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SUPPLEMENT TO "ON THE TOPOLOGICAL BOUNDARY OF THE RANGE OF SUPER-BROWNIAN MOTION"

BY JIELIANG HONG[*,§] LEONID MYTNIK[†,¶] AND EDWIN PERKINS[‡,‖]

University of British Columbia[§] and The Technion[¶] and University of British Columbia[‖]

This is the supplementary material to the paper [13]. It contains the proof of Proposition 5.1 and gives more details of the proof of Lemma 7.3 from [13].

CONTENTS

S.1. Proof of Proposition 5.1 . For $|x_i| \geq \varepsilon_0, i=1,2,$ and $\varepsilon \in (0,\varepsilon_0)$, if $|x_1-x_2| \leq 5\varepsilon$, then use $xe^{-x} \leq e^{-1}, \forall x \geq 0$ to get

$$\mathbb{E}_{\delta_0}\Big(\prod_{i=1}^2 \lambda\frac{X_{G_\varepsilon^{x_i}}(1)}{\varepsilon^2}\exp\big(-\lambda\frac{X_{G_\varepsilon^{x_i}}(1)}{\varepsilon^2}\big)\Big) \leq e^{-1}\mathbb{E}_{\delta_0}\Big(\lambda\frac{X_{G_\varepsilon^{x_1}}(1)}{\varepsilon^2}\exp\big(-\lambda\frac{X_{G_\varepsilon^{x_1}}(1)}{\varepsilon^2}\big)\Big).$$

Recall the definition of $F=F_{\varepsilon,x_1}$ in (4.17). For all $\lambda>0$, an integration by parts gives

$$\begin{aligned}
&\mathbb{E}_{\delta_0}\Big(\lambda\frac{X_{G_\varepsilon^{x_1}}(1)}{\varepsilon^2}\exp\big(-\lambda\frac{X_{G_\varepsilon^{x_1}}(1)}{\varepsilon^2}\big)\Big) = \int_0^\infty \lambda x e^{-\lambda x}dF(x)\\
=&\int_0^\infty \lambda(\lambda x-1)e^{-\lambda x}F(x)dx = \int_0^\infty (y-1)e^{-y}F(\frac{y}{\lambda})dy \leq F(2)+\int_{2\lambda}^\infty ye^{-y}F(\frac{y}{\lambda})dy\\
\leq & c_{4.9}2^{p-2}\varepsilon^{p-2}+\int_{2\lambda}^\infty ye^{-y}c_{4.9}(\frac{y}{\lambda})^{p-2}\varepsilon^{p-2}dy = C(\varepsilon_0,\lambda)\varepsilon^{p-2},
\end{aligned}$$

the last line by Proposition 4.9. Therefore

$$\begin{aligned}
\mathbb{E}_{\delta_0}\Big(\prod_{i=1}^2 \lambda\frac{X_{G_\varepsilon^{x_i}}(1)}{\varepsilon^2}\exp\big(-\lambda\frac{X_{G_\varepsilon^{x_i}}(1)}{\varepsilon^2}\big)\Big) &\leq e^{-1}C(\varepsilon_0,\lambda)\varepsilon^{p-2}\\
&\leq e^{-1}5^{p-2}C(\varepsilon_0,\lambda)|x_1-x_2|^{2-p}\varepsilon^{2(p-2)},
\end{aligned}$$

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provided $|x_1 - x_2| \le 5\varepsilon$. As a result,

throughout the rest of this Section we may fix $\varepsilon_0 > 0$, $|x_i| \ge \varepsilon_0$ **and** $\varepsilon \in (0, \varepsilon_0)$ **with** $|x_1 - x_2| > 5\varepsilon$. In this case, we have $B(x_1, 2\varepsilon) \cap B(x_2, 2\varepsilon) = \emptyset$.

Let $\vec{x} = (x_1, x_2)$, $G = G_\varepsilon^{x_1} \cap G_\varepsilon^{x_2}$, and $\vec{\lambda} = (\lambda_1, \lambda_2) \in [0,\infty)^2 \backslash \{(0,0)\}$. For $X_0 \in M_F(\mathbb{R}^d)$ such that $d(\mathrm{Supp}(X_0), G^c) > 0$, the decomposition (2.4) with $G = G_\varepsilon^{x_i}$, $i = 1, 2$, gives

$$\text{(S.1)} \qquad \mathbb{E}_{X_0}\Big(\exp\big(-\sum_{i=1}^2 \lambda_i \frac{X_{G_\varepsilon^{x_i}}(1)}{\varepsilon^2}\big)\Big) = \exp\Big(-\int U^{\vec{\lambda},\vec{x},\varepsilon}(x) X_0(dx)\Big),$$

where $U^{\vec{\lambda},\vec{x},\varepsilon} \ge 0$ is defined as

$$\text{(S.2)} \qquad U^{\vec{\lambda},\vec{x},\varepsilon}(x) \equiv \mathbb{N}_x\Big(1 - \exp\big(-\sum_{i=1}^2 \lambda_i \frac{X_{G_\varepsilon^{x_i}}(1)}{\varepsilon^2}\big)\Big), \quad \forall x \in G.$$

We use results from Chapter V of [17] to get the following lemma.

