse “long range” kernel voter models. Due to conditions (1) and (4) above, $J_N \rightarrow \infty$ and hence the impact factors $\rho/J_N \rightarrow 0$. It is this fact and the mass scaling condition (3) which make these SLFV processes analogous to the long range voter models in [6]. As for the duals, conditions (1) and (2) ensure that the single particle dual motion converges to Brownian motion, while the condition $J_N \rightarrow \infty$ ensures that the interactions between dual particles are weak.

If the sequence J_N were bounded, so that the impact factors $\rho = \rho/J_N$ do not vanish in the limit, the resulting SLFV processes would correspond to the “fixed” kernel voter models in Theorem 1.2 in [6]. In biological terms this corresponds to keeping the “neighbourhood size” finite in the scaling limit, while letting $J_N \rightarrow \infty$ effectively allows this parameter to become infinite; see the discussion in Section 2 of [14] and especially Definition 2.2 there. In that work they showed in this fixed neighbourhood size setting (Theorem 2.7 of [14]) that, with an appropriate selection term, the dual particle process converges to a branching Brownian motion in the scaling limit. The purpose of this paper is to prove that in this setting, with no selection, there is also a forwards limit theorem giving convergence to SBM.

Throughout this work we will assume $d \geq 2$, and

$$N \geq 3. \tag{1.13}$$

For our scaled SLFV processes w_t^N we then make the choices

$$J = 1, \quad M = \sqrt{N}, \quad K = \begin{cases} N^{\frac{d}{2}-1} & \text{for } d \geq 3, \\ \log N & \text{for } d = 2. \end{cases} \tag{1.14}$$

If we set $J \equiv 1$, and take $C_1 = C_2 = 1$ for simplicity, the conditions (1)-(3) in Theorem 1.0 suggest the choices for M and K above except for the logarithmic correction to K for $d = 2$. Without this correction, one can show that the limiting process in Theorem 1.2 would be nonrandom heat flow acting on X_0 , as is the case for the voter model [23]. We do not consider the case $d = 1$ in (1.14). For this case, the Wright-Fisher SPDE was obtained in [15] as an appropriate scaling limit of SLFV, but under that assumption that the scaled impact factors approach zero like $N^{-1/3}$ (see [21] for the corresponding scaling limit for the voter model). If the impact factors were bounded away from zero, the strong recurrence of one-dimensional random walk would lead to heavy clustering, resulting in scaling limits with segregation of types; the corresponding scaling limit for the voter model is the Arratia flow [1], not super-Brownian motion.

In order to state our limit theorem for scaled SLFV processes assuming (1.14), we must first identify certain constants $\gamma_e^{(d)}$ that appear in the limiting SBM branching rate. These constants are determined by the asymptotic tail behavior of the coalescence times τ for the unscaled two-particle dual process defined in (1.5). Introduce

$$\gamma_e(t) = \gamma_e^{(d)}(t) = \frac{1}{|B_r|^2} \int_{B_r} \int_{B_r} P_{\{x_1, x_2\}}(\tau > t) dx_1 dx_2.$$

Proposition 1.1. *There are constants $\gamma_e = \gamma_e^{(d)} > 0$ such that as $t \rightarrow \infty$,*

$$\lim_{t \rightarrow \infty} \gamma_e^{(d)}(t) = \gamma_e \text{ if } d \geq 3 \tag{1.15}$$

and

$$\lim_{t \rightarrow \infty} (\log t) \gamma_e^{(d)}(t) = \gamma_e \text{ if } d = 2. \tag{1.16}$$

Recall that when outside B_{2r} , $\xi_t^1 - \xi_t^2$ behaves like a rate $2\rho|B_r|$ random walk with jump distribution given in (1.3), and $\tau = \inf\{t \geq 0 : \xi_t^1 - \xi_t^2 = 0\}$. Therefore, if $d \geq 3$, the difference will escape to infinity with positive probability by transience, and so the limit in (1.15), which exists by monotonicity, will have a non-zero limit. For $d = 2$ the situation is more delicate. One can predict the $1/\log t$ behaviour of $\gamma_e(t)$ from the corresponding non-return probabilities for irreducible symmetric random walk on \mathbb{Z}^2 with diagonal covariance matrix (see, e.g., Lemma A.3(ii) of [6]), but the slowing rates when the difference $\xi^1 - \xi^2$ is in B_{2r} complicates things. The limit (1.16) can be derived from Lemma 4.10 in [14]. The analysis there is based on a construction using successive “inner” and “outer” excursions of $\xi^1 - \xi^2$ from certain balls before coalescence occurs. Our argument represents the difference process as a time change of a rate $2\rho|B_r|$ random walk with step distribution $h_{\bar{U}}$, and makes use of a reflection coupling. We feel the proof is of independent interest and so have included it in an Appendix. One advantage of the excursion approach in [14] is that it should also allow inclusion of a random “interaction radius”, that is

the driving Poisson point process Π is now on $\mathbb{R}^d \times [0, \infty) \times [0, r_{\max}]$ with intensity (1.17)

$$dx \otimes dt \otimes \mu(dr) \text{ for some finite measure } \mu \text{ on the compact interval } [0, r_{\max}].$$

However, as is discussed below, our time-change representation of the dual difference process in the fixed radius case will also play an important role in the analysis of the martingale square function which is the key ingredient in the proof of our main convergence result, Theorem 1.2 below.

With the choice of renormalization constants in (1.14) we now give a different description of the rescaled SLFV processes X^N , which will clarify the comparison with

Theorem 1.2 below of the fixed kernel voter model result in [6]. Assume $X_0 \in \mathcal{M}_F(\mathbb{R}^d)$ and the compactly supported initial conditions $\bar{w}_0^N : \mathbb{R}^d \rightarrow [0, 1]$ satisfy

$$K_N \bar{w}_0^N(\sqrt{N}x) dx \rightarrow X_0 \text{ in } \mathcal{M}_F(\mathbb{R}^d). \tag{1.18}$$

For each N , let \bar{w}^N be the (original, unscaled) SLFV process defined in (1.1) with fixed interaction radius r , fixed impact factor ρ and initial condition $w_0^N = \bar{w}_0^N$, and define the rescaled SLFV process by

$$w_t^N(x) = \bar{w}_{Nt}^N(\sqrt{N}x) \tag{1.19}$$

This process has the same law as w^N defined using Π^N right before Theorem 1.0, with J and M given in (1.14). (For example, if (x, s) is the first point in Π affecting \bar{w}^N , then $\bar{w}_s^N \neq \bar{w}_{s-}^N$ is only possible inside $B_r(x)$. Letting $Nt = s$, this means that $w_t^N(y) \neq w_{t-}^N(y)$ is possible only for $y\sqrt{N} \in B_r(x)$, or $y \in B_{r/\sqrt{N}}(x)$. Thus the interaction radius for w^N is r/\sqrt{N} .) Finally our approximating empirical measures are given by

$$X_t^N(dx) = K_N w_t^N(x) dx := X_t^N(x) dx \in \mathcal{M}_F(\mathbb{R}^d), \tag{1.20}$$

so that (1.18) just asserts that $X_0^N \rightarrow X_0$. A simple change of variables shows that in terms of the unscaled SLFV processes, \bar{w}^N , we have for any bounded Borel ϕ on \mathbb{R}^d ,

$$X_t^N(\phi) = \begin{cases} \frac{1}{N} \int_{\mathbb{R}^d} \bar{w}_{Nt}^N(y) \phi(yN^{-1/2}) dy & \text{if } d \geq 3, \\ \frac{\log N}{N} \int_{\mathbb{R}^d} \bar{w}_{Nt}^N(y) \phi(yN^{-1/2}) dy & \text{if } d = 2. \end{cases} \tag{1.21}$$

Here is our main result for the scaled SLFV process. For a measure or function H , we let $\text{supp}(H)$ denote its closed support. Recall the definition of $\bar{\sigma}^2$ from (1.4).

Theorem 1.2. *Suppose that $d \geq 2$, $\text{supp}(w_0^N)$ is compact for each N , and $\lim_{N \rightarrow \infty} X_0^N = X_0$ in $\mathcal{M}_F(\mathbb{R}^d)$. As $N \rightarrow \infty$, X^N converges weakly in $D([0, \infty), \mathcal{M}_F(\mathbb{R}^d))$ to $\text{SBM}(X_0, \sigma^2, b)$, where*

$$\sigma^2 = \rho |B_r| \bar{\sigma}^2 \text{ and } b = \rho^2 |B_r|^2 \gamma_e. \tag{1.22}$$

As is usual for SBM limit theorems, the scaling condition on the initial conditions in (1.18) corresponds to a regime where type 1's are scarce. For example, if, say, X_0 assigns no mass to the boundary of $[-1, 1]^d$, then taking $\phi = 1(\|x\|_\infty \leq 1)$ we have $X_0^N(\phi) \rightarrow X_0([-1, 1]^d)$ which implies

$$\frac{\int_{[-\sqrt{N}, \sqrt{N}]^d} \bar{w}_0^N(x) dx}{(2\sqrt{N})^d} \sim \frac{X_0([-1, 1]^d)}{2^{d/2} K_N} \rightarrow 0 \text{ as } N \rightarrow \infty.$$

It is important to note that in our scaling regime with $J = 1$, the original \bar{w}^N we are working with is an ordinary SLFV process with fixed interaction range r and impact factor ρ , but with an initial condition in which type 1's are scarce.

Equation (1.21) should be compared to the corresponding rescaled empirical measures in [6] associated with a sequence of voter models $\xi_t^{(N)}(x), x \in \mathbb{Z}^d$ whose rescaled initial states again converge to a limiting $X_0 \in \mathcal{M}_F(\mathbb{R}^d)$: if $N' = N$ for $d \geq 3$ and $N' = N/\log N$ if $d = 2$, let

$$\hat{X}_t^N = \frac{1}{N'} \sum_{x \in \mathbb{Z}^d} \xi_{Nt}^{(N)}(x) \delta_{xN^{-1/2}}.$$

In that reference it is shown that \hat{X}^N converges weakly in $D([0, \infty), \mathcal{M}_F(\mathbb{R}^d))$ to an appropriate SBM, whose branching rate is determined by the asymptotics of the escape probability (from 0) for a continuous time random walk starting at a uniformly chosen neighbour of 0 in the integer lattice through a two-particle dual calculation. This suggests

the same should hold (as it does) for the SLFV but now with the asymptotics of the non-coalescing probability of our two particle dual playing the role of the random walk escape probability.

The proof follows a familiar outline, based in part on methods in [6]. For appropriate test functions ϕ the semimartingale decomposition from [5], recalled in Section 2, states that

$$X_t^N(\phi) = X_0^N(\phi) + D_t^N(\phi) + M_t^N(\phi),$$

where $M^N(\phi)$ is a local martingale, and $D^N(\phi)$ is a drift term of bounded variation. In Section 2 we provide some elementary simplifications for the explicit expressions for both $D^N(\phi)$ and the predictable quadratic variation process $\langle M^N(\phi) \rangle_t$ from [5]. In Section 3 we use the above and the one- and two-particle duals to calculate the first moments of X_t^N and give uniform L^2 bounds on the total mass $X_t^N(\mathbf{1})$ (Corollary 3.2) which will be used throughout.

Assuming the key Proposition 4.1 which is proved in Section 7, tightness of $\{X^N\}$ is then established in Section 4, where Theorem 1.2 is also proved by showing that any weak limit satisfies the martingale problem for $\text{SBM}(X_0, \sigma^2, b)$. The term $D^N(\phi)$ is easy to handle (Lemma 2.3); it is the asymptotic behavior of the quadratic variation process $\langle M^N(\phi) \rangle$ which requires some work. The key result here is the aforementioned Proposition 4.1 which we present here for the discussion below.

Proposition (4.1). For all $A, T > 0$, and $\phi \in C_0^3$,

$$\sup_{X_0^N(\mathbf{1}) \leq A} E \left[\sup_{0 \leq t \leq T} \left| \langle M^N(\phi) \rangle_t - \int_0^t \rho^2 |B_r|^2 \gamma_e X_s^N(\phi^2) ds \right| \right] \rightarrow 0 \text{ as } N \rightarrow \infty.$$

After establishing preliminary random walk results in Section 5 and facts about two-particle duals in Section 6, it is proved in Section 7. Its proof uses Proposition 1.1 but the issues go well beyond this result.

The behavior of the quadratic variation process is the main difference in the proofs of Theorem 1.2 and its counterpart in [5], Theorem 1.0. Lemma 4.3 in [5] shows that a key term in the variation process is negligible in the limit $N \rightarrow \infty$. This fact is a consequence of the assumption $J \rightarrow \infty$. In our case, with $J \equiv 1$, this term is non-negligible, and in fact determines the limiting SBM branching rate. Its analysis is the main objective of Section 7. The analysis for $d \geq 3$ is straightforward; it is the 2-dimensional case (the most relevant from a biological perspective) that is the most interesting. In this setting the proof requires an extension of the arguments in [6] and [8] used to analyze the voter model and stochastic Lotka-Volterra models, respectively. The analogues of Proposition 4.1 in [6] ((I1) in that reference) and [8] (Proposition 4.7 in this work) involved L^2 and L^p ($p > 1$ is used) norms, respectively, instead of the L^1 norm in Proposition 4.1, but also had no supremum over time in the expectation. When the L^2 norm is expanded in the voter model paper this leads to a four-particle dual calculation, while for the more general stochastic Lotka-Volterra models considered in [8], a trick using the Markov property reduced this to a three-particle dual calculation. Here, because of the non-Markovian property of individual coordinates in the dual, similar calculations seem out of reach and we are led to the L^1 convergence in Proposition 4.1 which must be established using only one- and two-particle duals. The first issue here is that squares are easier to handle than absolute values (the $p > 1$ in [8] is bounded eventually by a square using a stopping argument), and here the innocuous looking Lemma 7.9 below allows one to handle the square (even with a supremum over time) by using a martingale argument. This then enables us to take absolute values inside the time integral where two-particle duals (albeit more complicated ones than those in [8]) can handle the calculation. Here a second issue arises as even in handling a second

moment calculation in Proposition 7.2 of [8], the use of stochastic calculus there leads to a three-particle calculation. We follow a more efficient path in its analogue, Lemma 7.8, in Section 7 which only involves the two-particle dual. A third issue is the fact that the weaker L^1 convergence in Proposition 4.1 will require some additional technical work to establish the local uniform integrability of the $\{M^N(\phi)_t^2 : N\}$, and hence identify the limiting square function. This is what occupies most of the proof of Theorem 1.2 in Section 4.

As a small bonus, the fact that Proposition 4.1 controls the square functions uniformly in time means it also allows one to establish tightness without any higher moments. The required properties of the two-particle dual are established in Section 6. Lemma 6.1 represents the difference of the coordinates of the dual as the time change of a continuous time random walk and this result is then used to obtain several probability estimates on the two-particle dual. These results (notably Lemmas 6.3 to 6.7) then play a central role in Section 7. The time-change is particularly useful when controlling the two-particle dual when the particles are close together and the dual motions slow down.

It would be interesting to see if it is possible to extend Theorem 1.2 to the variable but bounded radius case discussed above.

Constants. In proofs, C will denote a positive constant whose value may change from line to line. We will use C_T and C_ϕ for constants depending on $T > 0$ or functions ϕ in a similar way. In some cases constants will be numbered and dependence on various quantities indicated explicitly. Finally, most constants will have an implicit dependence on the impact radius r , this dependence will be pointed out in some cases for clarity.

2 Semimartingale characterization of the SLFV

Let $B_r^N(x) = B_{r/\sqrt{N}}(x)$, $B_r^N = B_r^N(0)$. For $d \geq 2$ and $\phi \in C_b(\mathbb{R}^d)$, let

$$d_s^N(\phi) = \frac{\rho N^{1+d}}{|B_r|} \int_{\mathbb{R}^d} \int_{B_r^N(x)} \left\{ X_s^N(z) \int_{B_r^N(x)} \phi(y) dy - \int_{B_r^N(x)} \phi(y) X_s^N(y) dy \right\} dz dx, \quad (2.1)$$

$$m_s^N(\phi) = \frac{\rho^2 N^{1+d}}{|B_r|} \int_{\mathbb{R}^d} \int_{B_r^N(x)} \left\{ \left(1 - \frac{X_s^N(z)}{K} \right) \left(\int_{B_r^N(x)} \phi(y) X_s^N(y) dy \right)^2 + \frac{X_s^N(z)}{K} \left(K \int_{B_r^N(x)} \phi(y) dy - \int_{B_r^N(x)} \phi(y) X_s^N(y) dy \right)^2 \right\} dz dx. \quad (2.2)$$

Lemma 2.1. Let $\phi \in C_b(\mathbb{R}^d)$ and $\Delta X_s^N(\phi) = |X_s^N(\phi) - X_{s-}^N(\phi)|$. Then with probability one,

$$|\Delta X_s^N(\phi)| \leq \|\phi\|_\infty \begin{cases} \rho |B_r| / N & \text{if } d \geq 3 \\ \rho |B_r| \log N / N & \text{if } d = 2 \end{cases} \quad \text{for all } s \geq 0. \quad (2.3)$$

Proof. By the dynamics (1.1), for $(x, s) \in \Pi^N$ (we may assume there is at most one such x), $w_s^N(y) = w_{s-}^N(y)$ for all $y \notin B_r^N(x)$, and for $y \in B_r^N(x)$,

$$w_s^N(y) - w_{s-}^N(y) = \begin{cases} -\rho w_{s-}^N(y) + \rho, & \text{or} \\ -\rho w_{s-}^N(y). \end{cases}$$

Thus, $|w_s^N(y) - w_{s-}^N(y)| \leq \rho 1_{B_r^N(x)}(y)$, and so,

$$\int_{\mathbb{R}^d} |w_s^N(y) - w_{s-}^N(y)| dy \leq \sup_{x \in \mathbb{R}^d} \int_{B_r^N(x)} \rho dy \leq \rho |B_r^N|.$$

Finally,

$$|\Delta X_s^N(\phi)| \leq K \int_{\mathbb{R}^d} |\phi(y)| |w_s^N(y) - w_{s-}^N(y)| dy \leq \|\phi\|_\infty \rho |B_r| K N^{-d/2},$$

which is (2.3). □

The martingale characterization below is provided by Lemma 3.1 of [5]. The filtration below is implicit in their argument. Although $\phi = 1$ is not included in that result it is easy to handle it by a localization argument using the stopping times $T_n = \inf\{t \geq 0 : \int_{\{|x| \geq n\}} w_t(x) dx > 0\}$. Recall \mathcal{F}_t^N is defined prior to Theorem 1.0.

Proposition 2.2. *Let $\phi \in C_0^3(\mathbb{R}^d)$ or $\phi = \mathbf{1}$ Then $X_t^N(\phi)$ has the semimartingale decomposition:*

$$X_t^N(\phi) = X_0^N(\phi) + D_t^N(\phi) + M_t^N(\phi), \tag{2.4}$$

where

$$D_s^N(\phi) = \int_0^s d_s^N(\phi) ds \tag{2.5}$$

and $M_t^N(\phi)$ is a local (\mathcal{F}_t^N) -martingale with predictable quadratic variation

$$\langle M^N(\phi) \rangle_t = \int_0^t m_s^N(\phi) ds. \tag{2.6}$$

Implicit in the above is the fact that the local martingale $M_t^N(\phi)$ is locally square integrable, but this is already clear from the fact that it has bounded jumps. The latter follows from Lemma 2.1 and (2.4) which imply

$$|\Delta M_s^N(\phi)| \leq \|\phi\|_\infty \begin{cases} \rho |B_r| / N & \text{if } d \geq 3 \\ \rho |B_r| \log N / N & \text{if } d = 2 \end{cases} \text{ for all } s \geq 0. \tag{2.7}$$

For the drift term $D_s^N(\phi)$ we will need only the following facts.

Lemma 2.3.

- (a) $d_s^N(\mathbf{1}) \equiv 0$.
- (b) For $\phi \in C_0^3(\mathbb{R}^d)$ there is a constant $C_{2.8} = C_{2.8}(\phi) > 0$ such that

$$d_s^N(\phi) = \rho |B_r| \sigma^2 X_s^N \left(\frac{1}{2} \Delta \phi \right) + \mathcal{E}_{2.8}^N(s), \text{ where} \tag{2.8}$$

$$|\mathcal{E}_{2.8}^N(s)| \leq C_{2.8} \frac{X_s^N(\mathbf{1})}{\sqrt{N}}.$$

Part (a) follows easily from (2.1), and (b) is the special case of Lemma 4.2 (and its proof) in [5] for our choices of J, M, K in (1.14). (We note that the constant $C(d)$ in Definition 4.1 in [5] is $\int_{B_1} (x_1)^2 dx$.) Turning next to the martingale square function, for $\phi \in C_b(\mathbb{R}^d)$ define

$$\bar{m}_s^N(\phi) = \rho^2 N^{1+d/2} K^2 \int_{\mathbb{R}^d} \phi^2(x) \int_{B_r^N(x)} \int_{B_r^N(x)} (1 - w_s^N(z_1)) w_s^N(z_2) dz_2 dz_1 dx \geq 0. \tag{2.9}$$

Lemma 2.4.

- (a) For $\phi \in C_0^3(\mathbb{R}^d)$, $\bar{m}_s^N(\phi) \vee m_s^N(\phi) \leq \|\phi\|_\infty^2 m_s^N(\mathbf{1})$.
- (b) $m_s^N(\mathbf{1}) = \bar{m}_s^N(\mathbf{1})$.
- (c) For $\phi \in C_0^3(\mathbb{R}^d)$ there is a constant $C_{2.10} = C_{2.10}(\phi) > 0$ such

$$m_s^N(\phi) = \bar{m}_s^N(\phi) + \mathcal{E}_{2.10}^N(s), \text{ where} \tag{2.10}$$

$$|\mathcal{E}_{2.10}^N(s)| \leq C_{2.10} \begin{cases} \frac{1}{\sqrt{N}} X_s^N(\mathbf{1}) & \text{if } d \geq 3 \\ \frac{\log N}{\sqrt{N}} X_s^N(\mathbf{1}) & \text{if } d = 2. \end{cases}$$

Proof. Define

$$\mathcal{I}(z_1, z_2, z_3) = (1 - w_s^N(z_1))w_s^N(z_2)w_s^N(z_3) + w_s^N(z_1)(1 - w_s^N(z_2))(1 - w_s^N(z_3)).$$

Then replacing X_s^N with Kw_s^N in (2.2), after expanding and rearranging, we find that

$$m_s^N(\phi) = \frac{\rho^2 N^{1+d}}{|B_r|} K^2 \int_{\mathbb{R}^d} \int_{B_r^N(x)} \int_{B_r^N(x)} \int_{B_r^N(x)} \phi(z_2)\phi(z_3)\mathcal{I}(z_1, z_2, z_3) dz_3 dz_2 dz_1 dx$$

On account of $0 \leq w_s^N \leq 1$, \mathcal{I} is nonnegative, hence (a) follows for m^N from the above expression, and is immediate for \bar{m}^n from (2.9) (integrate out z_3 in the first line on the right-hand side). If we define $\Delta_\phi(x, z_2, z_3) = \phi(z_2)\phi(z_3) - \phi^2(x)$, then

$$m_s^N(\phi) = \frac{\rho^2 N^{1+d}}{|B_r|} K^2 \int_{\mathbb{R}^d} \int_{B_r^N(x)} \int_{B_r^N(x)} \int_{B_r^N(x)} \phi^2(x)\mathcal{I}(z_1, z_2, z_3) dz_3 dz_2 dz_1 dx + \mathcal{E}_{2.10}^N(s), \tag{2.11}$$

where

$$\mathcal{E}_{2.10}^N(s) = \rho^2 \frac{N^{1+d}}{|B_r|} K^2 \int_{\mathbb{R}^d} \int_{B_r^N(x)} \int_{B_r^N(x)} \int_{B_r^N(x)} \Delta_\phi(x, z_2, z_3)\mathcal{I}(z_1, z_2, z_3) dz_3 dz_2 dz_1 dx.$$

Consider the integrals over $B_r^N(x)$ in (2.11). By a change of variables and order of integration,

$$\begin{aligned} & \int_{B_r^N(x)} \int_{B_r^N(x)} \int_{B_r^N(x)} \mathcal{I}(z_1, z_2, z_3) dz_3 dz_2 dz_1 \\ &= \int_{B_r^N(x)} w_s^N(z_3) \int_{B_r^N(x)} \int_{B_r^N(x)} (1 - w_s^N(z_1))w_s^N(z_2) dz_2 dz_1 dz_3 \\ & \quad + \int_{B_r^N(x)} (1 - w_s^N(z_3)) \int_{B_r^N(x)} \int_{B_r^N(x)} (1 - w_s^N(z_1))w_s^N(z_2) dz_2 dz_1 dz_3 \\ &= |B_r^N| \int_{B_r^N(x)} \int_{B_r^N(x)} (1 - w_s^N(z_1))w_s^N(z_2) dz_2 dz_1. \end{aligned}$$

Plugging this into (2.11), and using the definition of $\bar{m}_s^N(\phi)$, we now have $m_s^N(\phi) = \bar{m}_s^N(\phi) + \mathcal{E}_{2.10}^N(s)$.

In the case that $\phi = \mathbf{1}$, $\Delta_\mathbf{1} \equiv 0$ so that $\mathcal{E}_{2.10}^N(s) \equiv 0$, proving (b). More generally,

$$|\Delta_\phi(x, z_2, z_3)| \leq 2(r/\sqrt{N})\|\phi\|_\infty\|\phi\|_{\text{Lip}} \text{ for } z_2, z_3 \in B_r^N(x).$$

Using the fact that $|\mathcal{I}(z_1, z_2, z_3)| \leq w_s^N(z_2) + w_s^N(z_1)$, we have

$$\begin{aligned} |\mathcal{E}_{2.10}^N(s)| &\leq \rho^2 \frac{N^{1+d}}{|B_r|} K^2 2(r/\sqrt{N})\|\phi\|_\infty\|\phi\|_{\text{Lip}} \int_{\mathbb{R}^d} \int_{B_r^N(x)} \int_{B_r^N(x)} \int_{B_r^N(x)} (w_s^N(z_1) \\ & \quad + w_s^N(z_2)) dz_2 dz_1 dz_3 dx \\ &= 4\rho^2 \frac{r}{N^{1/2}}\|\phi\|_\infty\|\phi\|_{\text{Lip}} \frac{N^{1+d}}{|B_r|} |B_r^N|^2 K^2 \int_{\mathbb{R}^d} \int_{B_r^N(x)} w_s^N(z_1) dz_1 dx \\ &= 4r\rho^2 |B_r|^2 \|\phi\|_\infty\|\phi\|_{\text{Lip}} \begin{cases} \frac{1}{\sqrt{N}} X_s^N(\mathbf{1}) & \text{if } d \geq 3 \\ \frac{\log N}{\sqrt{N}} X_s^N(\mathbf{1}) & \text{if } d = 2. \end{cases} \end{aligned}$$

This proves (c). □

3 Total mass bounds

We start with the dual particle systems for the rescaled SLFV process w_t^N in (1.19). If η and (ξ^1, ξ^2) are as in (1.8), (1.9), introduce the rescaled duals,

$$P_{\{x\}}^N(\eta_t^N \in \cdot) = P_{\{\sqrt{N}x\}}\left(\frac{\eta_{Nt}}{\sqrt{N}} \in \cdot\right), \quad x \in \mathbb{R}^d,$$

$$P_{\{x_1, x_2\}}^N((\xi_t^{N,1}, \xi_t^{N,2}) \in \cdot) = P_{\{\sqrt{N}x_1, \sqrt{N}x_2\}}\left(\left(\frac{\xi_{Nt}^1}{\sqrt{N}}, \frac{\xi_{Nt}^2}{\sqrt{N}}\right) \in \cdot\right) \quad x_i \in \mathbb{R}^d,$$

and

$$\tau^N = \inf\{t : \xi_t^{N,1} = \xi_t^{N,2}\}. \tag{3.1}$$

Then (1.8) and (1.9) imply for Borel ψ_1 on \mathbb{R}^d , and Borel ψ_2 on $(\mathbb{R}^d)^2$, and $t \geq 0$ (recall (1.2)),

$$E_{w_0^N}[w_t^N(\psi_1)] = \int_{\mathbb{R}^d} \psi_1(x) E_{\{x\}}^N(w_0^N(\eta_t^N)) dx, \tag{3.2}$$

$$\begin{aligned} E_{w_0^N} \left[\int_{\mathbb{R}^d \times \mathbb{R}^d} \psi_2(x_1, x_2) w_t^N(x_1) w_t^N(x_2) dx_1 dx_2 \right] \\ = \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi_2(x_1, x_2) E_{\{x_1, x_2\}}^N [w_0^N(\xi_t^{N,1}) 1_{\{\tau^N \leq t\}} + w_0^N(\xi_t^{N,1}) w_0^N(\xi_t^{N,2}) 1_{\{\tau^N > t\}}] dx_1 dx_2. \end{aligned} \tag{3.3}$$

As before, either $\psi_i \geq 0$, or one side is integrable for the above to hold. A simple change of variables shows that (3.2) implies (for ψ_1 as above)

$$E(X_t^N(\psi_1)) = E_{X_0^N}(\psi_1(\eta_t^N)) := \int_{\mathbb{R}^d} E_{\{x\}}^N(\psi_1(\eta_t^N)) X_0^N(dx). \tag{3.4}$$

Proposition 3.1.

