Competing super-Brownian motions as limits of interacting particle systems

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Abstract. We study two-type branching random walks in which the birth or death rate of each type can depend on the number of neighbors of the opposite type. This competing species model contains variants of Durrett's predator-prey model and Durrett and Levin's colicin model as special cases. We verify in some cases convergence of scaling limits of these models to a pair of super-Brownian motions interacting through their collision local times, constructed by Evans and Perkins.

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1 Introduction

Consider the contact process on the fine lattice $\mathcal{Z}_N \equiv \mathbb{Z}^d/(\sqrt{N}M_N)$. Sites are either occupied by a particle or vacant.

- Particles die at rate N and give birth at rate $N + \theta$
- When a birth occurs at x the new particle is sent to a site $y \neq x$ chosen at random from $x + \mathcal{N}$ where $\mathcal{N} = \{z \in \mathcal{Z}_N : \|z\|_\infty \leq 1/\sqrt{N}\}$ is the set of neighbors of 0.
- If y is vacant a birth occurs there. Otherwise, no change occurs.

The $\sqrt{N}$ in the definition of $\mathcal{Z}_N$ scales space to take care of the fact that we are running time at rate N. The M_N serves to soften the interaction between a site and its neighbors so that we can get a nontrivial limit. From work of Bramson, Durrett, and Swindle (1989) it is known that one should take

$$M_N = \begin{cases} N^{3/2} & d = 1 \\ (N \log N)^{1/2} & d = 2 \\ N^{1/d} & d \geq 3 \end{cases}$$

Mueller and Tribe (1995) studied the case $d = 1$ and showed that if we assign each particle mass $1/N$ and the initial conditions converge to a continuous limiting density $u(x, 0)$, then the rescaled particle system converged to the stochastic PDE:

$$du = \left(\frac{u''}{6} + \theta u - u^2\right) dt + \sqrt{2u}\, dW$$

where dW is a space-time White noise.

Durrett and Perkins (1999) considered the case $d \geq 2$. To state their result we need to introduce super-Brownian motion with branching rate b, diffusion coefficient σ^2, and drift coefficient β. Let $\mathcal{M}_F = \mathcal{M}_F(\mathbb{R}^d)$ denote the space of finite measures on $\mathbb{R}^d$ equipped with the topology of weak convergence. Let C_b^∞ be the space of infinitely differentiable functions on $\mathbb{R}^d$ with bounded partial derivatives of all orders. Then the above super-Brownian motion is the $\mathcal{M}_F$-valued process X_t, which solves the following martingale problem:

For all $\phi \in C_b^\infty$, if $X_t(\phi)$ denotes the integral of ϕ with respect to X_t then

$$Z_t(\phi) = X_t(\phi) - X_0(\phi) - \int_0^t X_s(\sigma^2 \Delta\phi/2 + \beta\phi)\, ds \tag{1.1}$$

is a martingale with quadratic variation $< Z(\phi) >_t = \int_0^t X_s(b\phi^2)\, ds$.

Durrett and Perkins showed that if the initial conditions converge to a nonatomic limit then the rescaled empirical measures, formed by assigning mass $1/N$ to each site occupied by the rescaled contact processes, converge to the super-Brownian motion with $b = 2$, $\sigma^2 = 1/3$, and $\beta = \theta - c_d$. Here $c_2 = 3/2\pi$ and in $d \geq 3$, $c_d = \sum_{n=1}^\infty P(U_n \in [-1, 1]^d)/2^d$ with U_n a random walk that takes steps uniform on $[-1, 1]^d$. Note that the $-u^2$ interaction term in $d = 1$ becomes $-c_d u$ in $d \geq 2$. This occurs because the environments seen by well separated particles in a small macroscopic ball are almost independent, so by the law of large numbers mass is lost due to collisions (births onto occupied sites) at a rate proportional to the amount of mass there.

There has been a considerable amount of work constructing measure-valued diffusions with interactions in which the parameters b, σ^2 and β in (1.1) depend on X and may involve one or more interacting populations. State dependent σ's, or more generally state dependent spatial motions, can be characterized and constructed as solutions of a strong equation driven by a historical Brownian motion (see Perkins (1992), (2002)), and characterized as solutions of a martingale problem for historical superprocesses (Perkins (1995)) or more simply by the natural extension of (1.1) (see Donnelly and Kurtz (1999) Donnelly and Kurtz (1999)). (Historical superprocesses refers to a measure-valued process in which all the genealogical histories of the current population are recorded in the form of a random measure on path space.) State dependent branching in general seems more challenging. Many of the simple uniqueness questions remain open although there has been some recent progress in the case of countable state spaces (Bass and Perkins (2004)). In Dawson and Perkins (1998) and Dawson et al (2002), a particular case of a pair of populations exhibiting local interaction through their branching rates (called mutually catalytic or symbiotic branching) is analyzed in detail thanks to a couple of special duality relations. State dependent drifts (β) which are not "singular" and can model changes in birth and death rates within one or between several populations can be analyzed through the Girsanov techniques introduced by Dawson (1978) (see also Ch. IV of Perkins (2002)). Evans and Perkins (1994,1998) study a pair of interacting measure-valued processes which compete locally for resources through an extension of (1.1) discussed below (see remark after Theorem 1.1). In two or three dimensions these interactions involve singular drifts β for which it is believed the change of measure methods cited above will not work. In 3 dimensions this is known to be the case (see Theorem 4.14 of Evans and Perkins (1994)). Corresponding models with infinite variance branching mechanisms and stable migration processes have been constructed by Fleischmann and Mytnik (2003).

Given this work on interacting continuum models, it is natural to consider limits of multitype particle systems. The simplest idea is to consider a contact process with two types of particles for which births can only occur on vacant sites and each site can support at most one particle. However, this leads to a boring limit: independent super-processes. This can be seen from Section 5 in Durrett and Perkins (1999) which shows that in the single type contact process "collisions between distant relatives can be ignored."

To obtain an interesting interaction, we will follow Durrett and Levin (1998) and consider two types of particles that modify each other's death or birth rates. In order to concentrate on the new difficulties that come from the interaction, we will eliminate the restriction of at most one particle per site and let $\xi_t^{i,N}(x)$ be the number of particles of type i at x at time t. Having changed from a contact process to a branching process, we do not need to let $M_N \to \infty$, so we will again simplify by considering the case $M_N \equiv M$. Let σ^2 denote the variance of the uniform distribution on $(\mathbb{Z}/M) \cap [-1, 1]$.

Letting $x^+ = \max\{0, x\}$ and $x^- = \max\{0, -x\}$, the dynamics of our competing species model may be formulated as follows:

- When a birth occurs, the new particle is of the same type as its parent and is born at the same site.

- For $i = 1, 2$, let $n_i(x)$ be the number of individuals of type i in $x + \mathcal{N}$. Particles of type i give birth at rate $N + \gamma_i^+ 2^{-d} N^{d/2-1} n_{3-i}(x)$ and die at rate $N + \gamma_i^- 2^{-d} N^{d/2-1} n_{3-i}(x)$.

Here $3-i$ is the opposite type of particle. It is natural to think of the case in which $\gamma_1 < 0$ and $\gamma_2 < 0$ (resource competition), but in some cases the two species may have a synergistic effect: $\gamma_1 > 0$ and $\gamma_2 > 0$. Two important special cases that have been considered earlier are

(a) *the colicin model.* $\gamma_2 = 0$. In Durrett and Levin's paper, $\gamma_1 < 0$, since one type of *E. coli* produced a chemical (colicin) that killed the other type. We will also consider the case in which $\gamma_1 > 0$ which we will call colicin.

(b) *predator-prey model.* $\gamma_1 < 0$ and $\gamma_2 > 0$. Here the prey 1's are eaten by the predator 2's which have increased birth rates when there is more food.

Two related example that fall outside of the current framework, but for which similar results should hold:

(c) *epidemic model.* Here 1's are susceptible and 2's are infected. 1's and 2's are individually branching random walks. 2's infect 1's (and change them to 2's) at rate $\gamma 2^{-d} N^{d/2} n_2(x)$, while 2's revert to being 1's at rate 1.

(d) *voter model.* One could also consider branching random walks in which individuals give birth to their own types but switch type at rates proportional to the number of neighbours of the opposite type.

The scaling $N^{d/2-1}$ is chosen on the basis of the following heuristic argument. In a critical branching process that survives to time N there will be roughly N particles. In dimensions $d \geq 3$ if we tile the integer lattice with cubes of side 1 there will be particles in roughly N of the $N^{d/2}$ cubes within distance $\sqrt{N}$ of the origin. Thus there is probability $1/N^{d/2-1}$ of a cube containing a particle. To have an effect over the time interval $[0, N]$ a neighbor of the opposite type should produce changes at rate $N^{-1}N^{d/2-1}$ or on the speeded up time scale at rate $N^{d/2-1}$. In $d = 2$ an occupied square has about $\log N$ particles so there will be particles in roughly $N/(\log N)$ of the N squares within distance $\sqrt{N}$ of the origin. Thus there is probability $1/(\log N)$ of a square containing a particle, but when it does it contains $\log N$ particles. To have an effect interactions should produce changes at rate $1/N$ or on the speeded up time scale at rate $1 = N^{d/2-1}$. In $d = 1$ there are roughly $\sqrt{N}$ particles in each interval $[x, x+1]$ so each particle should produce changes at rate $N^{-1}N^{-1/2}$ or on the speeded up time scale at rate $N^{-1/2} = N^{d/2-1}$.

Our guess for the limit process comes from work of Evans and Perkins (1994, 1998) who studied some of the processes that will arise as a limit of our particle systems. We first need a concept that was introduced by Barlow, Evans, and Perkins (1991) for a class of measure-valued diffusions dominated by a pair of independent super-Brownian motions. Let (Y^1, Y^2) be an $\mathcal{M}_F{}^2$-valued process. Let $\mathrm{p}_s(x)$ $s \geq 0$ be the transition density function of Brownian motion with variance $\sigma^2 s$. For any $\phi \in B_b(\mathbb{R}^d)$ (bounded Borel functions on $\mathbb{R}^d$) and $\delta > 0$, let

$$(1.2) \quad L_t^\delta(Y^1, Y^2)(\phi) \equiv \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathrm{p}_\delta(x_1 - x_2)\phi((x_1 + x_2)/2) Y_s^1(dx_1) Y_s^2(dx_2)\, ds \quad t \geq 0.$$

The collision local time of (Y^1, Y^2) (if it exists) is a continuous non-decreasing $\mathcal{M}_F$-valued stochastic process $t \mapsto L_t(Y^1, Y^2)$ such that

$$L_t^\delta(Y^1, Y^2)(\phi) \to L_t(Y^1, Y^2)(\phi) \quad \text{as } \delta \downarrow 0 \text{ in probability},$$

for all $t > 0$ and $\phi \in C_b(\mathbb{R}^d)$, the bounded continuous functions on $\mathbb{R}^d$. It is easy to see that if $Y_s^i(dx) = y_s^i(x)dx$ for some Borel densities y_s^i which are uniformly bounded on compact time intervals, then $L_t(Y^1, Y^2)(dx) = \int_0^t y_s^1(x) y_s^2(x) ds dx$. However, the random measures we will be dealing with will not have densities for $d > 1$.