Lemma S.1.1. *$U^{\vec{\lambda},\vec{x},\varepsilon}$ is a C^2 function on G and solves*

$$\text{(S.3)} \qquad \Delta U^{\vec{\lambda},\vec{x},\varepsilon} = (U^{\vec{\lambda},\vec{x},\varepsilon})^2 \text{ on } G.$$

Moreover,

$$U^{\vec{\lambda},\vec{x},\varepsilon}(x) \le (\lambda_1 + \lambda_2)\varepsilon^{-2}, \ \forall x \in G.$$

Proof. Let

$$u(x) \equiv U^{\vec{\lambda},\vec{x},\varepsilon}(x) = \mathbb{N}_x\Big(1 - \exp\big(-\sum_{i=1}^2 \lambda_i \frac{X_{G_\varepsilon^{x_i}}(1)}{\varepsilon^2}\big)\Big).$$

Then use $1 - e^{-x} \le x$ to get

$$\text{(S.4)} \quad u(x) \le \mathbb{N}_x\Big(\sum_{i=1}^2 \lambda_i \frac{X_{G_\varepsilon^{x_i}}(1)}{\varepsilon^2}\Big) = \sum_{i=1}^2 \lambda_i \varepsilon^{-2} P_x(\tau_i < \infty) \le (\lambda_1 + \lambda_2)\varepsilon^{-2},$$

the equality by Proposition V.3 of [17], where (B_t) is d-dimensional Brownian motion starting from x under P_x and $\tau_i = \inf\{t \ge 0 : B_t \notin G_\varepsilon^{x_i}\}$.

Next, for any $x' \in G$, let D be an open ball that contains x', whose closure is in G. Use (S.1) with $X_0 = \delta_x$ and then Proposition 2.3(b)(i) to see that for $x \in D$,

$$e^{-u(x)} = \mathbb{E}_{\delta_x}\Big(\exp\big(-\sum_{i=1}^2 \lambda_i \frac{X_{G_\varepsilon^{x_i}}(1)}{\varepsilon^2}\big)\Big) = \mathbb{E}_{\delta_x}\Big(\mathbb{E}_{X_D}\Big(\exp\big(-\sum_{i=1}^2 \lambda_i \frac{X_{G_\varepsilon^{x_i}}(1)}{\varepsilon^2}\big)\Big)\Big)$$

$$=\mathbb{E}_{\delta_x}\Big(\exp\big(-\int u(x)X_D(dx)\big)\Big) = \exp\Big(-\mathbb{N}_x\big(1-\exp\big(-\int u(y)X_D(dy)\big)\big)\Big),$$

the third equality by (S.1) with $X_0 = X_D$, and the last by the decomposition (2.4). Therefore

$$u(x) = \mathbb{N}_x\big(1-\exp\big(-\int u(y)X_D(dy)\big)\big) \quad \forall x \in D.$$

Note u is bounded in G by (S.4), and hence on ∂D. Use Theorem V.6 of [17] to conclude

$$\Delta u(x) = (u(x))^2,\ \forall x \in D,\ \text{and, in particular, for } x = x'.$$

Since x' is arbitrary, it holds for all $x \in G$. ∎

Let $X_0 = \delta_x$ in (S.1) for $x \in G$ to get

$$\text{(S.5)} \qquad \mathbb{E}_{\delta_x}\Big(\exp\big(-\sum_{i=1}^2 \lambda_i \frac{X_{G_\varepsilon^{x_i}}(1)}{\varepsilon^2}\big)\Big) = \exp(-U^{\vec{\lambda},\vec{x},\varepsilon}(x)).$$

Monotone convergence and the convexity of e^{-ax} for $a, x > 0$ allow us to differentiate the left-hand side of (S.5) with respect to $\lambda_i > 0$ through the expectation and so conclude that for $i = 1, 2$, $U_i^{\vec{\lambda},\vec{x},\varepsilon}(x) = \frac{\partial}{\partial \lambda_i} U^{\vec{\lambda},\vec{x},\varepsilon}(x)$ exists and

$$\mathbb{E}_{\delta_x}\Big(\frac{X_{G_\varepsilon^{x_i}}(1)}{\varepsilon^2}\exp\big(-\sum_{i=1}^2 \lambda_i \frac{X_{G_\varepsilon^{x_i}}(1)}{\varepsilon^2}\big)\Big) = e^{-U^{\vec{\lambda},\vec{x},\varepsilon}(x)} U_i^{\vec{\lambda},\vec{x},\varepsilon}(x) \text{ for } \lambda_i > 0, \lambda_{3-i} \geq 0.$$