(a) There exists $C_{3.5} > 0$ such that for $s \geq 0$,

$$\bar{m}_s^N(\mathbf{1}) = m_s^N(\mathbf{1}) \leq C_{3.5} \begin{cases} X_s^N(\mathbf{1}) & \text{if } d \geq 3 \\ (\log N) X_s^N(\mathbf{1}) & \text{if } d = 2 \end{cases} \tag{3.5}$$

(b) Assume $d = 2$. If $0 < \alpha < 1$ then there exists $C_{3.6} = C_{3.6}(\alpha) > 0$ such that for $s \geq 0$,

$$E(m_s^N(\mathbf{1})) \leq C_{3.6} (1 + s^{-\alpha}) X_0^N(\mathbf{1}). \tag{3.6}$$

Proof. (a) The case $d \geq 3$ with $K = N^{d/2-1}$ is straightforward. Using the definition of \bar{m}_s^N , Lemma 2.4(b) and the fact that $0 \leq w_s^N \leq 1$,

$$\begin{aligned} m_s^N(\mathbf{1}) = \bar{m}_s^N(\mathbf{1}) &\leq \rho^2 N^d K |B_r^N| \int_{\mathbb{R}^d} \int_{B_r^N(x)} w_s^N(z_2) dz_2 dx = \rho^2 N^d |B_r^N|^2 X_s^N(\mathbf{1}) \\ &= \rho^2 |B_r|^2 X_s^N(\mathbf{1}). \end{aligned}$$

Now suppose $d = 2$, with $K = \log N$. Write $(1 - w_s^N(z_1))w_s^N(z_2) = w_s^N(z_2) - w_s^N(z_1)w_s^N(z_2)$, and so conclude that

$$m_s^N(\mathbf{1}) = \bar{m}_s^N(\mathbf{1}) = m_s^{N,1} - m_s^{N,2}, \tag{3.7}$$

where

$$\begin{aligned} m_s^{N,1} &= \rho^2 N^2 (\log N)^2 \int_{\mathbb{R}^2} \int_{B_r^N(x)} \int_{B_r^N(x)} w_s^N(z_2) dz_1 dz_2 dx \\ m_s^{N,2} &= \rho^2 N^2 (\log N)^2 \int_{\mathbb{R}^2} \int_{B_r^N(x)} \int_{B_r^N(x)} w_s^N(z_1) w_s^N(z_2) dz_1 dz_2 dx. \end{aligned}$$

Thus, $m_s^N(\mathbf{1}) \leq m_s^{N,1}$, and

$$\begin{aligned} m_s^{N,1} &= \rho^2 N^2 |B_r^N| (\log N) \int_{\mathbb{R}^2} \int_{B_r^N(x)} X_s^N(z_2) dz_2 dx \\ &= \rho^2 N^2 |B_r^N|^2 (\log N) X_s^N(\mathbf{1}) \\ &= \rho^2 |B_r|^2 (\log N) X_s^N(\mathbf{1}). \end{aligned} \tag{3.8}$$

This completes the proof of (3.5).

(b) For $d = 2$, using the two particle duality equation (3.3) with $\psi_2 \equiv 1$, we have

$$\begin{aligned} E[m_s^{N,2}] &= \rho^2 N^2 (\log N)^2 \int_{\mathbb{R}^2} \int_{B_r^N(x)} \int_{B_r^N(x)} E[w_s^N(z_1) w_s^N(z_2)] dz_1 dz_2 dx \\ &= \rho^2 N^2 (\log N)^2 \int_{\mathbb{R}^2} \int_{B_r^N(x)} \int_{B_r^N(x)} \left(E_{\{z_1, z_2\}}^N [w_0^N(\xi_s^{N,1}) 1\{\tau^N \leq s\}] \right. \\ &\quad \left. + E_{\{z_1, z_2\}}^N [w_0^N(\xi_s^{N,1}) w_0^N(\xi_s^{N,2}) 1\{\tau^N > s\}] \right) dz_1 dz_2 dx \\ &\geq \rho^2 N^2 (\log N) \int_{\mathbb{R}^2} \int_{B_r^N(x)} \int_{B_r^N(x)} E_{\{z_1, z_2\}}^N [X_0^N(\xi_s^{N,1}) 1\{\tau^N \leq s\}] dz_1 dz_2 dx \\ &= \rho^2 N^2 (\log N) \int_{\mathbb{R}^2} \int_{B_r^N} \int_{B_r^N} E_{\{x+z'_1, x+z'_2\}}^N [X_0^N(\xi_s^{N,1}) 1\{\tau^N \leq s\}] dz'_1 dz'_2 dx \\ &= \rho^2 N^2 (\log N) \int_{\mathbb{R}^2} \int_{B_r^N} \int_{B_r^N} E_{\{z'_1, z'_2\}}^N [X_0^N(x + \xi_s^{N,1}) 1\{\tau^N \leq s\}] dz'_1 dz'_2 dx, \end{aligned}$$

the last two equalities by a change of variables and translation invariance (1.7). Let $P_{\{U_1, U_2\}}(\cdot) = \int_{B_r} \int_{B_r} P_{\{z'_1, z'_2\}}(\cdot) dz'_1 dz'_2 / |B_r|^2$ so that $P_{\{U_1, U_2\}}(\tau > t) = \gamma_e(t)$. Integrating x out (inside the expectation), we obtain

$$\begin{aligned} E[m_s^{N,2}] &\geq \rho^2 N^2 (\log N) X_0^N(\mathbf{1}) \int_{B_r^N} \int_{B_r^N} P_{\{z'_1, z'_2\}}^N(\tau^N \leq s) dz'_1 dz'_2 \\ &= \rho^2 N^2 |B_r^N|^2 (\log N) P_{\{U_1, U_2\}}(\tau \leq Ns) X_0^N(\mathbf{1}) \\ &= \rho^2 |B_r|^2 (\log N) (1 - \gamma_e(Ns)) X_0^N(\mathbf{1}). \end{aligned}$$

If we plug this bound and (3.8) into (3.7), we get

$$E[m_s^N(\mathbf{1})] \leq \rho^2 |B_r|^2 (\log N) \gamma_e(Ns) X_0^N(\mathbf{1}). \tag{3.9}$$

If $s \leq (\log N)^{-\alpha}$, then $(\log N) \gamma_e(Ns) \leq \log N \leq s^{-1/\alpha}$. By Proposition 1.1, if $s \geq (\log N)^{-\alpha}$,

$$(\log N) \gamma_e(Ns) \leq (\log N) \gamma_e\left(\frac{N}{(\log N)^\alpha}\right) \rightarrow \gamma_e$$

as $N \rightarrow \infty$. This implies that there is a $C > 0$, depending on α , such that for all $s \geq (\log N)^{-\alpha}$, $(\log N) \gamma_e(Ns) \leq C$. Plugging these bounds into (3.9) we obtain (3.6). \square

Corollary 3.2. For $d \geq 2$ and $T > 0$ there exists $C_{3.10}(T) > 0$ such that

$$E\left[\sup_{t \leq T} X_t^N(\mathbf{1})^2\right] \leq 2X_0^N(\mathbf{1})^2 + C_{3.10}(T) X_0^N(\mathbf{1}). \tag{3.10}$$

Moreover $t \rightarrow X_t^N(\mathbf{1})$ is a non-negative square-integrable (\mathcal{F}_t^N) -martingale, and for any $\phi \in C_0^3(\mathbb{R}^d)$, $t \rightarrow M_t^N(\phi)$ is also a square-integrable (\mathcal{F}_t^N) -martingale.

Proof. By Proposition 2.2 and Lemma 2.3(a), $X_t^N(\mathbf{1}) = X_0^N(\mathbf{1}) + M_t^N(\mathbf{1})$, and so

$$X_t^N(\mathbf{1})^2 \leq 2X_0^N(\mathbf{1})^2 + 2M_t^N(\mathbf{1})^2.$$

Now $X_t^N(\mathbf{1})$ is a non-negative local martingale which by (1.10) satisfies $E[X_t^N(\mathbf{1})] = X_0^N(\mathbf{1})$, and so is a non-negative martingale. By Doob's L^2 submartingale inequality,

$$E\left[\sup_{t \leq T} M_t^N(\mathbf{1})^2\right] \leq 4E[\langle M^N(\mathbf{1}) \rangle_T] = 4 \int_0^T E[m_s^N(\mathbf{1})] ds.$$

As we don't know the square integrability yet, the first inequality holds by considering a sequence of localizing stopping times and applying monotone convergence. By Proposition 3.1(a), for $d \geq 3$,

$$\int_0^T E[m_s^N(\mathbf{1})] ds \leq C_{3.5} \int_0^T E[X_s^N(\mathbf{1})] ds = C_{3.5} T X_0^N(\mathbf{1}).$$

By Proposition 3.1(b), for $d = 2$ and taking $\alpha = 1/2$,

$$\int_0^T E[m_s^N(\mathbf{1})] ds \leq C_{3.6}(\frac{1}{2}) X_0^N(\mathbf{1}) \int_0^T (1 + s^{-1/2}) ds = C_{3.6}(\frac{1}{2})(T + 2T^{1/2}) X_0^N(\mathbf{1}).$$

Combining the above bounds we obtain (3.10) and hence the next to last statement as well.

It is easy to repeat the above reasoning using Lemma 2.4(a) and see that

$$E\left[\sup_{t \leq T} M_t^N(\phi)^2\right] \leq C_\phi [T + \sqrt{T}] X_0^N(\mathbf{1}). \tag{3.11}$$

This in turn shows that the local martingale $M^N(\phi)$ is in fact a square integrable martingale. □

4 Proof of Main Result

The proof of Theorem 1.2 proceeds by taking limits as $N \rightarrow \infty$ in Proposition 2.2 to derive the martingale problem for the limiting super-Brownian motion. The main issue is the identification of the square function of the limiting martingale part and the key here is the following result:

Proposition 4.1. *For all $A, T > 0$, and $\phi \in C_0^3$,*

$$\sup_{X_0^N(\mathbf{1}) \leq A} E \left[\sup_{0 \leq t \leq T} \left| \langle M^N(\phi) \rangle_t - \int_0^t \rho^2 |B_r|^2 \gamma_e X_s^N(\phi^2) ds \right| \right] \rightarrow 0 \text{ as } N \rightarrow \infty. \tag{4.1}$$

This will be proved in Section 7. In this section we will establish Theorem 1.2, assuming this result. If S is a metric space, recall that a sequence of laws on $D(\mathbb{R}_+, S)$ is C -tight iff it is tight and all limit laws are continuous. C -tightness on $D(\mathbb{R}_+, S) \times C(\mathbb{R}_+, S)$ is then defined in the obvious manner. The first step is to prove:

Lemma 4.2. *If $\phi \in C_0^3(\mathbb{R}^d)$, then $\{(X^N(\phi), \langle M^N(\phi) \rangle) : N \geq 3\}$ is C -tight in $D(\mathbb{R}_+, \mathbb{R}) \times C(\mathbb{R}_+, \mathbb{R})$.*

Proof. Let $\phi \in C_0^3(\mathbb{R}^d)$ and $A^N(t) = \int_0^t \rho^2 |B_r|^2 \gamma_e X_s^N(\phi^2) ds$. Then for $0 \leq s < t \leq T$,

$$E(|A_t^N - A_s^N|^2) \leq CE(\sup_{u \leq T} X_u^N(\mathbf{1})^2)(t - s)^2 \leq C(T)(t - s)^2,$$

by Corollary 3.2 and (1.18). Therefore, the collection of continuous increasing processes $\{A^N : N \geq 3\}$ is tight, and hence relatively compact, in $C(\mathbb{R}_+, \mathbb{R})$ by Prohorov's theorem. It then follows from Proposition 4.1 that the sequence of continuous (recall (2.6)) increasing processes $\{\langle M^N(\phi) \rangle : N \geq 3\}$ is relatively compact in $C(\mathbb{R}_+, \mathbb{R})$, and so also tight by Prohorov's theorem again.

Next, recall $D^N(\phi)$ from Proposition 2.2. Lemma 2.3 implies that $D_t^N(\phi) = \int_0^t d_s^N(\phi) ds$, where

$$|d_s^N(\phi)| \leq C_\phi X_s^N(\mathbf{1}) \quad \forall s \geq 0.$$

Therefore if $0 \leq s < t \leq T$, then by the above and Corollary 3.2,

$$E((D_t^N(\phi) - D_s^N(\phi))^2) \leq C_\phi^2 \int_s^t \int_s^t E(X_u^N(\mathbf{1})X_v^N(\mathbf{1})) dv du \leq C(\phi, T)(t - s)^2.$$

This implies $\{D^N(\phi) : N \geq 3\}$ is tight in $C(\mathbb{R}_+, \mathbb{R})$. Lemma 2.1 and Proposition 2.2 imply

$$\sup_{s \leq T} |\Delta M_s^N(\phi)| = \sup_{s \leq T} |\Delta X_s^N(\phi)| \rightarrow 0 \text{ a.s. as } N \rightarrow \infty. \tag{4.2}$$

Using (4.2) and the C -tightness of $\{\langle M^N(\phi) \rangle : N \geq 3\}$, established above, in Theorem VI.4.13 and Proposition IV.3.26 of [20], we see that $\{M^N(\phi) : N \geq 3\}$ is C -tight in $D(\mathbb{R}_+, \mathbb{R})$. C -tightness of $\{X^N(\phi)\}$ now follows from the above, our assumption on the initial conditions $\{X_0^N\}$ (i.e., (1.18)), and the semimartingale decomposition in Proposition 2.2. Having obtained C -tightness of each component, the result is now immediate. \square

Proposition 4.3. *For $d \geq 2$, under the assumptions of Theorem 1.2, the family $\{P(X^N \in \cdot) : N \geq 3\}$ is C -tight in $D(\mathbb{R}_+, \mathcal{M}_F(\mathbb{R}^d))$.*

Proof. By the Kurtz-Jakubowski theorem (e.g. see Proposition 3.1 in [6]) it suffices to show:

1. For each $T, \varepsilon > 0$ there is a compact set $K_{T,\varepsilon} \subset \mathbb{R}^d$ such that

$$\limsup_{N \rightarrow \infty} P\left(\sup_{t \leq T} X_t^N(K_{T,\varepsilon}^c) > \varepsilon\right) < \varepsilon. \tag{4.3}$$

2. For each $T > 0$,

$$\lim_{H \rightarrow \infty} \limsup_{N \rightarrow \infty} P\left(\sup_{t \leq T} X_t^N(\mathbf{1}) \geq H\right) = 0. \tag{4.4}$$

3. For each $\phi \in C_0^\infty(\mathbb{R}^d)$,

$$\{X^N(\phi), N \geq 3\} \text{ is } C\text{-tight in } D(\mathbb{R}_+, \mathbb{R}). \tag{4.5}$$

The last (4.5) holds by Lemma 4.2, and (4.4) is immediate from Corollary 3.2. The compact containment (4.3) is proved exactly as for the voter model in Lemma 3.3 of [6]. The argument there will now use the semimartingale decomposition in Proposition 2.2, the convergence of the initial states from (1.18), Lemma 2.3, and first moment bounds which are immediate from (3.4). This completes the proof. \square

We are ready to turn to the main result.

Proof of Theorem 1.2. By Proposition 4.3 it suffices to show that every weak subsequential limit is the super-Brownian motion described in the Theorem. Fix $\phi \in C_0^3(\mathbb{R}^d)$. By

Lemma 4.2 and Skorokhod’s theorem, and then taking a further subsequence, we may assume that we are on a probability space where

$$(X^{N_k}, \langle M^{N_k}(\phi) \rangle) \rightarrow (X, A^\phi) \text{ a.s. in } D(\mathbb{R}_+, \mathcal{M}_F(\mathbb{R}^d)) \times C(\mathbb{R}_+, \mathbb{R}). \quad (4.6)$$

Since the limit is continuous a.s. one has in fact a.s. uniform convergence on compact time intervals. It also follows from the above and Corollary 3.2 that

$$\begin{aligned} \sup_{t \leq T} [|X_t^{N_k}(\mathbf{1}) - X_t(\mathbf{1})| + |X_t^{N_k}(\phi) - X_t(\phi)| + |X_t^{N_k}(\phi^2) - X_t(\phi^2)| + |X_t^{N_k}(\Delta\phi) - X_t(\Delta\phi)|] \\ \rightarrow 0 \text{ a.s. and in } L^1 \text{ as } k \rightarrow \infty \text{ for all } T > 0. \end{aligned} \quad (4.7)$$

This and Proposition 4.1 show that

$$A_t^\phi = \int_0^t \rho^2 |B_r|^2 \gamma_e X_s(\phi^2) ds \text{ for all } t \geq 0. \quad (4.8)$$

It follows from (4.6), (4.7), and Proposition 4.1 that

$$\sup_{t \leq T} |\langle M^{N_k}(\phi) \rangle_t - A_t^\phi| \rightarrow 0 \text{ a.s. and in } L^1 \text{ as } k \rightarrow \infty \text{ for all } T > 0, \quad (4.9)$$

Lemma 2.3, Corollary 3.2, and (4.7) imply

$$\sup_{t \leq T} \left| D_t^{N_k}(\phi) - \int_0^t \rho |B_r| \bar{\sigma}^2 X_s(\Delta\phi/2) ds \right| \rightarrow 0 \text{ as } k \rightarrow \infty \text{ a.s. and in } L^1 \text{ as } k \rightarrow \infty. \quad (4.10)$$

Define an a.s. continuous process by $M_t(\phi) = X_t(\phi) - X_0(\phi) - \int_0^t \rho |B_r| \bar{\sigma}^2 X_s(\Delta\phi/2) ds$. Then the above, the convergence of the initial conditions in (1.18), and the semimartingale decomposition for $X^N(\phi)$ in Proposition 2.2 show that

$$\sup_{t \leq T} |M_t^{N_k}(\phi) - M_t(\phi)| \rightarrow 0 \text{ in } L^1 \text{ and a.s. as } k \rightarrow \infty \text{ for all } T > 0. \quad (4.11)$$

Since $M^{N_k}(\phi)$ is an $(\mathcal{F}_t^{X^{N_k}})$ -martingale by Corollary 3.2, it follows from the above that $M(\phi)$ is a continuous martingale and a standard argument (e.g. see the proof of Theorem 3.5 in [6]) shows it is in fact an (\mathcal{F}_t^X) -martingale.

Recalling (1.12), (4.8), and the value of b in Theorem 1.2, it remains to identify the square function of $M(\phi)$ as A^ϕ by showing

$$M_t(\phi)^2 - A_t(\phi) \text{ is a local martingale.} \quad (4.12)$$

For $d \geq 3$ this is fairly easy, but we give a stopping argument to include the more delicate 2-dimensional case. For $J \in \mathbb{N}$ define

$$T_J^N = \inf\{t : |M_t^N(\phi)| \geq J\}, \quad T_J = \inf\{t : |M_t(\phi)| \geq J\}.$$

The convergence in (4.11) readily shows that

$$\liminf_{k \rightarrow \infty} T_J^{N_k} \geq T_J \quad \forall J \in \mathbb{N} \text{ a.s.} \quad (4.13)$$

We claim that

$$\lim_{k \rightarrow \infty} \sup_{t \leq T} \left[|M_{t \wedge T_J^{N_k}}^{N_k}(\phi) - M_{t \wedge T_J}(\phi)| + |\langle M^{N_k}(\phi) \rangle_{t \wedge T_J^{N_k}} - A_{t \wedge T_J}^\phi| \right] = 0 \text{ for all } T > 0 \text{ a.s.} \quad (4.14)$$

The reason there is an issue here is that we do not know whether or not $\lim_k T_J^{N_k} = T_J$ a.s. It follows from (4.13) that for $t \leq T_J$ we have $\lim_k T_J^{N_k} \wedge t = t = T_J \wedge t$ (the convergence is uniform for $t \leq T_J \wedge T$ for any fixed T) and therefore by (4.11) and (4.9),

$$\lim_{k \rightarrow \infty} \sup_{t \leq T_J \wedge T} \left[|M_{t \wedge T_J^{N_k}}^{N_k}(\phi) - M_{t \wedge T_J}(\phi)| + |\langle M^{N_k}(\phi) \rangle_{t \wedge T_J^{N_k}} - A_{t \wedge T_J}^\phi| \right] = 0 \text{ for all } T > 0 \text{ a.s.} \tag{4.15}$$

A simple calculation using (4.13) shows that $(\sup \emptyset := 0)$ with probability one for any $T > 0$,

$$\begin{aligned} \limsup_{k \rightarrow \infty} \sup_{T_J^{N_k} \leq t \leq T} & \left[|M_{t \wedge T_J^{N_k}}^{N_k}(\phi) - M_{t \wedge T_J}(\phi)| + |\langle M^{N_k}(\phi) \rangle_{t \wedge T_J^{N_k}} - A_{t \wedge T_J}^\phi| \right] \\ &= \limsup_{k \rightarrow \infty} 1(T_J^{N_k} \leq T) \left[|M_{T_J^{N_k}}^{N_k}(\phi) - M_{T_J}(\phi)| + |\langle M^{N_k}(\phi) \rangle_{T_J^{N_k}} - A_{T_J}^\phi| \right]. \end{aligned} \tag{4.16}$$

In view of the above and (4.15), to prove (4.14) it suffices to show that for $T > 0$ fixed,

$$\limsup_{k \rightarrow \infty} \sup_{T_J < t \leq T_J^{N_k} \wedge T} \left[|M_t^{N_k}(\phi) - M_{T_J}(\phi)| + |\langle M^{N_k}(\phi) \rangle_t - A_{T_J}^\phi| \right] = 0 \text{ a.s.} \tag{4.17}$$

By (4.9) and (4.11) this would follow from

$$\limsup_{k \rightarrow \infty} \sup_{T_J < t \leq T_J^{N_k} \wedge T} \left[|M_t(\phi) - M_{T_J}(\phi)| + |A_t^\phi - A_{T_J}^\phi| \right] = 0 \text{ a.s.} \tag{4.18}$$

For this we will use the following lemma, whose proof is deferred to the end of this section.

Lemma 4.4. *With probability one, for all $0 \leq s < t$, $M_u(\phi) = M_s(\phi)$ for all $u \in [s, t]$ implies that $A_t^\phi = A_s^\phi$.*

By the Dubins-Schwarz theorem we may assume $M_t(\phi) = B(\langle M(\phi) \rangle_t)$ for some Brownian motion, B , on our probability space. We let T_J^B and T_{J+}^B denote the exit times of B from $(-J, J)$ and $[-J, J]$, respectively. On $\{\limsup_k T_J^{N_k} \wedge T \leq T_J\}$, (4.18) follows from the a.s. continuity of $M(\phi)$ and A^ϕ . So assume ω is in $\{\limsup_k T_J^{N_k} \wedge T > T_J\}$ and also outside of a null set so that:

- (i) (4.9) and (4.11) hold;
- (ii) for all $s < t$, $\langle M(\phi) \rangle_t = \langle M(\phi) \rangle_s$ implies $M_u(\phi) = M_s(\phi)$ for all $u \in [s, t]$;
- (iii) the conclusion of Lemmas 2.1 and 4.4 hold;
- (iv) $T_J^B = T_{J+}^B$.

Use the conclusion of Lemma 2.1 and our choice of ω to see that $\forall t \in [0, \limsup_k T_J^{N_k} \wedge T)$,

$$|B(\langle M(\phi) \rangle_t)| = |M_t(\phi)| = \lim_k |M_t^{N_k}(\phi)| \leq \limsup_k J + C \frac{\log N_k}{N_k} = J.$$

We easily see that $T_J^B = \langle (M(\phi)) \rangle_{T_J}$, and so the above shows that $|B(u)| \leq J$ on $[T_J^B, \langle M(\phi) \rangle(\limsup_k T_J^{N_k} \wedge T))$, so the fact that $T_{J+}^B = T_J^B$ (by our choice of ω) implies this interval must be empty. We conclude that $\langle M(\phi) \rangle(\limsup_k T_J^{N_k} \wedge T) = \langle (M(\phi)) \rangle(T_J)$, which by our choice of ω implies that $M_u(\phi) = M_{T_J}(\phi)$ for all $u \in [T_J, \limsup_k T_J^{N_k} \wedge T]$. Lemma 4.4 (and again our choice of ω) shows that this gives $A_u^\phi = A_{T_J}^\phi$ for all $u \in [T_J, \limsup_k T_J^{N_k} \wedge T]$. We have proved (4.18), and hence completed the derivation of (4.14).

Turning at last to (4.12), we see from Lemma 2.1 and Proposition 2.2 that

$$\sup_t |M^{N_k}(\phi)_{T_J^{N_k} \wedge t}| \leq J + C_\phi \frac{\log N_k}{N_k} \leq C(J, \phi) \text{ for all } k. \tag{4.19}$$

This, together with the L^1 convergence of $\langle M^{N_k}(\phi) \rangle_t$ to A_t^ϕ from (4.9), implies that $\{\sup_{t \leq T} |[M^{N_k}(\phi)_{T_j^{N_k \wedge t}}]^2 - \langle M^{N_k}(\phi) \rangle_{T_j^{N_k \wedge t}}| : k \in \mathbb{N}\}$ is uniformly integrable. From (4.14) and the above we can conclude that for each $t \geq 0$,

$$\begin{aligned} N_{T_j^{N_k \wedge t}}^{N_k} &:= (M^{N_k}(\phi)_{T_j^{N_k \wedge t}})^2 - \langle M^{N_k}(\phi) \rangle_{T_j^{N_k \wedge t}} \\ &\rightarrow M(\phi)_{T_j \wedge t}^2 - A_{T_j \wedge t}^\phi \text{ in } L^1 \text{ and a.s.} \end{aligned}$$

As $N_{T_j^{N_k \wedge t}}^{N_k}$ is a martingale (Corollary 3.2), this implies that $N_t = M(\phi)_{T_j \wedge t}^2 - A_{T_j \wedge t}^\phi$ is a martingale for all J which establishes (4.12) and so completes the proof. \square

Proof of Lemma 4.4. By continuity it suffices to prove the result for a fixed pair of times $0 \leq s < t$. For $J, k, n \in \mathbb{N}$, define

$$U_n^{J,k} = \inf\{u \geq s : |(M^{N_k}(\phi)(T_J^{N_k} \wedge u))^2 - (M^{N_k}(\phi)(T_J^{N_k} \wedge s))^2| \geq n^{-1}\} \quad (\inf \emptyset = \infty).$$

It follows from our jump bounds in (2.7) that

$$|\Delta(M_s^{N_k}(\phi)^2)| \leq 2J\|\phi\|_\infty C \frac{\log N_k}{N_k} \quad \text{for all } s \geq 0.$$

Recalling that $M^N(\phi)$ is a square integrable martingale (from Corollary 3.2), we have by optional stopping,

$$\begin{aligned} E\left(\langle M^{N_k}(\phi) \rangle_{U_n^{k,J} \wedge T_J^{N_k} \wedge t} - \langle M^{N_k}(\phi) \rangle_{T_J^{N_k} \wedge s}\right) & \tag{4.20} \\ &= E\left(M^{N_k}(\phi)(U_n^{k,J} \wedge T_J^{N_k} \wedge t)^2 - M^{N_k}(\phi)(U_n^{k,J} \wedge T_J^{N_k} \wedge s)^2\right) \\ &\leq \frac{1}{n} + 2J\|\phi\|_\infty C \frac{\log N_k}{N_k}. \end{aligned}$$

Next use (4.13), and the convergence in (4.9) and (4.11), together with Fatou's lemma, to see that

$$\begin{aligned} E\left((A_t^\phi - A_s^\phi)1(T_J > t)1\left(\sup_{s \leq u \leq t} |M_u(\phi)^2 - M_s(\phi)^2| < \frac{1}{2n}\right)\right) \\ &\leq E\left(\liminf_{k \rightarrow \infty} (\langle M^{N_k}(\phi) \rangle_t - \langle M^{N_k}(\phi) \rangle_s)1(T_J^{N_k} > t)1\left(\sup_{s \leq u \leq t} |M_u^{N_k}(\phi)^2 - M_s^{N_k}(\phi)^2| < \frac{1}{n}\right)\right) \\ &\leq \liminf_{k \rightarrow \infty} E\left((\langle M^{N_k}(\phi) \rangle_{t \wedge T_J^{N_k} \wedge U_n^{k,J}} - \langle M^{N_k}(\phi) \rangle_{s \wedge T_J^{N_k}})1(T_J^{N_k} > t) \right. \\ &\quad \left. \times 1\left(\sup_{s \leq u \leq t} |M_u^{N_k}(\phi)^2 - M_s^{N_k}(\phi)^2| < \frac{1}{n}\right)\right) \\ &\leq \frac{1}{n}, \end{aligned}$$

where the last is by (4.20). Let $J \rightarrow \infty$ and then $n \rightarrow \infty$ to prove the result for $s < t$ fixed, as required. \square

5 Random walk preliminaries

Recall that (ξ_t^1, ξ_t^2) is the two particle dual with initial state (x_1, x_2) . We will need to work with the difference process,

$$\tilde{\xi}_t = \xi_t^1 - \xi_t^2,$$

which is a Markov process starting at $x_1 - x_2$. When $|\tilde{\xi}_t| > 2r$, it makes the same transitions as a rate $2\rho|B_r|$ random walk with jump distribution that of \bar{U} , with density

$h_{\bar{U}}(z)$ given in (1.3). We will need basic information about this random walk, as well as a way to compare $\tilde{\xi}_t$ to it.