The final ingredient we need to state our theorem is the assumption on our initial conditions. Let $\mathbf{B}(x, r) = \{w \in \mathbb{R}^d : |w - x| \le r\}$, where $|z|$ is the L^∞ norm of z. For any $0 < \delta < 2 \wedge d$ we set

$$\varrho_\delta^N(\mu) \equiv \inf \left\{ \varrho : \sup_x \mu(\mathbf{B}(x, r)) \le \varrho r^{(2\wedge d)-\delta}, \text{ for all } r \in [N^{-1/2}, 1] \right\},$$

where the lower bound on r is being dictated by the lattice $\mathbb{Z}^d/(\sqrt{N}M)$.

We say that a sequence of measures $\{\mu^N,\ N \ge 1\}$ satisfies condition $\overline{\mathbf{UB}}_N$ if

$$\sup_{N\ge 1} \varrho_\delta^N(\mu^N) < \infty, \quad \text{for all } 0 < \delta < 2 \wedge d$$

We say that measure $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ satisfies condition $\mathbf{UB}$ if for all $0 < \delta < 2 \wedge d$

$$\varrho_\delta(\mu) \equiv \inf \left\{ \varrho : \sup_x \mu(\mathbf{B}(x, r)) \le \varrho r^{(2\wedge d)-\delta}, \text{ for all } r \in (0, 1] \right\} < \infty$$

If S is a metric space, C_S and D_S are the space of continuous S-valued paths and càdlàg S-valued paths, respectively, the former with the topology of uniform convergence on compacts and the latter with the Skorokhod topology. $C_b^k(\mathbb{R}^d)$ denotes the set of functions in $C_b(\mathbb{R}^d)$ whose partial derivatives of order k or less are also in $C_b(\mathbb{R}^d)$.

The main result of the paper is the following. If $X = (X^1, X^2)$, let $\mathcal{F}_t^X$ denote the right-continuous filtration generated by X.

Theorem 1.1 *Suppose $d \le 3$. Define measure-valued processes by*

$$X_t^{i,N}(\phi) = (1/N) \sum_x \xi_t^{i,N}(x)\phi(x)$$

Suppose $\gamma_1 \le 0$ and $\gamma_2 \in \mathbb{R}$. If $\{X_0^{i,N}\}, i = 1, 2$ satisfy $\overline{\mathbf{UB}}_N$ and converge to X_0^i in $\mathcal{M}_F$ for $i = 1, 2$, then $\{(X^{1,N}, X^{2,N}),\ N \ge 1\}$ is tight on $D_{\mathcal{M}_F{}^2}$. Each limit point $(X^1, X^2) \in C_{\mathcal{M}_F{}^2}$ and satisfies the following martingale problem $\mathbf{M}_{X_0^1, X_0^2}^{\gamma_1,\gamma_2}$: For $\phi_1, \phi_2 \in C_b^2(\mathbb{R}^d)$,

$$\begin{aligned} X_t^1(\phi_1) &= X_0^1(\phi_1) + M_t^1(\phi_1) + \int_0^t X_s^1\left(\frac{\sigma^2}{2}\Delta\phi_1\right) ds + \gamma_1 L_t(X^1, X^2)(\phi_1), \\ X_t^2(\phi_2) &= X_0^2(\phi_2) + M_t^2(\phi_2) + \int_0^t X_s^2\left(\frac{\sigma^2}{2}\Delta\phi_2\right) ds + \gamma_2 L_t(X^1, X^2)(\phi_2) \end{aligned} \tag{1.3}$$

where M^i are continuous $(\mathcal{F}_t^X)$-local martingales such that

$$\langle M^i(\phi_i), M^j(\phi_j)\rangle_t = \delta_{i,j} 2 \int_0^t X_s^i(\phi_i^2)\, ds$$

Remark. Barlow, Evans, and Perkins (1991) constructed the collision local time for two super-Brownian motions in dimensions $d \le 5$, but Evans and Perkins (1994) showed that no solutions to the martingale problem (1.3) exist in $d \ge 4$ for $\gamma_2 \le 0$.

Given the previous theorem, we will have convergence whenever we have a unique limit process. The next theorem gives uniqueness in the case of no feedback, i.e., $\gamma_1 = 0$. In this case, the first process provides an environment that alters the birth or death rate of the second one.

Theorem 1.2 *Let $\gamma_1 = 0$ and $\gamma_2 \in \mathbb{R}$, $d \le 3$, and X_0^i, $i = 1,2$, satisfy condition* **UB**. *Then there is a unique in law solution to the martingale problem ($\mathbf{MP}^{\gamma_1,\gamma_2}_{X_0^1,X_0^2}$).*

The uniqueness for $\gamma_1 = 0$ and $\gamma_2 \le 0$ above was proved by Evans and Perkins (1994) (Theorem 4.9) who showed that the law is the natural one: X^1 is a super-Brownian motion and conditional on X^1, X^2 is the law of a super ξ-process where ξ is Brownian motion killed according to an inhomogeneous additive functional with Revuz measure $X_s^1(dx)ds$. We prove the uniqueness for $\gamma_2 > 0$ in Section 5 below. Here X^1 is a super-Brownian motion and conditional on X^1, X^2 is the superprocess in which there is additional birthing according to the inhomogeneous additive functional with Revuz measure $X_s^1(dx)ds$. Such superprocesses are special cases of those studied by Dynkin (1994) and Kuznetsov (1994) although it will take a bit of work to connect their processes with our martingale problem.

Another case where uniqueness was already known is $\gamma_1 = \gamma_2 < 0$.

Theorem 1.3 *(Mytnik (1999)) Let $\gamma_1 = \gamma_2 < 0$, $d \le 3$, and X_0^i, $i = 1,2$, satisfy Condition* **UB**. *Then there is a unique in law solution to the martingale problem (1.3).*

Hence as an (almost) immediate Corollary to the above theorems we have:

Corollary 1.4 *Assume $d \le 3$, $\gamma_1 = 0$ or $\gamma_1 = \gamma_2 < 0$, and $\{X_0^{i,N}\}, i = 1,2$ satisfy $\overline{\mathbf{UB}}_N$ and converge to X_0^i in $\mathcal{M}_F$ for $i = 1,2$. If $X^{i,N}$ is defined as in Theorem 1.1, then $(X^{1,N}, X^{2,N})$ converges weakly in $D_{\mathcal{M}_F{}^2}$ to the unique in law solution of (1.3).*

Proof We only need point out that by elementary properties of weak convergence X_0^i will satisfy **UB** since $\{X_0^{i,N}\}$ satisfies $\mathbf{UB}_N$. The result now follows from the above three Theorems. ∎

For $d = 1$ uniqueness of solutions to (1.3) for $\gamma_i \le 0$ and with initial conditions satisfying

$$\int\int \log^+(1/|x_1 - x_2|)X_0^1(dx_1)X_0^2(dx_2) < \infty$$

(this is clearly weaker that each X_0^1 satisfying **UB**) is proved in Evans and Perkins (1994) (Theorem 3.9). In this case solutions can be bounded above by a pair of independent super-Brownian motions (as in Theorem 5.1 of Barlow, Evans and Perkins (1991)) from which one can readily see that $X_t^i(dx) = u_t^i(x)dx$ for $t > 0$ and $L_t(X^1, X^2)(dx) = \int_0^t u_s^1(x)u_s^2(x)dsdx$. In this case u^1, u^2 are also the unique in law solution of the stochastic partial differential equation

$$du^i = \left(\frac{\sigma^2 u^{i\prime\prime}}{2} + \theta u^i + \gamma_i u^1 u^2\right) dt + \sqrt{2u^i}\, dW^i \quad i = 1,2$$

where W^1 and W^2 are independent white noises. (See Proposition IV.2.3 of Perkins (2002).)

Turning next to $\gamma_2 > 0$ in one dimension we have the following result:

Theorem 1.5 *Assume $\gamma_1 \le 0 \le \gamma_2$, $X_0^1 \in \mathcal{M}_F$ has a continuous density on compact support and X_0^2 satisfies Condition* **UB**. *Then for $d = 1$ there is a unique in law solution to $\mathbf{M}^{\gamma_1,\gamma_2}_{X_0^1,X_0^2}$ which is absolutely continuous to the law of the pair of super-Brownian motions satisfying $\mathbf{M}^{0,0}_{X_0^1,X_0^2}$. In particular $X^i(t,dx) = u^i(t,x)dx$ for $u^i : (0,\infty) \to C_K$ continuous maps taking values in the space of continuous functions on $\mathbb{R}$ with compact support, $i = 1,2$.*

We will prove this result in Section 5 using Dawson's Girsanov Theorem (see Theorem IV. 1.6 (a) of Perkins (2002)). We have not attempted to find optimal conditions on the initial measures. As before, the following convergence theorem is then immediate from Theorem 1.1

Corollary 1.6 *Assume $d=1$, $\gamma_1 \le 0$, $\{X_0^{i,N}\}$ satisfy $\overline{\mathbf{UB}}_N$ and converge to $X_0^i \in \mathcal{M}_F$, $i=1,2$, where X_0^1 has a continuous density with compact support. If $X^{i,N}$ are as in Theorem 1.1, then $(X^{1,N}, X^{2,N})$ converges weakly in $D_{\mathcal{M}_F{}^2}$ to the unique solution of $\mathbf{MP}^{\gamma_1,\gamma_2}_{X_0^1,X_0^2}$.*

Having stated our results, the natural next question is: What can be said about uniqueness in other cases?

Conjecture 1.7 *Uniqueness holds in $d=2,3$ for any γ_1, γ_2.*

For $\gamma_i \le 0$ Evans and Perkins (1998) prove uniqueness of the historical martingale problem associated with (1.3). The particle systems come with an associated historical process as one simple puts mass N^{-1} on the path leading up to the current position of each particle at time t. It should be possible to prove tightness of these historical processes and show each limit point satisfies the above historical martingale problem. It would then follow that in fact one has convergence of empirical measures in Theorem 1.1 (for $\gamma_i \le 0$) to the natural projection of the unique solution to the historical martingale problem onto the space of continuous measure-valued processes.

Conjecture 1.8 *Theorem 1.1 continues to hold for $\gamma_1 > 0$ in $d=2,3$. There is no solution in $d \ge 4$. The solution explodes in finite time in $d=1$ when $\gamma_1, \gamma_2 > 0$.*

In addition to expanding the values of γ that can be covered, there is also the problem of considering more general approximating processes.

