An application of the voter model–super-Brownian motion invariance principle

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Abstract. We show that the asymptotics for the hitting time of 0 of the voter model started from a single 1 can be obtained from the invariance principle for voter models and super-Brownian motion.

1. Introduction and Summary. The voter model (see Chapter IV of Liggett (1985)) is one of the simplest *interacting particle systems*. It has been studied extensively since the 1970's. An invariance principle has recently been established (see Cox, Durrett and Perkins (2000) and Bramson, Cox and LeGall (2001)) which shows that appropriately rescaled voter models converge weakly to *super-Brownian motion*. Our purpose here is to use this invariance principle to give a new proof of a fundamental result of Bramson and Griffeath (1980) on the asymptotic behavior of the voter model started from a single 1.

We begin by describing the voter model. Let ξ_t denote the rate-1 voter model on $\mathbf{Z}^d$ with voting kernel $p(x,y)$ satisfying

$$\begin{aligned} &p(x,y) = p(0,y-x) \text{ is irreducible and symmetric, with } p(0,0)=0, \\ &\text{and for some } 0<\sigma^2<\infty, \quad \sum_{x\in\mathbf{Z}^d} p(0,x)x^i x^j = \delta(i,j)\sigma^2 \end{aligned} \tag{1.1}$$

($\delta(i,j)=1$ for $i=j$, and $\delta(i,j)=0$ otherwise). We think of $\xi_t(x)$ as the opinion, either 0 or 1, of a voter at site x at time t, where the dynamics of ξ_t are given by: independently, at each site x,

$$\begin{aligned} 0 \to 1 \text{ at rate } &\sum_y p(x,y) 1_{\{\xi_t(y)=1\}}, \\ 1 \to 0 \text{ at rate } &\sum_y p(x,y) 1_{\{\xi_t(y)=0\}}. \end{aligned}$$

We identify ξ_t with the set $\{x : \xi_t(x)=1\}$, and let ξ_t^A denote the voter model starting from 1's exactly on A, $\xi_0^A = A$. We write ξ_t^x for $\xi_t^{\{x\}}$, and make use of the usual additive construction of the voter model (see Section III.6 of Liggett (1985)),

$$\xi_t^A = \bigcup_{x\in A} \xi_t^x.$$

It is easy to see that $|\xi_t^0| = \sum_x \xi_t^0(x)$ is a martingale, and that $|\xi_t^0|$ hits 0 eventually with probability one. Letting $p_t = P(|\xi_t^0|>0)$, it follows that $p_t \to 0$ as $t\to\infty$. Determination of the rate at which $p_t \to 0$ is not simple, since the rate at which $|\xi_t^0|$ changes depends on the *spatial configuration* of the set ξ_t^0. In the one-dimensional nearest neighbor case, ξ_t^0 is always an interval, and it is straightforward to determine the asymptotic behavior of p_t. In higher dimensions, even in the nearest neighbor case, the situation is is far more complicated. Nevertheless, Bramson and Griffeath (1980) were able to obtain precise asymptotics.

To state their results, define (for $t > 0$)

$$m_t = \begin{cases} t/\log t & \text{in } d = 2, \\ t & \text{in } d \geq 3, \end{cases} \tag{1.2}$$

let $\gamma_2 = 2\pi\sigma^2$, and for $d \geq 3$, let γ_d be the probability that a random walk with jump kernel $p(x, y)$ starting at the origin never returns to the origin. The notation $f(t) \sim g(t)$ as $t \to \infty$ means that $\lim_{t\to\infty} f(t)/g(t) = 1$. Here is the Bramson and Griffeath result.

Theorem 1. *Assume $d \geq 2$. As $t \to \infty$,*

$$p_t \sim 1/\gamma_d m_t \tag{1.3}$$

and

$$P(p_t|\xi_t^0| > u \,\big|\, |\xi_t^0| > 0) = e^{-u}, \qquad u > 0. \tag{1.4}$$

(Although the proof given in Bramson and Griffeath (1980) was for the nearest-neighbor case $p(0, x) = (1/2d)$ for $|x| = 1$, as noted in Lemma 2 of Bramson, Cox and Le Gall (2001), it is easily modified to cover kernels $p(x, y)$ satisfying (1.1).)