Repeat the above to see that $U^{\vec{\lambda},\vec{x},\varepsilon}(x)$ is C^2 in $\lambda_1, \lambda_2 > 0$ and if $U_{1,2}^{\vec{\lambda},\vec{x},\varepsilon}(x) = \frac{\partial^2}{\partial \lambda_1 \partial \lambda_2} U^{\vec{\lambda},\vec{x},\varepsilon}(x)$, then

$$\text{(S.6)} \quad \mathbb{E}_{\delta_x}\Big(\frac{X_{G_\varepsilon^{x_1}}(1)}{\varepsilon^2}\frac{X_{G_\varepsilon^{x_2}}(1)}{\varepsilon^2}\exp\big(-\sum_{i=1}^2 \lambda_i \frac{X_{G_\varepsilon^{x_i}}(1)}{\varepsilon^2}\big)\Big)$$
$$= e^{-U^{\vec{\lambda},\vec{x},\varepsilon}(x)}\Big[U_1^{\vec{\lambda},\vec{x},\varepsilon}(x)U_2^{\vec{\lambda},\vec{x},\varepsilon}(x) - U_{1,2}^{\vec{\lambda},\vec{x},\varepsilon}(x)\Big],\ \text{for } \lambda_1, \lambda_2 > 0.$$

The next monotonicity result follows just as in the proof of Lemma 9.2 of [20].

LEMMA S.1.2.

(a) $U_i^{\vec{\lambda},\vec{x},\varepsilon}(x) > 0$ *is strictly decreasing in* $\vec{\lambda} \in \{(\lambda_1,\lambda_2) : \lambda_i > 0, \lambda_{3-i} \geq 0\}$, *for* $i = 1,2$.

(b) $-U_{1,2}^{\vec{\lambda},\vec{x},\varepsilon}(x) > 0$ *is strictly decreasing in* $\vec{\lambda} \in (0,\infty)^2$.

Note that

$$U^{\vec{\lambda},\vec{x},\varepsilon}(x) = U^{\lambda_i \varepsilon^{-2},\varepsilon}(x - x_i), \quad \text{for } \lambda_i > 0 \text{ and } \lambda_{3-i} = 0. \tag{S.7}$$

The above monotonicity results easily give the following, just as for Lemma 9.3 of [20].

LEMMA S.1.3. *(a) For all* $\lambda_i > 12$ *and* $\lambda_{3-i} \geq 0$,

$$\begin{aligned} U_i^{\vec{\lambda},\vec{x},\varepsilon}(x) &\leq \frac{2}{\lambda_i}(U^{\lambda_i\varepsilon^{-2},\varepsilon}(x_i - x) - U^{(\lambda_i/2)\varepsilon^{-2},\varepsilon}(x_i - x)) \\ &\leq \frac{2}{\lambda_i}\frac{2^p}{|x_i - x|^p} D^{\lambda_i/2}(2)\varepsilon^{p-2}, \ \forall |x_i - x| \geq 2\varepsilon. \end{aligned}$$

(b) For all $\lambda_1, \lambda_2 > 12$,

$$\begin{aligned} -U_{1,2}^{\vec{\lambda},\vec{x},\varepsilon}(x) &\leq \frac{4}{\lambda_1\lambda_2}\min_{i=1,2}(U^{\lambda_i\varepsilon^{-2},\varepsilon}(x_i - x) - U^{(\lambda_i/2)\varepsilon^{-2},\varepsilon}(x_i - x)) \\ &\leq \frac{4}{\lambda_1\lambda_2}2^p([D^{\lambda_1/2}(2)|x_1 - x|^{-p}] \wedge [D^{\lambda_2/2}(2)|x_2 - x|^{-p}])\varepsilon^{p-2}, \\ &\qquad \forall |x_i - x| \geq 2\varepsilon, \ i = 1,2. \end{aligned}$$

Let $r_\varepsilon = 2\varepsilon$ and assume $0 < r_\varepsilon < \min\{|x_i - x| : i = 1,2\}$. Set $T^i_{r_\varepsilon} = \inf\{t \geq 0 : |B_t - x_i| \leq r_\varepsilon\}$ and $T_{r_\varepsilon} = T^1_{r_\varepsilon} \wedge T^2_{r_\varepsilon}$, and let $(\mathcal{F}_t)$ denote the right-continuous filtration generated by the Brownian motion B, which starts at x under P_x.

LEMMA S.1.4. *Let* $\lambda_1, \lambda_2 > 12$.

(a) $U_1^{\vec{\lambda},\vec{x},\varepsilon}(B(t\wedge T_{r_\varepsilon})) - \int_0^{t\wedge T_{r_\varepsilon}} U^{\vec{\lambda},\vec{x},\varepsilon}(B(s))U_1^{\vec{\lambda},\vec{x},\varepsilon}(B(s))ds$ *is an* $(\mathcal{F}_t)$*-martingale.*

(b) For any $t > 0$,

$$U_1^{\vec{\lambda},\vec{x},\varepsilon}(x) = E_x\Big(U_1^{\vec{\lambda},\vec{x},\varepsilon}(B(t \wedge T_{r_\varepsilon})\exp\big(-\int_0^{t\wedge T_{r_\varepsilon}} U^{\vec{\lambda},\vec{x},\varepsilon}(B(s))ds\big)\Big).$$

This result follows from Lemmas S.1.1, S.1.3 and Itô's Lemma, exactly as for Lemma 9.4 in [20], and so the proof is omitted.