Conjecture 1.9 *Our results hold for the long-range contact process with modified birth and death rates.*

Returning to what we know, our final task in this Introduction is to outline the proofs of Theorems 1.1 and 1.2. Suppose $\gamma_1 \le 0$ and $\gamma_2 \in \mathbb{R}$, and set $\bar\gamma_1 = 0$ and $\bar\gamma_2 = \gamma_2^+$. Proposition 2.2 below will show that the corresponding measure-valued processes can be constructed on the same space so that $X^{i,N} \le \bar X^{i,N}$ for $i=1,2$. Here $(\bar X^{1,N}, \bar X^{2,N})$ are the sequence of processes corresponding to parameter values $(\bar\gamma_1, \bar\gamma_2)$. Tightness of our original sequence of processes then easily reduces to tightness of this sequence of bounding processes, because increasing the measures will both increase the mass far away (compact containment) and also increase the time variation in the integrals of test functions with respect to these measure-valued processes–see the approximating martingale problem (2.12) below. Turning now to $(\bar X^{1,N}, \bar X^{2,N})$, we first note that the tightness of the first coordinate (and convergence to super-Brownian motion) is well-known so let us focus on the second. The first key ingredient we will need is a bound on the mean measure, including of course its total mass. We will do this by conditioning on the branching environment $\bar X^{1,N}$. The starting point here will be the Feynman-Kac formula for this conditional mean measure given below in (2.17). In order to handle tightness of the discrete collision measure for $\bar X^{2,N}$ we will need a concentration inequality for the rescaled branching random walk $\bar X^{1,N}$, i.e., a uniform bound on the mass in small balls. A more precise result was given for super-Brownian motion in Theorem 4.7 of Barlow, Evans and Perkins (1991). The result we need is stated below as Proposition 2.4 and proved in Section 6.

Once tightness of $(X^{1,N}, X^{2,N})$ is established it is not hard to see that the limit points satisfy a martingale problem similar to our target, (1.3), but with some increasing continuous measure-valued process A in place of the collision local time. To identify A with the collision local time of the limits, we take limits in a Tanaka formula for the approximating discrete local times (Section 4 below) and derive the Tanaka formula for the limiting collision local time. As this will involve a number of singular integrals with respect to our random measures, the concentration inequality for $\bar{X}^{1,N}$ will again play an important role. This is reminiscent of the approach in Evans and Perkins (1994) to prove the existence of solutions to the limiting martingale problem when $\gamma_i \leq 0$. However the discrete setting here is a bit more involved, since requires checking the convergence of integrals of discrete Green functions with respect to the random mesures. The case of $\gamma_2 > 0$ forces a different approach as we have not been able to derive a concentration inequality for this process and so must proceed by calculation of second moments–Lemma 2.3 below is the starting point here. The Tanaka formula derived in Section 5 (see Remark 5.2) is new in this setting.

Theorem 1.2 is proved in Section 5 by using the conditional martingale problem of X^2 given X^1 to describe the Laplace functional of X^2 given X^1 in terms of an associated nonlinear equation involving a random semigroup depending on X^1. The latter shows that conditional on X^1, X^2 is a superprocess with immigration given by the collision local time of a Brownian path in the random field X^1.

Convention As our results only hold for $d \leq 3$, we will assume $d \leq 3$ throughout the rest of this work.

2 The Rescaled Particle System–Construction and Basic Properties

We first will write down a more precise description corresponding to the per particle birth and death rates used in the previous section to define our rescaled interacting particle systems. We let p^N denote the uniform distribution on $\mathcal{N}$, that is

$$p^N(z) = \frac{1(|z| \leq 1/\sqrt{N})}{(2M+1)^d}, \quad z \in \mathcal{Z}_N. \tag{2.1}$$

Let $P^N\phi(x) = \sum_y p^N(y-x)\phi(y)$ for $\phi : \mathcal{Z}_N \to \mathbb{R}$ for which the righthand side is absolutely summable. Set $M' = (M + (1/2))^d$. The per particle rates in Section 1 lead to a process $(\xi^1, \xi^2) \in \mathbb{Z}_+^{\mathcal{Z}_N} \times \mathbb{Z}_+^{\mathcal{Z}_N}$ such that for $i = 1, 2$,

$$\xi_t^i(x) \to \xi_t^i(x) + 1 \text{ with rate } N\xi_t^i(x) + N^{d/2-1}\xi_t^i(x)\gamma_i^+(M')^d \sum_y p^N(y-x)\xi_t^{3-i}(y),$$

$$\xi_t^i(x) \to \xi_t^i(x) - 1 \text{ with rate } N\xi_t^i(x) + N^{d/2-1}\xi_t(x)\gamma_i^-(M')^d \sum_y p^N(y-x)\xi_t^{3-i}(y),$$

and

$$(\xi_t^i(x), \xi_t^i(y)) \to (\xi_t^i(x)+1, \xi_t^i(y)-1) \text{ with rate } Np_N(x-y)\xi_t^i(y).$$

The factors of $(M')^d$ may look odd but they combine with the kernels p^N to get the factors of 2^{-d} in our interactive birth and death rates.

Such a process can be constructed as the unique solution of an SDE driven by a family of Poisson point processes. For $x, y \in \mathcal{Z}_N$, let $\Lambda_x^{i,+}, \Lambda_x^{i,-}, \Lambda_{x,y}^{i,+,c}, \Lambda_{x,y}^{i,-,c}, \Lambda_{x,y}^{i,m}, i = 1, 2$ be independent Poisson processes on $\mathbb{R}_+^2$, $\mathbb{R}_+^2$, $\mathbb{R}_+^3$, $\mathbb{R}_+^3$, and $\mathbb{R}_+^2$, respectively. Here $\Lambda_x^{i,\pm}$ governs the birth and death rates at x, $\Lambda_{x,y}^{i,\pm,c}$ governs the additional birthing or killing at x due to the influence of the other type at y and $\Lambda_{x,y}^{i,m}$ governs the migration of particles from y to x. The rates of $\Lambda_x^{i,\pm}$ are $N ds\, du$; the rates of $\Lambda_{x,y}^{i,\pm,c}$ are $N^{d/2-1}M'p^N(y-x)ds\, du\, dv$; the rates of $\Lambda_{x,y}^{i,m}$ are $Np_N(x-y)du$. Let $\mathcal{F}_t$ be the canonical right continuous filtration generated by this family of point processes and let $\mathcal{F}_t^i$ denote the corresponding filtrations generated by the point processes with superscript i, $i = 1, 2$. Let $\xi_0 = (\xi_0^1, \xi_0^2) \in \mathbb{Z}_+^{\mathcal{Z}_N} \times \mathbb{Z}_+^{\mathcal{Z}_N}$ be such that $|\xi_0^i| \equiv \sum_x \xi_0^i(x) < \infty$ for $i = 1, 2$–denote this set of initial conditions by S_F–and consider the following system of stochastic jump equations for $i = 1, 2$, $x \in \mathcal{Z}_N$ and $t \geq 0$:

$$
\begin{aligned}
\xi_t^i(x) = \xi_0^i(x) \quad &+ \quad \int_0^t \int 1(u \leq \xi_{s-}^i(x))\Lambda_x^{i,+}(ds, du) - \int_0^t \int 1(u \leq \xi_{s-}^i(x))\Lambda_x^{i,-}(ds, du) \\
&+ \quad \sum_y \int_0^t \int \int 1(u \leq \xi_{s-}^i(x), v \leq \gamma_i^+ \xi_{s-}^{3-i}(y))\Lambda_{x,y}^{i,+,c}(ds, du, dv) \\
&- \quad \sum_y \int_0^t \int \int 1(u \leq \xi_{s-}^i(x), v \leq \gamma_i^- \xi_{s-}^{3-i}(y))\Lambda_{x,y}^{i,-,c}(ds, du, dv) \\
&+ \quad \sum_y \int_0^t \int 1(u \leq \xi_{s-}^i(y))\Lambda_{x,y}^{i,m}(ds, du) - \sum_y \int_0^t \int 1(u \leq \xi_{s-}^i(x))\Lambda_{y,x}^{i,m}(ds, du).
\end{aligned}
\tag{2.2}
$$

Assuming for now that there is a unique solution to this system of equations, the reader can easily check that the solution does indeed have the jump rates described above. These equations are similar to corresponding systems studied in Mueller and Tribe (1994), but for completeness we will now show that (2.2) has a unique $\mathcal{F}_t$-adapted S_F-valued solution. Associated with (2.2) introduce the increasing $\mathcal{F}_t$-adapted $\mathbb{Z}_+ \cup \{\infty\}$-valued process

$$
\begin{aligned}
J_t = \sum_{i=1}^2 |\xi_0^i| \quad &+ \quad \sum_x \int_0^t \int 1(u \leq \xi_{s-}^i(x))\Lambda_x^{i,+}(ds, du) + \sum_x \int_0^t \int 1(u \leq \xi_{s-}^i(x))\Lambda_x^{i,-}(ds, du) \\
&+ \quad \sum_{x,y} \int_0^t \int \int 1(u \leq \xi_{s-}^i(x), v \leq \gamma_i^+ \xi_{s-}^{3-i}(y))\Lambda_{x,y}^{i,+,c}(ds, du, dv) \\
&+ \quad \sum_{x,y} \int_0^t \int \int 1(u \leq \xi_{s-}^i(x), v \leq \gamma_i^- \xi_{s-}^{3-i}(y))\Lambda_{x,y}^{i,-,c}(ds, du, dv) \\
&+ \quad \sum_{x,y} \int_0^t \int 1(u \leq \xi_{s-}^i(y))\Lambda_{x,y}^{i,m}(ds, du).
\end{aligned}
$$

Set $T_0 = 0$ and let T_1 be the first jump time of J. This is well-defined as any solution to (2.2) cannot jump until T_1 and so the solution is identically (ξ_0^1, ξ_0^2) until T_1. Therefore a short calculation shows that T_1 is exponential with rate at most

$$\sum_{i=1}^{2} 4N|\xi_0^i| + M'N^{d/2-1}|\gamma_i||\xi_0^i|. \tag{2.3}$$

At time T_1 (2.2) prescribes a unique single jump at a single site for any solution ξ and J increases by 1. Now proceed inductively, letting T_n be the nth jump time of J. Clearly the solution ξ to (2.2) is unique up until $T_\infty = \lim_n T_n$. Moreover

$$\sup_{s\le t} |\xi_s^1| + |\xi_s^2| \le J_t \text{ for all } t \le T_\infty. \tag{2.4}$$

Finally note that (2.3) and the corresponding bounds for the rates of subsequent times shows that J is stochastically dominated by a pure birth process starting at $|\xi_0^1| + |\xi_0^2|$ and with per particle birth rate $4N + M'N^{d/2-1}|(|\gamma_1| + |\gamma_2|)$. Such a process cannot explode and in fact has finite pth moments for all $p > 0$ (see Ex. 6.8.4 in Grimmett and Stirzaker (2001)). Therefore $T_\infty = \infty$ a.s. and we have proved (use (2.4) to get the moments below):

Proposition 2.1 *For each $\xi_0 \in S_F$, there is a unique $\mathcal{F}_t$-adapted solution (ξ^1, ξ^2) to (2.2). Moreover this process has càdlàg S_F-valued paths and satisfies*

$$E(\sup_{s\le t}(|\xi_s^1| + |\xi_s^2|)^p) < \infty \quad \textit{for all } p, t \ge 0. \tag{2.5}$$

The following "Domination Principle" will play an important role in this work.