The asymptotics in Theorem 1 have proved to be important tools in the study of the voter model and its variants. There were two key ingredients in Bramson and Griffeath's proof. The first was their derivation of the upper bound

$$p_t = O(\frac{1}{m_t}) \qquad \text{as } t \to \infty. \tag{1.5}$$

The second was Theorem 1.1 of Sawyer (1979), which gave asymptotics for the "patch of the origin" for a general stepping stone model. The proof of Sawyer's remarkable theorem proceded via the method of moments, using intricate calculations of transforms of coalescing random walk probabilites. It gave little insight into the theorem's conclusions. By combining the upper bound (1.5) and Sawyer's theorem, Bramson and Griffeath obtained (1.3) and (1.4).

Our purpose here is to give a new proof of these asymptotics which we feel is more probabilistic in nature and gives greater insight into why they hold. We make use of the upper bound (1.5), but avoid the use of Sawyer's result. Instead, we show that these asymptotics follow from an invariance principle showing that rescaled voter models converge to super-Brownian motion.

We begin by defining rescaled voter models ξ_t^N, which are rate-N voter models on $\mathbf{S_N} = \mathbf{Z}^d/\sqrt{N}$ with voting kernels $p_N(x, y) = p(x\sqrt{N}, y\sqrt{N})$ for $x, y \in \mathbf{S_N}$. We assume throughout that $|\xi_0^N| < \infty$. Let X_t^N denote the associated measure-valued processes

$$X_t^N = \frac{1}{m_N} \sum_{x\in\xi_t^N} \delta_x,$$

where δ_x is the unit point mass as x.

Now let X_t denote super-Brownian motion with branching rate $\gamma = 2\gamma_d$ and diffusion coefficient σ^2, taking values in $\mathcal{M}_F(\mathbf{R}^d)$, the space of finite measures on $\mathbf{R}^d$. X_t is obtained as the limit of rescaled critical branching random walks or Brownian motions, and can be defined via the following martingale problem (see Perkins (2002)): For all $\phi \in C_0^\infty(\mathbf{R}^d)$,

$$M_t(\phi) = X_t(\phi) - X_0(\phi) - \int_0^t X_s\left(\frac{\sigma^2\Delta\phi}{2}\right) ds$$

is a continuous L^2 martingale with $M_0(\phi) = 0$ and square function

$$\langle M(\phi)\rangle_t = \int_0^t X_s(b\phi^2)\, ds.$$

(For a measure μ on $\mathbf{R}^d$, $\mu(\phi) = \int \phi(x)\mu(dx)$.)

We will make use of the explicit formulas

$$P(X_t(\mathbf{1}) > 0) = 1 - \exp(-2X_0(\mathbf{1})/\gamma t) \tag{1.6}$$

and

$$E\exp(-\theta X_t(\mathbf{1})) = \exp(-\frac{2\theta X_0(\mathbf{1})}{2+\theta\gamma t}), \tag{1.7}$$

where $\mathbf{1}$ is the function identically 1 on $\mathbf{R}^d$. These formulas are not difficult to derive, since the total mass process, $X_t(\mathbf{1})$, is a Feller diffusion (see (II.5.11) and (II.5.12) of Perkins (2002)).

Here is the invariance principle, Theorem 1.2 of Cox, Durrett and Perkins (2000). The symbol $\Rightarrow$ denotes weak convergence, and $D(\mathbf{R}_+, \mathcal{M}_F(\mathbf{R}^d))$ is the Skorohod space of cadlag $\mathcal{M}_F(\mathbf{R}^d)$-valued paths.