Throughout the paper, $Y_t = Y_t^x$ will denote a rate $2\rho|B_r|$ random walk with jump distribution that of \bar{U} starting at x under P_x . That is, Y_t^x will be the pure-jump Markov process on \mathbb{R}^d with generator

$$\mathcal{A}^Y f(x) = 2\rho|B_r| \int_{B_{2r}(0)} (f(x+z) - f(x))h_{\bar{U}}(z)dz \tag{5.1}$$

defined for suitable f . We will often make use of the Poisson process construction

$$Y_t^0 = S_{N_t}, \quad t \geq 0,$$

where N_t is a rate $\lambda = 2\rho|B_r|$ Poisson process on $[0, \infty)$ which is independent of the iid random variables $\bar{U}_1, \bar{U}_2, \dots$ which have the same law as \bar{U} , and $S_n = \bar{U}_1 + \dots + \bar{U}_n, n \geq 1$ ($S_0 = 0$). We will often write Y_t for Y_t^0 and Y_t^x for $x + Y_t^0$, where x may be random. Recall from (1.4) that $E[|\bar{U}|^2] = d\bar{\sigma}^2$.

Lemma 5.1. (a) *There is a constant $C_{5.2} > 0$ such that for all $t \geq 0$ and Borel $B \subset \mathbb{R}^d$,*

$$P(Y_t^{\bar{U}} \in B) \leq \frac{C_{5.2}|B|}{1+t^{d/2}}, \quad \text{and} \tag{5.2}$$

$$P_x(Y_t \in B) \leq e^{-2\rho|B_r|t}1_B(x) + \frac{C_{5.2}|B|}{1+t^{d/2}} \quad \text{for all } x \in \mathbb{R}^d.$$

In particular, for all $x \in \mathbb{R}^d, t > 0$, and nonnegative Borel f ,

$$E_x \left[f\left(\frac{Y_t}{\sqrt{t}}\right) \right] \leq e^{-2\rho|B_r|t}\|f\|_\infty + C_{5.2} \int_{\mathbb{R}^d} f(z)dz \tag{5.3}$$

(b) *For all $t \geq 0, E_0[|Y_t|^2] = 2\rho|B_r|d\bar{\sigma}^2t$. For $k \in \mathbb{N}$ there is a constant $C_{5.4} = C_{5.4}(k) > 0$ such that*

$$E_0[\sup_{s \leq t} |Y_s|^{2k}] \leq C_{5.4}t^k \quad \text{for all } t \geq 1. \tag{5.4}$$

(c) *For $k \in \mathbb{N}$ there is a constant $C_{5.5} = C_{5.5}(k) > 0$*

$$P_0(\sup_{s \leq t} |Y_s| \geq a) \leq C_{5.5} \frac{t^k}{a^{2k}} \quad \text{for all } a > 0, t \geq 1. \tag{5.5}$$

Proof. (a) According to Theorem 19.1 of [3], there is a uniform bound on the densities of $S_n/\sqrt{n}, n = 1, 2, \dots$, so that

$$P\left(\frac{S_n}{\sqrt{n}} \in B\right) \leq C|B| \quad \forall n \geq 1, \text{ Borel } B \subset \mathbb{R}^d. \tag{5.6}$$

By a standard large deviations estimate, for $0 < \alpha < 1$,

$$P(N_t \leq \alpha\lambda t) \leq \exp(-c_\alpha\lambda t),$$

where $c_\alpha = 1 - \alpha + \alpha \log \alpha > 0$. Using $Y_t^x = x + S_{N_t}$ and the density bound (5.6),

$$\begin{aligned} P\left(Y_t^x \in B\right) &= e^{-\lambda t}1_B(x) + \sum_{n=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} P\left(\frac{S_n}{\sqrt{n}} \in \frac{B-x}{\sqrt{n}}\right) \\ &\leq e^{-\lambda t}1_B(x) + C \left\{ \sum_{1 \leq n < \lambda t/2} + \sum_{n \geq 1\vee(\lambda t/2)} \right\} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \frac{|B|}{n^{d/2}} \\ &\leq e^{-\lambda t}1_B(x) + C|B| \left(P(N_t < \lambda t/2) + E[1\{N_t \geq 1 \vee (\lambda t/2)\}N_t^{-d/2}] \right) \\ &\leq e^{-\lambda t}1_B(x) + C|B| \left(e^{-c_{1/2}\lambda t} + (1 \vee (\lambda t/2))^{-d/2} \right) \\ &\leq e^{-\lambda t}1_B(x) + C|B|(1+t)^{-d/2}, \end{aligned}$$

where we have used the large deviation bound with $\alpha = 1/2$. This proves (a) for Y starting at x . The result for $Y_t^{\bar{U}}$ follows from the observation that $Y_t^{\bar{U}}$ has the same law as $S_{N(t)+1}$ and a slight alteration in the above calculation.

(b) It is easy to see from the representation $Y_t^0 = S_{N_t}$ that $E_0[|Y_t|^2] = 2\rho|B_r|tE[|\bar{U}|^2]$. Now consider $k \in \mathbb{N}$. There is a constant $C_k > 0$ such that $E(\max_{m \leq n} |S_m|^{2k}) \leq C_k n^k$. To see this we switch to component notation, and write $\bar{U}_j = (\bar{U}_j^{(1)}, \dots, \bar{U}_j^{(d)})$ and $S_n = (S_n^{(1)}, \dots, S_n^{(d)})$, where $S_n^{(j)} = \sum_{i=1}^n \bar{U}_i^{(j)}$. Then

$$E[\max_{m \leq n} |S_m|^{2k}] = E\left[\max_{m \leq n} \left(\sum_{j=1}^d |S_m^{(j)}|^2\right)^k\right] \leq d^{k-1} E\left[\sum_{j=1}^d \max_{m \leq n} |S_m^{(j)}|^{2k}\right] = d^k E\left[\max_{m \leq n} |S_m^{(1)}|^{2k}\right].$$

Now $S_n^{(1)}$ is a sum of bounded, mean zero independent random variables, so a martingale square function argument (e.g. see Theorem 21.1 of [4]) shows that for each $k \geq 1$ there is a constant $C_k = C_k(r) > 0$ such that $E[\max_{m \leq n} |S_m^{(1)}|^{2k}] \leq C_k n^k$ for all $n \in \mathbb{N}$. This implies

$$E[|Y_t|^{2k}] = \sum_{n=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} E[|S_n|^{2k}] \leq C d^{k+1} \left(\sum_{n=1}^{k-1} e^{-\lambda t} \frac{(\lambda t)^n}{n!} n^k + \sum_{n=k}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} n^k \right).$$

The first sum is bounded by $(k-1)^k$. The second sum is bounded by

$$(\lambda t)^k \sum_{n=k}^{\infty} e^{-\lambda t} \frac{(\lambda t)^{n-k}}{(n-k)!} \prod_{j=1}^k \frac{n}{n-j+1} \leq (\lambda t)^k k^k.$$

This proves (5.4) for $t \geq 1$.

(c) This is immediate from (b) and Markov's inequality. □

For $a, A > 0$, define the hitting times

$$t_a = \inf\{s \geq 0 : |Y_s| \leq a\} \text{ and } T_A = \inf\{s \geq 0 : |Y_s| \geq A\}. \tag{5.7}$$

Proposition 5.2. Assume $d \geq 3$ and $2r < a < |x|$. Then

$$P_x(t_a < \infty) \leq \left(\frac{a}{|x|}\right)^{d-2} \tag{5.8}$$

Proof. Let $A > |x|$. By radial symmetry and (5.1), $f(x) = |x|^{2-d}$ is a harmonic function for Y . If we let $\sigma = t_a \wedge T_A$, then $|Y_{s \wedge \sigma}|^{2-d}$ is a bounded martingale (recall $a > 2r$), and so

$$|x|^{2-d} = E_x[|Y_{t_a}|^{2-d} 1(t_a < T_A)] + E_x[|Y_{T_A}|^{2-d} 1(T_A < t_a)]. \tag{5.9}$$

Clearly $|Y_{T_A}|^{2-d} \leq A^{2-d}$ if $T_A < t_a$, and $|Y_{t_a}|^{2-d} \geq a^{2-d}$ if $t_a < T_A$. This means that if we let $A \rightarrow \infty$ in (5.9), then

$$|x|^{2-d} \geq E_x[a^{2-d} 1(t_a < \infty)] = a^{2-d} P_x(t_a < \infty),$$

proving (5.8). □

Lemma 5.3. Assume $d = 2$. If $A > 2$ and $2r < a < |x| < A$, then

$$P_x(T_A < t_a) \leq \frac{\log |x| - \log(a - 2r)}{\log A - \log(a - 2r)}, \tag{5.10}$$

$$P_x(t_a < T_A) \leq \frac{\log(A + 2r) - \log |x|}{\log(A + 2r) - \log a}, \tag{5.11}$$

and

$$\lim_{A \rightarrow \infty} (\log A) P_x(T_A < t_a) = \log |x| - E_x[\log |Y_{t_a}|]. \tag{5.12}$$

Proof. By radial symmetry and (5.1), $\log|x|$ is a harmonic function for Y . If $\sigma = t_a \wedge T_A$ as before then $\log|Y_{s \wedge \sigma}|$ is a bounded martingale, and

$$\log|x| = E_x[\log|Y_{t_a}|; t_a < T_A] + E_x[\log|Y_{T_A}|; T_A < t_a]. \tag{5.13}$$

Using $|Y_{T_A}| \geq A$ and $|Y_{t_a}| > a - 2r$ in the above gives

$$\log|x| \geq (\log(a - 2r))(1 - P_x(T_A < t_a)) + (\log A)P_x(T_A < t_a).$$

Rearranging gives (5.10). A similar argument yields (5.11).

For (5.12), rearranging (5.13) gives

$$\begin{aligned} & \left| \log|x| - E_x[\log|Y_{t_a}|; t_a < T_A] - (\log A)P_x(T_A < t_a) \right| \\ &= E_x[(\log|Y_{T_A}| - \log A)1(T_A < t_a)] \leq \log\left(\frac{A + 2r}{A}\right) \rightarrow 0 \end{aligned}$$

as $A \rightarrow \infty$. Also, by recurrence and bounded convergence, $E_x[\log|Y_{t_a}|; t_a < T_A] \rightarrow E_x[\log|Y_{t_a}|]$ as $A \rightarrow \infty$, which means that (5.12) must hold. \square

Lemma 5.4. Assume $d = 2$. There is a constant $C_{5.14} > 0$ such that for $A > 2$ and $|x| < A/2$,

$$P_x\left(T_A \notin \left[\frac{A^2}{\log A}, A^2 \log A\right]\right) \leq C_{5.14}/(\log A)^2. \tag{5.14}$$

Proof. By (5.5) with $k = 2$, for all x, A as in the Lemma,

$$\begin{aligned} P_x(T_A \leq \frac{A^2}{\log A}) &= P_0\left(\sup_{s \leq A^2/\log A} |x + Y_s| \geq A\right) \leq P_0\left(\sup_{s \leq A^2/\log A} |Y_s| > A/2\right) \\ &\leq C_{5.5} \frac{(A^2/\log A)^2}{(A/2)^4} = 16C_{5.5}/(\log A)^2. \end{aligned} \tag{5.15}$$

To handle $P_x(T_A \geq A^2 \log A)$ we must first estimate $E_x(T_A^2)$. Let $\sigma_2 = E(|\bar{U}|^2)$, $\sigma_4 = E(|\bar{U}|^4)$ and $\lambda = 2\rho|B_r|$, and define the functions

$$\begin{aligned} u_2(t, y) &= |y|^2 - \lambda\sigma_2 t, \\ u_4(t, y) &= |y|^4 - 4\lambda\sigma_2|y|^2 t + 2(\lambda\sigma_2)^2 t^2 - \lambda\sigma_4 t. \end{aligned}$$

It is a straightforward calculation to check that both $u = u_2$ and $u = u_4$ satisfy

$$\frac{\partial u}{\partial t} + \mathcal{A}^Y u \equiv 0.$$

This and the fact that for $p = 2, 4$, $|y|^p$, and $\mathcal{A}^Y(|y|^p)$ are bounded on $\{|y| \leq A + 2r\}$, so that $u_p(t \wedge T_A, Y_{t \wedge T_A})$ is uniformly bounded for $t \leq t_0$, easily imply that both $u_2(t \wedge T_A, Y_{t \wedge T_A})$ and $u_4(t \wedge T_A, Y_{t \wedge T_A})$ are martingales. Therefore $E_0[u_2(t \wedge T_A, Y_{t \wedge T_A})] = 0$, and since $|Y_{T_A}| \leq A + 2r$,

$$\lambda\sigma_2 E_0[T_A] = \lim_{t \rightarrow \infty} \lambda\sigma_2 E_0[T_A \wedge t] = \lim_{t \rightarrow \infty} E_0[(Y_{T_A \wedge t})^2] \leq (A + 2r)^2. \tag{5.16}$$

Now, since $E_0[u_4(T_A \wedge t, Y_{T_A \wedge t})] = 0$,

$$\begin{aligned} 2(\lambda\sigma_2)^2 E_0[(T_A \wedge t)^2] &= -E_0[|Y_{T_A \wedge t}|^4] + 4\lambda\sigma_2 E_0[(T_A \wedge t)|Y_{T_A \wedge t}|^2] + \lambda\sigma_4 E_0[T_A \wedge t] \\ &\leq 4\lambda\sigma_2(A + 2r)^2 E_0[T_A] + \lambda\sigma_4 E_0[T_A] \\ &\leq 4(A + 2r)^4 + \frac{\sigma_4}{\sigma_2}(A + 2r)^2, \end{aligned}$$

where we have used (5.16). Let $t \rightarrow \infty$ on the left-hand side of the above to conclude that

$$E_0[T_A^2] \leq C(A + 2r)^4 \quad \text{for all } A > 1. \tag{5.17}$$

On account of this bound and Markov's inequality, we have for $|x| < A/2$ and $A > 2$,

$$P_x(T_A > A^2 \log A) \leq P_0(T_{3A/2} > A^2 \log A) \leq E_0[T_{3A/2}^2]/(A^2 \log A)^2 \leq C/(\log A)^2.$$

Together with (5.15) this proves (5.14). □

The following technical result will play a key role in the proof of Lemma 6.5 below.

Lemma 5.5. *Assume $d = 2$. For $\alpha, \beta > 0$ and $t > 0$ there is a constant $C_{5.18} = C_{5.18}(\alpha, \beta, t) \geq 1$ such that if $(\log N)^{-\beta} \leq s \leq t$ and $w \in \mathbb{R}^2$ satisfies $0 < |w| \leq (\log N)^{-\alpha}$, then*

$$P_{w\sqrt{N}}(t_{3r} < Ns) \leq C_{5.18} \frac{\log(1/|w|)}{\log N}. \tag{5.18}$$

Proof. We may suppose $|w| > 3r/\sqrt{N}$, because otherwise $C_{5.18}$ can be chosen large enough so that the right side of (5.18) is at least one. Now for any $A > |w|\sqrt{N}$,

$$P_{w\sqrt{N}}(t_{3r} < Ns) \leq P_{w\sqrt{N}}(t_{3r} < T_A) + P_{w\sqrt{N}}(T_A \leq Ns). \tag{5.19}$$

To handle the first term, we apply Lemma 5.3 with $x = w\sqrt{N}$ and $a = 3r$,

$$P_{w\sqrt{N}}(t_{3r} < T_A) \leq \frac{\log\left(\frac{A+2r}{w\sqrt{N}}\right)}{\log\left(\frac{A+2r}{3r}\right)}. \tag{5.20}$$

Now set $A = |w|\sqrt{N} + \sqrt{Ns}(\log N)^\alpha$. Using $s \leq t$ and $|w| \leq (\log N)^{-\alpha}$, and taking $N \geq N_0(t)$, we see that for some $C(t) > 0$,

$$\log\left(\frac{A+2r}{|w|\sqrt{N}}\right) \leq \log\left(1 + \frac{2\sqrt{t}(\log N)^\alpha}{|w|}\right) \leq \log\left(1 + \frac{2\sqrt{t}}{|w|^2}\right) \leq C(t) \log(1/|w|).$$

Using $s \geq (\log N)^{-\beta}$ gives for $N \geq N_1(\alpha, \beta)$,

$$\begin{aligned} \log\left(\frac{A+2r}{3r}\right) &\geq \log\left(\frac{\sqrt{Ns}(\log N)^\alpha}{3r}\right) \geq \log\left(\frac{\sqrt{N}(\log N)^{\alpha-(\beta/2)}}{3r}\right) \\ &= \frac{1}{2} \log N + \left(\alpha - \frac{\beta}{2}\right) \log \log N - \log(3r) \geq \frac{1}{4} \log N. \end{aligned}$$

Plug the above bounds in (5.20) to see that for $N \geq N_0(t) \vee N_1(\alpha, \beta)$,

$$P_{w\sqrt{N}}(t_{3r} < T_A) \leq 4C(t) \frac{\log(1/|w|)}{\log N}. \tag{5.21}$$

For the second term in (5.19), take $k \geq 1/\alpha$ and use (5.5) to get

$$\begin{aligned} P_{w\sqrt{N}}(T_A \leq Ns) &= P_0\left(\sup_{u \leq Ns} |w\sqrt{N} + Y_u| \geq A\right) \leq P_0\left(\sup_{u \leq Ns} |Y_u| \geq A - |w|\sqrt{N}\right) \\ &= P_0\left(\sup_{u \leq Ns} |Y_u| \geq \sqrt{Ns}(\log N)^\alpha\right) \leq C_{5.5}(k) \frac{(Ns)^k}{(\sqrt{Ns}(\log N)^\alpha)^{2k}} \\ &\leq C(\log N)^{-2k\alpha} \leq C(\log N)^{-2} \end{aligned}$$

for $Ns \geq N(\log N)^{-\beta} \geq 1$. Plugging this bound and (5.21) into (5.19) we get (5.18) for $N \geq N_2$ depending only on α, β, t . By the upper bound on $|w|$ we may increase $C_{5.18}$ to get (5.18) for all $N \geq 3$ (recall (1.13)). □

6 The two particle dual

In this section we collect some properties of the two-particle dual which will be needed in our analysis of the martingale square functions. Our main focus will be on the difference of the two particles. Define

$$\psi_r(a) = \begin{cases} \rho^2 |B_r \cap B_r(a)| & \text{if } a \neq 0 \\ \rho |B_r| & \text{if } a = 0, \end{cases}$$

and observe that $\psi_r(a)$ is decreasing in $|a|$ and $0 \leq \psi_r(a)/\rho |B_r| \leq 1$. Consider the two-particle dual (ξ_t^1, ξ_t^2) starting at (x_1, x_2) , $x = x_1 - x_2$, the difference process $\tilde{\xi}_t = \tilde{\xi}_t^x = \xi_t^1 - \xi_t^2$, and the coalescence time τ defined in (1.6). By the dynamics defining the two-particle dual (recall (1.5)), the fact that $|B_r(a) \cap B_r(b)| = |B_r(0) \cap B_r(a - b)|$ shows that for $y \neq 0$, $\tilde{\xi}$ makes transitions

$$y \rightarrow \begin{cases} y + \bar{U} & \text{at rate } 2\rho |B_r| - 2\psi_r(y) \\ 0 & \text{at rate } \psi_r(y), \end{cases} \tag{6.1}$$

while 0 is a trap for $\tilde{\xi}$. Let

$$\tilde{\tau} = \tilde{\tau}(x) = \inf\{t \geq 0 : \tilde{\xi}_t^x = 0\} (= \tau) \tag{6.2}$$

be the time at which $\tilde{\xi}_t^x$ jumps to 0. By standard results (see Sections 4.2 and 4.4 of [19]) $\tilde{\xi}^x$ is a pure jump Feller process which is the unique in law solution of the martingale problem for its generator \tilde{A} on the space $B(\mathbb{R}^d)$ of bounded Borel measurable functions. \tilde{A} is given by

$$\tilde{A}f(x) = (2\rho |B_r| - 2\psi_r(x)) \int_{\mathbb{R}^d} (f(x + u) - f(x)) h_{\bar{U}}(u) du + \psi_r(x)(f(0) - f(x)). \tag{6.3}$$

Recall from Section 5 that Y_t^x is the rate $2\rho |B_r|$ random walk starting at $x \in \mathbb{R}^d$ under P_x , and with jump distribution that of \bar{U} and generator A^Y given in (5.1) for $f \in B(\mathbb{R}^d)$. For a random variable V we let Y^V denote the same random walk with initial law that of V , and will use this notation with other Markov processes below.

We will construct a version of $\tilde{\xi}_t^x$ by absorbing a random time change of Y^x at 0. Define $\beta(y) = 1 - \frac{\psi_r(y)}{\rho |B_r|}$ and

$$I(t) = \int_0^t \frac{1}{\beta(Y_s^x)} ds. \tag{6.4}$$

Note that for $x \neq 0$,

$$\sup_{s \leq t} \frac{\psi_r(Y_s^x)}{\rho |B_r|} = \sup_{s \leq t} \frac{\rho |B_r(Y_s^x) \cap B_r|}{|B_r|} < 1 \text{ a.s.}$$

and thus $\inf_{s \leq t} \beta(Y_s^x) > 0$ a.s. This implies that $I(t)$ is finite and strictly increasing a.s. for all t . Evidently $I(t) = \infty$ for all $t > 0$ if $x = 0$. We will allow $x = 0$ later, but until otherwise indicated we will take our initial point $x \neq 0$. From the definition of I we see that for $0 < s < t$,

$$t - s \leq I(t) - I(s). \tag{6.5}$$

Therefore $I^{-1}(t)$ exists for all t a.s., and

$$\int_0^{I^{-1}(t)} \frac{1}{\beta(Y_s^x)} ds = t. \tag{6.6}$$

If we define $\tilde{Y}_t^x = Y_{I^{-1}(t)}^x$, then it follows from (6.6) that for all but countably many t ,

$$(I^{-1})'(t) = \beta(Y_{I^{-1}(t)}^x)$$

and therefore that

$$I^{-1}(t) = \int_0^t \beta(\tilde{Y}_s^x) ds. \tag{6.7}$$

Clearly, $I^{-1}(t) \leq t$. For $x = 0$ it is natural to define $I^{-1}(t) = 0$ for all $t \geq 0$, which means that $\tilde{Y}_t^0 := Y_{I^{-1}(t)}^0 = 0$ for all $t \geq 0$. Thus (6.7) holds for all x and

$$\tilde{Y}_t^x = Y_{I^{-1}(t)}^x = Y^x \left(\int_0^t \beta(\tilde{Y}_s^x) ds \right) \quad \forall x \in \mathbb{R}^d. \tag{6.8}$$

We may apply Theorems 1.1 and 1.3 of Sec. 6.1 of [19] to see that \tilde{Y}^x is the unique solution of the martingale problem for

$$A^{\tilde{Y}} f(x) = \beta(x) A^Y f(x) = (2\rho|B_r| - 2\psi_r(x)) \int_{\mathbb{R}^d} (f(x+u) - f(x)) h_{\bar{Y}}(u) du, \quad f \in B(\mathbb{R}^d).$$

Here we note that the continuity of f is not needed for Theorem 1.3 of [19] in our jump process setting as the proof there shows. Uniqueness of the martingale problem is classical for such bounded jump generators, e.g., see Theorem 4.1 in Chapter 4 of [19], and so \tilde{Y}^x is the unique Feller process with generator $A^{\tilde{Y}}$, and in particular is strong Markov. Finally we send \tilde{Y}^x to its absorbing state, 0 according to the continuous additive functional

$$C_t = C_t^x = \int_0^t \psi_r(\tilde{Y}_s^x) ds.$$

For an independent mean one exponential random variable, e , define the absorbing time

$$\kappa = \kappa_x = \inf\{t \geq 0 : C_t^x > e\}, \tag{6.9}$$

and the absorbed process

$$\tilde{\xi}_t^x = \begin{cases} \tilde{Y}^x(t) & \text{if } t < \kappa \\ 0 & \text{if } t \geq \kappa. \end{cases}$$

Then $\tilde{\xi}^x$ is a Feller jump process and an elementary calculation shows that it solves the martingale problem for $A^{\tilde{Y}} f(x) + \psi_r(x)(f(0) - f(x)) = \tilde{A} f(x)$, $f \in B(\mathbb{R}^d)$ (from (6.3)). From (6.1) we see that the two-particle dual difference, $\tilde{\xi}$, is the Feller jump process satisfying the same well-posed martingale problem, and so, as the notation suggests, $\tilde{\xi}^x$ has the same law as $\tilde{\xi}^x$.

We have proved:

Lemma 6.1. *If $(x_1, x_2) \in \mathbb{R}^d \times \mathbb{R}^d$, and $x = x_1 - x_2$ define $I^{-1}(t)$ by (6.6) and set $\tilde{Y}^x(t) = Y^x(I^{-1}(t))$. If*

$$\tilde{\xi}_t^x = \begin{cases} \tilde{Y}^x(t) & \text{if } t < \kappa, \\ 0 & \text{if } t \geq \kappa, \end{cases}$$

where $\kappa = \kappa_x$ is as in (6.9), then

$$I^{-1}(t) = \int_0^t \beta(\tilde{Y}_s^x) ds$$

and $\tilde{\xi}^x$ is a version of the dual difference $\xi_t^1 - \xi_t^2$ under $P_{\{x_1, x_2\}}$. Moreover

$$\tilde{\tau}(x) = \kappa_x \text{ for all } x \neq 0. \tag{6.10}$$

We often denote the starting point x of $\tilde{\xi}$ in the underlying probability as P_x . The tail behaviour of the coalescing time κ_x will be important for us. Introduce

$$k(a) = \rho|B_r| \frac{\psi_r(a)}{\rho|B_r| - \psi_r(a)}, \quad a \in \mathbb{R}^d. \tag{6.11}$$

Lemma 6.2. *If $x \in \mathbb{R}^d \setminus \{0\}$, then*

$$P_x(\kappa > t) = E_x \left[\exp \left(- \int_0^{I^{-1}(t)} k(Y_s) ds \right) \right]. \tag{6.12}$$

Proof. By definition of κ ,

$$P_x(\kappa > t) = E_x[e^{-Ct}] = E_x \left[\exp \left(- \int_0^t \psi_r(\tilde{Y}_s) ds \right) \right] = E_x \left[\exp \left(- \int_0^t \psi_r(Y_{I^{-1}(s)}) ds \right) \right].$$

Now change variables with $I(u) = s$ and use $I'(u) = 1/\beta(Y_u)$ to get the required expression. \square

The following result shows that $I^{-1}(t)$ is close to t , and so Y_t^x is a good approximation to \tilde{Y}_t^x .