Proposition 2.2 *Assume $\gamma_i^+ \le \bar{\gamma}_i$, $i = 1, 2$ and let ξ, respectively $\bar{\xi}$, denote the corresponding unique solutions to (2.2) with initial conditions $\xi_0^i \le \bar{\xi}_0^i$, $i = 1, 2$. Then $\xi_t^i \le \bar{\xi}_t^i$ for all $t \ge 0$, $i = 1, 2$ a.s.*

Proof. Let J and T_n be as in the previous proof but for $\bar{\xi}$. One then argues inductively on n that $\xi_t^i \le \bar{\xi}_t^i$ for $t \le T_n$. Assuming the result for n ($n = 0$ holds by our assumption on the initial conditions), then clearly neither process can jump until T_{n+1}. To extend the comparison to T_{n+1} we only need consider the cases where ξ^i jumps upward at a single site x for which $\xi_{T_{n+1}-}^i(x) = \bar{\xi}_{T_{n+1}-}^i(x)$ or $\bar{\xi}^i$ jumps downward at a single site x for which the same equality holds. As only one type and one site can change at any given time we may assume the processes do not change in any other coordinates. It is now a simple matter to analyze these cases using (2.2) and show that in either case the other process (the one not assumed to jump) must in fact mirror the jump taken by the jumping process and so the inequality is maintained at T_{n+1}. As we know $T_n \to \infty$ a.s. this completes the proof. ∎

Denote dependence on N by letting $\xi^N = (\xi^{1,N}, \xi^{2,N})$ be the unique solution to (2.2) with a given initial condition ξ_0^N and let

$$X_t^{i,N} = \frac{1}{N} \sum_{x \in \mathcal{Z}_N} \delta_x \xi_t^{i,N}(x), \quad i = 1, 2 \tag{2.6}$$

denote the associated pair of empirical measures, each taking values in $\mathcal{M}_F$. We will not be able to deal with the case of symbiotic systems where both $\gamma_i > 0$ so we will assume from now on that $\gamma_1 \le 0$. As we prefer to write positive parameters we will in fact replace γ_1 with $-\gamma_1$ and therefore assume $\gamma_1 \ge 0$. We will let $\bar{\xi}^{i,N}$ and $\bar{X}^{i,N}$ denote the corresponding particle system and empirical measures with $\bar{\gamma} = (0, \gamma_2^+)$. We call $(X^{1,N}, X^{2,N})$ a positive colicin process, as $\bar{X}^{1,N}$ is just a rescaled branching random walk which has a non-negative local influence on $\bar{X}^{2,N}$. The above Domination Principle implies

$$X^{i,N} \le \bar{X}^{i,N} \quad \text{for } i = 1, 2. \tag{2.7}$$

In order to obtain the desired limiting martingale problem we will need to use a bit of jump calculus to derive the martingale properties of $X^{i,N}$. Define the discrete collision local time for $X^{i,N}$ by

$$L_t^{i,N}(\phi) = 2^{-d} \int_0^t \int_{\mathbb{R}^d} \phi(x) N^{d/2} X_s^{3-i,N}(\mathbf{B}(x, N^{-1/2})) X_s^{i,N}(dx)\, ds \tag{2.8}$$

We denote the corresponding quantity for our bounding positive colicin process by $\bar{L}^{i,N}$. These integrals all have finite means by (2.5) and, in particular, are a.s. finite. Let $\tilde{\Lambda}$ denote the predictable compensator of a Poisson point process Λ and let $\hat{\Lambda} = \Lambda - \tilde{\Lambda}$ denote the associated martingale measure. If $\psi^i : \mathbb{R}_+ \times \Omega \times \mathcal{Z}_N \to \mathbb{R}$ are $\mathcal{F}_t$-predictable define a discrete inner product by

$$\nabla_N \psi_s^1 \cdot \nabla_N \psi_s^2(x) = N \sum_y p_N(y - x)(\psi^1(s, y) - \psi^1(s, x))(\psi^2(s, y) - \psi^2(s, x))$$

and write $\nabla_N^2 \psi_s^i(x)$ for $\nabla_N \psi_s^i \cdot \nabla_N \psi_s^i(x)$. Next define

$$\begin{aligned} M_t^{i,N}(\psi^i) \quad = \quad & \sum_x \frac{1}{N} \Big[\int_0^t \int \psi^i(s,x) 1(u \le \xi_{s-}^i(x)) \hat{\Lambda}_x^{i,+}(ds, du) \\ - \quad & \sum_x \int_0^t \int \psi^i(s,x) 1(u \le \xi_{s-}^i(x)) \hat{\Lambda}_x^{i,-}(ds, du) \\ + \quad & \sum_{x,y} \int_0^t \int \int \psi^i(s,x) 1(u \le \xi_{s-}^i(x), v \le \gamma_i^+ \xi_{s-}^{3-i}(y)) \hat{\Lambda}_{x,y}^{i,+,c}(ds, du, dv) \\ - \quad & \sum_{x,y} \int_0^t \int \int \psi^i(s,x) 1(u \le \xi_{s-}^i(x), v \le \gamma_i^- \xi_{s-}^{3-i}(y)) \hat{\Lambda}_{x,y}^{i,-,c}(ds, du, dv) \\ + \quad & \sum_{x,y} \int_0^t \int \psi^i(s,x) 1(u \le \xi_{s-}^i(y)) \hat{\Lambda}_{x,y}^{i,m}(ds, du) \\ - \quad & \sum_{x,y} \int_0^t \int \psi^i(s,x) 1(u \le \xi_{s-}^i(x)) \hat{\Lambda}_{y,x}^{i,m}(ds, du) \Big]. \end{aligned} \tag{2.9}$$

To deal with the convergence of the above sum note that its predictable square function is

$$\langle M^{i,N}(\psi^i) \rangle_t \quad = \quad \int_0^t X_s^{i,N}(2(\psi_s^i)^2)\, ds + \frac{|\gamma_i|}{N} L_t^{i,N}((\psi^i)^2) + \int_0^t \frac{1}{N} X_s^{i,N}(\nabla_N^2 \psi_s^i)\, ds. \tag{2.10}$$

If ψ is bounded, the above is easily seen to be square integrable by (2.5), and so $M^{i,N}(\psi^i)_t$ is an L^2 $\mathcal{F}_t$-martingale. More generally whenever the above expression is a.s. finite for all $t > 0$, $M_t^{i,N}(\psi)$ is an $\mathcal{F}_t$-local martingale. The last two terms are minor error terms. We write $\bar{M}^{i,N}$ for the corresponding martingale measures for our dominating positive colicin processes.

Let Δ^N be the generator of the "motion" process $B^N_\cdot$ which takes steps according to p^N at rate N:

$$\Delta^N\phi(x) = N\sum_{y\in\mathcal{Z}_N}(\phi(y)-\phi(x))p^N(y-x).$$

Let $\Pi^N_{s,x}$ be the law of this process which starts from x at time s. We will adopt the convention $\Pi^N_x = \Pi^N_{0,x}$. It follows from Lemma 2.6 of Cox, Durrett and Perkins (2000) that if σ^2 is as defined in Section 1 then for $\phi \in C_b^{1,3}([0,T]\times\mathbb{R}^d)$

$$\Delta^N\phi(s,x) \to \frac{\sigma^2}{2}\Delta\phi(s,x), \qquad \text{uniformly in } s\le T \text{ and } x\in\mathbb{R}^d \text{ as } N\to\infty. \tag{2.11}$$

Let $\phi_1,\phi_2 \in C_b([0,T]\times\mathcal{Z}_N)$ with $\dot{\phi}_i = \frac{\partial\phi_i}{\partial t}$ also in $C_b([0,T]\times\mathcal{Z}_N)$. It is now fairly straightforward to multiply (2.2) by $\phi_i(t,x)/N$, sum over x, and integrate by parts to see that $(X^{1,N},X^{2,N})$ satisfies the following martingale problem $\mathbf{M}^{N,\gamma_1,\gamma_2}_{X_0^{1,N},X_0^{2,N}}$:

$$\begin{aligned}
X_t^{1,N}(\phi_1(t)) &= X_0^{1,N}(\phi_1(0)) + M_t^{1,N}(\phi_1) + \int_0^t X_s^{1,N}(\Delta^N\phi_1(s)+\dot{\phi}_1(s))\,ds \\
&\quad - \gamma_1 L_t^{1,N}(\phi_1),\ t\le T,\\
X_t^{2,N}(\phi_2(t)) &= X_0^{2,N}(\phi_2(0)) + M_t^{2,N}(\phi_2) + \int_0^t X_s^{2,N}(\Delta^{2,N}\phi_2(s)+\dot{\phi}_2(s))\,ds \\
&\quad + \gamma_2 L_t^{2,N}(\phi_2),\ t\le T,
\end{aligned} \tag{2.12}$$

where $M^{i,N}$ are $\mathcal{F}_t-$martingales, such that

$$\begin{aligned}
\langle M^{i,N}(\phi_i), M^{j,N}(\phi_j)\rangle_t &= \delta_{i,j}\Bigg(\int_0^t X_s^{i,N}(2\phi_i(s)^2)\,ds \\
&\quad + \frac{|\gamma_i|}{N}L_t^{i,N}\left(\phi_i^2\right) + \int_0^t \frac{1}{N}X_s^{i,N}(\nabla_N^2\phi_s^i)\,ds.\Bigg)
\end{aligned}$$

Let

$$g_N(\bar{X}_s^{1,N},x) = 2^{-d}N^{d/2}\bar{X}_s^{1,N}(\mathbf{B}(x,N^{-1/2})).$$

To derive the conditional mean of $\bar{X}^{2,N}$ given $\bar{X}^{1,N}$ we first note that $\bar{\xi}^{1,N}$ is in fact $\mathcal{F}_t^1$-adapted as the equations for $\bar{\xi}^{1,N}$ are autonomous since $\bar{\gamma}_1 = 0$ and so the pathwise unique solution will be adapted to the smaller filtration. Note also that if $\bar{\mathcal{F}}_t = \mathcal{F}^1_\infty \vee \mathcal{F}^2_t$, then $\hat{\Lambda}^{2,\pm}, \hat{\Lambda}^{2,\pm,c}, \hat{\Lambda}^{2,m}$ are all $\bar{\mathcal{F}}_t$-martingale measures and so $\bar{M}^{2,N}(\psi)$ will be a $\bar{\mathcal{F}}_t$-martingale whenever $\psi:[0,T]\times\Omega\times\mathcal{Z}_N\to\mathbb{R}$ is bounded and $\bar{\mathcal{F}}_t$-predictable. Therefore if $\psi,\dot{\psi}:[0,T]\times\Omega\times\mathcal{Z}_N\to\mathbb{R}$ are bounded, continuous in the first and third variables for a.a. choices of the second, and $\bar{\mathcal{F}}_t$-predictable in the first two variables for each point in $\mathcal{Z}_{\mathcal{N}}$, then we can repeat the derivation of the martingale problem for $(X^{1,N}, X^{2,N})$ and see that

$$\bar{X}_t^{2,N}(\psi_t) \;\;=\;\; \bar{X}_0^{2,N}(\psi_0) + \bar{M}_t^{2,N}(\psi) + \int_0^t \bar{X}_s^{2,N}\left(\Delta^N\psi_s + \gamma_2^+ g_N(\bar{X}_s^{1,N},\cdot)\psi_s + \dot{\psi}_s\right) ds,\;\; t \le T,$$

where $\bar{M}_t^{2,N}(\psi)$ is now an $\bar{\mathcal{F}}_t$-local martingale because the right-hand side of (2.10) is a.s. finite for all $t > 0$.