Theorem 2. *Assume $d \ge 2$, and $X_0^N \to X_0 \in \mathcal{M}_F(\mathbf{R}^d)$ as $N \to \infty$. Then $X_\bullet^N \Rightarrow X_\bullet$ in $D(\mathbf{R}_+, \mathcal{M}_F(\mathbf{R}^d))$.*

Let us consider the case $d \ge 3$ and see why Theorem 2 and the formulas (1.6) and (1.7) suggest that (1.3) and (1.4) should hold. Let $\mathcal{L}$ denote law, and let $\approx$ denote "approximate equality." Let $\xi_0^N = \{0\}$. Then $\mathcal{L}(|\xi_t^0|) = \mathcal{L}(|\xi_{t/N}^N|)$, and in view of Theorem 2, we expect that $\mathcal{L}(|\xi_t^N|) \approx \mathcal{L}(NX_t(\mathbf{1}))$ for large N, where $X_0(\mathbf{1}) = |\xi_0^N|/N = 1/N$. Setting $t = 1$ and using (1.6), it follows that

$$p_N = P(|\xi_N^0| > 0) \approx P(NX_1(\mathbf{1}) > 0) = 1 - \exp(-1/N\gamma_d) \sim 1/N\gamma_d$$

as $N \to \infty$. This is (1.3). Similarly, for $\theta > 0$, we have

$$\begin{aligned} E\left(1 - e^{-\theta p_N|\xi_N^0|}\,\middle|\, |\xi_N^0| > 0\right) &= p_N^{-1} E\left(1 - e^{-\theta p_N|\xi_N^0|}\right) \\ &\approx p_N^{-1} E\left(1 - e^{-\theta p_N N X_1(\mathbf{1})}\right) \\ &= p_N^{-1}(1 - \exp(-\frac{\theta p_N}{1+\theta N p_N \gamma_d}))\,, \end{aligned}$$

where we have used (1.7) and the fact that $X_0(\mathbf{1}) = 1/N$. It is easy to see, since $p_N \sim 1/N\gamma_d$ as $N \to \infty$, that the last expression converges to $\theta/(1+\theta)$, which implies (1.4).

In order to make these arguments rigorous, we make use of the upper bound (1.5) and ideas from Bramson, Cox and LeGall (2001). We also require a corollary to Theorem 2, which says that the hitting times of 0 for X_t^N converge weakly to the hitting time of 0 for X_t. With these ingredients, we give a proof of Theorem 1 which avoids the use of Sawyer's theorem.

We close the introduction by stating our hitting time result. For $a \geq 0$ let τ_a^N and τ_a be the hitting times

$$\tau_a^N = \inf\{t > 0 : X_t^N(\mathbf{1}) \leq a\} \quad \text{and} \quad \tau_a = \inf\{t > 0 : X_t(\mathbf{1}) \leq a\}.$$

Corollary 3. *Assume that $d \geq 2$, and $X_0^N \to X_0 \in \mathcal{M}_F(\mathbf{R}^d)$. Then*

$$\lim_{N\to\infty} P(\tau_0^N > t) = P(\tau_0 > t) = 1 - \exp(-\frac{X_0(\mathbf{1})}{t\gamma_d}), \qquad t > 0. \tag{1.8}$$

The reason that Corollary 3 does not follow immediately from Theorem 2 is that there is no "soft" way to ensure that once $X_t^N(\mathbf{1})$ reaches a level $a > 0$ very close to 0, it doesn't linger there rather than reaching 0 fairly quickly. We use (1.5) to take care of this problem.

2. Proofs. We first prove Corollary 3, then Theorem 1.

Proof of Corollary 3. The second equality is immediate from (1.6), so we only need prove the first equality. For $t > 0$ define

$$I_t^N = \inf\{X_s^N(\mathbf{1}) : 0 \leq s \leq t\}, \qquad I_t = \inf\{X_s(\mathbf{1}) : 0 \leq s \leq t\}.$$

It follows from Theorem 2 that $X_\bullet^N(\mathbf{1}) \Rightarrow X_\bullet(\mathbf{1})$ as $N \to \infty$, and since the infimum over a path is a continuous function on the space of continuous paths, we also have, for fixed $t > 0$, $I_t^N \Rightarrow I_t$. More specifically, as $X_\bullet(\mathbf{1})$ is continuous, this follows from Theorem 3.10.2 of Ethier and Kurtz (1986). For any $a \geq 0$, $\{I_t^N > a\} = \{\tau_a^N > t\}$ and $\{I_t > a\} = \{\tau_a > t\}$. Consequently,

$$\liminf_{N\to\infty} P(\tau_0^N > t) \geq P(\tau_0 > t). \tag{2.1}$$

By (1.5) there is a constant C such that $p_t \leq C/m_t$. Consequently, for any initial state ξ_0^N,

$$P(\xi_t^N > 0) \leq C|\xi_0^N|/m_{Nt}\,. \tag{2.2}$$

This is because, by additivity,

$$P(\xi_t^N > 0) = P(|\bigcup_{x\in\xi_0^N} \xi_t^{N,x}| > 0) \leq \sum_{x\in\xi_0^N} P(|\xi_t^{N,x}| > 0) \leq C|\xi_0^N|/m_{Nt}.$$