Lemma 6.3. *There is a constant $C_{6.3} > 0$ such that for all $0 < \alpha < 1$ and $t > 1$, and $Y_0 = x, |x| > 2r$, or $Y_0 = \bar{U}$,*

$$P_{Y_0}(I^{-1}(t) \notin [t - t^\alpha, t]) \leq C_{6.3} \begin{cases} \log(1+t)t^{-\alpha} & \text{if } d = 2, \\ t^{-\alpha} & \text{if } d \geq 3. \end{cases} \tag{6.13}$$

Proof. Let $Y_0 = x, |x| > 2r$. By (6.4) and $Y_s \neq 0$ for all s ,

$$0 \leq I(t) - t = \int_0^t \frac{\rho|B_r|}{\rho|B_r| - \psi_r(Y_s)} - 1 ds = \int_0^t \frac{\rho|B_r \cap B_r(Y_s)|}{|B_r| - \rho|B_r \cap B_r(Y_s)|} ds. \tag{6.14}$$

By an elementary argument, there is a constant $C_{6.15} = C_{6.15}(d, r) > 0$ such that

$$\frac{|B_r \cap B_r(a)|}{|B_r| - |B_r \cap B_r(a)|} \leq \frac{|B_r|}{|B_r| - |B_r \cap B_r(a)|} 1(|a| \leq 2r) \leq C_{6.15} \frac{1}{|a|} 1\{|a| \leq 2r\}, \quad a \in \mathbb{R}^d. \tag{6.15}$$

We are assuming $|x| > 2r$, so using the density bound (5.2), we see that

$$\begin{aligned} E_x \left[\frac{1}{|Y_s|} 1\{|Y_s| \leq 2r\} \right] &= \int_0^\infty \frac{1}{u^2} P_x(|Y_s| \leq u \wedge 2r) du \\ &= \int_0^{2r} \frac{1}{u^2} P_x(Y_s \in B_u) du + \int_{2r}^\infty \frac{1}{u^2} P_x(Y_s \in B_{2r}) du \\ &\leq \frac{C_{5.2}}{s^{d/2} + 1} \left[\int_0^{2r} \frac{1}{u^2} |B_u| du + \int_{2r}^\infty \frac{1}{u^2} |B_{2r}| du \right] \\ &= \frac{C(r)}{s^{d/2} + 1}. \end{aligned}$$

On account of (6.15), plugging this bound into (6.14) gives

$$E_x(I(t) - t) \leq C_{6.15} \int_0^t \frac{C(r)}{s^{d/2} + 1} ds \leq C \begin{cases} \log(1+t) & \text{if } d = 2 \\ 1 & \text{if } d \geq 3. \end{cases}$$

Applying Markov's inequality we obtain

$$P_x(I(t) - t \geq t^\alpha) \leq C_{6.3} \begin{cases} \log(1+t)t^{-\alpha} & \text{if } d = 2, \\ t^{-\alpha} & \text{if } d \geq 3. \end{cases} \quad (6.16)$$

This proves (6.13), because by (6.5), $P_x(t - I^{-1}(t) \geq t^\alpha) \leq P_x(I(t) - t \geq t^\alpha)$, and we also have $I^{-1}(t) \leq t$ by $I(t) \geq t$. The proof for $Y^{\bar{U}}$ is essentially the same. \square

Lemma 6.4. For $\beta \in (\frac{1}{4}, \frac{1}{2})$ there exists a constant $C_{6.4} = C_{6.4}(\beta) > 0$ such that

$$P(|\tilde{Y}_t^{\bar{U}}| \leq r_0) \leq C_{6.4}t^{2\beta-1} \quad \text{for all } t > 0, r_0 \leq t^\beta.$$

Proof. The bounds on β imply that $0 < 1 - 2\beta < 2\beta < 1$. This means we can choose $\alpha \in (0, 1)$ such that $1 - 2\beta < \alpha < 2\beta$. For this α choose t large enough so that $\log(1+t) \geq 1$ and $t - t^\alpha > t/2$. By (6.13),

$$\begin{aligned} P(|\tilde{Y}_t^{\bar{U}}| \leq r_0) &= P(|Y_{I^{-1}(t)}^{\bar{U}}| \leq r_0) \\ &\leq P(I^{-1}(t) \notin [t - t^\alpha, t]) + P\left(\inf_{s \in [t - t^\alpha, t]} |Y_s^{\bar{U}}| \leq r_0\right) \\ &\leq C_{6.3} \log(1+t)t^{-\alpha} + P(|Y_{t-t^\alpha}^{\bar{U}}| \leq r_0 + t^\beta) + \\ &\quad + P(|Y_{t-t^\alpha}^{\bar{U}}| > r_0 + t^\beta, \inf_{s \in [t - t^\alpha, t]} |Y_s^{\bar{U}}| \leq r_0). \end{aligned} \quad (6.17)$$

By $r_0 \leq t^\beta$, (5.2), and the above choice of t ,

$$P(|Y_{t-t^\alpha}^{\bar{U}}| \leq r_0 + t^\beta) \leq C \frac{(r_0 + t^\beta)^d}{(t - t^\alpha)^{d/2}} \leq C \frac{(2t^\beta)^d}{(t/2)^{d/2}} \leq Ct^{\frac{d}{2}(2\beta-1)} \leq Ct^{2\beta-1}.$$

For the last term in (6.17), the Markov property and (5.5) imply that (recall $t > 1$) for all $k \geq 1$,

$$P(|Y_{t-t^\alpha}^{\bar{U}}| > r_0 + t^\beta, \inf_{s \in [t - t^\alpha, t]} |Y_s^{\bar{U}}| \leq r_0) \leq P\left(\sup_{s \leq t^\alpha} |Y_s^0| \geq t^\beta\right) \leq C_{5.5}(k)t^{k(\alpha-2\beta)}.$$

Plugging these bounds into (6.17) gives

$$P(|\tilde{Y}_t^{\bar{U}}| \leq r_0) \leq C(t^{-\alpha} \log(1+t) + t^{2\beta-1} + t^{k(\alpha-2\beta)}).$$

Recalling that $\alpha > 1 - 2\beta$ and $\alpha - 2\beta < 0$, and choosing k large such that $k(\alpha - 2\beta) < 2\beta - 1$, it follows that for large t , $P(|\tilde{Y}_t^{\bar{U}}| \leq r_0) \leq Ct^{2\beta-1}$. Increasing C appropriately to handle small t completes the proof. \square

For $0 < a < b$ define

$$G(a, b) = \left\{ |Y_{I^{-1}(u)}| > 2r \ \forall u \in [a, b] \right\}, \quad (6.18)$$

and for $q \geq 1$ introduce

$$s_N = (\log N)^{-q}. \quad (6.19)$$

Lemma 6.5. There is a constant $C_{6.20}(q) > 0$ such that

$$P_x\left(G\left(\frac{Ns_N}{4}, 2Ns_N\right)^c\right) \leq C_{6.20} \frac{\log \log N}{\log N} \quad \text{for all } |x| > 2r. \quad (6.20)$$

Proof. Recall from (1.13) that $N \geq 3$. Let $u_N = Ns_N/4$ and $u'_N = u_N - u_N^{1/2}$. Then since $I^{-1}(u) \leq u$,

$$P_x\left(G(u_N, 2Ns_N)^c\right) \leq P_x\left(I^{-1}(u_N) \leq u'_N\right) + P_x(|Y_u| \leq 2r \text{ for some } u \in [u'_N, 2Ns_N]). \tag{6.21}$$

By (6.13) with $\alpha = 1/2$ and $N \geq N_0(q)$ (recall $|x| > 2r$),

$$P_x\left(I^{-1}(u_N) \leq u'_N\right) \leq C \frac{\log(1 + u_N)}{\sqrt{u_N}} \leq \frac{C}{N^{1/4}}. \tag{6.22}$$

Next, using the Markov property at time u'_N , we have for $N \geq N_0(q)$,

$$\begin{aligned} &P_x(|Y_u| \leq 2r \text{ for some } u \in [u'_N, 2Ns_N]) \\ &\leq P_x(|Y_{u'_N}| \leq s_N\sqrt{N}) + \sup_{|y| \geq s_N} P_{y\sqrt{N}}(t_{2r} \leq 2Ns_N - u'_N) \\ &\leq \exp(-2\rho|B_r|u'_N) + C \frac{(s_N\sqrt{N})^d}{(u'_N)^{d/2}} + C \frac{\log(1/s_N)}{\log N} \\ &\quad \text{(by (5.2), (5.18) (if } d = 2) \text{ and (5.8) (if } d = 3)) \\ &\leq \frac{C}{\log N} + C \frac{\log \log N}{\log N}. \end{aligned}$$

In the next to last line we have used the $d = 2$ bound; if $d \geq 3$, (5.8) gives a much smaller bound. Here we have also used the strong Markov property and applied (5.18) (if $d = 2$) with $w\sqrt{N}$ equal to the location of the first jump into $B_{s_N\sqrt{N}}$. Use this bound and (6.22) in (6.21) derive (6.20) for $N \geq N_0(q)$. Now adjust $C_{6.20}$ to handle the remaining values of N . \square

We will also need a bound on the two-particle dual $\xi_t = (\xi_t^1, \xi_t^2)$ after the coalescing time κ for any $d \geq 2$. In this setting assume

$$W^{1,x_1}, W^{2,x_2} \text{ and } W^{3,0} \text{ are independent rate } \rho|B_r| \text{ random walks in } \mathbb{R}^d \text{ with step} \tag{6.23}$$

distribution \bar{U} (now in \mathbb{R}^d) and starting at points $x_1, x_2, 0 \in \mathbb{R}^d$, respectively.

Define $W_t = (W_t^{1,x_1}, W_t^{2,x_2})$, and

$$\psi_W(y_1, y_2) = k(|y_1 - y_2|)1(y_1 \neq y_2) = \frac{\rho|B_r|\psi_r(y_1 - y_2)}{\rho|B_r| - \psi_r(y_1 - y_2)}1(y_1 \neq y_2),$$

and $D(t) = \int_0^t \psi_W(W_s)ds$. Although $\psi_W(y_1 - y_2)$ becomes unbounded (if $\rho = 1$) as $y_1 - y_2 \rightarrow 0$, as for $I(t)$, $D(t) < \infty$ for all $t > 0$ a.s. Let \mathbf{e} be an independent exponential mean 1 random variable, and introduce $\bar{\kappa} = \inf\{t \geq 0 : D(t) > \mathbf{e}\} \leq \infty$ (it will be a.s. finite if $d = 2$). Assume also that conditional on $(W, W^{3,0}, \mathbf{e})$, $U_{W(\bar{\kappa})}$ has a uniform distribution on $B_r(W_{\bar{\kappa}}^{1,x_1}) \cap B_r(W_{\bar{\kappa}}^{2,x_2})$ (the intersection is non-empty a.s. by the definition of $\bar{\kappa}$ because $\psi_W(y_1, y_2) = 0$ if $|y_1 - y_2| > 2r$), and given $(W, W^{3,0}, \mathbf{e}, U_{W(\bar{\kappa})})$, U is an independent r.v. uniformly distributed on B_r . We can use the above to define a version of our two-particle dual but now "run at a constant rate" by

$$\bar{W}_t = \begin{cases} (W_t^{1,x_1}, W_t^{1,x_1}) & \text{if } x_1 = x_2, \\ W_t & \text{if } t < \bar{\kappa} \text{ and } x_1 \neq x_2, \\ (U + U_{W_{\bar{\kappa}}} + W_{t-\bar{\kappa}}^{3,0}, U + U_{W_{\bar{\kappa}}} + W_{t-\bar{\kappa}}^{3,0}) & \text{if } t \geq \bar{\kappa} \text{ and } x_1 \neq x_2. \end{cases} \tag{6.24}$$

Note that when \bar{W} is at (y_1, y_2) , $y_1 \neq y_2$, \bar{W} jumps to the diagonal in $\mathbb{R}^d \times \mathbb{R}^d$ at rate $\psi_W(y_1, y_2)$.

We now extend our earlier time change and define

$$\beta(y_1, y_2) = \left[1 - \frac{\psi_r(y_1 - y_2)}{\rho|B_r|}\right]1(y_1 \neq y_2) + 1(y_1 = y_2) \leq 1, \quad \forall y \in \mathbb{R}^d, \tag{6.25}$$

$$\bar{I}(t) = \int_0^t \beta(\bar{W}_s)^{-1} ds < \infty \quad \forall t > 0, \tag{6.26}$$

$$\bar{\xi}_t = \bar{W}(\bar{I}^{-1}(t)), \tag{6.27}$$

where \bar{I}^{-1} is the inverse of the strictly increasing continuous function \bar{I} . As in (6.7), one sees that

$$\bar{I}^{-1}(t) = \int_0^t \beta(\bar{\xi}_s) ds \quad \forall t \geq 0. \tag{6.28}$$

The following result shows that $\bar{\xi}_t$ is a version of the two particle dual.

Lemma 6.6. *If $(x_1, x_2) \in \mathbb{R}^d \times \mathbb{R}^d$ define \bar{W} by (6.24), β by (6.25), \bar{I} by (6.26), and $\bar{\xi}$ by (6.27). Then*

$$\bar{I}^{-1}(t) = \int_0^t \beta(\bar{\xi}_s) ds,$$

and $\bar{\xi}$ is a version of the two-particle dual described in (1.5) under $P_{\{x_1, x_2\}}$.

Proof. The jump rate of \bar{W} to the diagonal becomes unbounded as it approaches the diagonal (for $\rho = 1$), so we proceed more carefully than in the proof of Lemma 6.1, making use of optional stopping. Let

$$R_n = \{(y_1, y_2) \in \mathbb{R}^d \times \mathbb{R}^d : 0 < |y_1 - y_2| < n^{-1}\}$$

and

$$T_n^W = \inf\{t \geq 0 : \bar{W}_t \in R_n\} \leq \infty.$$

Then $\bar{W}_n(t) = \bar{W}(t \wedge T_n^W)$ is a pure jump process on $\mathbb{R}^d \times \mathbb{R}^d$ with bounded jump rates and generator

$$\begin{aligned} \bar{A}_n f(y) = & \rho|B_r|[E(f(y_1 + \bar{U}, y_2) + f(y_1, y_2 + \bar{U}) - 2f(y))]1(|y_1 - y_2| \geq 1/n) \\ & + \psi_W(y)[E(f(U + U_y, U + U_y) - f(y))]1(|y_1 - y_2| \geq 1/n) \\ & + \rho|B_r|[E(f(y + (\bar{U}, \bar{U})) - f(y))]1(y_1 = y_2). \end{aligned}$$

Here U_y is uniformly distributed on $B_r(y_1) \cap B_r(y_2)$ and is independent of the uniform (on B_r) r.v. U . It is easy to check that \bar{W}_n solves the martingale problem for \bar{A}_n on the domain $B(\mathbb{R}^d \times \mathbb{R}^d)$ of bounded Borel functions on $\mathbb{R}^d \times \mathbb{R}^d$.

Let $\bar{T}_n = \inf\{t \geq 0 : \bar{\xi}_t \in R_n\} \leq \infty$. Using the properties of \bar{I}^{-1} and (6.28), it is easy to check that

$$T_n^W = \bar{I}^{-1}(\bar{T}_n) = \int_0^{\bar{T}_n} \beta(\bar{\xi}_s) ds.$$

It follows that

$$I^{-1}(t \wedge \bar{T}_n) = I^{-1}(t) \wedge T_n^W = \int_0^t \beta(\bar{\xi}_s) ds \wedge T_n^W = \int_0^t \beta(\bar{\xi}(s \wedge \bar{T}_n)) ds \wedge T_n^W.$$

If we define $\bar{\xi}_t^{\bar{T}_n} = \bar{\xi}(t \wedge \bar{T}_n)$, the above implies $\bar{\xi}_t^{\bar{T}_n} = \bar{W}_n\left(\int_0^t \beta(\bar{\xi}_s^{\bar{T}_n}) ds\right)$, and thus we may apply Theorems 1.1 and 1.3 in Chapter 6 of [19] to conclude that $\bar{\xi}_t^{\bar{T}_n}$ solves the martingale problem for

$$G_n f(y) = \beta(y)\bar{A}_n f(y), \quad f \in B(\mathbb{R}^d \times \mathbb{R}^d).$$

Here we recall again that the continuity of f assumed in Ch. 6 Theorem 1.3 of [19] is not needed in our jump process setting. A bit of arithmetic shows

$$\begin{aligned} G_n f(y) &= (\rho|B_r| - \psi_r(y_1 - y_2)) [E(f(y_1 + \bar{U}, y_2) + f(y_1, y_2 + \bar{U}) - 2f(y))] 1(|y_1 - y_2| \geq 1/n) \\ &\quad + \psi_r(y_1 - y_2) [E(f(U + U_y, U + U_y) - f(y))] 1(|y_1 - y_2| \geq 1/n) \\ &\quad + \rho|B_r| [E(f(y + (\bar{U}, \bar{U})) - f(y))] 1(y_1 = y_2). \end{aligned}$$

If $\xi = (\xi^1, \xi^2)$ is the two-particle dual process, as described in (1.5), the above is the generator of the Feller pure jump process $\xi(t \wedge T_n)$, where $T_n = \inf\{t \geq 0 : \xi_t \in R_n\}$ and so $\xi^{T_n}(t) = \xi(t \wedge T_n)$ also solves the martingale problem for G_n ($f \in B(\mathbb{R}^d \times \mathbb{R}^d)$). By well-posedness of this martingale problem ((Section 2 and Thm. 4.1 of Chapter 4 of [19]) we concluded that ξ^{T_n} and $\bar{\xi}^{T_n}$ are identical in law for all $n \in \mathbb{N}$. Since $R_n \downarrow \emptyset$ and $\xi(T_n), \bar{\xi}(T_n) \in R_n$ when these times are finite, it follows that $T_n, \bar{T}_n \uparrow \infty$ a.s. as $n \rightarrow \infty$ (in fact for large n they will be infinite a.s.), and therefore ξ and $\bar{\xi}$ are identical in law. \square

The following result is now an easy consequence of (6.24), Lemma 6.6 and the bound $\bar{I}^{-1}(t) \leq t$ for all $t \geq 0$.

Lemma 6.7. *Assume ξ^x is the two-particle dual in $\mathbb{R}^d \times \mathbb{R}^d$, starting at $x = (x_1, x_2)$. Then we may assume there are random walks W^{i,x_i} ($i = 0, 1, 2, x_0 = 0$) as in (6.23) such that*

$$\sup_{s \leq t} |\xi_s^x| \leq \left[\sum_{i=0}^3 \sup_{s \leq t} |W_s^{i,x_i}| \right] + 2\sqrt{2}r. \tag{6.29}$$

7 Analysis of the square function: Proof of Proposition 4.1

In this section we analyze the martingale square function $\langle M^N(\phi) \rangle_t$ for $\phi \in C_0^3(\mathbb{R}^d)$, and in particular give the Proof of Proposition 4.1. We recall from Proposition 2.2 and Lemma 2.4 that

$$\langle M^N(\phi) \rangle_t = \int_0^t m_s^N(\phi) ds = \int_0^t (\bar{m}_s^N(\phi) + \mathcal{E}_{2,10}^N(\phi, s)) ds,$$

and that the integral of $\mathcal{E}_{2,10}^N(\phi, s)$ is negligible for all $d \geq 2$ on account of (2.10). We can thus focus on $\bar{m}_s^N(\phi)$ defined in (2.9), which we write in the form

$$\bar{m}_s^N(\phi) = \bar{m}_s^{N,1}(\phi) - \bar{m}_s^{N,2}(\phi), \tag{7.1}$$

where

$$\begin{aligned} \bar{m}_s^{N,1}(\phi) &= \rho^2 N^{1+d/2} |B_r^N| K^2 \int_{\mathbb{R}^d} \phi^2(x) dx \int_{B_r^N(x)} w_s^N(z) dz, \\ \bar{m}_s^{N,2}(\phi) &= \rho^2 N^{1+d/2} K^2 \int_{\mathbb{R}^d} \phi^2(x) dx \int_{B_r^N(x)} \int_{B_r^N(x)} w_s^N(z_1) w_s^N(z_2) dz_1 dz_2. \end{aligned} \tag{7.2}$$

Recall that $K = K' N^{d/2-1}$ where $K' = 1$ if $d \geq 3$ and $K' = \log N$ if $d = 2$, and define

$$\bar{\gamma}^N(s) = K' \gamma_e(Ns). \tag{7.3}$$

Lemma 7.1. *There is a constant $C_{7.4} = C_{7.4}(\phi) > 0$ such that for $s \geq 0$,*

$$E(\bar{m}_s^{N,1}(\phi)) = \rho^2 |B_r|^2 K' X_0^N(\phi^2) + \mathcal{E}_{7.4}(\phi, s) \tag{7.4}$$

where

$$|\mathcal{E}_{7.4}(\phi, s)| \leq C_{7.4} K' \left(\frac{1}{\sqrt{N}} + \sqrt{s} \right) X_0^N(\mathbf{1}).$$

Proof. By a change of variables,

$$\begin{aligned} \int_{\mathbb{R}^d} \phi^2(x) \int_{B_r^N(x)} w_s^N(z) dz dx &= \int_{\mathbb{R}^d} w_s^N(z) \int_{B_r^N(z)} \phi^2(x) dx dz \\ &= \int_{\mathbb{R}^d} w_s^N(z) \int_{B_r^N(z)} [\phi^2(z) + (\phi^2(x) - \phi^2(z))] dx dz \\ &= |B_r^N| \int_{\mathbb{R}^d} \phi^2(z) w_s^N(z) dz + \varepsilon_s^N, \end{aligned}$$

where we have set $\varepsilon_s^N = \int_{\mathbb{R}^d} \int_{B_r^N(x)} w_s^N(z) [\phi^2(x) - \phi^2(z)] dx dz$. For $z \in B_r^N(x)$ and $C_\phi = 2\|\phi\|_\infty \|\phi\|_{\text{Lip}}$, $|\phi^2(x) - \phi^2(z)| \leq C_\phi 2r/\sqrt{N}$. Thus

$$|\varepsilon_s^N| \leq C_\phi \frac{2r}{\sqrt{N}} \int_{\mathbb{R}^d} \int_{B_r^N(x)} w_s^N(z) dx dz = C_\phi \frac{2r}{\sqrt{N}} |B_r^N| w_s^N(\mathbf{1}).$$

Returning to the definition of $\bar{m}_s^{N,1}(\phi)$ we see that

$$E(\bar{m}_s^{N,1}(\phi)) = \rho^2 N^{1+d/2} |B_r^N| K^2 \left(|B_r^N| E(w_s^N(\phi^2)) + E(\varepsilon_s^N) \right) = \rho^2 |B_r|^2 K' E(X_s^N(\phi^2)) + \mathcal{E}, \tag{7.5}$$

with

$$|\mathcal{E}| = \rho^2 N^{1+d/2} |B_r^N| K^2 E(|\varepsilon_s^N|) \leq C_\phi \rho^2 |B_r|^2 \frac{2rK'}{\sqrt{N}} E(X_s^N(\mathbf{1})) = \frac{CK'}{\sqrt{N}} X_0^N(\mathbf{1}) \tag{7.6}$$

by the martingale property of $X_s^N(\mathbf{1})$ (Corollary 3.2).