Fix $t > 0$, and a map $\phi : \mathcal{Z}_N \times \Omega \to \mathbb{R}$ which is $\mathcal{F}^1_\infty$-measurable in the second variable and satisfies

$$\sup_{x\in\mathcal{Z}_N} |\phi(x)| < \infty \text{ a.s.} \tag{2.13}$$

Let ψ satisfy

$$\begin{aligned} \frac{\partial\psi_s}{\partial s} &= -\Delta^N\psi_s - \gamma_2^+ g_N(\bar{X}_s^{1,N},\cdot)\psi_s,\; 0 \le s \le t, \\ \psi_t &= \phi. \end{aligned}$$

One can check that $\psi_s\,, s \le t$ is given by

$$\psi_s(x) = P_{s,t}^{g_N}(\phi)(x) \equiv \Pi_{s,x}^N\left[\phi(B_t^N)\exp\left\{\int_s^t \gamma_2^+ g_N(\bar{X}_r^{1,N}, B_r^N)\,dr\right\}\right], \tag{2.14}$$

which indeed does satisfy the above conditions on ψ. Therefore for ψ, ϕ as above

$$\bar{X}_t^{2,N}(\phi) \;\;=\;\; \bar{X}_0^{2,N}(P_{0,t}^{g_N}(\phi)) + \bar{M}_t^{2,N}(\psi). \tag{2.15}$$

For each $K > 0$,

$$E(\bar{M}_t^{2,N}(\psi)|\bar{\mathcal{F}}_0) = E(\bar{M}_t^{2,N}(\psi\wedge K)|\bar{\mathcal{F}}_0) = 0 \text{ a.s. on } \{\sup_{s\le t}|\psi_s| \le K|\} \in \mathcal{F}_0$$

and hence, letting $K \to \infty$, we get

$$E(\bar{M}_t^{2,N}(\psi)|\bar{\mathcal{F}}_0) = 0 \text{ a.s.} \tag{2.16}$$

This and (2.15) imply

$$\begin{aligned} E\left[\bar{X}_t^{2,N}(\phi)|\bar{X}^{1,N}\right] &= \bar{X}_0^{2,N}\left(P_{0,t}^{g_N}(\phi)(\cdot)\right) \\ &= \int_{\mathbb{R}^d}\Pi_{0,x}\left[\phi(B_t^N)\exp\left\{\int_0^t \gamma_2^+ g_N(\bar{X}_r^{1,N}, B_r^N)\,dr\right\}\right]\bar{X}_0^{2,N}(dx). \end{aligned} \tag{2.17}$$

It will also be convenient to use (2.15) to prove a corresponding result for conditional second moments.

Lemma 2.3 *Let $\phi_i : \mathcal{Z}_N \times \Omega \to \mathbb{R}$, $(i = 1, 2)$ be $\mathcal{F}^1_\infty$-measurable in the second variable and satisfy (2.13). Then*

$$\begin{aligned}
(2.18)\qquad & E\left[\bar{X}^{2,N}_t(\phi_1)\bar{X}^{2,N}_t(\phi_2)|\bar{X}^{1,N}\right] = \bar{X}^{2,N}_0\left(P^{g_N}_{0,t}(\phi_1)(\cdot)\right)\bar{X}^{2,N}_0\left(P^{g_N}_{0,t}(\phi_2)(\cdot)\right) \\
&+ E\left[\int_0^t\int_{\mathbb{R}^d} 2P^{g_N}_{s,t}(\phi_1)(x)P^{g_N}_{s,t}(\phi_2)(x)\bar{X}^{2,N}_s(dx)\,ds\Big|\bar{X}^{1,N}\right] \\
&+ E\Bigg[\int_0^t\int_{\mathbb{R}^d}\Big(\sum_{y\in\mathcal{Z}_N}(P^{g_N}_{s,t}(\phi_1)(y) - P^{g_N}_{s,t}(\phi_1)(x)) \\
&\qquad \times (P^{g_N}_{s,t}(\phi_2)(y) - P^{g_N}_{s,t}(\phi_2)(x))p^N(x-y)\Big)\bar{X}^{2,N}_s(dx)\,ds\Big|\bar{X}^{1,N}\Bigg] \\
&+ E\left[\frac{\gamma_2^+}{N}\bar{L}^{2,N}_t\left(P^{g_N}_{\cdot,t}(\phi_1)(\cdot)P^{g_N}_{\cdot,t}(\phi_2)(\cdot)\right)\Big|\bar{X}^{1,N}\right].
\end{aligned}$$

Proof. Argue just as in the derivation of (2.16) to see that

$$E(\bar{M}^{2,N}_t(\phi_1)\bar{M}^{2,N}_t(\phi_2)|\bar{\mathcal{F}}_0) = E(\langle\bar{M}^{2,N}(\phi_1), \bar{M}^{2,N}(\phi_2)\rangle_t|\bar{\mathcal{F}}_0) \text{ a.s.}$$

The result is now immediate from this, (2.15) and (2.10).

Now we will use the Taylor expansion for the exponential function in (2.14) to see that for $\phi : \mathcal{Z}_N \times \Omega \to [0, \infty)$ as above, and $0 \le s < t$,

$$\begin{aligned}
P^{g_N}_{s,t}(\phi)(x) &= \sum_{n=0}^\infty \frac{(\gamma_2^+)^n}{n!}\Pi^N_{s,x}\left[\phi(B^N_t)\int_s^t\ldots\int_s^t\prod_{i=1}^n g_N(\bar{X}^{1,N}_{s_i}, B^N_{s_i})\,ds_1\ldots ds_n\right] \\
(2.19)\qquad &= \sum_{n=0}^\infty(\gamma_2^+)^n\Bigg[\int_{\mathbb{R}^n_+}1(s < s_1 < s_2 < \ldots < s_n \le t) \\
&\quad \times\left(\int_{\mathbb{R}^{dn}}p^{(n)}_x(s_1,\ldots,s_n,t,y_1,\ldots,y_n,\phi)\prod_{i=1}^n\bar{X}^{1,N}_{s_i}(dy_i)\right)ds_1\ldots ds_n\Bigg].
\end{aligned}$$

Here

$$\begin{aligned}
&p^{(n)}_x(s_1,\ldots,s_n,t,y_1,\ldots,y_n,\phi) \\
&\quad = 2^{-dn}N^{dn/2}\Pi^N_x\left(\phi(B^N_t)1(|y_i - B^N_{s_i}| \le 1/\sqrt{N},\ i = 1,\ldots,n)\right).
\end{aligned}$$

We now state the concentration inequality for our rescaled branching random walks $\bar{X}^{1,N}$ which will play a central role in our proofs. The proof is given in Section 6.

Proposition 2.4 *Assume that the non-random initial measure $\{\bar{X}^{1,N}_0\}$ satisfies $\overline{\mathbf{UB}}_N$. For $\delta > 0$, define*

$$\mathcal{H}_{\delta,N} \equiv \sup_{t\ge 0}\varrho^N_\delta(\bar{X}^{1,N}_t).$$

Then for any $\delta > 0$, $\mathcal{H}_{\delta,N}$ is bounded in probability uniformly in N, that is, for any $\epsilon > 0$, there exists $M(\epsilon)$ such that

$$P(\mathcal{H}_{\delta,N} \ge M(\epsilon)) \le \epsilon,\ \forall N \ge 1.$$

Throughout the rest of the paper we will assume

Assumption 2.5 *The sequences of measures* $\{X_0^{i,N},\ N \geq 1\}$, $i = 1,2$, *satisfy condition* $\overline{\mathbf{UB}}_N$, *and for each* i, $X_0^{i,N} \to X_0^i$ *in* $\mathcal{M}_F$ *as* $N \to \infty$.

It follows from $(\mathbf{M}^{N,0,0}_{X_0^{1,N},X_0^{2,N}})$ and the above assumption that $\sup_s \bar{X}_s^{1,N}(1)$ is bounded in probability uniformly in N. For example, it is a non-negative martingale with mean $X_0^{1,N}(1) \to X_0^1(1)$ and so one can apply the weak L^1 inequality for non-negative martingales. It therefore follows from Proposition 2.4 that (suppressing dependence on $\delta > 0$)

$$R_N = \mathcal{H}_{\delta,N} + \sup_s \bar{X}_s^{1,N}(1) \tag{2.20}$$

is also bounded in probability uniformly in N, that is

(2.21) for any $\varepsilon > 0$ there is an $M_\varepsilon > 0$ such that $P(R_N \geq M_\varepsilon) \leq \varepsilon$ for all N.

The next two Sections will deal with the issues of tightness and Tanaka's formula, respectively. In the course of the proofs we will use some technical Lemmas which will be proved in Sections 7 and 8, and will involve a non-decreasing $\sigma(\bar{X}^{1,N})$-measurable process $\bar{R}_N(t,\omega)$ whose definition (value) may change from line to line and which also satisfies

(2.22) for each $t > 0$, $\bar{R}_N(t)$ is bounded in probability uniformly in N.

3 Tightness of the Approximating Systems

It will be convenient in Section 4 to also work with the symmetric collision local time defined by

$$L_t^N(\phi) = 2^{-d}\int_0^t \int_{\mathbb{R}^d\times\mathbb{R}^d} 1\left(|x_1 - x_2| \leq N^{-1/2}\right) N^{d/2}\phi((x_1+x_2)/2) X_s^{1,N}(dx_1) X_s^{2,N}(dx_2)\, ds.$$

This section is devoted to the proof of the following proposition.