Now choose s, a such that $s < t$ and $0 < a < X_0(\mathbf{1})$. Then, making use of (2.2) and the Markov property, for N large enough so that $X_0^N(\mathbf{1}) > a$, we have

$$\begin{aligned} P(\tau_0^N > t) &\leq P(\tau_a^N > s) + P(\tau_a^N \leq s, \tau_0^N > t) \\ &\leq P(\tau_a^N > s) + \sup\{P(|\xi_{t-s}^N| > 0) \colon |\xi_0^N| \leq am_N\} \\ &\leq P(\tau_a^N > s) + Cam_N/m_{N(t-s)} \\ &= P(I_s^N > a) + Cam_N/m_{N(t-s)}. \end{aligned}$$

We now take a to be a continuity point for the distribution function of I_s, so that $P(I_s^N > a) \to P(I_s > a)$ as $N \to \infty$. Since $P(I_s > a) = P(\tau_a > s) \le P(\tau_0 > s)$, using the definition of m_t we therefore have

$$\limsup_{N\to\infty} P(\tau_0^N > t) \le P(\tau_0 > s) + Ca/(t-s).$$

We may now let $s \uparrow t$ and $a \downarrow 0$ such that $a/(t-s) \to 0$, to obtain (recall from (1.6) that τ_0 has a continuous distribution function)

$$\limsup_{N\to\infty} P(\tau_0^N > t) \le P(\tau_0 > t). \tag{2.3}$$

Together, (2.1) and (2.3) imply (1.8). □

Proof of Theorem 1. For $\varepsilon > 0$, let $B_{N,\varepsilon}$ be the box in $\mathbf{S_N}$ centered at the origin of side length $(\varepsilon m_N)^{1/d}/N^{1/2}$, so that $|B_{N,\varepsilon}| \sim \varepsilon m_N$ as $N \to \infty$. Let $\xi_t^{N,B_{N,\varepsilon}}$ denote the rate-N voter model with $\xi_0^{N,B_{N,\varepsilon}} = B_{N,\varepsilon}$, with corresponding measure-valued process $X_t^{N,\varepsilon}$, and let $\xi_t^{N,x}$ denote the process with initial state $\xi_0^{N,x} = \{x\}$. Let X_t^ε denote super-Brownian motion with $X_0 = \varepsilon\delta_0$, branching rate $\gamma = 2\gamma_d$, and diffusion coefficient σ^2. Since $X_0^N \to X_0$, by Corollary 3 it follows that

$$\lim_{N\to\infty} P(|\xi_t^{N,B_{N,\varepsilon}}| > 0) = P(X_t^\varepsilon > 0) = 1 - e^{-\varepsilon/\gamma_d t}. \tag{2.4}$$

Since

$$P(|\xi_t^{N,B_{N,\varepsilon}}| > 0) \le \sum_{x\in B_{N,\varepsilon}} P(|\xi_t^{N,x}| > 0) = |B_{N,\varepsilon}| P(|\xi_t^{N,0}| > 0),$$

it follows that

$$\liminf_{N\to\infty} m_N P(|\xi_t^{N,0}| > 0) \ge \frac{1 - e^{-\varepsilon/\gamma_d t}}{\varepsilon}.$$

Letting $\varepsilon \to 0$, we obtain

$$\liminf_{N\to\infty} m_N P(|\xi_t^{N,0}| > 0) \ge 1/\gamma_d t. \tag{2.5}$$

For a bound in the other direction, we appeal to additivity and inclusion-exclusion,

$$P(|\xi_t^{N,B_{N,\varepsilon}}| > 0) \ge \sum_{x\in B_{N,\varepsilon}} P(|\xi_t^{N,x}| > 0) - \sum_{\substack{x\ne y \\ x,y\in B_{N,\varepsilon}}} P(|\xi_t^{N,x}| > 0, |\xi_t^{N,y}| > 0)\,.$$