Next, we bound the difference $|E(X_s^N(\phi^2)) - X_0^N(\phi^2)|$. By the single particle duality equation (3.2) and a change of variables,

$$\begin{aligned} E(X_s^N(\phi^2)) &= \int_{\mathbb{R}^d} \phi^2(x) E_{\{x\}}^N(X_0^N(\eta_s^N)) dx = \int_{\mathbb{R}^d} \phi^2(x) E_{\{0\}}^N(X_0^N(x + \eta_s^N)) dx \\ &= \int_{\mathbb{R}^d} E_{\{0\}}^N(\phi^2(x' - \eta_s^N)) X_0^N(x') dx' \\ &= X_0^N(\phi^2) + \int_{\mathbb{R}^d} E_{\{0\}}^N \left[\phi^2(x' - \eta_s^N) - \phi^2(x') \right] X_0^N(x') dx'. \end{aligned}$$

Using the smoothness of ϕ and scaling, we see that Lemma 5.1(b) (it applies to the rate $\rho|B_r|$ walk η as well) implies

$$\int_{\mathbb{R}^d} E_{\{0\}}^N \left[\left| \phi^2(x' - \eta_s^N) - \phi^2(x') \right| \right] X_0^N(x') dx' \leq C_\phi E_{\{0\}}^N \left(\frac{|\eta_s^N|}{\sqrt{N}} \right) X_0^N(\mathbf{1}) \leq C_\phi C \sqrt{s} X_0^N(\mathbf{1}).$$

Combining this bound with (7.5) and (7.6) gives (7.4). □

To handle $\bar{m}_s^{N,2}(\phi)$ we apply the two-particle duality equation (3.3) and then split the resulting expression into two pieces, obtaining

$$E(\bar{m}_s^{N,2}(\phi)) = J_s^{N,1}(\phi) + J_s^{N,2}(\phi), \tag{7.7}$$

with

$$\begin{aligned} J_s^{N,1}(\phi) &= \rho^2 N^{1+d/2} K^2 \int_{\mathbb{R}^d} \phi^2(x) \int_{B_r^N(x)} \int_{B_r^N(x)} E_{\{z_1, z_2\}}^N [w_0^N(\xi_s^{N,1}) \\ &\quad \times 1\{\tau^N \leq s\}] dz_1 dz_2 dx \end{aligned} \tag{7.8}$$

$$\begin{aligned} J_s^{N,2}(\phi) &= \rho^2 N^{1+d/2} K^2 \int_{\mathbb{R}^d} \phi^2(x) \int_{B_r^N(x)} \int_{B_r^N(x)} E_{\{z_1, z_2\}}^N [(w_0^N(\xi_s^{N,1})) w_0^N(\xi_s^{N,2}) \\ &\quad 1\{\tau^N > s\}] dz_1 dz_2 dx. \end{aligned} \tag{7.9}$$

Lemma 7.2. *There is a constant $C_{7.10} = C_{7.10}(\phi) > 0$ such that for $s \geq 0$,*

$$\begin{aligned}
 J_s^{N,1}(\phi) &= \rho^2 |B_r|^2 (K' - \bar{\gamma}^N(s)) X_0^N(\phi^2) + \mathcal{E}_{7.10}, \text{ where} \\
 |\mathcal{E}_{7.10}| &\leq C_{7.10} K' \left(\frac{1}{\sqrt{N}} + \sqrt{s} \right) X_0^N(\mathbf{1}).
 \end{aligned}
 \tag{7.10}$$

Proof. By translation invariance, changing of variables and order of integration, we see

$$\begin{aligned}
 &\int_{\mathbb{R}^d} \int_{B_r^N(x)} \int_{B_r^N(x)} \phi^2(x) E_{\{z_1, z_2\}}^N \left[w_0^N(\xi_s^{N,1}) 1\{\tau^N \leq s\} \right] dz_1 dz_2 dx \\
 &= \int_{\mathbb{R}^d} \int_{B_r^N} \int_{B_r^N} \phi^2(x) E_{\{x+z'_1, x+z'_2\}}^N \left[w_0^N(\xi_s^{N,1}) 1\{\tau^N \leq s\} \right] dz'_1 dz'_2 dx \\
 &= \int_{B_r^N} \int_{B_r^N} E_{\{z'_1, z'_2\}}^N \left[\int_{\mathbb{R}^d} \phi^2(x) w_0^N(x + \xi_s^{N,1}) 1\{\tau^N \leq s\} dx \right] dz'_1 dz'_2.
 \end{aligned}$$

Changing variables again with $x' = x + \xi_s^{N,1}$ and adding and subtracting $\phi^2(x')$, the right-side above equals

$$\begin{aligned}
 &\int_{B_r^N} \int_{B_r^N} E_{\{z'_1, z'_2\}}^N \left[\int_{\mathbb{R}^d} \phi^2(x' - \xi_s^{N,1}) w_0^N(x') 1\{\tau^N \leq s\} dx' \right] dz'_1 dz'_2 \\
 &= \int_{B_r^N} \int_{B_r^N} \int_{\mathbb{R}^d} \phi^2(x') w_0^N(x') P_{\{z'_1, z'_2\}}^N(\tau^N \leq s) dx' dz'_1 dz'_2 \\
 &\quad + \int_{B_r^N} \int_{B_r^N} E_{\{z'_1, z'_2\}}^N \left[\int_{\mathbb{R}^d} \left(\phi^2(x' - \xi_s^{N,1}) - \phi^2(x') \right) 1\{\tau^N \leq s\} w_0^N(x') dx' \right] dz'_1 dz'_2 \\
 &= w_0^N(\phi^2) \int_{B_r^N} \int_{B_r^N} P_{\{z'_1, z'_2\}}^N(\tau^N \leq s) dz'_1 dz'_2 + \varepsilon_s^N \\
 &= w_0^N(\phi^2) N^{-d} \int_{B_r} \int_{B_r} P_{\{z_1, z_2\}}(\tau \leq Ns) dz_1 dz_2 + \varepsilon_s^N \\
 &= |B_r|^2 N^{-d} w_0^N(\phi^2) (1 - \gamma_e(Ns)) + \varepsilon_s^N,
 \end{aligned}$$

where

$$\begin{aligned}
 |\varepsilon_s^N| &= \left| \int_{B_r^N} \int_{B_r^N} \int_{\mathbb{R}^d} E_{\{z'_1, z'_2\}}^N \left[\left(\phi^2(x' - \xi_s^{N,1}) - \phi^2(x') \right) 1\{\tau^N \leq s\} \right] w_0^N(x') dx' dz'_1 dz'_2 \right| \\
 &\leq C_\phi \int_{B_r^N} \int_{B_r^N} \int_{\mathbb{R}^d} E_{\{z'_1, z'_2\}}^N \left[|\xi_s^{N,1}| 1\{\tau^N \leq s\} \right] w_0^N(x') dx' dz'_1 dz'_2 \\
 &= C_\phi w_0^N(\mathbf{1}) \int_{B_r^N} \int_{B_r^N} E_{\{z'_1, z'_2\}}^N (|\xi_s^{N,1}| 1\{\tau^N \leq s\}) dz'_1 dz'_2 \\
 &= C_\phi w_0^N(\mathbf{1}) N^{-d} \frac{1}{\sqrt{N}} \int_{B_r} \int_{B_r} E_{\{z_1, z_2\}} (|\xi_{Ns}^1| 1\{\tau \leq Ns\}) dz_1 dz_2.
 \end{aligned}
 \tag{7.11}$$

For fixed $z_1, z_2 \in B_r$, letting $z_3 = 0$, Lemma 6.7 implies that

$$E_{\{z_1, z_2\}}(|\xi_{Ns}^1|) \leq 2\sqrt{2}r + \sum_{i=1}^3 E \left[\sup_{t \leq Ns} |W_t^{i, z_i}| \right] \leq 2\sqrt{2}r + 3 \left(r + E \left(\sup_{t \leq Ns} |Y_t^0| \right) \right) \leq C_r + C\sqrt{Ns}$$

using Lemma 5.1(b) for the last inequality. Plugging this bound into (7.11), we obtain

$$|\varepsilon_s^N| \leq C \left(\frac{1}{\sqrt{N}} + \sqrt{s} \right) N^{-d} w_0^N(\mathbf{1}).$$

Returning to $J_s^{N,1}$, we now have

$$\begin{aligned} J_s^{N,1}(\phi) &= \rho^2 N^{1+\frac{d}{2}} K^2 \left(|B_r|^2 N^{-d} w_0^N(\phi^2) (1 - \gamma_e(Ns)) + \varepsilon_s^N \right) \\ &= \rho^2 |B_r|^2 K' X_0^N(\phi^2) (1 - \gamma_e(Ns)) + \mathcal{E} \\ &= \rho^2 |B_r|^2 X_0^N(\phi^2) [K' - \bar{\gamma}^N(s)] + \mathcal{E}, \end{aligned}$$

where

$$\begin{aligned} |\mathcal{E}| &\leq CN^{1+\frac{d}{2}} K^2 |\varepsilon_s^N| \leq CN^{1+\frac{d}{2}} KN^{-d} \left(\frac{1}{\sqrt{N}} + \sqrt{s} \right) X_0^N(\mathbf{1}) \\ &= CK' \left(\frac{1}{\sqrt{N}} + \sqrt{s} \right) X_0^N(\mathbf{1}), \end{aligned}$$

which proves (7.10). □

Using Lemmas 7.1 and 7.2 in (7.1) and (7.7), we arrive at the following:

Corollary 7.3. For $s \geq 0$,

$$\left| E(\bar{m}_s^N(\phi)) - \rho^2 |B_r|^2 \bar{\gamma}^N(s) X_0^N(\phi^2) \right| \leq J_s^{N,2}(\phi) + C_{7.12} K' \left(\frac{1}{\sqrt{N}} + \sqrt{s} \right) X_0^N(\mathbf{1}). \quad (7.12)$$

We turn now to the analysis of $J_s^{N,2}(\phi) \leq \|\phi\|_\infty^2 J_s^{N,2}(\mathbf{1})$. Recall Y_t^x and $\tilde{Y}_t^x = Y^x(I^{-1}(t))$ from Section 6, $\tilde{\tau}$ from (6.2), and the process $\xi_t^x = \tilde{Y}_t^x 1\{\tilde{\tau} > t\}$, which by Lemma 6.1 has the same law as $\xi_t^1 - \xi_t^2$ under $P_{\{z_1, z_2\}}$ when $z_1 - z_2 = x$. As in Lemma 6.3 we will write $Y_t^{\bar{U}}$ when the initial law of Y_t is the law of \bar{U} .

Lemma 7.4. For $s \geq 0$,

$$J_s^{N,2}(\mathbf{1}) = \rho^2 |B_r|^2 N^{1-\frac{d}{2}} \int_{\mathbb{R}^d} X_0^N(y) E \left[X_0^N \left(y - \frac{\tilde{Y}_{Ns}^{\bar{U}}}{\sqrt{N}} \right) 1\{\tilde{\tau} > Ns\} \right] dy. \quad (7.13)$$

Proof. By changing variables and orders of integration, and using the difference process $\tilde{\xi}_s^N = \xi_s^{N,1} - \xi_s^{N,2}$, we have

$$\begin{aligned} &J_s^{N,2}(\mathbf{1}) \\ &= \rho^2 N^{1+d/2} K^2 \int_{\mathbb{R}^d} \int_{B_r^N} \int_{B_r^N} E_{\{z_1, z_2\}}^N \left[w_0^N(x + \xi_s^{N,1}) w_0^N(x + \xi_s^{N,2}) 1\{\tau^N > s\} \right] dz_1 dz_2 dx \\ &= \rho^2 N^{1+d/2} K^2 \int_{B_r^N} \int_{B_r^N} \int_{\mathbb{R}^d} w_0^N(y) E_{\{z_1, z_2\}}^N \left[w_0^N(y - \xi_s^{N,1} + \xi_s^{N,2}) 1\{\tau^N > s\} \right] dy dz_1 dz_2 \\ &= \rho^2 N^{1+d/2} K^2 \int_{\mathbb{R}^d} w_0^N(y) \int_{B_r^N} \int_{B_r^N} E_{\{z_1, z_2\}}^N \left(w_0^N(y - \tilde{\xi}_s^N) 1\{\tau^N > s\} \right) dz_1 dz_2 dy \\ &= \rho^2 N^{1-d/2} K^2 \int_{\mathbb{R}^d} w_0^N(y) \int_{B_r} \int_{B_r} E_{\{z_1, z_2\}} \left[w_0^N \left(y - \frac{\tilde{\xi}_{Ns}^N}{\sqrt{N}} \right) 1\{\tilde{\tau} > Ns\} \right] dy dz_1 dz_2 \\ &= \rho^2 N^{1-d/2} K^2 \int_{\mathbb{R}^d} w_0^N(y) \int_{B_r} \int_{B_r} E \left[w_0^N \left(y - \frac{\tilde{Y}_{Ns}^{z_1 - z_2}}{\sqrt{N}} \right) 1\{\tilde{\tau} > Ns\} \right] dy dz_1 dz_2 \end{aligned}$$

which is (7.13). □

Recall s_N from (6.19) and define δ_N so that

$$s_N = (\log N)^{-q}, \quad \delta_N = (\log N)^\delta s_N = (\log N)^{\delta-q}, \quad (7.14)$$

where $\delta \in (0, \frac{1}{2}]$ and $q > 4$ are fixed constants.

Lemma 7.5. *There is a constant $C_{7.15} > 0$ such that*

$$\sup_{s \in [s_N, 2s_N]} J_s^{N,2}(\mathbf{1}) \leq C_{7.15} \left[\frac{\log \log N}{\log N} X_0^N(\mathbf{1}) + \frac{N^{\frac{1}{2} - \frac{d}{4}}}{s_N \log N} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} X_0^N(y_1) X_0^N(y_2) 1\{|y_1 - y_2| \leq \sqrt{\delta_N}\} dy_1 dy_2 \right]. \quad (7.15)$$

Proof. Let $s \in [s_N, 2s_N]$, let $t_N = Ns/4$, and define

$$\begin{aligned} \mathcal{E}_1 &= N^{1-\frac{d}{2}} \int_{\mathbb{R}^d} X_0^N(y) E \left[X_0^N \left(y - \frac{\tilde{Y}_{Ns}^{\bar{U}}}{\sqrt{N}} \right) 1 \left\{ \frac{|\tilde{Y}_{Ns}^{\bar{U}}|}{\sqrt{N}} > \sqrt{\delta_N}, \tilde{\tau} > Ns \right\} \right] dy \\ \mathcal{E}_2 &= N^{1-\frac{d}{2}} \int_{\mathbb{R}^d} X_0^N(y) E \left[X_0^N \left(y - \frac{\tilde{Y}_{Ns}^{\bar{U}}}{\sqrt{N}} \right) 1 \left\{ |\tilde{Y}_{t_N}^{\bar{U}}| \leq 2r, \frac{|\tilde{Y}_{Ns}^{\bar{U}}|}{\sqrt{N}} \leq \sqrt{\delta_N}, \tilde{\tau} > Ns \right\} \right] dy \\ \mathcal{E}_3 &= N^{1-\frac{d}{2}} \int_{\mathbb{R}^d} X_0^N(y) E \left[X_0^N \left(y - \frac{\tilde{Y}_{Ns}^{\bar{U}}}{\sqrt{N}} \right) 1 \left\{ |\tilde{Y}_{t_N}^{\bar{U}}| > 2r, \frac{|\tilde{Y}_{Ns}^{\bar{U}}|}{\sqrt{N}} \leq \sqrt{\delta_N}, \tilde{\tau} > Ns \right\} \right] dy \end{aligned}$$

To prove (7.15), by Lemma 7.4 it suffices to show that each \mathcal{E}_i is uniformly bounded in $s \in [s_N, 2s_N]$ by terms in the right side of (7.15).

Since $I^{-1}(Ns) \leq Ns \leq 2Ns_N$, for $k > 2/\delta$ and $N \geq N_0(q, r)$ large enough,

$$\begin{aligned} P \left(\frac{|\tilde{Y}_{Ns}^{\bar{U}}|}{\sqrt{N}} > \sqrt{\delta_N}, \tau > Ns \right) &\leq P \left(\sup_{t \leq 2Ns_N} |Y_t^{\bar{U}}| \geq \sqrt{N\delta_N} \right) \\ &\leq P \left(\sup_{t \leq 2Ns_N} |Y_t^0| \geq \sqrt{N\delta_N} - 2r \right) \leq C \frac{(2Ns_N)^k}{(\sqrt{N\delta_N} - 2r)^{2k}} \leq C (s_N/\delta_N)^k \leq C/(\log N)^2, \end{aligned}$$

where we have used (5.5). With this bound and $X_0^N(\cdot) \leq K$ we obtain from the definition of \mathcal{E}_1 that

$$\mathcal{E}_1 \leq N^{1-\frac{d}{2}} K \frac{C}{(\log N)^2} \int_{\mathbb{R}^d} X_0^N(y) dy \leq \frac{C}{\log N} X_0^N(\mathbf{1}).$$

The above bound is then extended to all $N \geq 3$ by increasing C .

Using $X_0^N(\cdot) \leq K$ again, we have

$$\mathcal{E}_2 \leq N^{1-\frac{d}{2}} K \int_{\mathbb{R}^d} X_0^N(y) P \left(|\tilde{Y}_{Ns/4}^{\bar{U}}| \leq 2r \right) dy = K' X_0^N(\mathbf{1}) P \left(|\tilde{Y}_{Ns/4}^{\bar{U}}| \leq 2r \right).$$

It follows from Lemma 6.4, taking $\beta = 1/3$, that for $N \geq N_0(q)$,

$$\mathcal{E}_2 \leq C \frac{K'}{(Ns/4)^{1/3}} X_0^N(\mathbf{1}) \leq C \frac{\log N}{(Ns_N/4)^{1/3}} X_0^N(\mathbf{1}) \leq C \frac{(\log N)^{1+q/3}}{N^{1/3}} X_0^N(\mathbf{1}).$$

As before, the above bound is then valid for all $N \geq 3$ by increasing C .

We split \mathcal{E}_3 into two parts, letting $G(a, b) = \{|Y_{I^{-1}(u)}^{\bar{U}}| > 2r \text{ for all } u \in [a, b]\}$. If we let $G_N = G(Ns/2, Ns)$ then we can write $\mathcal{E}_3 = \mathcal{E}'_3 + \mathcal{E}''_3$, where

$$\begin{aligned} \mathcal{E}'_3 &= N^{1-\frac{d}{2}} \int_{\mathbb{R}^d} X_0^N(y) E \left[X_0^N \left(y - \frac{\tilde{Y}_{Ns}^{\bar{U}}}{\sqrt{N}} \right) 1_{G_N^c} 1 \left\{ |\tilde{Y}_{t_N}^{\bar{U}}| > 2r, \frac{|\tilde{Y}_{Ns}^{\bar{U}}|}{\sqrt{N}} \leq \sqrt{\delta_N}, \tilde{\tau}(\bar{U}) > Ns \right\} \right] dy \\ \mathcal{E}''_3 &= N^{1-\frac{d}{2}} \int_{\mathbb{R}^d} X_0^N(y) E \left[X_0^N \left(y - \frac{\tilde{Y}_{Ns}^{\bar{U}}}{\sqrt{N}} \right) 1_{G_N} 1 \left\{ |\tilde{Y}_{t_N}^{\bar{U}}| > 2r, \frac{|\tilde{Y}_{Ns}^{\bar{U}}|}{\sqrt{N}} \leq \sqrt{\delta_N}, \tilde{\tau}(\bar{U}) > Ns \right\} \right] dy \end{aligned}$$

Applying the Markov property at time $t_N = Ns/4$, and letting $P_{\tilde{Y}_{t_N}^{\bar{U}}(\omega)}$ denote the law of

Y with $Y_0 = \tilde{Y}_{t_N}^{\bar{U}}(\omega)$,

$$\begin{aligned} \mathcal{E}'_3 &\leq KN^{1-\frac{d}{2}} \int_{\mathbb{R}^2} X_0^N(y) P\left(\left\{|\tilde{Y}_{t_N}^{\bar{U}}| > 2r, \tilde{\tau}(\bar{U}) > t_N\right\} \cap G_N^c\right) dy \\ &= K'E \left[1\left\{|\tilde{Y}_{t_N}^{\bar{U}}| > 2r, \tilde{\tau}(\bar{U}) > t_N\right\}\right. \\ &\quad \times \left.P_{\tilde{Y}_{t_N}^{\bar{U}}}\left(|Y(I^{-1}(u))| \leq 2r \text{ for some } u \in [Ns/4, 3Ns/4]\right)\right] X_0^N(\mathbf{1}) \\ &\leq K'P\left(\tilde{\tau}(\bar{U}) > Ns_N/4\right) \\ &\quad \times \sup_{|x|>2r} P(Y^x(I^{-1}(u)) \leq 2r \text{ for some } u \in [Ns_N/4, 3Ns_N/2]) X_0^N(\mathbf{1}) \\ &\leq C \frac{\log \log N}{\log N} X_0^N(\mathbf{1}), \end{aligned}$$

where we have used Lemma 6.5, $s_N = (\log N)^{-q}$, and the fact that $K'P(\tau(\bar{U}) > Ns_N/4)$ is bounded in N for $d = 2$ by Proposition 1.1, and at most one if $d > 2$.

Set $t'_N = 3Ns/4$ and $t''_N = t'_N - t_N = Ns/2$. By the Markov property of $\tilde{Y}^{\bar{U}}$ at time $t_N = Ns/4$,

$$\begin{aligned} \mathcal{E}''_3 &= N^{1-\frac{d}{2}} \int_{\mathbb{R}^d} X_0^N(y) E\left[1\left\{|\tilde{Y}_{t_N}^{\bar{U}}| > 2r, \tilde{\tau}(\bar{U}) > t_N\right\}\right. \\ &\quad \times \left.E_{\tilde{Y}_{t_N}^{\bar{U}}}\left(X_0^N\left(y - \frac{\tilde{Y}_{t'_N}}{\sqrt{N}}\right) 1\left\{\frac{|\tilde{Y}_{t'_N}}{\sqrt{N}}| \leq \sqrt{\delta_N}\right\} 1\left\{G(t_N, t'_N) \cap \{\tilde{\tau} > t'_N\}\right\}\right)\right] dy. \end{aligned} \tag{7.16}$$

On the event $G(t_N, t'_N) \cap \{\tilde{\tau} > t'_N\}$, for all $u \in [t_N, t'_N]$,

$$I^{-1}(u) = I^{-1}(t_N) + \int_{t_N}^u \left[1 - \frac{\psi_r(\tilde{Y}_s)}{\rho|B_r|}\right] ds = I^{-1}(t_N) + u - t_N.$$

Using the above and Lemma 6.1, we see that on the above event and for u as above, under $P_{\tilde{Y}_{t_N}^{\bar{U}}}$,

$$\tilde{Y}(u) - \tilde{Y}(t_N) = Y(I^{-1}(t_N) + (u - t_N)) - Y(I^{-1}(t_N)) := \hat{Y}^0(u - t_N). \tag{7.17}$$

Set $\tilde{\mathcal{F}}_s = \mathcal{F}_{I^{-1}(s)}^Y$, where we recall that \mathcal{F}_t^Y is the right-continuous filtration generated by the random walk Y . Then $I^{-1}(s)$ is an (\mathcal{F}_t^Y) -stopping time. By the strong Markov property of Y , \hat{Y}^0 is a copy of Y starting at 0 and is independent of $\tilde{\mathcal{F}}_{t_N}$. Since \tilde{Y}_{t_N} is $\tilde{\mathcal{F}}_{t_N}$ -measurable, we may conclude from (7.17) and (7.16) that

$$\begin{aligned} \mathcal{E}''_3 &\leq N^{1-\frac{d}{2}} \int_{\mathbb{R}^2} X_0^N(y) E\left[1\left\{|\tilde{Y}_{t_N}^{\bar{U}}| > 2r, \tilde{\tau}(\bar{U}) > t_N\right\}\right. \\ &\quad \times \left.E_{\tilde{Y}_{t_N}^{\bar{U}}}\left(X_0^N\left(y - \left(\frac{\tilde{Y}_{t_N} + \hat{Y}_{t'_N}^0}{\sqrt{t''_N}}\right) \sqrt{\frac{t''_N}{N}}\right) 1\left\{\left|\frac{\tilde{Y}_{t_N} + \hat{Y}_{t'_N}^0}{\sqrt{t''_N}}\right| \sqrt{\frac{t''_N}{N}} \leq \sqrt{\delta_N}\right\}\right)\right] dy \\ &\leq N^{1-\frac{d}{2}} \int_{\mathbb{R}^2} X_0^N(y) E\left[1\left\{\tilde{\tau}(\bar{U}) > t_N, |\tilde{Y}_{t_N}^{\bar{U}}| > 2r\right\}\right. \\ &\quad \times \left.\sup_x E\left[X_0^N\left(y - \frac{\hat{Y}_{t''_N}^x}{\sqrt{t''_N}} \sqrt{\frac{t''_N}{N}}\right) 1\left\{\left|\frac{\hat{Y}_{t''_N}^x}{\sqrt{t''_N}}\right| \sqrt{\frac{t''_N}{N}} \leq \sqrt{\delta_N}\right\}\right]\right] dy, \end{aligned} \tag{7.18}$$

where $\hat{Y}^x = x + \hat{Y}^0$. Apply (5.3) to $f(z) = X_0^N\left(y - z\sqrt{\frac{t''_N}{N}}\right) 1\{|z|\sqrt{\frac{t''_N}{N}} \leq \sqrt{\delta_N}\}$, to see that,

uniformly in x ,

$$\begin{aligned} E \left[X_0^N \left(y - \frac{\hat{Y}_{t_N''}^x}{\sqrt{t_N''}} \right) \sqrt{\frac{t_N''}{N}} \mathbf{1} \left\{ \left| \frac{\hat{Y}_{t_N''}^x}{\sqrt{t_N''}} \right| \sqrt{\frac{t_N''}{N}} \leq \sqrt{\delta_N} \right\} \right] \\ \leq e^{-2\rho|B_r|t_N''} K + C \int_{\mathbb{R}^d} X_0^N(y - z \sqrt{\frac{t_N''}{N}}) \mathbf{1} \left\{ |z| \sqrt{\frac{t_N''}{N}} \leq \sqrt{\delta_N} \right\} dz \\ \leq e^{-\rho|B_r|N s_N} K + C \left(\frac{N}{t_N''} \right)^{d/2} \int_{\mathbb{R}^d} X_0^N(y - x') \mathbf{1} \left\{ |x'| \leq \sqrt{\delta_N} \right\} dx' \\ \leq \frac{C}{\log N} + \frac{C}{s_N^{d/2}} \int_{\mathbb{R}^d} X_0^N(y - x') \mathbf{1} \left\{ |x'| \leq \sqrt{\delta_N} \right\} dx'. \end{aligned}$$

Use this and the fact that $P(\tilde{\tau}(\bar{U}) > t_N) \leq \frac{C}{K'}$ (from Proposition 1.1 if $d = 2$) in (7.18), to see that

$$\begin{aligned} \mathcal{E}_3'' &\leq C(s_N N)^{1-\frac{d}{2}} P(\tilde{\tau}(\bar{U}) > t_N) \left[\frac{X_0^N(\mathbf{1})}{\log N} \right. \\ &\quad \left. + \frac{1}{s_N} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} X_0^N(y) X_0^N(y - x') \mathbf{1} \left\{ |x'| \leq \sqrt{\delta_N} \right\} dx' dy \right] \\ &\leq C \left[\frac{X_0^N(\mathbf{1})}{(\log N)} + \frac{(s_N N)^{1-\frac{d}{2}}}{s_N K'} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} X_0^N(y_1) X_0^N(y_2) \mathbf{1} \left\{ |y_1 - y_2| \leq \sqrt{\delta_N} \right\} dy_1 dy_2 \right] \\ &\leq C \left[\frac{X_0^N(\mathbf{1})}{(\log N)} + \frac{N^{\frac{1}{2}-\frac{d}{4}}}{s_N \log N} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} X_0^N(y_1) X_0^N(y_2) \mathbf{1} \left\{ |y_1 - y_2| \leq \sqrt{\delta_N} \right\} dy_1 dy_2 \right], \end{aligned}$$

where the last line is very crude if $d \geq 3$, and is an equality if $d = 2$. Combining the bounds for $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3', \mathcal{E}_3''$, we establish (7.15). \square

Corollary 7.3, Lemma 7.5, and $q > 4$ imply the following:

Lemma 7.6. *There is a constant $C_{7.19} = C_{7.19}(\phi)$ such that*

$$\begin{aligned} \sup_{s \in [s_N, 2s_N]} \left| E(\bar{m}_s^N(\phi)) - \rho^2 |B_r|^2 \bar{\gamma}^N(s) X_0^N(\phi^2) \right| &\leq C_{7.19} \left[\frac{\log \log N}{\log N} X_0^N(\mathbf{1}) \right. \\ &\quad \left. + \frac{N^{\frac{1}{2}-\frac{d}{4}}}{s_N \log N} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} X_0^N(y_1) X_0^N(y_2) \mathbf{1} \left\{ |y_1 - y_2| \leq \sqrt{\delta_N} \right\} dy_1 dy_2 \right]. \end{aligned} \quad (7.19)$$

Remark 7.7. To identify the square function of $M^N(\phi)$ we will need to use the above and the Markov property to bound

$$\left| E(\bar{m}_s^N(\phi) \mid \mathcal{F}_{s-s_N}^N) - \rho^2 |B_r|^2 \bar{\gamma}^N(s_N) X_{s-s_N}^N(\phi^2) \right|.$$

This means we will need to bound the expected value of the last term in (7.19) with $X_{s-s_N}^N$ replacing X_0^N . For $d \geq 3$ we only need to bound the resulting double integral on the right-hand side of (7.19) by $X_{s-s_N}^N(\mathbf{1})^2$, but for $d = 2$ we require the following additional result.