LEMMA S.1.5. *For all $\lambda_1, \lambda_2 > 12$,*

$$
\begin{aligned}
-U_{1,2}^{\vec{\lambda},\vec{x},\varepsilon}(x) =& E_x\Big(\int_0^{T_{r_\varepsilon}} \prod_{i=1}^{2} U_i^{\vec{\lambda},\vec{x},\varepsilon}(B(t)) \exp\big(-\int_0^t U^{\vec{\lambda},\vec{x},\varepsilon}(B(s))ds\big)dt\Big) \\
&+ E_x\Big(\exp\big(-\int_0^{T_{r_\varepsilon}} U^{\vec{\lambda},\vec{x},\varepsilon}(B(s))ds\big)1(T_{r_\varepsilon}<\infty)(-U_{1,2}^{\vec{\lambda},\vec{x},\varepsilon}(B(T_{r_\varepsilon})))\Big).
\end{aligned}
$$

This follows from Lemmas S.1.3 and S.1.4, as in the proof of Lemma 9.5 of [20].

PROOF OF PROPOSITION 5.1. Recall $r_\varepsilon = 2\varepsilon$. For the case $\varepsilon \in [\varepsilon_0/2, \varepsilon_0)$, the result follows immediately by letting $c_{5.1} \geq e^{-2}2^{2(p-2)}\varepsilon_0^{-2(p-2)}$ and by using $xe^{-x} \leq e^{-1}$, for $x \geq 0$, so we assume

$$
r_\varepsilon = 2\varepsilon < \varepsilon_0. \tag{S.8}
$$

Recall that $T_{r_\varepsilon}^i = \inf\{t \geq 0 : |B_t - x_i| \leq r_\varepsilon\}$ and $T_{r_\varepsilon} = T_{r_\varepsilon}^1 \wedge T_{r_\varepsilon}^2$. Since $|x_i| \geq \varepsilon_0$, we have $T_{r_\varepsilon} > 0, P_0$-a.s.. We set $\vec{\lambda} = (\lambda, \lambda)$, $\vec{x} = (x_1, x_2)$, and $\Delta = |x_1 - x_2|$, where the constant $\lambda > 0$ will be chosen large below.

Apply (S.6) and Lemma S.1.3(a) to see that for $\lambda > 12$,

$$
\begin{aligned}
&\mathbb{E}_{\delta_0}\Big(\lambda^2 \frac{X_{G_\varepsilon^{x_1}}(1)}{\varepsilon^2}\frac{X_{G_\varepsilon^{x_2}}(1)}{\varepsilon^2}\exp\big(-\lambda\sum_{i=1}^{2}\frac{X_{G_\varepsilon^{x_i}}(1)}{\varepsilon^2}\big)\Big) \\
&= \lambda^2 e^{-U^{\vec{\lambda},\vec{x},\varepsilon}(x)}\Big[U_1^{\vec{\lambda},\vec{x},\varepsilon}(0)U_2^{\vec{\lambda},\vec{x},\varepsilon}(0) - U_{1,2}^{\vec{\lambda},\vec{x},\varepsilon}(0)\Big] \\
&\leq 2^{2p+2}(D^{\lambda/2}(2))^2|x_1|^{-p}|x_2|^{-p}\varepsilon^{2(p-2)} - \lambda^2 U_{1,2}^{\vec{\lambda},\vec{x},\varepsilon}(0) \\
&\leq c\varepsilon_0^{-2p}\varepsilon^{2(p-2)} + \lambda^2(-U_{1,2}^{\vec{\lambda},\vec{x},\varepsilon}(0)).
\end{aligned} \tag{S.9}
$$

To bound the last term, use Lemma S.1.5 to get

$$
\begin{aligned}
&\lambda^2(-U_{1,2}^{\vec{\lambda},\vec{x},\varepsilon}(0)) \\
&=\lambda^2 E_0\Big(\int_0^{T_{r_\varepsilon}} \prod_{i=1}^{2} U_i^{\vec{\lambda},\vec{x},\varepsilon}(B(t)) \exp\big(-\int_0^t U^{\vec{\lambda},\vec{x},\varepsilon}(B(s))ds\big)dt\Big) \\
&+\lambda^2 E_0\Big(\exp\big(-\int_0^{T_{r_\varepsilon}} U^{\vec{\lambda},\vec{x},\varepsilon}(B(s))ds\big)1(T_{r_\varepsilon}<\infty)(-U_{1,2}^{\vec{\lambda},\vec{x},\varepsilon}(B(T_{r_\varepsilon})))\Big) \\
&\equiv K_1 + K_2.
\end{aligned} \tag{S.10}
$$

We first consider K_2. On $\{T_{r_\varepsilon} < \infty\}$ we may set $x_\varepsilon(\omega) = B(T_{r_\varepsilon})$ and choose $i(\omega)$ so that $|x_i - x_\varepsilon| \geq \Delta/2$. By the definition of T_{r_ε}, $|x_i - x_\varepsilon| \geq r_\varepsilon = 2\varepsilon$, and so $|x_i - x_\varepsilon| \geq \frac{1}{2}(\Delta \vee r_\varepsilon)$. Lemma S.1.3(b) and the above imply

$$\lambda^2(-U_{1,2}^{\vec{\lambda},\vec{x},\varepsilon}(B(T_{r_\varepsilon}))) \leq 4 \cdot 2^p (D^{\lambda/2}(2)(\Delta \vee r_\varepsilon)^{-p} 2^p)\varepsilon^{p-2} \leq c(\Delta \vee r_\varepsilon)^{-p}\varepsilon^{p-2}.$$