By a correlation inequality, Lemma 1 of Arratia (1981), for $x \ne y$,

$$P(|\xi_t^{N,x}| > 0, |\xi_t^{N,y}| > 0) \le P(|\xi_t^{N,x}| > 0) P(|\xi_t^{N,y}| > 0) = P(|\xi_t^{N,0}| > 0)^2. \tag{2.6}$$

It therefore follows that

$$P(|\xi_t^{N,B_{N,\varepsilon}}| > 0) \ge |B_{N,\varepsilon}| P(|\xi_t^{N,0}| > 0) - |B_{N,\varepsilon}|^2 P(|\xi_t^{N,0}| > 0)^2.$$

Rearranging this inequality and using the bound $p_t \le C/m_t$, we obtain

$$|B_{N,\varepsilon}| P(|\xi_t^{N,0}| > 0) \le P(|\xi_t^{N,B_{N,\varepsilon}}| > 0) + |B_{N,\varepsilon}|^2 \frac{C^2}{m_{tN}^2}.$$

Since $|B_{N,\varepsilon}| \sim \varepsilon m_N$ and $m_N/m_{Nt} \to 1/t$ as $N \to \infty$, (2.4) implies that

$$\limsup_{N\to\infty} m_N P(|\xi_t^{N,0}| > 0) \le \frac{1 - e^{-\varepsilon/\gamma_d t}}{\varepsilon} + C^2\varepsilon/t^2.$$

Letting $\varepsilon \to 0$ now gives

$$\limsup_{N\to\infty} m_N P(|\xi_t^{N,0}| > 0) \le 1/\gamma_d t. \tag{2.7}$$

Together, (2.5) and (2.7) imply $m_N P(|\xi_t^{N,0}| > 0) \to 1/\gamma_d t$ as $N \to \infty$. Setting $t = 1$ we obtain (1.3).

To prove (1.4), we fix $\theta > 0$ and set $\psi(u) = 1 - e^{-\theta u}$, $u \ge 0$. We will use several times without comment the simple fact that $\psi(0) = 0$. By Theorem 2,

$$\lim_{N\to\infty} E\psi(X_t^{N,\varepsilon}(\mathbf{1})) = E\psi(X_t^{\varepsilon}(\mathbf{1})).$$

In view of (1.7), this shows that

$$\lim_{N\to\infty} E\psi(\frac{|\xi_t^{N,B_{N,\varepsilon}}|}{m_N}) = 1 - \exp(-\frac{\theta\varepsilon}{1+\theta\gamma_d t}). \tag{2.8}$$

We will show that

$$|B_{N,\varepsilon}| E\psi(\frac{|\xi_t^{N,0}|}{m_N}) = E\psi(\frac{|\xi_t^{N,B_{N,\varepsilon}}|}{m_N}) + O(\varepsilon^2/t^2) \qquad \text{as } N \to \infty\,. \tag{2.9}$$

By (2.8) and the fact that $p_{tN}|B_{N,\varepsilon}| \to \varepsilon/\gamma_d t$ as $N \to \infty$, (2.9) implies

$$\limsup_{N\to\infty} \left| E(\psi(\frac{|\xi_t^{N,0}|)}{m_N}) \,\big|\, |\xi_t^{N,0}| > 0) - \gamma_d t \frac{1 - \exp(-\frac{\theta\varepsilon}{1+\theta\gamma_d t})}{\varepsilon} \right| = O(\varepsilon/t)\,.$$

Letting $\varepsilon \to 0$ gives

$$\lim_{N\to\infty} E(\psi(\frac{|\xi_t^{N,0}|}{m_N}) \,\big|\, |\xi_t^{N,0}| > 0) = \frac{\theta\gamma_d t}{1+\theta\gamma_d t}.$$

That is, conditional on $|\xi_t^{N,0}| > 0$, $|\xi_t^{N,0}|/m_N \Rightarrow \gamma_d t\mathcal{E}(1)$, where $\mathcal{E}(1)$ denotes an exponential random variable with mean 1. Since $p_{tN} m_N \to 1/\gamma_d t$, this implies that, conditional on $|\xi_t^{N,0}| > 0$, $p_{tN}|\xi_t^{N,0}| \Rightarrow \mathcal{E}(1)$ as $N \to \infty$. This proves (1.4).