Proposition 3.1 *Let* $\gamma_1 \geq 0, \gamma_2 \in \mathbb{R}$ *and* $\{(X^{1,N}, X^{2,N}) : N \in \mathbb{N}\}$ *be as in (2.6) with* $\{X_0^{i,N},\ N \in \mathbb{N}\}$, $i = 1,2$, *satisfying Assumption 2.5. Then* $\{(X^{1,N}, X^{2,N}, L^{2,N}, L^{1,N}, L^N),\ N \in \mathbb{N}\}$ *is tight on* $D_{\mathcal{M}_F{}^5}$ *and each limit point* $(X^1, X^2, A, A, A) \in C_{\mathcal{M}_F{}^5}$ *and satisfies the following martingale problem* $\mathbf{M}^{-\gamma_1,\gamma_2,A}_{X_0^1,X_0^2}$: *For* $\phi_i \in C_b^2(\mathbb{R}^d)$, $i = 1,2$,

$$\begin{aligned}
X_t^1(\phi_1) &= X_0^1(\phi_1) + M_t^1(\phi_1) + \int_0^t X_s^1(\frac{\sigma^2}{2}\Delta\phi_1)\, ds - \gamma_1 A_t(\phi_1), \\
X_t^2(\phi_2) &= X_0^2(\phi_2) + M_t^2(\phi_2) + \int_0^t X_s^2(\frac{\sigma^2}{2}\Delta\phi_2)\, ds + \gamma_2 A_t(\phi_2)
\end{aligned}$$

where M^i are continuous local martingales such that

$$\langle M^i(\phi_i), M^j(\phi_j)\rangle_t = \delta_{i,j} 2\int_0^t X_s^i(\phi_i^2)\, ds.$$

As has alreeay been noted, the main step will be to establish Proposition 3.1 for the positive colicin process $(\bar{X}^{1,N}, \bar{X}^{2,N})$ which bounds $(X^{1,N}, X^{2,N})$. Recall that this process solves the following martingale problem: For bounded $\phi : \mathcal{Z}_N \to \mathbb{R}$,

$$\bar{X}_t^{1,N}(\phi) = \bar{X}_0^{1,N}(\phi) + \bar{M}_t^{1,N}(\phi) + \int_0^t \bar{X}_s^{1,N}(\Delta^N \phi)\, ds, \tag{3.1}$$

$$\bar{X}_t^{2,N}(\phi) = \bar{X}_0^{2,N}(\phi) + \bar{M}_t^{2,N}(\phi) + \int_0^t \bar{X}_s^{2,N}(\Delta^N \phi)\, ds + \gamma_2^+ \int_0^t \int_{\mathbb{R}^d} \phi(x) \bar{L}^{2,N}(dx, ds), \tag{3.2}$$

where

$$\langle \bar{M}^{1,N}(\phi) \rangle_t = 2\int_0^t \bar{X}_s^{1,N}(\phi^2)\, ds + \int_0^t \int_{\mathbb{R}^d} \left(\sum_{x \in \mathcal{Z}_N} (\phi(x) - \phi(y))^2 p^N(x-y) \right) \bar{X}_s^{1,N}(dy)\, ds \tag{3.3}$$

$$\langle \bar{M}^{2,N}(\phi) \rangle_t = 2\int_0^t \bar{X}_s^{2,N}(\phi^2)\, ds + \int_0^t \int_{\mathbb{R}^d} \left(\sum_{y \in \mathcal{Z}_N} (\phi(y) - \phi(x))^2 p^N(x-y) \right) \bar{X}_s^{2,N}(dx)\, ds + \frac{\gamma_2^+}{N} \bar{L}_t^{2,N}\left(\phi^2\right) \tag{3.4}$$

Proposition 3.2 *The sequence $\{\bar{X}^{1,N},\ N \geq 1\}$ converges weakly in $D_{\mathcal{M}_F}$ to super-Brownian motion with parameters $b = 2$, σ^2, and $\beta = 0$.*

Proof This result is standard in the super-Brownian motion theory, see e.g. Theorem 15.1 of Cox, Durrett, and Perkins (1999).

■

Most of the rest of this section is devoted to the proof of the following proposition.

Proposition 3.3 *The sequence $\{\bar{X}^{2,N},\ N \geq 1\}$ is tight in $D_{\mathcal{M}_F}$ and each limit point is supported by $C_{\mathcal{M}_F}$.*

Recall the following lemma (Lemmas 2.7, 2.8 of Durrett and Perkins (1999)) which gives conditions for tightness of a sequence of measure-valued processes.

Lemma 3.4 *Let $\{P^N\}$ be a sequence of probabilities on $D_{\mathcal{M}_F}$ and let Y_t denote the coordinate variables on $D_{\mathcal{M}_F}$. Assume $\Phi \subset C_b(\mathbb{R}^d)$ be a separating class which is closed under addition.*

(a) *$\{P^N\}$ is tight in $D_{\mathcal{M}_F}$ if and only if the following conditions holds.*

(i) *For each $T, \epsilon > 0$, there is a compact set $K_{T,\epsilon} \subset \mathbb{R}^d$ such that*

$$\limsup_N P^N \left(\sup_{t \leq T} Y_t(K_{T,\epsilon}^c) > \epsilon \right) < \epsilon.$$

(ii) $\lim_{M\to\infty}\sup_N P^N(\sup_{t\le T} Y_t(1) > M) = 0$.

(iii) *If* $P^{N,\phi}(A) \equiv P^N(Y_\cdot(\phi)\in A)$, *then for each* $\phi\in\Phi$, $\{P^{N,\phi}, N\ge 1\}$ *is tight in* $D_{\mathbb{R}}$.

(b) *If* P^N *satisfies* **(i), (ii), and (iii)**, *and for each* $\phi\in\Phi$ *all limit points of* $P^{N,\phi}$ *are supported by* $C_{\mathbb{R}}$, *then* P^N *is tight in* $D_{\mathcal{M}_F}$ *and all limit points are supported on* $C_{\mathcal{M}_F}$.

Notation. We choose the following constants:

$$0<\delta<\tilde\delta<1/6, \tag{3.5}$$

and define

$$\begin{aligned} l_d &\equiv (d/2-1)^+ ,\\ \hat\varrho_\delta^N(\mu) &\equiv \varrho_\delta^N(\mu)+\mu(1) . \end{aligned} \tag{3.6}$$

By Proposition 2.4

$$\sup_{x,t}\bar X_t^{1,N}(B(x,r))\le \mathcal{H}_{\delta,N}r^{(2\wedge d)-\delta},\ \ \forall r\in[1/\sqrt{N},1],$$

and hence for $\phi:\mathbb{R}_+\times\mathcal{Z}_N\to[0,\infty)$,

$$\frac{1}{N}\bar L_t^{2,N}(\phi) \ \le\ \mathcal{H}_{\delta,N}N^{-1+l_d+\delta/2}2^{-d}\int_0^t\int_{\mathbb{R}^d}\phi(x)\bar X_s^{2,N}(dx)\,ds . \tag{3.7}$$

Recall our convention with respect to $\bar R_N(t,\omega)$ from the end of Section 2. The proof of the following bound on the semigroup $P_{s,t}^{g_N}$ is deferred until Section 7.

Proposition 3.5 *Let* $\phi:\mathcal{Z}_N\to[0,\infty)$. *Then for* $0\le s<t$

(a)

$$\begin{aligned}\int P_{s,t}^{g_N}(\phi)(x_1)\mu^N(dx_1) &\le \int \Pi_{s,x_1}^N\left[\phi(B_t^N)\right]\mu^N(dx_1)\\ &\quad+\hat\varrho_\delta^N(\mu^N)\bar R_N(t)\int_s^t(s_n-s)^{-l_d-\tilde\delta}\int\sup_{|z_n|\le N^{-1/2}}\Pi_{s_n,y_n+z_n}^N\left[\phi(B_t^N)\right]\bar X_{s_n}^{1,N}(dy_n)\,ds_n .\end{aligned}$$

(b)

$$P_{s,t}^{g_N}(\phi)(x_1)\ \le\ \|\phi\|_\infty\bar R_N(t),\ x_1\in\mathcal{Z}_N .$$

As simple consequences of the above we have the following bounds on the conditional mean measures of $\bar X^{2,N}$.

Corollary 3.6 *If* $\phi:\mathcal{Z}_N\to[0,\infty)$, *then*

(a)

$$\begin{aligned}
&E\left[\bar{X}_t^{2,N}(\phi)|\bar{X}^{1,N}\right] \\
&\le \int \Pi_{x_1}^N\left[\phi(B_t^N)\right]\bar{X}_0^{2,N}(dx_1) \\
&\quad + \hat{\rho}_\delta(\bar{X}_0^{2,N})\bar{R}_N(t)\int_0^t s_n^{-\tilde{\delta}-l_d}\int \sup_{|z_n|\le N^{-1/2}} \Pi_{y_n+z_n}^N\left[\phi(B_{t-s_n}^N)\right]\bar{X}_{s_n}^{1,N}(dy_n)\,ds_n\,.
\end{aligned}$$

(b)

$$E\left[\bar{X}_t^{2,N}(\phi)|\bar{X}^{1,N}\right] \le \|\phi\|_\infty \bar{X}_0^{2,N}(1)\bar{R}_N(t).$$

Proof Immediate from (2.17) and Proposition 3.5. ∎

The next lemma gives a bound for a particular test function ϕ and is essential for bounding the first moments of the approximate local times.

Lemma 3.7 **(a)**

$$\begin{aligned}
&\int_{\mathcal{Z}_N} P_{0,t}^{g_N}\left(N^{d/2}1\left(|\cdot - y|\le 1/\sqrt{N}\right)\right)(x_1)\mu^N(dx_1) \\
&\le \hat{\varrho}_\delta^N(\mu^N)\bar{R}_N(t)t^{-l_d-\tilde{\delta}},\ \ \forall t>0, y\in\mathcal{Z}_N,\ N\ge 1.
\end{aligned}$$

(b)

$$\begin{aligned}
&\int_{\mathcal{Z}_N} P_{0,t}^{g_N}\left(N^{d/2}1\left(|\cdot - y|\le 1/\sqrt{N}\right)\right)(x_1)\mu^N(dy) \\
&\le \hat{\varrho}_\delta^N(\mu^N)\bar{R}_N(t)t^{-l_d-\tilde{\delta}},\ \ \forall t>0, x_1\in\mathcal{Z}_N,\ N\ge 1.
\end{aligned}$$

Proof Deferred to Section 7.

Lemma 3.8 *For any $\epsilon>0$, $T>0$, there exist r_1 such that*

$$P\left(\sup_{t\le T} E\left[\bar{X}_t^{2,N}(\mathbf{B}(0,r)^c)|\bar{X}^{1,N}\right]\le \epsilon^3\right)\ge 1-\epsilon_1,\ \forall N,\ \forall r\ge r_1. \tag{3.8}$$

Proof Corollary 3.6(a) implies

$$\begin{aligned}
&E\left[\bar{X}_t^{2,N}(\mathbf{B}(0,r)^c)|\bar{X}^{1,N}\right] \\
&\le \int \Pi_{x_1}^N\left(\left|B_t^N\right|>r\right)\bar{X}_0^{2,N}(dx_1) \\
&\quad + \hat{\rho}_\delta(\bar{X}_0^{2,N})\bar{R}_N(t)\int_0^t s_n^{-\tilde{\delta}-l_d}\int \sup_{|z_n|\le N^{-1/2}} \Pi_{s_n,y_n+z_n}^N\left(\left|B_t^N\right|>r\right)\bar{X}_{s_n}^{1,N}(dy_n)\,ds_n \\
&\equiv I_t^{1,N}(r)+I_t^{2,N}(r)\,.
\end{aligned}$$

Now we have

$$I_t^{1,N}(r) \quad \leq \quad \bar{X}_0^{2,N}(\mathbf{B}(0,r/2)^c) + \bar{X}_0^{2,N}(1)\Pi_0^N\left(\left|B_t^N\right| > r/2\right).$$

Clearly,

$$\text{For any compact } K \subset \mathbb{R}^d, \ \left\{\Pi_y^N : \, y \in K, \, N \geq 1\right\} \text{ is tight on } D_{\mathbb{R}^d}. \tag{3.9}$$

By (3.9) and Assumption 2.5 we get that for all r sufficiently large and all $N \in \mathbb{N}$,

$$\sup_{t \leq T} I_t^{1,N}(r) \leq \frac{1}{2}\epsilon^3. \tag{3.10}$$

Arguing in a similar manner for $I^{2,N}$, we get

$$\begin{aligned} \sup_{t\leq T} I_t^{2,N}(r) \quad &\leq \quad \hat{\rho}_\delta(\bar{X}_0^{2,N})\bar{R}_N(T)\int_0^T s^{-\tilde{\delta}-l_d}\,ds \left(\sup_{s\leq T}\bar{X}_s^{1,N}(\mathbf{B}(0,r/2)^c)\right. \\ &\left. + \sup_{s\leq T}\bar{X}_s^{1,N}(1)\Pi_0^N\left(\left|B_t^N\right| > r/2 - N^{-1/2}\right)\right). \end{aligned}$$

Again, by (3.9), our assumptions on $\{\bar{X}_0^{2,N}, \ N \geq 1\}$ and tightness of $\{\bar{R}_N(T), \ N \geq 1\}$ and $\{\bar{X}^{1,N}, \ N \geq 1\}$ we get that for all r sufficiently large and all N,

$$P\left(\sup_{t\leq T} I_t^{2,N}(r) \leq \frac{1}{2}\epsilon^3\right) \geq 1 - \epsilon_1, \tag{3.11}$$

and we are done.