This shows that

$$K_2 \leq c(\Delta \vee r_\varepsilon)^{-p}\varepsilon^{p-2} \sum_{i=1}^{2} E_0\Big(1(T_{r_\varepsilon}^i < \infty) \exp\big(-\int_0^{T_{r_\varepsilon}^i} U^{\vec{\lambda},\vec{x},\varepsilon}(B(s))ds\big)\Big). \tag{S.11}$$

Use (S.7) and Corollary 4.7(a) with $|B(s) - x_i| \geq r_\varepsilon = 2\varepsilon$ and $R = 2$ to see that

$$\begin{aligned} U^{\vec{\lambda},\vec{x},\varepsilon}(B(s)) \geq U^{\lambda\varepsilon^{-2},\varepsilon}(B(s) - x_i) \geq & U^{\infty,\varepsilon}(B(s) - x_i) - 2^p|B(s) - x_i|^{-p} D^\lambda(2)\varepsilon^{p-2} \\ \geq & V^\infty(B(s) - x_i) - 2^p|B(s) - x_i|^{-p} D^\lambda(2)\varepsilon^{p-2}, \end{aligned} \tag{S.12}$$

where the last follows by using (4.1) and scaling to see that $U^{\infty,\varepsilon}(x) = \varepsilon^{-2}U^{\infty,1}(x/\varepsilon) \geq \varepsilon^{-2}V^\infty(x/\varepsilon) = V^\infty(x)$ for all $|x|/\varepsilon > 1$. Let $\tau_{r_\varepsilon} = \inf\{t : |B_t| \leq r_\varepsilon\}$ and let μ, ν be as in (4.9). Use the above in (S.11) and then use Brownian scaling to see that for $i = 1, 2$,

$$\begin{aligned} & E_0\Big(1(T_{r_\varepsilon}^i < \infty) \exp\Big(-\int_0^{T_{r_\varepsilon}^i} U^{\vec{\lambda},\vec{x},\varepsilon}(B(s))ds\Big)\Big) \\ \leq & E_{-x_i}\Big(1(\tau_{r_\varepsilon} < \infty) \exp\Big(\int_0^{\tau_{r_\varepsilon}} \frac{2^p D^\lambda(2)\varepsilon^{p-2}}{|B(s)|^p} ds\Big) \exp\Big(-\int_0^{\tau_{r_\varepsilon}} \frac{2(4-d)}{|B(s)|^2} ds\Big)\Big) \\ \leq & E_{-x_i/r_\varepsilon}\Big(1(\tau_1 < \infty) \exp\Big(\int_0^{\tau_1} \frac{2^p D^\lambda(2)\varepsilon^{p-2} r_\varepsilon^{2-p}}{|B(s)|^p} ds\Big) \exp\Big(-\int_0^{\tau_1} \frac{2(4-d)}{|B(s)|^2} ds\Big)\Big) \\ = & E_{|x_i|/r_\varepsilon}^{(2+2\nu)}\Big(\exp\Big(\int_0^{\tau_1} \frac{4D^\lambda(2)}{\rho_s^p} ds\Big) \Big| \tau_1 < \infty\Big)(|x_i|/r_\varepsilon)^{-p}, \end{aligned} \tag{S.13}$$

where we have used Lemma 4.5 in the last line, and recalled that $p = \nu + \mu$. Choose $\lambda > 12$ large such that

$$2\gamma \equiv 2 \cdot 4D^\lambda(2) \leq 2(4-d) < \nu^2,$$

and then apply Lemma 4.4 to conclude that (S.13) is bounded by

$$c_{4.4}(p,\nu)(|x_i|/r_\varepsilon)^{-p} \leq c_{4.4}(p,\nu)\varepsilon_0^{-p} r_\varepsilon^p.$$

So (S.11) becomes

$$\begin{aligned} K_2 \leq c(\Delta \vee r_\varepsilon)^{-p}\varepsilon^{p-2} 2c_{4.4}(p,\nu)\varepsilon_0^{-p} r_\varepsilon^p \leq & c(\varepsilon_0)\Delta^{2-p}\varepsilon^{p-2} r_\varepsilon^{p-2} \\ = & 2^{p-2} c(\varepsilon_0)\Delta^{2-p}\varepsilon^{2(p-2)}. \end{aligned} \tag{S.14}$$

In view of (S.9), (S.10) and (S.14), it remains to prove

$$K_1 \le C(\varepsilon_0)\Delta^{2-p}\varepsilon^{2(p-2)}. \tag{S.15}$$

Apply Lemma S.1.3(a) to K_1 defined in (S.10) to get

$$\begin{aligned} K_1 \le& \lambda^2 \frac{1}{\lambda^2}(2^{p+1}\varepsilon^{p-2}D^{\lambda/2}(2))^2 \\ &\times E_0\Big(\int_0^{T_{r_\varepsilon}} \prod_{i=1}^2 |B_t - x_i|^{-p} \exp\Big(-\int_0^t U^{\vec{\lambda},\vec{x},\varepsilon}(B(s))ds\Big)dt\Big). \end{aligned} \tag{S.16}$$