Lemma 7.8. *Assume $d = 2$. For $T > 0$ there exists $C_{7.20}(T) > 0$ such that for $\delta_N \leq s \leq T$,*

$$\begin{aligned} E \left[\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathbf{1} \left\{ |y_1 - y_2| \leq \sqrt{\delta_N} \right\} X_s^N(y_1) X_s^N(y_2) dy_1 dy_2 \right] \\ \leq C_{7.20}(T) \left(\frac{\delta_N}{s} X_0^N(\mathbf{1})^2 + \delta_N \log(1/\delta_N) X_0^N(\mathbf{1}) \right). \end{aligned} \quad (7.20)$$

Proof. By the duality equation (3.3) and then a change of variables,

$$\begin{aligned} E \left[\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} 1\{|y_1 - y_2| \leq \sqrt{\delta_N}\} X_s^N(y_1) X_s^N(y_2) dy_1 dy_2 \right] \\ = (\log N) \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} 1\{|y_1 - y_2| \leq \sqrt{\delta_N}\} E_{\{y_1, y_2\}}^N \left[X_0^N(\xi_s^{N,1}) 1\{\tau^N \leq s\} \right] dy_1 dy_2 \\ + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} 1\{|y_1 - y_2| \leq \sqrt{\delta_N}\} E_{\{y_1, y_2\}}^N \left[X_0^N(\xi_s^{N,1}) X_0^N(\xi_s^{N,2}) 1\{\tau^N > s\} \right] dy_1 dy_2 \\ = \mathcal{E}_1 + \mathcal{E}_2. \end{aligned}$$

Here

$$\begin{aligned} \mathcal{E}_1 &= (\log N) \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} 1\{|y_1 - y_2| \leq \sqrt{\delta_N}\} E_{\{y_1, y_2\}}^N \left[X_0^N(\xi_s^{N,1}) 1\{\tau^N \leq s\} \right] dy_1 dy_2 \\ &= (\log N) \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} 1\{|y_1 - y_2| \leq \sqrt{\delta_N}\} E_{\{y_1 - y_2, 0\}}^N \left[X_0^N(y_2 + \xi_s^{N,1}) 1\{\tau^N \leq s\} \right] dy_2 dy_1 \\ &= (\log N) \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} 1\{|y| \leq \sqrt{\delta_N}\} E_{\{y, 0\}}^N \left[X_0^N(y_2 + \xi_s^{N,1}) 1\{\tau^N \leq s\} \right] dy_2 dy, \end{aligned}$$

and

$$\mathcal{E}_2 = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} 1\{|y| \leq \sqrt{\delta_N}\} E_{\{y, 0\}}^N \left[X_0^N(y_2 + \xi_s^{N,1}) X_0^N(y_2 + \xi_s^{N,2}) 1\{\tau^N > s\} \right] dy_2 dy.$$

First, integrating y_2 out in \mathcal{E}_1 yields

$$\begin{aligned} \mathcal{E}_1 &\leq (\log N) X_0^N(\mathbf{1}) \int_{\mathbb{R}^2} 1\{|y| \leq \sqrt{\delta_N}\} P_{\{y, 0\}}^N(\tau^N \leq s) dy \\ &= (\log N) X_0^N(\mathbf{1}) \int_{\mathbb{R}^2} 1\{|y| \leq \sqrt{\delta_N}\} P_{\{y\sqrt{N}, 0\}}(\tau \leq Ns) dy. \end{aligned}$$

By Lemma 6.1, switching to $\tilde{\xi}^{y\sqrt{N}}$, and using $I^{-1}(u) \leq u$,

$$\begin{aligned} P_{\{y\sqrt{N}, 0\}}(\tau \leq Ns) &= P(\tilde{\xi}_{N_s}^{y\sqrt{N}} = 0) \leq P(\inf_{u \leq Ns} |\tilde{Y}_u^{y\sqrt{N}}| \leq 3r) \\ &\leq P(\inf_{u \leq Ns} |Y_u^{y\sqrt{N}}| \leq 3r) = P_{y\sqrt{N}}(t_{3r} \leq Ns). \end{aligned}$$

By Lemma 5.5, which is applicable because $T \geq s \geq \delta_N$ and we consider only $|y| \leq \sqrt{\delta_N}$, the last probability above is bounded by $C \frac{\log(1/|y|)}{\log N}$ ($C = C(T)$). It follows that

$$\begin{aligned} \mathcal{E}_1 &\leq C X_0^N(\mathbf{1}) \int_{\mathbb{R}^2} 1\{|y| \leq \sqrt{\delta_N}\} \log(1/|y|) dy \\ &= C X_0^N(\mathbf{1}) \int_0^{\sqrt{\delta_N}} u \log(1/u) du \\ &\leq C X_0^N(\mathbf{1}) \delta_N \log(1/\delta_N). \end{aligned}$$

By definition,

$$\begin{aligned} \mathcal{E}_2 &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} 1\{|y| \leq \sqrt{\delta_N}\} E_{\{y\sqrt{N}, 0\}} \left[X_0^N\left(y_2 + \frac{\xi_s^1}{\sqrt{N}}\right) X_0^N\left(y_2 + \frac{\xi_s^2}{\sqrt{N}}\right) 1\{\tau > Ns\} \right] dy_2 dy \\ &= \mathcal{E}'_2 + \mathcal{E}''_2, \end{aligned}$$

where, with $G_N = \{|\xi_u^1 - \xi_u^2| \geq 3r \text{ for all } u \leq Ns\}$,

$$\begin{aligned} \mathcal{E}'_2 &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} 1\{|y| \leq \sqrt{\delta_N}\} E_{\{y\sqrt{N}, 0\}} \left[X_0^N \left(y_2 + \frac{\xi_{sN}^1}{\sqrt{N}} \right) X_0^N \left(y_2 + \frac{\xi_{sN}^2}{\sqrt{N}} \right) \right. \\ &\quad \left. \times 1_{G_N} 1\{\tau > Ns\} \right] dy_2 dy, \\ \mathcal{E}''_2 &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} 1\{|y| \leq \sqrt{\delta_N}\} E_{\{y\sqrt{N}, 0\}} \left[X_0^N \left(y_2 + \frac{\xi_{sN}^1}{\sqrt{N}} \right) X_0^N \left(y_2 + \frac{\xi_{sN}^2}{\sqrt{N}} \right) \right. \\ &\quad \left. \times 1_{G_N^c} 1\{\tau > Ns\} \right] dy_2 dy. \end{aligned}$$

We use the representation of $\xi = (\xi^1, \xi^2)$ in Lemma 6.6 and the fact that on $G_N \cap \{\tau > Ns\}$, $\xi_{sN} = W_{sN}$. Then, dropping the indicator of $G_N \cap \{\tau > Ns\}$, we see that

$$\begin{aligned} \mathcal{E}'_2 &\leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} 1\{|y| \leq \sqrt{\delta_N}\} E \left[X_0^N \left(y_2 + \frac{W_{sN}^{y\sqrt{N}}}{\sqrt{N}} \right) X_0^N \left(y_2 + \frac{W_{sN}^0}{\sqrt{N}} \right) \right] dy_2 dy \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} 1\{|y| \leq \sqrt{\delta_N}\} E \left[X_0^N \left(y_2 + \frac{W_{sN}^{y\sqrt{N}}}{\sqrt{N}} \right) \right] E \left[X_0^N \left(y_2 + \frac{W_{sN}^0}{\sqrt{N}} \right) \right] dy_2 dy, \end{aligned}$$

the last equality by independence of the walks $W^{y\sqrt{N}}, W^0$. By the density bound (5.3) applied to $f(z) = X_0^N(y_2 + z\sqrt{s})$, and $s \geq s_N$, for all y_2 ,

$$\begin{aligned} E \left[X_0^N \left(y_2 + \frac{W_{sN}^0}{\sqrt{N}} \right) \right] &= E \left[f \left(\frac{W_{sN}^0}{\sqrt{N}} \right) \right] \leq e^{-2\rho|B_r|Ns} \log N + C \int_{\mathbb{R}^2} X_0^N(y_2 - z\sqrt{s}) dz \\ &\leq \frac{C}{\log N} + \frac{C}{s} \int_{\mathbb{R}^2} X_0^N(y_2 - z') dz' = \frac{C}{\log N} + \frac{C}{s} X_0^N(\mathbf{1}). \end{aligned}$$

With this bound, integrating out y_2 first and then y , it follows that

$$\begin{aligned} \mathcal{E}'_2 &\leq C \left(\frac{1}{\log N} + \frac{1}{s} X_0^N(\mathbf{1}) \right) \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} 1\{|y| \leq \sqrt{\delta_N}\} E \left[X_0^N \left(y_2 + \frac{W_{sN}^{y\sqrt{N}}}{\sqrt{N}} \right) \right] dy_2 dy \\ &= C \left(\frac{1}{\log N} + \frac{1}{s} X_0^N(\mathbf{1}) \right) X_0^N(\mathbf{1}) |B_{\sqrt{\delta_N}}| \leq C \left[\frac{1}{\log N} + \frac{1}{s} X_0^N(\mathbf{1}) \right] X_0^N(\mathbf{1}) \delta_N. \end{aligned}$$

Using $X_0^N(\cdot) \leq K' = \log N$ and integrating out y_2 gives

$$\mathcal{E}''_2 \leq (\log N) X_0^N(\mathbf{1}) \int_{\mathbb{R}^2} 1\{|y| \leq \sqrt{\delta_N}\} P_{\{y\sqrt{N}, 0\}} \left(G_N^c \cap \{\tau > Ns\} \right) dy.$$

By the representation of $\xi = (\xi^1, \xi^2)$ given in Lemma 6.6, for any y ,

$$P_{\{y\sqrt{N}, 0\}} \left(G_N^c \cap \{\tau > Ns\} \right) \leq P(|W_u^{y\sqrt{N}} - W_u^0| \leq 3r \text{ for some } u \leq Ns).$$

Using Lemma 5.5 again, for $\delta_N \leq s \leq T$, we have

$$\begin{aligned} \mathcal{E}''_2 &\leq (\log N) X_0^N(\mathbf{1}) \int_{\mathbb{R}^2} 1\{|y| \leq \sqrt{\delta_N}\} P \left(|Y_u^{y\sqrt{N}}| \leq 3r \text{ for some } u \leq Ns \right) dy \\ &\leq C(T) X_0^N(\mathbf{1}) \int_{\mathbb{R}^2} 1\{|y| \leq \sqrt{\delta_N}\} \log(1/|y|) dy \\ &\leq C(T) X_0^N(\mathbf{1}) \delta_N \log(1/\delta_N). \end{aligned}$$

Combining the bounds on $\mathcal{E}_1, \mathcal{E}'_2, \mathcal{E}''_2$ gives (7.20). □

We are almost ready for the proof of Proposition 4.1. The proof is lengthy, so we separate out one of its key steps in the following lemma. For $s \geq s_N$ define

$$[s]_N = (j - 1)s_N \text{ if } s \in [js_N, (j + 1)s_N), \quad j \geq 1.$$

For $T > 0$ define $j_T = j_T(N) \in \mathbb{N}$ by

$$(j_T - 1)s_N \leq T < j_T s_N.$$

Lemma 7.9. Assume $\phi \in C_0^3(\mathbb{R}^d)$ and $T > 0$. There exists $\varepsilon_{7.9}^N = \varepsilon_{7.9}^N(\phi, T) \rightarrow 0$ as $N \rightarrow \infty$ such that

$$E \left[\sup_{s_N \leq t \leq T} \left| \int_{s_N}^t (\bar{m}_s^N(\phi) - E(\bar{m}_s^N(\phi) | \mathcal{F}_{[s]_N}^N)) ds \right|^2 \right] \leq \varepsilon_{7.9}^N (X_0^N(\mathbf{1})^2 + 1) \tag{7.21}$$

Proof. Define

$$\Delta_j^N = \int_{(j-1)s_N}^{js_N} (\bar{m}_s^N(\phi) - E(\bar{m}_s^N(\phi) | \mathcal{F}_{[s]_N}^N)) ds, \quad j \geq 2.$$

Then

$$\int_{s_N}^{js_N} (\bar{m}_s^N(\phi) - E(\bar{m}_s^N(\phi) | \mathcal{F}_{[s]_N}^N)) ds = \sum_{i=2}^j \Delta_i^N = Q_j^{N,o} + Q_j^{N,e}$$

where

$$Q_j^{N,o} = \sum_{i=2}^j \Delta_i^N \mathbf{1}_{\{i \text{ is odd}\}} \text{ and } Q_j^{N,e} = \sum_{i=2}^j \Delta_i^N \mathbf{1}_{\{i \text{ is even}\}}.$$

Noting that $\mathcal{F}_{[s]_N}^N = \mathcal{F}_{(2i-1)s_N}^N$ for all $s \in [2is_N, (2i + 1)s_N)$, we see that for $i \geq 1$,

$$\begin{aligned} E(\Delta_{2i+1}^N | \mathcal{F}_{(2i-1)s_N}^N) &= \int_{2is_N}^{(2i+1)s_N} E \left[(\bar{m}_s^N(\phi) - E(\bar{m}_s^N(\phi) | \mathcal{F}_{[s]_N}^N)) | \mathcal{F}_{(2i-1)s_N}^N \right] ds \\ &= \int_{2is_N}^{(2i+1)s_N} \left(E(\bar{m}_s^N(\phi) | \mathcal{F}_{(2i-1)s_N}^N) - E(\bar{m}_s^N(\phi) | \mathcal{F}_{(2i-1)s_N}^N) \right) ds \\ &= 0, \end{aligned} \tag{7.22}$$

which shows that $(Q_{2j-1}^{N,o}, j \geq 2)$ is an $(\mathcal{F}_{(2j-1)s_N}^N)$ martingale. Similarly, $(Q_{2j}^{N,e}, j \geq 1)$ is an $(\mathcal{F}_{2js_N}^N)$ martingale. Using Doob's L^2 inequality, we have

$$\begin{aligned} E \left[\sup_{2 \leq j \leq j_T} \left| \int_{s_N}^{js_N} (\bar{m}_s^N(\phi) - E(\bar{m}_s^N(\phi) | \mathcal{F}_{[s]_N}^N)) ds \right|^2 \right] \\ &= E \left[\sup_{2 \leq j \leq j_T} \left| Q_j^{N,o} \mathbf{1}_{\{j \text{ odd}\}} + Q_j^{N,e} \mathbf{1}_{\{j \text{ even}\}} \right|^2 \right] \\ &\leq 2E \left[\sup_{2 \leq j \leq j_T, j \text{ odd}} (Q_j^{N,o})^2 + \sup_{2 \leq j \leq j_T, j \text{ even}} (Q_j^{N,e})^2 \right] \leq CE \left[(Q_{j_T}^{N,o})^2 + (Q_{j_T}^{N,e})^2 \right]. \end{aligned} \tag{7.23}$$

Now if $j_T = 2k - 1$, then by (7.22),

$$\begin{aligned} E \left[(Q_{j_T}^{N,o})^2 \right] &= \sum_{i=2}^k E \left[(\Delta_{2i-1}^N)^2 \right] \leq \sum_{i=2}^{j_T} E \left[(\Delta_i^N)^2 \right] \\ &\leq 2 \sum_{i=2}^{j_T} E \left[\left(\int_{(i-1)s_N}^{is_N} \bar{m}_s^N(\phi) ds \right)^2 \right] \\ &\quad + 2 \sum_{i=2}^{j_T} E \left[\left(\int_{(i-1)s_N}^{is_N} E(\bar{m}_s^N(\phi) | \mathcal{F}_{(i-1)s_N}^N) ds \right)^2 \right], \end{aligned} \tag{7.24}$$

which also holds if j_T is even. Define $\bar{X}_T^N(\mathbf{1}) = \sup_{t \leq T} X_t^N(\mathbf{1})$. By Lemma 2.4(a), (3.5) and (3.10) we have

$$\begin{aligned} \sum_{i=2}^{j_T} E \left[\left(\int_{(i-1)s_N}^{is_N} \bar{m}_s^N(\phi) ds \right)^2 \right] &\leq \sum_{i=2}^{j_T} E \left[\left(\int_{(i-1)s_N}^{is_N} C_\phi(\log N) X_s^N(\mathbf{1}) ds \right)^2 \right] \\ &\leq C_\phi(\log N)^2 \sum_{i=2}^{j_T} s_N^2 E \left[\bar{X}_{T+1}^N(\mathbf{1})^2 \right] \\ &\leq C(\phi, T)(X_0^N(\mathbf{1})^2 + 1)(j_T s_N) s_N (\log N)^2 \\ &\leq C(\phi, T)(X_0^N(\mathbf{1})^2 + 1)(\log N)^{2-q}. \end{aligned} \tag{7.25}$$

This and (7.24) imply $E \left[(Q_{j_T}^{N,o})^2 \right] \leq C(\phi, T)(X_0^N(\mathbf{1})^2 + 1)(\log N)^{2-q}$, and it is clear this bound also holds for $E \left[(Q_{j_T}^{N,e})^2 \right]$. Plugging into (7.23) we obtain

$$E \left[\sup_{2 \leq j \leq j_T} \left| \int_{s_N}^{js_N} (\bar{m}_s^N(\phi) - E(\bar{m}_s^N(\phi) | \mathcal{F}_{[s]_N}^N)) ds \right|^2 \right] \leq C(\phi, T)(X_0^N(\mathbf{1})^2 + 1)(\log N)^{2-q}. \tag{7.26}$$

Now if $(i-1)s_N \leq t \leq is_N$, then

$$\begin{aligned} &\left(\int_{s_N}^t (\bar{m}_s^N(\phi) - E(\bar{m}_s^N(\phi) | \mathcal{F}_{[s]_N}^N)) ds \right)^2 \\ &\leq \left(\left| \int_{s_N}^{(i-1)s_N} (\bar{m}_s^N(\phi) - E(\bar{m}_s^N(\phi) | \mathcal{F}_{[s]_N}^N)) ds \right| \right. \\ &\quad \left. + \int_{(i-1)s_N}^{is_N} (\bar{m}_s^N(\phi) - E(\bar{m}_s^N(\phi) | \mathcal{F}_{[s]_N}^N)) ds \right)^2 \\ &\leq 2 \left(\int_{s_N}^{(i-1)s_N} (\bar{m}_s^N(\phi) - E(\bar{m}_s^N(\phi) | \mathcal{F}_{[s]_N}^N)) ds \right)^2 + 2 \left(2C_\phi(\log N) s_N \bar{X}_{T+1}^N(\mathbf{1}) \right)^2 \\ &\leq 2 \sup_{2 \leq j \leq j_T} \left(\int_{s_N}^{js_N} (\bar{m}_s^N(\phi) - E(\bar{m}_s^N(\phi) | \mathcal{F}_{[s]_N}^N)) ds \right)^2 + C_\phi(\log N)^{2-2q} \bar{X}_{T+1}^N(\mathbf{1})^2, \end{aligned}$$

where we have used Lemma 2.4(a), (3.5), and the martingale property of $X_s^N(\mathbf{1})$ (Corollary 3.2). Thus from the above, (7.26), and Corollary 3.2, the left side of (7.21) is bounded above by

$$\begin{aligned} 2E \left[\sup_{2 \leq j \leq j_T} \left(\int_{s_N}^{js_N} (\bar{m}_s^N(\phi) - E(\bar{m}_s^N(\phi) | \mathcal{F}_{[s]_N}^N)) ds \right)^2 \right] + C_\phi(\log N)^{2-2q} E \left[\bar{X}_{T+1}^N(\mathbf{1})^2 \right] \\ \leq C(\phi, T)(X_0^N(\mathbf{1})^2 + 1)(\log N)^{2-q}. \end{aligned} \tag{7.27}$$

□

Define $\tilde{\gamma}^N(s) = \bar{\gamma}^N(s - [s]_N) = K' \gamma_e(N(s - [s]_N))$. Note that for $s \geq s_N$, $s - [s]_N \geq s_N$, and so by Proposition 1.1,

$$\tilde{\gamma}^N(s) \rightarrow \gamma_e \text{ uniformly in } s \geq s_N \text{ as } N \rightarrow \infty, \tag{7.28}$$

and there is a constant $C_\gamma > 0$ such that

$$\sup_{N \geq 3, s \geq s_N} \tilde{\gamma}^N(s) \leq C_\gamma. \tag{7.29}$$

Proof of Proposition 4.1. It follows from Proposition 2.2, Lemma 2.4, and Corollary 3.2 that

$$E \left[\sup_{0 \leq t \leq T} \left| \langle M^N(\phi) \rangle_t - \int_0^t \bar{m}_s^N(\phi) ds \right| \right] \leq C \frac{\log N}{\sqrt{N}} \int_0^T E(X_s^N(\mathbf{1})) ds = CT \frac{\log N}{\sqrt{N}} X_0^N(\mathbf{1}),$$

and so to prove (4.1) it suffices to show

$$\sup_{X_0^N(\mathbf{1}) \leq A} E \left[\sup_{0 \leq t \leq T} \left| \int_0^t (\bar{m}_s^N(\phi) - \rho^2 |B_r|^2 \gamma_e X_s^N(\phi^2)) ds \right| \right] \rightarrow 0 \text{ as } N \rightarrow \infty. \quad (7.30)$$

Furthermore, by Lemma 2.4(a), Proposition 3.1(a) and Corollary 3.2,

$$E(\bar{m}_s^N(\phi) + \rho^2 |B_r|^2 \gamma_e X_s^N(\phi^2)) \leq E(CK' X_s^N(\mathbf{1})) = CK' X_0^N(\mathbf{1}), \quad (7.31)$$

and therefore

$$E \left[\int_0^{3\delta_N} (\bar{m}_s^N(\phi) + \rho^2 |B_r|^2 \gamma_e X_s^N(\phi^2)) ds \right] \leq CK' \delta_N X_0^N(\mathbf{1}) \leq C(\log N)^{1+\delta-q} X_0^N(\mathbf{1}).$$

So by our choice of δ, q (recall (7.14)), we need only integrate over $[3\delta_N, T]$ in (7.30).

Let us write

$$E \left[\sup_{3\delta_N \leq t \leq T} \left| \int_{3\delta_N}^t (\bar{m}_t^N(\phi) - \rho^2 |B_r|^2 \gamma_e X_s^N(\phi^2)) ds \right| \right] \leq \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4, \quad (7.32)$$

where

$$\begin{aligned} \mathcal{E}_1 &= E \left[\sup_{3\delta_N \leq t \leq T} \left| \int_{3\delta_N}^t (\bar{m}_s^N(\phi) - E(\bar{m}_s^N(\phi) | \mathcal{F}_{[s]_N})) ds \right| \right], \\ \mathcal{E}_2 &= E \left[\int_{3\delta_N}^T \left| E(\bar{m}_s^N(\phi) | \mathcal{F}_{[s]_N}) - \rho^2 |B_r|^2 \tilde{\gamma}^N(s) X_{[s]_N}^N(\phi^2) \right| ds \right], \\ \mathcal{E}_3 &= \rho^2 |B_r|^2 E \left[\int_{3\delta_N}^T \tilde{\gamma}^N(s) |X_{[s]_N}^N(\phi^2) - X_s^N(\phi^2)| ds \right], \\ \mathcal{E}_4 &= \rho^2 |B_r|^2 E \left[\int_{3\delta_N}^T |\tilde{\gamma}^N(s) - \gamma_e| X_s^N(\phi^2) ds \right]. \end{aligned}$$

By Lemma 7.9,

$$\mathcal{E}_1^2 \leq \varepsilon_{7.9}^N (X_0^N(\mathbf{1})^2 + 1). \quad (7.33)$$

The error term \mathcal{E}_4 is simple:

$$\mathcal{E}_4 \leq C \sup_{s \geq 3\delta_N} |\tilde{\gamma}^N(s) - \gamma_e| \rho^2 |B_r|^2 \int_{3\delta_N}^T E(X_s^N(\mathbf{1})) ds \leq CT \sup_{s \geq 3\delta_N} |\tilde{\gamma}^N(s) - \gamma_e| X_0^N(\mathbf{1}) \rightarrow 0 \quad (7.34)$$

as $N \rightarrow \infty$, by (7.28) and $s \geq 3\delta_N \geq s_N$.

We now consider \mathcal{E}_3 . Let j_T be as defined before Lemma 7.9, and let $j_0 \in \mathbb{N}$ be defined by

$$(j_0 - 1)s_N \leq 3\delta_N < j_0 s_N.$$

By the martingale decomposition in Proposition 2.2 and then Cauchy-Schwarz,

$$\begin{aligned} E(|X_s^N(\phi^2) - X_{[s]_N}^N(\phi^2)|) &\leq E(|D_s^N(\phi^2) - D_{[s]_N}^N(\phi^2)|) + E(|M_s^N(\phi^2) - M_{[s]_N}^N(\phi^2)|) \\ &\leq \int_{[s]_N}^s E(|d_u^N(\phi^2)|) du + \left[E(\langle M^N(\phi^2) \rangle_s - \langle M^N(\phi^2) \rangle_{[s]_N}) \right]^{1/2}. \end{aligned} \quad (7.35)$$

By Lemma 2.3 and Corollary 3.2,

$$\int_{[s]_N}^s E(|d_u^N(\phi^2)|)du \leq C_\phi \int_{[s]_N}^s X_0^N(\mathbf{1})du \leq C_\phi 2s_N X_0^N(\mathbf{1}).$$

By Proposition 2.2, Lemma 2.4 and (7.31),

$$\begin{aligned} E(\langle M^N(\phi^2) \rangle_s - \langle M^N(\phi^2) \rangle_{[s]_N}) &= E\left[E_{X_{[s]_N}^N} \left(\int_0^{s-[s]_N} m_u^N(\phi^2)du \right)\right] \\ &\leq C_\phi(\log N) \int_0^{2s_N} E(X_u^N(\mathbf{1}))du \\ &= C_\phi(\log N)s_N X_0^N(\mathbf{1}). \end{aligned}$$

Plugging these bounds into (7.35), and using (7.29) and (7.14), we obtain

$$\mathcal{E}_3 \leq CT \left((\log N)^{-q} X_0^N(\mathbf{1}) + (\log N)^{\frac{1-q}{2}} X_0^N(\mathbf{1})^{\frac{1}{2}} \right) \tag{7.36}$$

Turning to \mathcal{E}_2 , first use the Markov property and Lemma 7.6, and then Corollary 3.2, to conclude that

$$\begin{aligned} \mathcal{E}_2 &\leq E \left[\int_{3\delta_N}^T C \left[\frac{\log \log N}{\log N} X_{[s]_N}^N(\mathbf{1}) \right. \right. \\ &\quad \left. \left. + \frac{N^{\frac{1}{2}-\frac{d}{4}}}{s_N \log N} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1\{|y_1 - y_2| \leq \sqrt{\delta_N}\} X_{[s]_N}^N(y_1) X_{[s]_N}^N(y_2) dy_1 dy_2 \right] ds \right] \\ &\leq CT \frac{\log \log N}{\log N} X_0^N(\mathbf{1}) \\ &\quad + \frac{CN^{\frac{1}{2}-\frac{d}{4}}}{s_N \log N} \int_{3\delta_N}^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1\{|y_1 - y_2| \leq \sqrt{\delta_N}\} E(X_{[s]_N}^N(y_1) X_{[s]_N}^N(y_2)) dy_1 dy_2 ds. \end{aligned} \tag{7.37}$$

If $d \geq 3$, use Corollary 3.2 to bound the last term by

$$\frac{CTN^{\frac{1}{2}-\frac{d}{4}}}{s_N \log N} (X_0(\mathbf{1}) + X_0^N(\mathbf{1})^2). \tag{7.38}$$

If $d = 2$, then by Lemma 7.8,

$$\begin{aligned} &\int_{3\delta_N}^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1\{|y_1 - y_2| \leq \sqrt{\delta_N}\} E(X_{[s]_N}^N(y_1) X_{[s]_N}^N(y_2)) dy_1 dy_2 ds \\ &\leq C_T \int_{3\delta_N}^T \left[\frac{\delta_N}{[s]_N} X_0^N(\mathbf{1})^2 + \delta_N \log(1/\delta_N) X_0^N(\mathbf{1}) \right] ds \\ &\leq C_T \delta_N X_0^N(\mathbf{1})^2 \int_{\delta_N}^T \frac{1}{u} du + C_T T \delta_N \log(1/\delta_N) X_0^N(\mathbf{1}) \\ &= C_T \delta_N \log(T/\delta_N) (X_0^N(\mathbf{1})^2 + X_0^N(\mathbf{1})). \end{aligned}$$

Now insert the above or (7.38) into (7.37) to arrive at

$$\begin{aligned} \mathcal{E}_2 &\leq C_T \frac{\log \log N}{\log N} X_0^N(\mathbf{1}) + C_T \frac{\delta_N \log(1/\delta_N) + N^{\frac{1}{2}-\frac{d}{4}}}{s_N \log N} (X_0^N(\mathbf{1})^2 + 1) \\ &\leq C_T \frac{\log \log N}{\log N} X_0^N(\mathbf{1}) + C_T \frac{(\log N)^\delta \log \log N}{\log N} (X_0^N(\mathbf{1})^2 + 1), \end{aligned} \tag{7.39}$$

where the last inequality is rather crude if $d \geq 3$. The required result now follows from (7.32), (7.33), (7.34), (7.36) and (7.39) \square

8 Appendix: Proof of Proposition 1.1

From the discussion following the statement of Proposition 1.1 we may assume $d = 2$ throughout. Recall the definitions and time change construction from Section 6 (especially Lemmas 6.1 and 6.2), using the rate $2\rho|B_r|$ random walk Y_t , the difference process $\tilde{\xi}_t^x$, and absorption time $\tilde{\tau} = \kappa$. By Lemma 6.2, if $x = x_1 - x_2$,

$$P_x(\tilde{\tau} > t) = P_x(\kappa > t) = E_x \left[\exp\left(-\int_0^{I^{-1}(t)} k(Y_s) ds\right) \right]. \tag{8.1}$$

Thus, to prove (1.16) in Proposition 1.1 we need to show existence and positivity of

$$\lim_{t \rightarrow \infty} (\log t) E_x \left[\exp\left(-\int_0^{I^{-1}(t)} k(Y_s) ds\right) \right], \quad x \neq 0. \tag{8.2}$$

As the exact form of the killing rate $k(x)$ will not be important in our arguments, we will replace it with a general radial function $\phi : \mathbb{R}^2 \rightarrow [0, \infty]$ satisfying

$$\phi(x) < \infty \text{ for } x \neq 0, \phi(x) \text{ is } \downarrow \text{ in } |x| \text{ and } \phi(x) = 0 \text{ for } |x| > 2r,$$

where \downarrow in $|x|$ means non-increasing in $|x|$, and similarly for \uparrow . We assume throughout that ϕ has these properties. Recall the stopping times t_A and T_A from (5.7).