To prove (2.9), we introduce the set $S_t^N = \{x \in B_{N,\varepsilon} : |\xi_t^{N,x}| > 0\}$ of surviving families in $\xi_t^{N,B_{N,\varepsilon}}$. Clearly,

$$E\psi(\frac{|\xi_t^{N,\varepsilon}|}{m_n}) = E(\psi(\frac{|\xi_t^{N,\varepsilon}|}{m_n}); |S_t^N| = 1) + E(\psi(\frac{|\xi_t^{N,\varepsilon}|}{m_n}); |S_t^N| \ge 2). \tag{2.10}$$

We use the correlation inequality (2.6) to handle the the second term on the right side of (2.10). By additivity and the fact that $\psi(0) = 0$,

$$\begin{aligned}
E(\psi(\frac{|\xi_t^{N,\varepsilon}|}{m_N}); |S_t^N| \ge 2) &\le P(\bigcup_{\substack{x\ne y\\ x,y\in B_{N,\varepsilon}}} \{|\xi_t^{N,x}| > 0, |\xi_t^{N,y}| > 0\}) \\
&\le \sum_{\substack{x\ne y\\ x,y\in B_{N,\varepsilon}}} P(|\xi_t^{N,x}| > 0, |\xi_t^{N,y}| > 0) \\
&\le |B_{N,\varepsilon}|^2 P(|\xi_t^{N,0}| > 0)^2.
\end{aligned}$$

Since the definition of $B_{N,\varepsilon}$ and (1.5) imply that

$$(2.11), \qquad |B_{N,\varepsilon}|p_{tN} = O(\varepsilon/t) \text{ as } N \to \infty$$

we have shown that

$$(2.12) \qquad E(\psi(\frac{|\xi_t^{N,\varepsilon}|}{m_N}); |S_t^N| \geq 2) = O(\varepsilon^2/t^2) \qquad \text{as } N \to \infty.$$

Consider now the first term on the right side of (2.10). The event $\{|\xi_t^{N,x}| > 0\}$ is the disjoint union $\{S_t^N = \{x\}\} \cup \{|\xi_t^{N,x}| > 0, |S_t^N| > 1\}$, and

$$\begin{aligned} P(|\xi_t^{N,x}| > 0, |S_t^N| > 1\}) &\leq \sum_{\substack{y \neq x \\ y \in B_{N,\varepsilon}}} P(|\xi_t^{N,x}| > 0, |\xi_t^{N,y}| > 0) \\ (2.13) \qquad &\leq |B_{N,\varepsilon}| P(|\xi_t^{N,0}| > 0)^2, \end{aligned}$$

where we have again used the inequality (2.6). Consequently,

$$\begin{aligned} &E(\psi(\frac{|\xi_t^{N,B_{N,\varepsilon}}|}{m_N}); |S_t^N| = 1) \\ &\quad = \sum_{x \in B_{N,\varepsilon}} E(\psi(\frac{|\xi_t^{N,x}|}{m_N}); S_t^N = \{x\}) \\ (2.14) \qquad &\quad = \sum_{x \in B_{N,\varepsilon}} [E(\psi(\frac{|\xi_t^{N,x}|}{m_N}); |\xi_t^{N,x}| > 0) - E(\psi(\frac{|\xi_t^{N,x}|}{m_N}); |\xi_t^{N,x}| > 0, |S_t^N| > 1)] \\ &\quad = |B_{N,\varepsilon}| E(\psi(\frac{|\xi_t^{N,0}|}{m_N})) + O((|B_{N,\varepsilon}|p_{Nt})^2) \\ &\quad = |B_{N,\varepsilon}| E(\psi(\frac{|\xi_t^{N,0}|}{m_N})) + O(\varepsilon^2/t^2) \end{aligned}$$

as $N \to \infty$, where (2.13) is used in the next to last equality and (2.11) is used in the last equality. Plugging (2.12) and (2.14) into (2.10) yields (2.9), and we are done. □

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