Let $\Delta_i = x_{3-i} - x_i$, so that $|\Delta_i| = \Delta$. Let $T^{',i}_{r_\varepsilon} = \inf\{t : |B_t| \le r_\varepsilon$ or $|B_t - \Delta_i| \le r_\varepsilon\}$. Apply (S.12) to see that (S.16) becomes

$$\begin{aligned} K_1 \le& c\varepsilon^{2(p-2)} \sum_{i=1}^2 E_{-x_i}\Big(\int_0^{T^{',i}_{r_\varepsilon}} |B_t|^{-p}|B_t - \Delta_i|^{-p} 1(|B_t| \le |B_t - \Delta_i|) \\ &\times \exp\Big(\int_0^t \frac{2^p D^\lambda(2)\varepsilon^{p-2}}{|B(s)|^p}ds\Big) \exp\Big(-\int_0^t \frac{2(4-d)}{|B(s)|^2}ds\Big)dt\Big). \end{aligned} \tag{S.17}$$

On $\{|B_t| \le |B_t - \Delta_i|\}$, we have

$$\Delta = |\Delta_i| \le |B_t - \Delta_i| + |B_t| \le 2|B_t - \Delta_i|,$$

and hence

$$|B_t - \Delta_i|^{-p} \le \Big(\frac{1}{2}\Delta \vee |B_t|\Big)^{-p} \le 2^p(\Delta^{-p} \wedge |B_t|^{-p}).$$

Use $T^{',i}_{r_\varepsilon} \le \tau_{r_\varepsilon}$ and Brownian scaling to see that

$$\begin{aligned} K_1 \le\ & c\varepsilon^{2(p-2)} \sum_{i=1}^2 E_{-x_i}\Big(\int_0^{\tau_{r_\varepsilon}} |B_t|^{-p}(|B_t|^{-p} \wedge \Delta^{-p}) \\ &\qquad \times \exp\Big(\int_0^t \frac{2^p D^\lambda(2)\varepsilon^{p-2}}{|B(s)|^p}ds\Big) \exp\Big(-\int_0^t \frac{2(4-d)}{|B(s)|^2}ds\Big)dt\Big) \\ \le\ & c\varepsilon^{2(p-2)} \sum_{i=1}^2 E_{-x_i/r_\varepsilon}\Big(\int_0^{\tau_1} r_\varepsilon^{2-2p}|B_t|^{-p}(|B_t|^{-p} \wedge (\Delta/r_\varepsilon)^{-p}) \\ &\qquad \times \exp\Big(\int_0^t \frac{2^p D^\lambda(2)\varepsilon^{p-2}r_\varepsilon^{2-p}}{|B(s)|^p}ds\Big) \exp\Big(-\int_0^t \frac{2(4-d)}{|B(s)|^2}ds\Big)dt\Big) \\ =\ & c\varepsilon^{-2} \sum_{i=1}^2 \int_0^\infty E_{-x_i/r_\varepsilon}\Big(1(t<\tau_1)|B(t\wedge\tau_1)|^{-p}(|B(t\wedge\tau_1)|^{-p} \wedge (\Delta/r_\varepsilon)^{-p}) \\ &\qquad \times \exp\Big(\int_0^{t\wedge\tau_1} \frac{4D^\lambda(2)}{|B(s)|^p}ds\Big) \exp\Big(-\int_0^{t\wedge\tau_1} \frac{2(4-d)}{|B(s)|^2}ds\Big)\Big)dt. \end{aligned} \tag{S.18}$$

Now let $\delta = 4D^{\lambda}(2)$, μ, ν be as in (4.9), and use Lemma A.1 to get

(S.19)

$$\begin{aligned}
K_1 \le c\varepsilon^{-2} \sum_{i=1}^{2} \int_0^{\infty} (|x_i|/r_\varepsilon)^{\nu-\mu} E^{(2+2\nu)}_{|x_i|/r_\varepsilon} \Big(1(t<\tau_1)\rho(t\wedge\tau_1)^{-p} \\
\times (\rho(t\wedge\tau_1)^{-p} \wedge (\Delta/r_\varepsilon)^{-p}) \exp\Big(\int_0^{t\wedge\tau_1} \delta\rho_s^{-p} ds\Big)\rho(t\wedge\tau_1)^{-\nu+\mu}\Big) dt \\
= c\varepsilon^{\mu-\nu-2} \sum_{i=1}^{2} |x_i|^{\nu-\mu} E^{(2+2\nu)}_{|x_i|/r_\varepsilon} \Big(\int_0^{\tau_1} \rho_t{}^{-p-\nu+\mu} (\rho_t^{-p} \wedge (\Delta/r_\varepsilon)^{-p}) \exp\Big(\int_0^t \delta\rho_s^{-p} ds\Big) dt\Big).
\end{aligned}$$

We interrupt the proof of the proposition for another auxiliary result from [20].