Proposition 8.1. *Let $Y_0 = x \in \mathbb{R}^2, |x| > 2r$, or $Y_0 = \bar{U}$. If there is a constant $c_\phi(Y_0) > 0$ such that the limit*

$$c_\phi(Y_0) = \lim_{A \rightarrow \infty} \log(A^2) E_{Y_0} \left[\exp\left(-\int_0^{T_A} \phi(Y_s) ds\right) \right] \tag{8.3}$$

exists, then

$$\lim_{t \rightarrow \infty} (\log t) E_{Y_0} \left[\exp\left(-\int_0^{I^{-1}(t)} \phi(Y_s) ds\right) \right] = c_\phi(Y_0). \tag{8.4}$$

Proof. Suppose (8.3) holds. We first prove that this implies

$$\lim_{t \rightarrow \infty} (\log t) E_{Y_0} \left[\exp\left(-\int_0^t \phi(Y_s) ds\right) \right] = c_\phi(Y_0). \tag{8.5}$$

If we let Γ_A be the event $\{A^2/\log A \leq T_A \leq A^2 \log A\}$, then by Lemma 5.4, for $A > (2|Y_0|) \vee 2$, $P_{Y_0}(\Gamma_A^c) \leq C/(\log A)^2$. Thus

$$\begin{aligned} E_{Y_0} \left[\exp\left(-\int_0^{T_A} \phi(Y_s) ds\right) \right] &\leq E_{Y_0} \left[1_{\Gamma_A} \exp\left(-\int_0^{T_A} \phi(Y_s) ds\right) \right] + P_{Y_0}(\Gamma_A^c) \\ &\leq E_{Y_0} \left[\exp\left(-\int_0^{A^2/\log A} \phi(Y_s) ds\right) \right] + \frac{C}{(\log A)^2}. \end{aligned}$$

It follows that

$$\begin{aligned} &\log\left(\frac{A^2}{\log A}\right) E_{Y_0} \left[\exp\left(-\int_0^{A^2/\log A} \phi(Y_s) ds\right) \right] \\ &\geq \log\left(\frac{A^2}{\log A}\right) \left(E_{Y_0} \left[\exp\left(-\int_0^{T_A} \phi(Y_s) ds\right) \right] - \frac{C}{(\log A)^2} \right) \rightarrow c_\phi(Y_0) \text{ as } A \rightarrow \infty. \end{aligned}$$

This proves

$$\liminf_{t \rightarrow \infty} (\log t) E_{Y_0} \left[\exp\left(-\int_0^t \phi(Y_s) ds\right) \right] \geq c_\phi(Y_0). \tag{8.6}$$

Similarly,

$$\begin{aligned} E_{Y_0} \left[\exp \left(- \int_0^{T_A} \phi(Y_s) ds \right) \right] &\geq E_{Y_0} \left[1_{\Gamma_A} \exp \left(- \int_0^{T_A} \phi(Y_s) ds \right) \right] \\ &\geq E_{Y_0} \left[\exp \left(- \int_0^{A^2 \log A} \phi(Y_s) ds \right) \right] - P_{Y_0}(\Gamma_A^c). \end{aligned}$$

It follows that

$$\begin{aligned} \log \left(A^2 \log A \right) E_{Y_0} \left[\exp \left(- \int_0^{A^2 \log A} \phi(Y_s) ds \right) \right] \\ \leq \log \left(A^2 \log A \right) \left(E_{Y_0} \left[\exp \left(- \int_0^{T_A} \phi(Y_s) ds \right) \right] + \frac{C}{(\log A)^2} \right) \\ \rightarrow c_\phi(Y_0) \text{ as } A \rightarrow \infty. \end{aligned} \tag{8.7}$$

Along with (8.6), this proves (8.5).

We can now prove (8.4). By (8.1), (8.5) and $I^{-1}(t) \leq t$,

$$\liminf_{t \rightarrow \infty} (\log t) E_x \left[\exp \left(- \int_0^{I^{-1}(t)} \phi(Y_s) ds \right) \right] \geq \lim_{t \rightarrow \infty} (\log t) E_{Y_0} \left[\exp \left(- \int_0^t \phi(Y_s) ds \right) \right] = c_\phi(Y_0).$$

On the other hand, by Lemma 6.3, taking $\alpha = 1/2$,

$$\begin{aligned} (\log t) E_x \left[\exp \left(- \int_0^{I^{-1}(t)} \phi(Y_s) ds \right) \right] &\leq (\log t) E_{Y_0} \left[1_{\{I^{-1}(t) \geq t - t^{1/2}\}} \right. \\ &\quad \times \exp \left(- \int_0^{I^{-1}(t)} \phi(Y_s) ds \right) \left. \right] + P_{Y_0}(I^{-1}(t) < t - t^{1/2}) \\ &\leq (\log t) E_{Y_0} \left[\exp \left(- \int_0^{t - \sqrt{t}} \phi(Y_s) ds \right) \right] + (\log t) C_{6.3} \frac{\log(1+t)}{\sqrt{t}} \\ &\rightarrow c_\phi(Y_0) \text{ as } t \rightarrow \infty \end{aligned}$$

by (8.5). Along with the previous lim inf bound this proves (8.4). □

To prove (8.3) we first establish a number of properties of

$$\Phi(x, A) = E_x \left[\exp \left(- \int_0^{T_A} \phi(Y_s) ds \right) \right]. \tag{8.8}$$

It is elementary that $0 \leq \Phi(x, A) \leq 1$ and that by recurrence, $\Phi(x, A) \rightarrow 0$ as $A \rightarrow \infty$. The next two results will show that $\Phi(x, A)$ is increasing in $|x|$.

Lemma 8.2. *Let $N \in \mathbb{N}$. If $0 = s_0 < s_1 < \dots < s_N$ and $f : \mathbb{R}^{N+1} \rightarrow \mathbb{R}$ is bounded and \uparrow in each coordinate then*

$$y \mapsto E_y[f(|Y_{s_0}|, \dots, |Y_{s_N}|)] \text{ is } \uparrow \text{ in } |y|. \tag{8.9}$$

Proof. Let U, U_1, U_2, \dots be iid rv's uniform on B_r , and let $S_m = U_1 + \dots + U_m$. The first step is to prove that if $N = 1$ then for $m = 1, 2, \dots$,

$$E[f(|y|, |y + S_m|)] \text{ is increasing in } |y|. \tag{8.10}$$

Let $u \geq 0$, and define

$$h_u(y) = P(|y + U| \leq u) = \frac{|B_r \cap B_u(-y)|}{|B_r|} = \frac{|B_r \cap B_u(y)|}{|B_r|}. \tag{8.11}$$

It is easy to see that $h_u(y)$ is decreasing in $|y|$, so that $|y + S_1|$ is stochastically increasing in $|y|$, which proves (8.10) for $m = 1$. Now suppose $m = 2$, and consider

$$P(|y + S_2| \leq u) = P(|y + S_1 + U_2| \leq u) = E[h_u(y + S_1)].$$

Clearly $h_u(y)$ depends only on $|y|$, and having established it is decreasing in $|y|$, the $m = 1$ case of (8.10) implies that $E[h_u(y + S_2)]$ is decreasing in $|y|$, which shows $|y + S_2|$ is stochastically increasing in $|y|$, proving (8.10) for $m = 2$. The general inductive step for (8.10) is similar.

Consider next, the $N = 1$ case of (8.9). With $\lambda = 2\rho|B_r|$,

$$E_{y_0}[f(|Y_{s_0}|, |Y_{s_1}|)] = \sum_{m=0}^{\infty} e^{-\lambda s_1} \frac{(\lambda s_1)^m}{m!} E[f(|y_0|, |y_0 + S_{2m}|)]$$

By (8.10), this shows $E_{y_0}[f(|Y_{s_0}|, |Y_{s_1}|)]$ is increasing in $|y_0|$, proving (8.9) for $N = 1$. Now suppose $N > 1$, let (Y'_t) under P' be an independent copy of (Y_t) , and define $\tilde{f} : \mathbb{R}^N \rightarrow \mathbb{R}$ by

$$\tilde{f}(|y_0|, |y_1|, \dots, |y_{N-1}|) = E'_{y_{N-1}}[f(|y_0|, \dots, |y_{N-1}|, |Y'_{s_N - s_{N-1}}|)]. \tag{8.12}$$

Then \tilde{f} is increasing in $|y_0|, \dots, |y_{N-2}|$ by definition, and is increasing in $|y_{N-1}|$ by the $N = 1$ case of (8.9) just established. By the Markov property applied at time s_{N-1} ,

$$\begin{aligned} E_{y_0}[f(|Y_{s_0}|, \dots, |Y_{s_{N-1}}|, |Y_{s_N}|)] &= E_{y_0}[E'_{Y_{s_{N-1}}}[f(|Y_{s_0}|, \dots, |Y_{s_{N-1}}|, |Y'_{s_N - s_{N-1}}|)]] \\ &= E_{y_0}[\tilde{f}(|Y_{s_0}|, \dots, |Y_{s_{N-1}}|)]. \end{aligned}$$

This provides the inductive step to complete the proof of (8.9) for general N . □

Lemma 8.3. *Let $g : (0, \infty) \rightarrow [0, \infty)$ be continuous and \downarrow . Then for all $A, t \in [0, \infty)$,*

$$y \mapsto E_y \left[\exp \left(- \int_0^{T_A \wedge t} g(|Y_s|) ds \right) \right] \text{ is } \uparrow \text{ in } |y|. \tag{8.13}$$

In particular, $x \rightarrow \Phi(x, A)$ is increasing in $|x|$.

Proof. By monotone convergence, we may assume A, t are finite and g is bounded. Let $g(0) = \lim_{s \downarrow 0} g(s)$. For $N \in \mathbb{N}$ let $M_N \in \mathbb{N}$ and $0 = s_0^N < s_1^N < \dots < s_{M_N}^N = t$ satisfy $s_{i+1}^N - s_i^N < 2^{-N}$ for $0 \leq i < M_N$, and define

$$\tau^N = \min\{s_i^N : |Y_{s_i^N}| > A\} \wedge t.$$

By right-continuity of $|Y_s|$, $\tau^N \downarrow T_A$ a.s. as $N \rightarrow \infty$. By continuity of g on $[0, \infty)$ and dominated convergence,

$$\begin{aligned} E_y \left[\exp \left(- \int_0^{T_A \wedge t} g(|Y_s|) ds \right) \right] &= \lim_{N \rightarrow \infty} E_y \left[\exp \left(- \sum_{i=1}^{M_N-1} 1_{\{s_i^N < \tau^N\}} g(|Y_{s_i^N}|) (s_{i+1}^N - s_i^N) \right) \right] \\ &= \lim_{N \rightarrow \infty} E_y \left[\prod_{i=0}^{M_N-1} G_i^N(|Y_{s_0^N}|, \dots, |Y_{s_i^N}|) \right] \end{aligned}$$

where

$$G_i^N(|Y_{s_0^N}|, \dots, |Y_{s_i^N}|) = \exp \left(- 1_{\{s_i^N < \tau^N\}} g(|Y_{s_i^N}|) (s_{i+1}^N - s_i^N) \right).$$

It is easy to check that G_i^N is increasing in each of its variables, and hence applying Lemma 8.2 to their product, $E_y \left[\prod_{i=0}^{M_N-1} G_i^N(|Y_{s_0}|, \dots, |Y_{s_i}|) \right]$ is increasing in $|y|$. The result (8.13) now follows from the above. \square

A consequence of the strong Markov property we will use repeatedly is

$$\Phi(x, A) = P_x(T_A < t_a) + E_x \left[1\{t_a < T_A\} \Phi(Y_{t_a}, A) \right] \text{ if } 2r < a < |x| < A. \tag{8.14}$$

Lemma 8.4. *There exists $C_{8.15} = C_{8.15}(r) > 1$ such that for all $k \geq 2$ and $0 < |x| \leq k < A$,*

$$\Phi(x, A) \leq C_{8.15} \frac{\log k}{\log(A^2)}. \tag{8.15}$$

Proof. By the monotonicity in Lemma 8.3, it suffices to prove (8.15) for $x = x_k = (k, 0)$. Assume additionally $k > 6r \vee r^{-1}$ and $A > r^2$. By (8.14), $|Y_{t_{3r}}| \leq 3r$, and monotonicity, we have

$$\begin{aligned} \Phi(x_k, A) &= P_{x_k}(T_A < t_{3r}) + E_{x_k} \left[1\{t_{3r} < T_A\} \Phi(Y_{t_{3r}}, A) \right] \\ &\leq P_{x_k}(T_A < t_{3r}) + P_{x_k}(t_{3r} < T_A) \Phi((3r, 0), A). \end{aligned} \tag{8.16}$$

Using the strong Markov property at time T_{4r} , and noting $|Y_{T_{4r}}| \leq 6r \leq |x_k| < A$, we again have from Lemma 8.3,

$$\Phi((3r, 0), A) = E_{(3r,0)} \left[\exp \left(- \int_0^{T_{4r}} \phi(Y_s) ds \right) \Phi(Y_{T_{4r}}, A) \right] \leq \alpha(r) \Phi(x_k, A),$$

where we have set $\alpha(r) = \Phi(x_{3r}, 4r) < 1$. Insert this into (8.16) and rearrange to conclude

$$\Phi(x_k, A) \leq \frac{P_{x_k}(T_A < t_{3r})}{1 - \alpha(r)}. \tag{8.17}$$

By Lemma 5.3, taking $a = 3r$,

$$P_{x_k}(T_A < t_{3r}) \leq \frac{\log(k/r)}{\log(A/r)} \leq \frac{\log(k^2)}{\log(\sqrt{A})} = \frac{8 \log k}{\log(A^2)} \tag{8.18}$$

where the second inequality uses $k > 1/r$ and $A > r^2$. In view of (8.17), letting $C = 8/(1 - \alpha(r))$, we now have

$$\Phi(x_k, A) \leq C \frac{\log k}{\log(A^2)} \tag{8.19}$$

for all $k > 6r \vee r^{-1} \vee 2$ and $A > k \vee r^2$. It is easy to see that C can be increased so that (8.19) will hold for all $k \geq 2$ and $A > k$, completing the proof of (8.15). \square

We will construct a coupling of the random walks Y_t started at $x' \neq x$ in order to obtain good bounds on the difference $\Phi(x', A) - \Phi(x, A)$. We start in discrete time. Let $\{U_i\}$ be iid r.v.'s which are uniformly distributed over B_r , and for $x' \in H_r = \{(x_1, x_2) : x_1 > 0\}$ define

$$S_n^{x'} = x' + \sum_{i=1}^n U_i.$$

Let π denote the reflection mapping $\pi(x_1, x_2) = (-x_1, x_2)$ and set $x = \pi(x')$. We will use a reflection coupling to define $(S_n^x : n \geq 0)$. Let $H_\ell = \{(x_1, x_2) : x_1 \leq 0\}$, and define

$$N_c = N_c^{x,x'} = \min\{n \geq 1 : S_n^{x'} \in B_r(\pi(S_{n-1}^x))\}.$$

Lemma 8.5. $N_c \leq N_\ell := \min\{n \geq 0 : S_n^{x'} \in H_\ell\}$ a.s., and so $S_n^{x'} \in H_r$ for all $0 \leq n < N_c$ a.s.

Proof. $S_{N_\ell}^{x'} \in H_\ell$ and $\pi(S_{N_\ell-1}^{x'}) \in H_\ell$ imply that

$$|S_{N_\ell}^{x'} - \pi(S_{N_\ell-1}^{x'})| \leq |\pi(S_{N_\ell}^{x'}) - \pi(S_{N_\ell-1}^{x'})| = |S_{N_\ell}^{x'} - S_{N_\ell-1}^{x'}| < r.$$

The result follows. □

We now define $(S_n^x)_{n \geq 0}$ by

$$S_n^x = \begin{cases} \pi(S_n^{x'}) & \text{if } n < N_c \\ S_n^{x'} & \text{if } n \geq N_c. \end{cases} \tag{8.20}$$

Then $S_0^x = x$, and it follows from Lemma 8.5 that for $n < N_c$, $S_n^{x'}$ is in H_r and so $S_n^{x'} \neq S_n^x$, which implies that

$$N_c = \min\{n \geq 0 : S_n^{x'} = S_n^x\}. \tag{8.21}$$

That is, N_c is the coupling time of (S_n^x) and $(S_n^{x'})$. If we let $\mathcal{F}_n^{S^{x'}} = \sigma(S_m^{x'}, m \leq n)$, then N_c is an $(\mathcal{F}_n^{S^{x'}})$ -stopping time, and S^x is $(\mathcal{F}_n^{S^{x'}})$ -adapted. We next show that S_n^x is an $(\mathcal{F}_n^{S^{x'}})$ -random walk starting at x with step distribution U_1 , as the notation suggests.

Lemma 8.6. For any Borel $A \subset \mathbb{R}^2$, $P(S_{n+1}^x \in A \mid \mathcal{F}_n^{S^{x'}})(\omega) = P(S_n^x(\omega) + U_{n+1} \in A)$ a.s.

Proof. This is obvious on $\{N_c \leq n\}$ (in $\mathcal{F}_n^{S^{x'}}$) since then S_n^x and S_{n+1}^x equal $S_n^{x'}$ and $S_{n+1}^{x'}$, respectively. Suppose now that $N_c > n$, and define $\hat{B} = \hat{B}(\omega) = B_r(S_n^{x'}) \cap B_r(\pi(S_n^{x'}))$, so that

$$\pi(\hat{B}) = \hat{B}, \tag{8.22}$$

and

$$\hat{B} \subset B_r(S_n^x). \tag{8.23}$$

This last inclusion holds because $S_n^x = S_n^{x'}$ or $\pi(S_n^{x'})$ for all n .

For simplicity we will write \mathcal{F}_n for $\mathcal{F}_n^{S^{x'}}$ in the rest of this proof. By the definition of S_n^x ,

$$\begin{aligned} & P(S_{n+1}^x \in A \mid \mathcal{F}_n) 1(N_c > n) \\ &= P(S_{n+1}^{x'} \in B_r(\pi(S_n^{x'})), N_c > n, S_n^{x'} + U_{n+1} \in A \mid \mathcal{F}_n) \\ &\quad + P(S_{n+1}^{x'} \notin B_r(\pi(S_n^{x'})), N_c > n, \pi(S_{n+1}^{x'}) \in A \mid \mathcal{F}_n) \\ &= P(S_{n+1}^{x'} \in \hat{B} \cap A, N_c > n \mid \mathcal{F}_n) + P(S_{n+1}^{x'} \notin \hat{B}, N_c > n, \pi(S_{n+1}^{x'}) \in A \mid \mathcal{F}_n) \\ &= P(\pi(S_{n+1}^{x'}) \in \hat{B} \cap \pi(A), N_c > n \mid \mathcal{F}_n) + P(\pi(S_{n+1}^{x'}) \in \hat{B}^c \cap A, N_c > n \mid \mathcal{F}_n) \text{ (by (8.22))} \\ &= [P(S_n^x + \pi(U_{n+1}) \in \hat{B} \cap \pi(A) \mid \mathcal{F}_n) + P(S_n^x + \pi(U_{n+1}) \in \hat{B}^c \cap A \mid \mathcal{F}_n)] 1(N_c > n). \end{aligned}$$

Next introduce the dependence on ω in the above, and use the fact that, conditionally on \mathcal{F}_n , $S_n^x(\omega) + \pi(U_{n+1})$ is uniformly distributed over $B_r(S_n^x(\omega))$ to see that if $|C|$ is the Lebesgue measure of C , then the above evaluated at ω is a.s. equal to

$$\begin{aligned} & [|\pi(A) \cap \hat{B}(\omega) \cap B_r(S_n^x(\omega))| + |A \cap \hat{B}^c(\omega) \cap B_r(S_n^x(\omega))|] 1(N_c(\omega) > n) / |B_r| \\ &= [|A \cap \hat{B}(\omega)| + |A \cap \hat{B}^c(\omega) \cap B_r(S_n^x(\omega))|] 1(N_c(\omega) > n) / |B_r| \\ &\quad \text{(by } |\pi(C)| = |C|, \text{ (8.22), (8.23))} \\ &= [|A \cap \hat{B}(\omega) \cap B_r(S_n^x(\omega))| + |A \cap \hat{B}^c(\omega) \cap B_r(S_n^x(\omega))|] 1(N_c(\omega) > n) / |B_r| \text{ (by (8.23))} \\ &= |A \cap B_r(S_n^x(\omega))| 1(N_c(\omega) > n) / |B_r| \\ &= P(S_n^x(\omega) + U_{n+1} \in A) 1(N_c(\omega) > n). \end{aligned}$$

The result follows. □

Let $S_n = (S_n^{(1)}, S_n^{(2)})$ denote a copy of the random walk starting at x' under $P_{x'}$.

Lemma 8.7. *There is a constant $C_{8.7}$ so that for all x' in the positive x_1 -axis and all $n \in \mathbb{N}$,*

$$P(N_c^{x,x'} \geq n) \leq \frac{C_{8.7}}{\sqrt{n}} \left(1 + \frac{|x'|}{2r}\right).$$

Proof. Use Lemma 8.5 and then the reflection principle to see that

$$\begin{aligned} P(N_c^{x,x'} \geq n) &\leq P_{x'}(N_\ell \geq n) = 1 - P_{x'}(N_\ell < n) \\ &= 1 - 2P_{x'}(S_n^{(1)} < S_{N_\ell}^{(1)}, N_\ell < n) \\ &\leq 1 - 2P_0(S_n^{(1)} < -|x'| - r) \\ &= P_0(|S_n^{(1)}| \leq r + |x'|). \end{aligned}$$

The step distribution of $(S_n^{(1)})$ has density $f(u) = 2\sqrt{r^2 - u^2}/|B_r| \leq 1/r$ on $[-r, r]$. It follows from the $d = 1$ version of (5.6) applied to random variables with this distribution that for a constant $C = C(r)$,

$$P_0(|S_n^{(1)}| \leq r + |x'|) \leq C \frac{2(r + |x'|)}{\sqrt{n}} \leq \frac{4Cr}{\sqrt{n}} \left(1 + \frac{|x'|}{2r}\right),$$

so we are done. □

We now use translation invariance to extend the above to points $x, x' \in \{(x_1, 0) : x_1 \geq 0\}$ such that $0 \leq |x| < |x'|$, where now $\frac{x+x'}{2} = (m, 0)$ plays the role of the origin, $H_\ell^m = \{x : x_1 \leq m\}$, and π^m is reflection in the plane $\{x_1 = m\}$. So we define

$$N_c^{x,x'} = N_c = \min\{n \geq 1 : S_n^{x'} \in B_r(\pi^m(S_{n-1}^{x'}))\} \leq N_\ell^{x'} = \min\{n \geq 0 : S_n^{x'} \in H_\ell^m\}, \quad (8.24)$$

where the inequality is by Lemma 8.5, and

$$S_n^x = \begin{cases} \pi^m(S_n^{x'}) & \text{if } n < N_c \\ S_n^{x'} & \text{if } n \geq N_c. \end{cases} \quad (8.25)$$

The above results imply that both S^x and $S^{x'}$ are $(\mathcal{F}_n^{S^{x'}})$ -random walks with step distribution U_1 ,

$$N_c^{x,x'} = \min\{n \geq 0 : S_n^x = S_n^{x'}\} \quad (8.26)$$

is their coupling time, and

$$P(N_c^{x,x'} \geq n) \leq \frac{C_{8.7}}{\sqrt{n}} \left(1 + \frac{|x' - x|}{2r}\right). \quad (8.27)$$

Next define coupled copies of the discrete time random walk with step distribution $U_1 + U_2$ by $\hat{Y}_n^x = S_{2n}^x$ and $\hat{Y}_n^{x'} = S_{2n}^{x'}$, and also set $\hat{\mathcal{F}}_n^{x'} = \mathcal{F}_{2n}^{S^{x'}}$. We will write $\hat{\mathcal{F}}_n$ for $\hat{\mathcal{F}}_n^{x'}$ if there is no ambiguity. Then it follows from Lemma 8.6 that both \hat{Y}_n^x and $\hat{Y}_n^{x'}$ are $(\hat{\mathcal{F}}_n)$ -random walks with step distribution $U_1 + U_2$, that is, they are $(\hat{\mathcal{F}}_n)$ -adapted and

$$P(\hat{Y}_{n+1}^x \in A | \hat{\mathcal{F}}_n)(\omega) = P(\hat{Y}_n^x(\omega) + U_1 + U_2 \in A) \text{ a.s.}, \quad (8.28)$$

and similarly for $\hat{Y}^{x'}$. It follows easily from (8.26) that

$$\hat{N}_c^{x,x'} := \min\{n \geq 0 : \hat{Y}_n^x = \hat{Y}_n^{x'}\} = \left\lceil \frac{N_c^{x,x'}}{2} \right\rceil. \quad (8.29)$$

Next use (8.25) and the fact that $2n \geq N_c^{x,x'}$ iff $n \geq \lceil \frac{N_c^{x,x'}}{2} \rceil$ to conclude that

$$\hat{Y}_n^x = \begin{cases} \pi^m(\hat{Y}_n^{x'}) & \text{if } n < \hat{N}_c^{x,x'} \\ \hat{Y}_n^{x'} & \text{if } n \geq \hat{N}_c^{x,x'}. \end{cases} \tag{8.30}$$

Letting $\hat{Y}_n^{x',(1)}$ be the first coordinate of $\hat{Y}_n^{x'}$, define

$$\hat{N}^{x'}(m) = \min\{n \geq 0 : \hat{Y}_n^{x',(1)} \leq m\}.$$

Lemma 8.8. *With x, x', m as above, $\hat{N}_c^{x,x'} \leq \hat{N}^{x'}(m)$ a.s.*

Proof. It follows from Lemma 8.5 that for all n , $S_{2n}^x \neq S_{2n}^{x'}$ implies that $S_{2n}^{(1),x'} > m$. This shows that $n < \hat{N}_c^{x,x'}$ implies $n < \hat{N}^{x'}(m)$ which clearly gives the required result. \square

Lemma 8.9. *With x, x', m as above, and for all $n \in \mathbb{N}$,*

$$P(\hat{N}_c^{x,x'} \geq n) \leq C_{8.7} n^{-1/2} \left(1 + \frac{|x' - x|}{2r}\right).$$

Proof. By (8.29)

$$P(\hat{N}_c^{x,x'} \geq n) = P\left(\lceil \frac{N_c^{x,x'}}{2} \rceil \geq n\right) \leq P(N_c^{x,x'} \geq 2n - 1).$$

The result follows from (8.27). \square

We move now to the continuous time random walks. Let $N(t)$ be an independent Poisson process with rate $\lambda = 2\rho|B_r|$ and jump time sequence $(s_n)_{n \in \mathbb{Z}_+}$, i.e., $s_n = \inf\{t \geq 0 : N_t = n\}$. For $K > 0$ put $x' = (K + 2r, 0)$, and let $x \in [K, K + 2r) \times \{0\}$. Define coupled continuous time rate λ random walks with step distribution $U_1 + U_2$, starting at x' and x , respectively, by

$$Y_t^{x'} = \hat{Y}_{N_t}^{x'} = \hat{Y}_n^{x'} \text{ if } s_n \leq t < s_{n+1},$$

and

$$Y_t^x = \hat{Y}_{N_t}^x = \hat{Y}_n^x \text{ if } s_n \leq t < s_{n+1}.$$

The coupling time of these random walks is

$$\begin{aligned} S_c^{x,x'} &:= \inf\{t \geq 0 : Y_t^x = Y_t^{x'}\} = \inf\{t \geq 0 : \hat{Y}_{N_t}^x = \hat{Y}_{N_t}^{x'}\} \\ &= \inf\{t \geq 0 : N_t = \hat{N}_c^{x,x'}\} = s_{\hat{N}_c^{x,x'}}. \end{aligned} \tag{8.31}$$

Note that $t < S_c^{x,x'} = s_{\hat{N}_c^{x,x'}}$ iff $N_t < \hat{N}_c^{x,x'}$, and so by setting $n = N_t$ in (8.30), we have

$$Y_t^x = \begin{cases} \pi^m(Y_t^{x'}) & \text{if } t < S_c^{x,x'} \\ Y_t^{x'} & \text{if } t \geq S_c^{x,x'}. \end{cases} \tag{8.32}$$

Let \mathcal{F}_t be the right-continuous filtration generated by $(Y^x, Y^{x'}, N)$, and let Y_t (respectively \hat{Y}_n) denote a generic rate λ continuous time (respectively, discrete time) random walk with step distribution $U_1 + U_2$, starting at 0 under P_0 .