■

Lemma 3.9 *For any $\epsilon, \epsilon_1 > 0$, $T > 0$, there exists r_1 such that*

$$P\left(E\left[\bar{L}_T^{2,N}\left(\mathbf{B}(0,r)^c\right)|\bar{X}^{1,N}\right] \leq \epsilon^2\right) \geq 1 - \epsilon_1, \ \forall N \in \mathbb{N}, \ \forall r \geq r_1. \tag{3.12}$$

Proof

$$\begin{aligned} &E\left[\bar{L}_T^{2,N}\left(\mathbf{B}(0,r)^c\right)|\bar{X}^{1,N}\right] \\ &\quad\leq \quad 2^{-d}\int_0^T\int_{z\in\mathbb{R}^d} E\left[N^{d/2}\int_{|x|\geq r} 1\left(|x-z| \leq N^{-1/2}\right)\bar{X}_s^{2,N}(dx)|\bar{X}^{1,N}\right]\bar{X}_s^{1,N}(dz)\,ds \\ &\quad= \quad 2^{-d}\int_0^T\int_{|z|\geq r-N^{-1/2}}\int_{\mathcal{Z}_N} P_{0,s}^{g_N}\left(N^{d/2}1\left(|\cdot - z| \leq 1/\sqrt{N}\right)\right)(x_1)\bar{X}_0^{2,N}(dx_1)\bar{X}_s^{1,N}(dz)\,ds \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by (2.17))} \\ &\quad\leq \quad \bar{R}_N(T)\hat{\varrho}_\delta^N(\bar{X}_0^{2,N})\sup_{s\leq T}\bar{X}_s^{1,N}\left(\mathbf{B}(0,r-N^{-1/2})^c\right)\int_0^T s^{-l_d-\tilde{\delta}}\,ds \qquad \text{(by Lemma 3.7}(a)) \\ &\quad= \quad \bar{R}_N(T)\hat{\varrho}_\delta^N(\bar{X}_0^{2,N})\sup_{s\leq T}\bar{X}_s^{1,N}\left(\mathbf{B}(0,r-N^{-1/2})^c\right). \end{aligned}$$

Now recalling Assumption 2.5 and the tightness of $\{\bar{R}_N(T),\ N \geq 1\}$ and $\{\bar{X}^{1,N},\ N \geq 1\}$, we complete the proof as in Lemma 3.8.

■

Notation For any $r > 1$ let $f_r : \mathbb{R}^d \mapsto [0,1]$ be a C^∞ function such that

$$\mathbf{B}(0,r) \subset \{x :\ f_r(x) = 0\} \subset \{x :\ f_r(x) < 1\} \subset \mathbf{B}(0,r+1)$$

Lemma 3.10 *For each $T, \epsilon > 0$, there is an $r = r_2 > 0$ sufficiently large, such that*

$$\limsup_{N\to\infty} P\left(\sup_{t\leq T} \bar{X}^{2,N}_t(\mathbf{B}(0,r_2)^c) > \epsilon\right) \leq \epsilon. \tag{3.13}$$

Proof Apply Chebychev's inequality on each term of the martingale problem (3.2) for $\bar{X}^{2,N}$ and then Doob's inequality to get

$$\begin{aligned} & P\left(\sup_{t\leq T} \bar{X}^{2,N}_t(f_r) > \epsilon | \bar{X}^{1,N}\right) \\ \leq\ & \Big\{1\left(\bar{X}^{2,N}_0(f_r) > \epsilon/4\right) + \frac{c}{\epsilon^2} E\left[\langle \bar{M}^{2,N}_\cdot(f_r)\rangle_T | \bar{X}^{1,N}\right] \\ & + \frac{c}{\epsilon}\int_0^T E\left[\bar{X}^{2,N}_s(\mathbf{B}(0,r)^c)|\bar{X}^{1,N}\right]\, ds \\ & + \frac{c}{\epsilon}\gamma E\left[\int_0^T \int_{\mathbb{R}^d} f_r(x)\bar{L}^{2,N}(dx,ds)|\bar{X}^{1,N}\right]\Big\} \wedge 1 \end{aligned} \tag{3.14}$$

Hence by tightness of $\{\bar{X}^{2,N}_0\}$, (3.4) and Lemmas 3.8, 3.9 we may take r sufficiently large such that the right-hand side of (3.14) is less than $\epsilon/2$ with probability at least $1 - \epsilon/2$ for all N. This completes the proof.

■

Lemma 3.11 *For any $\phi \in B_{b,+}(\mathbb{R}^d),\ t > 0$,*

$$E\left[\bar{L}^{2,N}_t(\phi)\,|\bar{X}^{1,N}\right] \ \leq\ \|\phi\|_\infty\, \bar{R}_N(t)\bar{X}^{2,N}_0(1)\int_0^t s^{-l_d-\tilde{\delta}}\, ds. \tag{3.15}$$

Proof By (2.17),

$$\begin{aligned} & E\left[\bar{L}^{2,N}_t(\phi)\,|\bar{X}^{1,N}\right] \\ \leq\ & \|\phi\|_\infty\, 2^{-d}\int_0^t \int_{x_1\in\mathbb{R}^d}\left(\int_{z\in\mathbb{R}^d} P^{g_N}_{0,t}\left(N^{d/2}1\left(|\cdot - z| \leq 1/\sqrt{N}\right)\right)(x_1)\bar{X}^{1,N}_s(dz)\right)\bar{X}^{2,N}_0(dx_1)\, ds \\ \leq\ & \|\phi\|_\infty \sup_{s\leq t}\hat{\varrho}^N_{\tilde{\delta}}(\bar{X}^{1,N}_s)\bar{X}^{2,N}_0(1)\int_0^t s^{-l_d-\tilde{\delta}}\, ds, \end{aligned}$$

where the last inequality follows by Lemma 3.7(b) with $\mu^N = \bar{X}^{1,N}_s$. As we may assume

$$\sup_{s\le t} \hat{\varrho}^N_\delta(\bar{X}^{1,N}_s) \le \bar{R}_N(t)$$

(the left-hand side is bounded in probability uniformly in N by Propostions 2.4 and 3.2), the result follows.

■

Lemma 3.12 *For any $T > 0$,*

$$\lim_{K\to\infty} \sup_N P(\sup_{t\le T} \bar{X}^{2,N}_t(1) > K) = 0.$$

Proof Applying Chebychev's inequality on each term of the martingale problem (3.2) for $\bar{X}^{2,N}$ and then Doob's inequality, one sees that

$$(3.16)\ \ P\left(\sup_{t\le T} \bar{X}^{2,N}_t(1) > K|\bar{X}^{1,N}\right) \le \Big\{1\left(\bar{X}^{2,N}_0(1) > K/3\right) + \frac{c}{K^2}E\left[\langle \bar{M}^{2,N}_{\cdot}(1)\rangle_T|\bar{X}^{1,N}\right] + \frac{c}{K}\gamma_2^+ E\left[\bar{L}^{2,N}_T(1)|\bar{X}^{1,N}\right]\Big\} \wedge 1.$$

Now apply Assumption 2.5, (3.4), Lemma 3.11, and Corollary 3.6(b) to finish the proof.

■

Lemma 3.13 *The sequence $\{\bar{L}^{2,N},\ N \ge 1\}$ is tight in $C_{\mathcal{M}_F}$.*

Proof Lemmas 3.9 and 3.11 imply conditions (i) and (ii), respectively in Lemma 3.4. Now let us check (iii). Let $\Phi \subset C_b(\mathbb{R}^d)$ be a separating class of functions. We will argue by Aldous' tightness criterion (see Theorem 6.8 of Walsh (1986)). First by (2.22) and Lemma 3.11 we immediately get that for any $\phi \in \Phi,\ t \ge 0,\ \{\bar{L}^{2,N}_t(\phi) : N \in \mathbb{N}\}$ is tight. Next, let $\{\tau_N\}$ be arbitrary sequence of stopping times bounded by some $T > 0$ and let $\{\epsilon_N,\ N \ge 1\}$ be a sequence such that $\epsilon_N \downarrow 0$ as $N \to \infty$. Then arguing as in Lemma 3.11 it is easy to verify

$$\begin{aligned} &E\left[\bar{L}^{2,N}_{\tau_N+\epsilon_N}(\phi) - \bar{L}^{2,N}_{\tau_N}(\phi)\,|\bar{X}^{1,N},\mathcal{F}_{\tau_N}\right] \\ &\le \|\phi\|_\infty \bar{R}_N(T)\bar{X}^{2,N}_{\tau_N}(1)\int_{\tau_N}^{\tau_N+\epsilon_N}(s-\tau_N)^{-l_d-\tilde{\delta}}\,ds, \end{aligned}$$

Then by (2.22) and Lemma 3.12 we immediately get that

$$\left|\bar{L}^{2,N}_{\tau_N+\epsilon_N}(\phi) - \bar{L}^{2,N}_{\tau_N}(\phi)\right| \to 0$$

in probability as $N \to \infty$. Hence by Aldous' criterion for tightness we get that $\{\bar{L}^{2,N}(\phi)\}$ is tight in $D_{\mathbb{R}}$ for any $\phi \in \Phi$. Note $\bar{L}^{2,N}(\phi) \in C_{\mathbb{R}}$ for all N, and so $\{\bar{L}^{2,N}(\phi)\}$ is tight in $C_{\mathbb{R}}$, and we are done.

■

The next lemma will be used for the proof of Proposition 3.1. The processes $X^{i,N}$, $L^{i,N}$ and L^N are all as in that result.

Lemma 3.14 *The sequences $\{L^{i,N},\ N\geq 1\}$, $i=1,2$, and $\{L^{N},\ N\geq 1\}$ are tight in $C_{\mathcal{M}_F}$, and moreover for any uniformly continuous function ϕ on $\mathbb{R}^d$ and $T>0$*

$$\sup_{t\leq T}\left|L^{i,N}_t(\phi)-L^{N}_t(\phi)\right|\to 0,\ \text{in probability as } N\to\infty,\ i=1,2. \tag{3.17}$$

Proof First, by Proposition 2.2

$$L^{2,N}_t\leq \bar{L}^{2,N}_t,\ \ L^{2,N}_t-L^{2,N}_s\leq \bar{L}^{2,N}_t-\bar{L}^{2,N}_s,\ \ \forall 0\leq s\leq t, \tag{3.18}$$

where $\bar{L}^{2,N}$ is the approximating collision local time for the $(\bar{X}^{1,N},\bar{X}^{2,N})$ solving $\mathbf{M}^{N,0,\gamma_2^+}_{X^{1,N}_0,X^{2,N}_0}$. By Lemma 3.13 $\bar{L}^{2,N}_t$ is tight in $C_{\mathcal{M}_F}$, and hence, by (3.18), $L^{2,N}$ is tight as well (see the proof of Lemma 3.13).