Lemma S.1.6. *There is some universal constant $c_{S.1.6} > 0$ such that for any $r > 0$ with $r < (|x_i| \wedge \Delta)$ and $0 < \delta < (p-2)(2-\mu)$, we have*

$$\begin{aligned}
E^{(2+2\nu)}_{|x_i|/r} \Big(\int_0^{\tau_1} \rho_t{}^{-p-\nu+\mu} (\rho_t^{-p} \wedge (\Delta/r)^{-p}) \exp\Big(\int_0^t \delta\rho_s^{-p} ds\Big) dt\Big) \\
\le c_{S.1.6} r^{-2+2p+\nu-\mu} |x_i|^{-2\nu} \Delta^{2-p}.
\end{aligned}$$

Proof. This is included in the proof of Proposition 6.1 of [20] with $r = r_\lambda$. In particular, the above expectation appears in (9.23) of [20] and is bounded by eJ_i in (9.27) of that paper. Following the inequalities in that work, noting we only need consider Case 1 or Case 3 (the latter with $r \le |x_i| \le \Delta$) at the end of the proof, we arrive at the above bound. ∎

Returning now to the proof of Proposition 5.1. Pick $\lambda > 12$ large such that $\delta < (p-2)(2-\mu)$. Note we assumed $|x_i| \ge \varepsilon_0 > r_\varepsilon$ by (S.8) and $\Delta = |x_1 - x_2| > 5\varepsilon > r_\varepsilon$ at the very beginning of this section. So use Lemma S.1.6 applied with $r = r_\varepsilon$ to see that

$$\begin{aligned}
K_1 \le & c\varepsilon^{\mu-\nu-2} \sum_{i=1}^{2} |x_i|^{\nu-\mu} c_{S.1.6} r_\varepsilon^{-2+2p+\nu-\mu} |x_i|^{-2\nu} \Delta^{2-p} \\
= & C\varepsilon^{2p-4} \Delta^{2-p} \sum_{i=1}^{2} |x_i|^{-p}. \qquad \text{(S.20)}
\end{aligned}$$

Use $|x_i| \ge \varepsilon_0$ to conclude

$$K_1 \le 2C\varepsilon_0^{-p} \Delta^{2-p} \varepsilon^{2p-4}.$$

This gives (S.15), and so the proof is complete. ∎

S.2. Proof of Lemma 7.3. We work under Q_{x_0} where $|x_0| \geq 2r_0$. Recall the definitions of η_s^G and $\mathcal{E}_G$ from Section 2. For $0 \leq r < r_0$, introduce

$$A_t^r = \int_0^t 1(\zeta_u \leq S_{G_{r_0-r}}(W_u))\, du,$$

so that

$$\eta_s^r := \eta_s^{G_{r_0-r}} = \inf\{t : A_t^r > s\}.$$

LEMMA S.2.1. *(a) Q_{x_0}-a.s. for all $t \geq 0$ we have*

$$A_t^r = \int_0^t 1(\inf_{v \leq \zeta_u} |W_u(v)| > r_0 - r)\, du \quad \forall r \in [0, r_0),$$

and

$$r \mapsto A_t^r \text{ is left-continuous on } [0, r_0).$$

(b) $\lim_{r' \uparrow r} \eta_s^{r'} = \eta_s^r$ for all $r \in (0, r_0)$, $s \geq 0$ Q_{x_0}-a.s.
(c) If T is an $(\mathcal{E}_r^+)$-stopping time, then $W_{\eta_s^T}$ is $\mathcal{E}_T^+$-measurable.

PROOF. The proof is a straightforward modification of that of Lemma 7.4 in [20], where shrinking half spaces have now been replaced with shrinking balls. ■

Proof of Lemma 7.3. By (7.23) (with a different radii) and Lemma 2.1(a) there are Borel maps $\tilde{\psi}$ on $\mathcal{K}$ and ψ on $C(\mathbb{R}_+, \mathcal{W})$ such that

$$1_{D_{r_0}} = \tilde{\psi}(\mathcal{R}) = \lim_{N \to \infty} \tilde{\psi}(\{\hat{W}(s) : s \leq N\}) = \psi(W),$$

where we have used (2.2) in the second equality. In the last equality we have also called on the continuity of $W \mapsto \{\hat{W}(s) : s \leq N\}$ from $C([0, \infty), \mathcal{W})$ to $\mathcal{K}$. Therefore a monotone class argument shows it suffices to fix $s \geq 0$ and show that if $\phi : \mathcal{W} \to \mathbb{R}$ is bounded Borel then

$$\phi(W_s) \text{ is } \mathcal{E}_{T_0-}^+ - \text{measurable.} \tag{S.21}$$

Lemma S.2.1(b) implies that $W_{\eta_s^{T_0}} = \lim_{n \to \infty} W_{\eta_s^{T_0 - n^{-1}}}$ Q_{x_0}-as. and so by Lemma S.2.1(c) and (7.20), $W_{\eta_s^{T_0}}$ is $\mathcal{E}_{T_0-}^+$-measurable. So to prove (S.21) it suffices to show

$$W_s = W_{\eta_s^{T_0}} \quad Q_{x_0} - \text{a.s..}$$

This, in turn, would follow from $A_t^{T_0} = t$ for all $t \geq 0$ Q_{x_0}-a.s., or equivalently by Lemma S.2.1(a),

$$\int_0^\sigma 1(\inf_{v \leq \zeta_u} |W_u(v)| \leq r_0 - T_0)\, du = 0 \quad Q_{x_0} - \text{a.s..} \tag{S.22}$$