Lemma 8.10. *(a) Both Y^x and $Y^{x'}$ are rate λ continuous time (\mathcal{F}_t) -random walks (and (\mathcal{F}_t) -strong Markov processes) with jump distribution $U_1 + U_2$. That is for $y = x$ or x' , $t > 0$, and any a.s. finite (\mathcal{F}_t) -stopping time S ,*

$$P(Y_{S+t}^y \in A | \mathcal{F}_S)(\omega) = P_0(Y_S^y(\omega) + Y_t \in A) \text{ a.s. for any Borel } A \subset \mathbb{R}^2. \tag{8.33}$$

(b) $S_c^{x,x'}$ is an (\mathcal{F}_t) -stopping time.

Proof. (b) is obvious from the definition of $S_c^{x,x'}$.

(a) This is an easy and standard consequence of (8.28). □

For $y = x$ or x' and $2r \leq \delta < A$ we let

$$t_\delta^y = \inf\{t \geq 0 : |Y_s^y| \leq \delta\}, \quad T_A^y = \inf\{t \geq 0 : |Y_t^y| \geq A\},$$

and also set

$$t_\delta^{x,x'} = t_\delta^x \wedge t_\delta^{x'}, \quad T_A^{x,x'} = T_A^x \wedge T_A^{x'}.$$

We define $\hat{t}_\delta^y, \hat{T}_A^y, \hat{t}_\delta^{x,x'}$, and $\hat{T}_A^{x,x'}$ in a similar way, using the discrete time random walks $\hat{Y}^x, \hat{Y}^{x'}$, for example,

$$\hat{t}_\delta^y = \min\{n \geq 0 : |\hat{Y}_n^y| \leq \delta\}.$$

Lemma 8.11. *Let $K > 3r$ and $x' = (K + 2r, 0)$.*

(a) *For all $x \in [K, K + 2r) \times \{0\}$, $t_{3r}^{x'} \geq S_c^{x,x'} \vee t_{3r}^x$.*

(b) *For all $x \in [K, K + 2r) \times \{0\}$, if $t_{3r}^x < S_c^{x,x'}$, then $|Y_{t_{3r}^{x'}}^{x'}| \leq 2K + 10r$.*

(c) *For $\varepsilon \in (0, 1)$ and $\delta \geq 3r$ there is a constant $C_{8.34} = C_{8.34}(\delta, \varepsilon) > 0$ such that for all $K > \delta \vee 1$,*

$$\inf_{x \in [K, K+2r) \times \{0\}} P(S_c^{x,x'} < t_\delta^{x,x'} \wedge T_{2K}^{x,x'}) \geq 1 - \frac{C_{8.34}}{K^{1-\varepsilon}}. \quad (8.34)$$

Proof. (a) First consider the discrete time walks. For $x \in [K, K + 2r) \times \{0\}$, we have $m = \frac{|x+x'|}{2} > K > 3r$. This shows that $B_{3r} \subset \{x_1 < m\}$ and so $\hat{Y}^{x'}$ must first enter $\{x_1 \leq m\}$ before it can enter B_{3r} . That is, $\hat{t}_{3r}^{x'} \geq \hat{N}^{x'}(m)$. Therefore by Lemma 8.8, $\hat{N}_c^{x,x'} \leq \hat{t}_{3r}^{x'}$ and hence

$$S_c^{x,x'} = s_{\hat{N}_c^{x,x'}} \leq s_{\hat{t}_{3r}^{x'}} = t_{3r}^{x'}.$$

Since the random walks must couple before $Y^{x'}$ can enter B_{3r} , we also have $t_{3r}^x \leq t_{3r}^{x'}$ and (a) follows.

(b) Let x and m be as in (a). If $t_{3r}^x < S_c^{x,x'}$, then by the coupling definition (8.32),

$$Y_{t_{3r}^{x'}}^{x'} = \pi^m(Y_{t_{3r}^x}^x). \quad (8.35)$$

For any $a = (a_1, a_2) \in B_{3r}$, $|\pi^m(a)| = |(2m - a_1, a_2)| \leq 2m + 3r + 3r \leq 2K + 10r$, so by (8.35),

$$|Y_{t_{3r}^{x'}}^{x'}| = |\pi^m(Y_{t_{3r}^x}^x)| \leq 2K + 10r.$$

This proves (b).

(c) Let $\hat{Y}^{(1)}$ be the first coordinate of \hat{Y} , and let x, x', m be as above, with $K > \delta \geq 3r$, so that $\delta < |x| \leq |x'| < 2K$. Let $n = \lceil K^{2-2\varepsilon} \rceil$. Then, using Lemma 8.9 for the second inequality and symmetry for the second to last inequality, we have

$$\begin{aligned} & P(\hat{N}_c^{x,x'} < \hat{t}_\delta^{x,x'} \wedge \hat{T}_{2K}^{x,x'}) \\ & \geq P(\hat{N}_c^{x,x'} \leq n) - P(\hat{t}_\delta^{x'} \wedge \hat{T}_{2K}^{x'} \leq n) - P(\hat{t}_\delta^x \wedge \hat{T}_{2K}^x \leq n) \\ & \geq 1 - \frac{C_{8.7}}{\sqrt{n}} \left(1 + \frac{|x' - x|}{2r}\right) - P_0(\max_{k \leq n} |\hat{Y}_k| \geq (K - 2r) \wedge (K + 2r - \delta)) \\ & \quad - P_0(\max_{k \leq n} |\hat{Y}_k| \geq (K - 2r) \wedge (K - \delta)) \\ & \geq 1 - \frac{2C_{8.7}}{\sqrt{n}} - 2P_0(\max_{k \leq n} |\hat{Y}_k| \geq K - \delta) \\ & \geq 1 - \frac{C}{K^{1-\varepsilon}} - 4P_0\left(\max_{k \leq n} |\hat{Y}_k^{(1)}| \geq \frac{K - \delta}{\sqrt{2}}\right) \\ & \geq 1 - \frac{C}{K^{1-\varepsilon}} - 4P_0\left(\max_{k \leq n} |\hat{Y}_k^{(1)}| \geq K/2\right), \end{aligned} \quad (8.36)$$

provided K is larger than some $K_0(\delta) > 0$. We recall Theorem 21.1 in [4], which in the present context implies

$$E_0 \left[\max_{k \leq n} |\hat{Y}_k^{(1)}|^p \right] \leq c_p [(nE[|\bar{U}^{(1)}|^2])^{p/2} + (2r)^p] \leq C_p n^{p/2} \tag{8.37}$$

for a constant $C_p > 0$. By Markov's inequality,

$$P_0 \left(\max_{k \leq n} |\hat{Y}_k^{(1)}| \geq K/2 \right) \leq \frac{C_p n^{p/2}}{(K/2)^p} \leq \frac{C}{K^{p\varepsilon}}.$$

If we take $p = p_0(\varepsilon)$ large enough so that $K^{p\varepsilon} > K^{1-\varepsilon}$, substituting into (8.36) we obtain for a constant $C > 0$ depending on ε ,

$$P(\hat{N}_c^{x,x'} < \hat{t}_\delta^{x,x'} \wedge \hat{T}_{2K}^{x,x'}) \geq 1 - \frac{C}{K^{1-\varepsilon}},$$

provided $K \geq K_0(\delta)$. Multiplication of C by a large enough constant depending on δ allows us to remove the restriction $K > K_0(\delta)$. That is, for some $C_{8.34}(\delta, \varepsilon) > 0$,

$$\inf_{x \in [K, K+2r] \times \{0\}} P(\hat{N}_c^{x,x'} < \hat{t}_\delta^{x,x'} \wedge \hat{T}_{2K}^{x,x'}) \geq 1 - \frac{C_{8.34}(\delta, \varepsilon)}{K^{1-\varepsilon}} \tag{8.38}$$

for all $K > \delta \vee 1$.

The corresponding inequality for the continuous time walks follows at once. First, note that

$$t_\delta^x = \inf\{t : |\hat{Y}_{N_t}^x| \leq \delta\} = \inf\{t : N_t = \hat{t}_\delta^x\} = s_{\hat{t}_\delta^x},$$

and similarly for the other hitting times. So in view of (8.31), for every $x \in [K, K + 2r] \times \{0\}$,

$$\begin{aligned} P(T_c^{x,x'} < t_\delta^{x,x'} \wedge T_{2K}^{x,x'}) &= P(s_{\hat{N}_c^{x,x'}} < s_{\hat{t}_\delta^{x,x'}} \wedge s_{\hat{T}_{2K}^{x,x'}}) \\ &= P(\hat{N}_c^{x,x'} < \hat{t}_\delta^{x,x'} \wedge \hat{T}_{2K}^{x,x'}). \end{aligned}$$

So (c) is now immediate from (8.38). □

Lemma 8.12. *There is a constant $C_{8.39} = C_{8.39}(r, \phi) > 0$ such that for $K > 5r \vee 2$, $A > 2K + 2r$, $x, x' \in [K, K + 2r] \times \{0\}$ with $|x| \leq |x'|$, and $\varepsilon_K = (\log K)/\sqrt{K}$,*

$$0 \leq \Phi(x', A) - \Phi(x, A) \leq C_{8.39} \frac{\varepsilon_K}{\log(A^2)}. \tag{8.39}$$

Proof. The first inequality in (8.39) follows from monotonicity (Lemma 8.3). For the second, it suffices to take $x' = (K + 2r, 0)$ and $x \in [K, K + 2r] \times \{0\}$. Recall the times $t_\delta^{x,x'} = t_\delta^x \wedge t_\delta^{x'}$, $T_A^{x,x'} = T_A^x \wedge T_A^{x'}$, etc. for the coupled walks $(Y^x, Y^{x'})$, and write S_c for the coupling time $S_c^{x,x'}$. By Lemma 8.11(a),

$$t_{3r}^{x,x'} = t_{3r}^x. \tag{8.40}$$

If $S_c \leq t_{3r}^x \wedge T_A^{x,x'}$, then

$$\int_0^{T_A^{x,x'}} \phi(Y_s^{x'}) ds = \int_0^{T_A^x} \phi(Y_s^x) ds.$$

As a consequence,

$$\begin{aligned} \Delta(x, x', A) &:= \Phi(x', A) - \Phi(x, A) \\ &= E \left[\exp \left(- \int_0^{T_A^{x'}} \phi(Y_s^{x'}) ds \right) - \exp \left(- \int_0^{T_A^x} \phi(Y_s^x) ds \right) \right] \\ &\leq E \left[1_{\{S_c > t_{3r}^x \wedge T_A^{x,x'}\}} \exp \left(- \int_0^{T_A^{x'}} \phi(Y_s^{x'}) ds \right) \right]. \end{aligned}$$

We now introduce

$$\begin{aligned} \Delta_1(x, x', A) &= E \left[1_{\{T_A^{x,x'} < S_c, T_A^{x,x'} \leq t_{3r}^x\}} \exp \left(- \int_0^{T_A^{x'}} \phi(|Y_s^{x'}|) ds \right) \right], \\ \Delta_2(x, x', A) &= E \left[1_{\{t_{3r}^x < S_c, t_{3r}^x < T_A^{x,x'}\}} \exp \left(- \int_0^{T_A^{x'}} \phi(|Y_s^{x'}|) ds \right) \right], \end{aligned}$$

so that $\Delta = \Delta_1 + \Delta_2$, and bound Δ_1, Δ_2 separately. For Δ_1 , using (8.40) and $T_A^{x'} \neq t_{3r}^{x'}$,

$$\Delta_1(x, x', A) \leq P(T_A^x < S_c \wedge t_{3r}^x) + P(T_A^{x'} < S_c \wedge t_{3r}^{x'}). \tag{8.41}$$

It suffices to consider the first term, as the second follows in the same way. By the strong Markov property

$$P(T_A^x < S_c \wedge t_{3r}^x) \leq E[1_{\{T_{2K}^x < S_c\}} P_{Y_{T_{2K}^x}^x}(T_A < t_{3r})]. \tag{8.42}$$

Now taking $a = 3r$ in Lemma 5.3, and noting that $3r < 2K \leq |Y_{T_{2K}^x}^x| \leq 2K + 2r < A$, we have

$$P_{Y_{2K}^x}(T_A < t_{3r}) \leq \frac{\log(|Y_{T_{2K}^x}^x|) - \log r}{\log A - \log r} \leq \frac{\log\left(\frac{2K}{r} + 2\right)}{\log(A/r)} \text{ a.s.}$$

Choose K_0 large enough so that $K > K_0$ implies $K^2 > \frac{2K}{r} + 2$. If, in addition we have $K > K_0$ and $A > r^2$, then

$$P_{Y_{2K}^x}(T_A < t_{3r}) \leq \frac{\log(K^2)}{\log \sqrt{A}} = \frac{8 \log K}{\log(A^2)}.$$

By replacing 8 with a sufficiently large constant C we may drop the additional conditions $K > K_0$ and $A > r^2$, and so obtain for all $K > 5r \wedge 2$ and $A > 2K + 2r$,

$$P_{Y_{2K}^x}(T_A < T_{3r}) \leq C \frac{\log K}{\log(A^2)}.$$

Plug this bound into (8.42) and use the coupling bound Lemma 8.11(c) with $\delta = 3r, \varepsilon = 1/2$ to obtain

$$P(T_A^x < S_c \wedge t_{3r}^x) \leq C \frac{\log K}{\log(A^2)} P(T_{2K}^x < S_c) \leq C_{8.34}(3r, \frac{1}{2}) C \frac{\log K}{\sqrt{K} \log(A^2)}.$$

The above and (8.41) imply

$$\Delta_1(x, x', A) \leq C \frac{\log K}{\sqrt{K} \log(A^2)}. \tag{8.43}$$

Now consider Δ_2 . Recalling from (8.40) that $t_{3r}^x \leq t_{3r}^{x'}$, Δ_2 is bounded by the sum of

$$\begin{aligned} \Delta_{2a}(x, x', A) &= E \left[1\{t_{3r}^x < S_c, t_{3r}^x \leq t_{3r}^{x'} < T_A^{x'}\} \exp \left(- \int_0^{T_A^{x'}} \phi(|Y_s^{x'}|) ds \right) \right], \\ \Delta_{2b}(x, x', A) &= E \left[1\{t_{3r}^x < S_c, t_{3r}^x < T_A^{x'} < t_{3r}^{x'}\} \exp \left(- \int_0^{T_A^{x'}} \phi(|Y_s^{x'}|) ds \right) \right]. \end{aligned}$$

In Δ_{2a} , the event in the indicator function belongs to $\mathcal{F}_{t_{3r}^{x'}}$, and so by the strong Markov property, $|Y_{t_{3r}^{x'}}^{x'}| \leq 5r \leq K$, monotonicity from Lemma 8.3, the coupling bound (8.34) with $\varepsilon = 1/2$ and $\delta = 3r$, and Lemma 8.4,

$$\begin{aligned} \Delta_{2a}(x, x', A) &= E \left[1\{t_{3r}^x < S_c, t_{3r}^x \leq t_{3r}^{x'} < T_A^{x'}\} E_{Y_{t_{3r}^{x'}}^{x'}} \left[\exp \left(- \int_0^{T_A} \phi(|Y_s|) ds \right) \right] \right] \\ &\leq P(t_{3r}^x < S_c) \Phi((K, 0), A) \\ &\leq C \frac{\log K}{\sqrt{K} \log(A^2)}. \end{aligned} \tag{8.44}$$

Finally, consider Δ_{2b} . Dropping the exponential and applying the strong Markov property to $Y^{x'}$ at time $t_{3r}^{x'}$, we have

$$\begin{aligned} \Delta_{2b}(x, x', A) &\leq P(t_{3r}^x < S_c, t_{3r}^x \leq T_A^{x'} < t_{3r}^{x'}) \\ &= E \left[1\{t_{3r}^x < S_c \wedge T_A^{x'}\} P_{Y_{t_{3r}^{x'}}^{x'}}(T_A < t_{3r}) \right]. \end{aligned} \tag{8.45}$$

By Lemma 8.11(b), on the event $\{t_{3r}^x < S_c\}$, $|Y_{t_{3r}^{x'}}^{x'}| \leq 2K + 10r$. Let K_0 be large enough so that $K > K_0$ implies $K^2 > (2K/r) + 10$, and assume additionally that $K > K_0$ and $A > (2K + 10r) \vee r^2$. By the hitting probability bound (5.11), with $a = 3r$ we see that if $|Y_{t_{3r}^{x'}}^{x'}| > 3r$, then

$$P_{Y_{t_{3r}^{x'}}^{x'}}(T_A < t_{3r}) \leq \frac{\log(\frac{2K}{r} + 10)}{\log \frac{A}{r}} \leq \frac{\log(K^2)}{\log(\sqrt{A})} = \frac{8 \log K}{\log(A^2)}.$$

The same bound holds if $|Y_{t_{3r}^{x'}}^{x'}| \leq 3r$ because then the left-hand side is zero. Now the additional restrictions on K, A can be dropped by replacing 8 with a larger constant C , so we may conclude that for A, K as in the Lemma and on $\{t_{3r}^x < S_c\}$,

$$P_{Y_{t_{3r}^{x'}}^{x'}}(T_A < t_{3r}) \leq C \frac{\log K}{\log(A^2)}.$$

Insert this into (8.45), apply Lemma 8.11(c) as before, to obtain

$$\Delta_{2b}(x, x', A) \leq P(t_{3r}^x < S_c) \frac{C \log K}{\log(A^2)} \leq C \frac{\log K}{\sqrt{K} \log(A^2)}. \tag{8.46}$$

Combine (8.43), (8.44) and (8.46) to complete the proof with ε_K as claimed. □

Theorem 8.13. *For all $x \neq 0$ there exists $c_\phi(x) > 0$ such that*

$$\lim_{A \rightarrow \infty} \log(A^2) \Phi(x, A) = c_\phi(x) \tag{8.47}$$

Proof. We may assume x is on the positive x_1 -axis, and for now that $|x| > 3r$. Assume $K > (|x| + 2r) \vee 2$ and $A > 2K + 2r$, put $x_K = (K, 0)$ and $\varepsilon_K = (\log K)/\sqrt{K}$ as in Lemma 8.12. By the strong Markov property,

$$\Phi(x, A) = E_x \left[\exp \left(- \int_0^{T_K} \phi(Y_s) ds \right) \Phi(Y_{T_K}, A) \right]. \tag{8.48}$$

By Lemma 8.12, noting that $|Y_{T_K}| \in [K, K + 2r]$,

$$|\Phi(Y_{T_k}, A) - \Phi(x_K, A)| \leq C_{8.39} \frac{\varepsilon_K}{\log(A^2)}. \tag{8.49}$$

Using this bound in (8.48) we obtain

$$\Phi(x, A) = \Phi(x, K)\Phi(x_K, A) + \mathcal{E}_1 \tag{8.50}$$

where $|\mathcal{E}_1| \leq C\varepsilon_K/\log(A^2)$. Now consider $\Phi(x_K, A)$. On account of $\phi(Y_s^{x_K}) = 0$ for $s > t_{3r}$, (8.14) and the strong Markov property, we see that

$$\begin{aligned} \Phi(x_K, A) &= P_{x_K}(T_A < t_{3r}) + E_{x_K} \left[1\{t_{3r} < T_A\} E_{Y_{t_{3r}}} \left[\exp \left(- \int_0^{T_K} \phi(Y_s) ds \right) \Phi(Y_{T_K}, A) \right] \right] \\ &= P_{x_K}(T_A < t_{3r}) + \Phi(x_K, A) E_{x_K} \left[1\{t_{3r} < T_A\} \Phi(Y_{t_{3r}}, K) \right] + \mathcal{E}_2 \end{aligned} \tag{8.51}$$

where $|\mathcal{E}_2| \leq C \frac{\varepsilon_K}{\log(A^2)}$ by (8.49). If we set

$$\alpha(x_K, A) = E_{x_K} \left[1\{t_{3r} < T_A\} \Phi(Y_{t_{3r}}, K) \right] < 1$$

then (8.51) can be rearranged to obtain

$$\Phi(x_K, A) = \frac{P_{x_K}(T_A < t_{3r}) + \mathcal{E}_2}{1 - \alpha(x_K, A)}. \tag{8.52}$$

Plugging this into (8.50) gives

$$\Phi(x, A) = \Phi(x, K) \frac{P_{x_K}(T_A < t_{3r}) + \mathcal{E}_2}{1 - \alpha(x_K, A)} + \mathcal{E}_1,$$

and thus

$$\log(A^2)\Phi(x, A) = \frac{\Phi(x, K)}{1 - \alpha(x_K, A)} \log(A^2)P_{x_K}(T_A < t_{3r}) + \mathcal{E}_3 \tag{8.53}$$

where $|\mathcal{E}_3| \leq C\varepsilon_K/(1 - \alpha(x_K, A))$.

Now consider $\alpha(x_K, A)$. By recurrence, $1\{t_{3r} < T_A\} \rightarrow 1$ P_{x_K} -a.s. as $A \rightarrow \infty$, which implies the limit

$$\alpha(x_K, \infty) := \lim_{A \rightarrow \infty} \alpha(x_K, A) = E_{x_K} [\Phi(Y_{t_{3r}}, K)]$$

exists. By monotonicity (Lemma 8.3), $|Y_{t_{3r}}| \leq 3r$, and recurrence, we have $\alpha(x_K, \infty) \leq \Phi((3r, 0), K) \rightarrow 0$ as $K \rightarrow \infty$. If we let K_0 satisfy $\alpha(x_K, \infty) \leq 1/2$ for all $K \geq K_0$, then $|\mathcal{E}_3| \leq 2C\varepsilon_K$. Thus, assuming in addition that $K > K_0$, we have from (8.53) and the above,

$$\limsup_{A \rightarrow \infty} \left| \log(A^2)\Phi(x, A) - \Phi(x, K) \frac{\log(A^2)P_{x_K}(T_A < t_{3r})}{1 - \alpha(x_K, A)} \right| \leq C\varepsilon_K. \tag{8.54}$$

By (5.12), $(\log A^2)P_{x_K}(T_A < t_{3r})$ is bounded above for large A and approaches $p(3r, x_K) := 2(\log|x_K| - E_{x_K}[\log|Y_{t_{3r}}|]) > 0$ as $A \rightarrow \infty$. It follows from (8.54) that

$$\limsup_{A \rightarrow \infty} \left| \log(A^2)\Phi(x, A) - \Phi(x, K) \frac{p(3r, x_K)}{1 - \alpha(x_K, \infty)} \right| \leq C\varepsilon_K. \tag{8.55}$$

The fact that $\varepsilon_K \rightarrow 0$ as $K \rightarrow \infty$ now implies $A \mapsto \log(A^2)\Phi(x, A)$ as $A \rightarrow \infty$ is Cauchy, hence there exists $c_\phi(x) \in [0, \infty)$ such that $\lim_{A \rightarrow \infty} \log(A^2)\Phi(x, A) = c_\phi(x)$.

To check that $c_\phi(x) > 0$, note that (for K large as above) (8.55) implies

$$\liminf_{A \rightarrow \infty} \log(A^2)\Phi(x, A) \geq p(3r, x_K) \frac{\Phi(x, K)}{1 - \alpha(x_K, \infty)} - C\varepsilon_K \geq p(3r, x_K)\Phi(x, K) - C\varepsilon_K.$$

The right-hand side above is positive if, for large K , $\Phi(x, K)/\varepsilon_K \rightarrow \infty$ as $K \rightarrow \infty$ (note that $p(3r, x_K) \rightarrow \infty$ as $K \rightarrow \infty$). Using $\Phi(x, K) \geq P_x(T_K < t_{3r})$, we have

$$\frac{\Phi(x, K)}{\varepsilon_K} \geq \frac{\sqrt{K}}{\log K} P_x(T_K < t_{3r}) = \frac{\sqrt{K}}{(\log K)^2} \log(K) P_x(T_K < t_{3r}) \rightarrow \infty \text{ as } K \rightarrow \infty$$

by Lemma 5.3. Hence, $c_\phi(x) > 0$.

Finally, suppose $0 < |x| \leq 3r$. By the strong Markov property, for $A > 6r$,

$$\Phi(x, A) = E_x \left[\exp\left(-\int_0^{T_{4r}} \phi(Y_s) ds\right) \Phi(Y_{T_{4r}}, A) \right]$$

By monotonicity (Lemma 8.3) and Lemma 8.4, $\log(A^2)\Phi(Y_{T_{4r}}, A) \leq \log(A^2)\Phi((6r \vee 2, 0), A) \leq C_{8.15} \log(6r \vee 2)$, and also converges to $c_\phi(Y_{T_{4r}}) > 0$ a.s. as $A \rightarrow \infty$, by the above and $|Y_{T_{4r}}| > 3r$. Apply bounded convergence to obtain existence of the limit

$$\lim_{A \rightarrow \infty} \log(A^2)\Phi(x, A) = E_x \left[\exp\left(-\int_0^{T_{4r}} \phi(Y_s) ds\right) c_\phi(Y_{T_{4r}}) \right] > 0.$$

This completes the proof of (8.47). □

Proof of Proposition 1.1 ($d = 2$). Let $\phi(x) = k(x)$, so $\Phi(x, A) = E_x \left[\exp\left(-\int_0^{T_A} k(Y_s) ds\right) \right]$. By Theorem 8.13, for $x \neq 0$, the positive limit $c_k(x) = \lim_{A \rightarrow \infty} \log(A^2)\Phi(x, A)$ exists. By Lemma 8.4, there is a constant $C > 0$ such that for all large A , $(\log A^2)\Phi(x, A) \leq C$ for all $|x| \leq 2r$. Therefore, by bounded convergence, if $Y_0 = \bar{U}$, the limit

$$\lim_{A \rightarrow \infty} (\log(A^2))E_{Y_0} \left[\exp\left(-\int_0^{T_A} k(Y_s) ds\right) \right] = \lim_{A \rightarrow \infty} E(\log(A^2)\Phi(Y_0, A)) = E(c_k(Y_0)) > 0 \tag{8.56}$$

exists. That is, (8.3) holds for $Y_0 = \bar{U}$, and by Proposition 8.1, the limit

$$\lim_{t \rightarrow \infty} (\log t)P_{Y_0}(\kappa > t) = E(c_k(Y_0)) > 0$$

exists, which is exactly (1.16). □

Remark 8.14. Theorem 8.13 is proved for continuous time rate $2\rho|B_r|$ random walks Y_t with step distribution (1.3). The result is needed in our proof of Proposition 1.1 because the dual particle difference $\tilde{\xi}_t$ behaves like Y_t when $|\tilde{\xi}_t| > 2r$ in our fixed radius case of the SLFV. With a view to the variable radius case in (1.17), we remark that Theorem 8.13 holds for more general radially symmetric step distributions. Assume $r_{\max} > 0$ and suppose ν is a probability measure on $[0, r_{\max}]$ with a bounded density such that $\int_0^{r_{\max}} \rho^{-2}\nu(d\rho) < \infty$, and consider the step distribution on \mathbb{R}^2 given by

$$\int_0^{r_{\max}} \nu(d\rho)P(\rho U \in \cdot), \tag{8.57}$$

where U is uniform on $B_1(0)$. A bit of calculus shows that in the variable radius case (1.17), $\tilde{\xi}_t$ behaves like a fixed rate random walk with step distribution satisfying (8.57) for appropriate ν when $|\tilde{\xi}_t| > 2r_{\max}$. Theorem 8.13 holds when Y_t is a continuous time rate $\lambda > 0$ with this step distribution (8.57). The modifications needed to prove this are minor. Of course this will not immediately give Proposition 1.1 as we no longer have an identity like (8.1) which arose from our time-change representation for the difference of the dual components.

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