Here we have truncated the integral at σ since $\zeta_u = 0$ and $|W_u(0)| = |x_0| \geq 2r_0$ for $u \geq \sigma$. If $0 \leq u < \zeta_s$ and $s' < s$ is the last time before s that $\zeta_{s'} = u$, then $\inf_{t\in[s',s]} \zeta_t = \zeta_{s'} = u$ and so (e.g., see p. 66 of [17]) $W_s(u) = \hat{W}(s')$ Q_{x_0}-a.s. This and Lemma 7.1 (recall also (7.1)) imply

$$\inf_{u\leq\sigma} \inf_{v\leq\zeta_u} |W_u(v)| = \hat{T}_0 = \inf\{|x| : x \in \mathcal{R}\} = r_0 - T_0 \quad Q_{x_0} - \text{a.s..} \tag{S.23}$$

Therefore (S.22) is equivalent to

$$\int_0^\sigma 1(\inf_{v\leq\zeta_u} |W_u(v)| = \hat{T}_0)\, du = 0 \quad Q_{x_0} - \text{a.s..} \tag{S.24}$$

The historical process, $(H_t, t \geq 0)$ is an inhomogeneous Markov process under $\mathbb{N}_{x_0}$ taking values in $M_F(C(\mathbb{R}_+, \mathbb{R}^d))$–see [4] or p. 64 of [17] to see how it is easily defined from the snake W. The latter readily implies

$$\int_0^\infty H_t(\phi)dt = \int_0^\sigma \phi(W_u)\, du \quad \text{for all non-negative Borel } \phi, \tag{S.25}$$

where we have extended W_u to $\mathbb{R}_+$ in the obvious manner. Recalling (7.1) and letting X be the SBM under $\mathbb{N}_{x_0}$ as usual, we have

(S.26)

$$\begin{aligned}
&\mathbb{N}_{x_0}\Big(\int_0^\infty 1(\inf_{v\leq\zeta_u} |W_u(v)| = \hat{T}_0)du\Big) \\
&\quad\leq \mathbb{N}_{x_0}\Big(\int_0^\infty \int 1(\inf_{t'} |y_{t'}| = \hat{T}_0) H_t(dy)dt\Big) \quad \text{(by (S.25))}
\end{aligned}$$

$$\leq \int_0^\infty \mathbb{N}_{x_0}\Big(\int 1\Big(\int_0^\infty X_s(\{x : |x| < \inf_{t'\leq t} |y(t')|\})ds = 0\Big) H_t(dy)\Big)dt, \tag{S.27}$$

where in the last line we use (S.23) and $y(\cdot) = y(\cdot \wedge t)$ $H_t -$a.a. y $\forall t \geq 0$ $\mathbb{N}_{x_0}$-a.e. Below we will let B denote a d-dimensional Brownian motion starting at x_0 under $P^B_{x_0}$, $m_t = \inf_{t'\leq t} |B_{t'}| = |B_{\tau_t}|$ (for some $\tau_t < t$), and L^x be the local time of the SBM X (at time infinity). Fix $t > 0$ and use the Palm measure formula for H_t (e.g. Proposition 4.1.5 of [4]) to see that (cf. (7.22) in [20])

$$\begin{aligned}
&\mathbb{N}_{x_0}\Big(\int 1\Big(\int_0^\infty X_s(\{x : |x| < \inf_{t'\leq t} |y(t')|\})ds = 0\Big) H_t(dy)\Big) \\
&\quad= E^B_{x_0}\Big(\exp\Big(-\int_0^t \int 1\Big(\int_0^\infty X_s(\{x : |x| < m_t\})ds > 0\Big) d\mathbb{N}_{B_u} du\Big)\Big)
\end{aligned}$$

$$\leq E^B_{x_0}\Big(\exp\Big(-\int_0^t \mathbb{N}_{B_u}(L^{B_{\tau_t}} > 0)du\Big)\Big). \tag{S.28}$$

It follows from (1.13), (1.14) and $\mathbb{P}_{\delta_x}(L^y = 0) = \exp(-\mathbb{N}_x(L^y > 0))$ (see, e.g., (2.12) in [20]) that

$$\mathbb{N}_x(L^y > 0) = 2(4-d)|x-y|^{-2}.$$

Use this to bound (S.28) by

$$E^B_{x_0}\Big(\exp\Big(-\int_0^t \frac{2(4-d)}{|B_s - B_{\tau_t}|^2} ds\Big)\Big).$$

A simple application of Lévy's modulus for B shows the above integral is infinite a.s. and so proves that (S.26) equals zero. This implies (S.24), as required.

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Department of Mathematics
University of British Columbia
Vancouver, BC, Canada V6T1Z2
E-mail: jlhong@math.ubc.ca

Faculty of Industrial Engineering and Management,
Technion – Israel Institute of Technology
Haifa 32000, Israel
E-mail: leonid@ie.technion.ac.il

Department of Mathematics
University of British Columbia
Vancouver, B.C., Canada V6T 1Z2
E-mail: perkins@math.ubc.ca