To finish the proof it is enough to show that for any uniformly continuous function ϕ on $\mathbb{R}^d$ and $T>0$

$$\sup_{t\leq T}\left|L^{1,N}_t(\phi)-L^{2,N}_t(\phi)\right|\to 0,\ \text{as } N\to\infty,\ \ \text{in probability}, \tag{3.19}$$

$$\sup_{t\leq T}\left|L^{N}_t(\phi)-L^{2,N}_t(\phi)\right|\to 0,\ \text{as } N\to\infty,\ \ \text{in probability}. \tag{3.20}$$

We will check only (3.19), since the proof of (3.20) goes along the same lines. By trivial calculations we get

$$\begin{aligned}\sup_{t\leq T}\left|L^{1,N}_t(\phi)-L^{2,N}_t(\phi)\right| &\leq \sup_{|x-y|\leq N^{-1/2}}|\phi(x)-\phi(y)|\,L^{2,N}_T(1),\\ &\leq \sup_{|x-y|\leq N^{-1/2}}|\phi(x)-\phi(y)|\,\bar{L}^{2,N}_T(1),\ \forall N\geq 1,\end{aligned}$$

where the last inequality follows by (3.18). The result follows by the uniform continuity assumption on ϕ and Lemma 3.13. ∎

Now we are ready to present the

Proof of Proposition 3.3 We will check conditions (i)–(iii) of Lemma 3.4. By Lemmas 3.10 and 3.12, conditions (i) and (ii) of Lemma 3.4 are satisfied. Turning to (iii), fix a $\phi\in C_b^3(\mathbb{R}^d)$. Then using the Aldous criterion for tightness along with Lemma 3.12 and (2.11), and arguing as in Lemma 3.13, it is easy to verify that $\{\int_0^\cdot \bar{X}^{2,N}_s(\Delta^N\phi)\,ds\}$ is a tight sequence of processes in $C_{\mathbb{R}}$. By Lemma 3.13 and the uniform convergence of $P^N\phi$ to ϕ we also see that $\{\gamma_2^+\bar{L}^{2,N}_\cdot(P^N\phi)\}$ is a tight sequence of processes in $C_{\mathbb{R}}$.

Turning now to the local martingale term in (3.2), arguing as above, now using $|\nabla\phi|^2\leq C_\phi<\infty$ and Lemma 3.13 as well, we see from (3.4) that $\{\langle\bar{M}^{2,N}(\phi)\rangle_\cdot\,,N\geq 1\}$ is a tight sequence of processes in $C_{\mathbb{R}}$. Note also that by definition,

$$\sup_{t\leq T}\left|\Delta\bar{M}^{2,N}_t(\phi)\right|\leq 2\left\|\phi\right\|_\infty N^{-1}.$$

Theorem VI.4.13 and Proposition VI.3.26 of Jacod and Shiryaev Jacod and Shiryaev (1987) show that $\{\bar{M}^{2,N}_t(\phi)\,,N\geq 1\}$ is a tight sequence in $D_{\mathbb{R}}$ and all limit point are supported in $C_{\mathbb{R}}$. The above results with (3.2) and Corollary VI.3.33 of Jacod and Shiryaev Jacod and Shiryaev (1987) show that $\bar{X}^{2,N}(\phi)$ is tight in $D_{\mathbb{R}}$ and all the limit points are supported in $C_{\mathbb{R}}$. Lemma 3.4(b) now completes the proof.

∎

Proof of Proposition 3.1 Arguing as in the proof of Proposition 3.3 and using Proposition 2.2 and Lemma 3.14, we can easily show that $\{(X^{1,N}, X^{2,N}, L^{2,N}, L^{1,N}, L^N),\ N \geq 1\}$ is tight on $D_{\mathcal{M}_F{}^5}$, and any limit point belongs to $C_{\mathcal{M}_F{}^5}$. Let $\{(X^{1,N_k}, X^{2,N_k}, L^{2,N_k}, L^{1,N_k}, L^{N_k}),\ k \geq 1\}$ be any convergent subsequence of $\{(X^{1,N}, X^{2,N}, L^{2,N}, L^{1,N}, L^N),\ N \geq 1\}$. By Lemma 3.14, if (X^1, X^2, A) is the limit of $\{(X^{1,N_k}, X^{2,N_k}, L^{2,N_k}),\ k \geq 1\}$, then

$$(X^{1,N_k}, X^{2,N_k}, L^{2,N_k}, L^{1,N_k}, L^{N_k}) \Rightarrow (X^1, X^2, A, A, A), \tag{3.21}$$

as $k \to \infty$. By Skorohod's theorem, we may assume that convergence in (3.21) is a.s. in $D_{\mathcal{M}_F{}^5}$ to a continuous limit. To complete the proof we need to show that (X^1, X^2) satisfies the martingale problem $\mathbf{M}^{-\gamma_1,\gamma_2,A}_{X_0^1,X_0^2}$. Let $\phi_i \in C_b^3(\mathbb{R}^d)$, $i = 1, 2$. Recalling from (2.11), that

$$\Delta^N \phi_i \to \frac{\sigma^2}{2}\Delta\phi_i \ \text{ uniformly on } \mathbb{R}^d, \tag{3.22}$$

we see that all the terms in $\mathbf{M}^{N_k,-\gamma_1,\gamma_2}_{X_0^{1,N_k},X_0^{2,N_k}}$ converge to the corresponding terms in $\mathbf{M}^{-\gamma_1,\gamma_2,A}_{X_0^1,X_0^2}$, except perhaps the local martingale terms. By convergence of the other terms in $\mathbf{M}^{N_k,\gamma_1,\gamma_2}_{X_0^{1,N_k},X_0^{2,N_k}}$ we see that

$$M_t^{i,N_k}(\phi_i) \to M_t^i(\phi_i) = X_t^i(\phi_i) - X_0^i(\phi_i) - \int_0^t X_s^i(\frac{\sigma^2\Delta\phi_i}{2})\, ds - (-1)^i\gamma_i A_t(\phi_i) \text{ a.s. in } D_{\mathbb{R}},\ i = 1, 2.$$

These local martingales have jumps bounded by $\frac{2}{N_k}\|\phi_i\|_\infty$, and square functions which are bounded in probability uniformly in N_k by Proposition 3.2 and Lemma 3.12. Therefore they are locally bounded using stopping times $\{T_n^{N_k}\}$ which become large in probability as $n \to \infty$ uniformly in N_k. One can now proceed in a standard manner (see, e.g. the proofs of Lemma 2.10 and Proposition 2 in Durrett and Perkins (1999)) to show that $M^i(\phi)$ have the local martingale property and square functions claimed in $\mathbf{M}^{-\gamma_1,\gamma_2,A}_{X_0^1,X_0^2}$. Finally we need to increase the class of test functions from C_b^3 to C_b^2. For $\phi_i \in C_b^2$ apply the martingale problem with $P_\delta\phi_i$ (P_δ is the Brownian semigroup) and let $\delta \to 0$. As $P_\delta\Delta\phi_i \to \Delta\phi_i$ in the bounded pointwise sense, we do get $\mathbf{M}^{-\gamma_1,\gamma_2,A}_{X_0^1,X_0^2}$ for ϕ_i in the limit and so the proof is complete. ∎

4 Convergence of the approximating Tanaka formulae

Define $K_N = N^{d/2}(M + \frac{1}{2})^d$. Then for any $\phi : \mathcal{Z} \to \mathbb{R}$ bounded or non-negative define

$$G_N^\alpha\phi(x_1\,, x_2) = \Pi_{x_1}^N \times \Pi_{x_2}^N \left[\int_0^\infty e^{-\alpha s} K_N p^N(B_s^{1,N} - B_s^{2,N})\phi\left(\frac{B_s^{1,N} + B_s^{2,N}}{2}\right) ds\right], \tag{4.1}$$

where $\alpha \geq 0$ for $d = 3$ and $\alpha > 0$ for $d \leq 2$. These conditions on α will be implicitly assumed in what follows. Note that for any bounded ϕ we have

$$\begin{aligned} G_N^\alpha \phi(x_1\,, x_2) &\leq \|\phi\|_\infty \,\Pi_{x_1}^N \times \Pi_{x_2}^N \left[\int_0^\infty e^{-\alpha s} K_N p^N(B_s^{1,N} - B_s^{2,N})\,ds\right] \\ &\equiv \|\phi\|_\infty \,G_N^\alpha \mathbf{1}(x_1\,, x_2) \\ &= \|\phi\|_\infty \,N^{d/2} 2^{-d} \sum_{|z|\leq N^{-1/2}} \int_0^\infty e^{-\alpha s} \mathrm{p}_{2s}^N(x_1 - x_2 - z)\,ds, \end{aligned} \tag{4.2}$$

where $\mathrm{p}_\cdot^N$ is the transition probability function of the continuous time random walk B^N with generator Δ^N.

For $0 < \epsilon < 1$, define

$$\begin{aligned} \psi_N^\epsilon(x_1\,, x_2) &\equiv G_N^\alpha \mathbf{1}(x_1\,, x_2) 1(|x_1 - x_2| \leq \epsilon), \\ h_d(t) &\equiv \begin{cases} 1, & \text{if } d = 1, \\ 1 + \ln_+(1/t), & \text{if } d = 2, \\ t^{1-d/2}, & \text{if } d = 3. \end{cases} \end{aligned} \tag{4.3}$$

Let $(X^{1,N}, X^{2,N})$ be as in (2.6) as usual. Recall

$$\begin{aligned} L_t^{2,N}(\phi) &= 2^{-d} \int_0^t \int_{\mathbb{R}^d} \phi(x) N^{d/2} X_s^{1,N}(\mathbf{B}(x, N^{-1/2})) X_s^{2,N}(dx)\,ds, \\ L_t^{1,N}(\phi) &= 2^{-d} \int_0^t \int_{\mathbb{R}^d} \phi(x) N^{d/2} X_s^{2,N}(\mathbf{B}(x, N^{-1/2})) X_s^{1,N}(dx)\,ds, \\ L_t^{N}(\phi) &= 2^{-d} \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} 1\left(|x_1 - x_2| \leq N^{-1/2}\right) N^{d/2} \phi((x_1 + x_2)/2) X_s^{1,N}(dx_1) X_s^{2,N}(dx_2)\,ds. \end{aligned}$$

We introduce

$$\mathbf{X}_t^N = X_t^{1,N} \times X_t^{2,N}\,, \quad \forall t \geq 0. \tag{4.4}$$

Then arguing as in Lemma 5.2 of Barlow, Evans, and Perkins (1991) where an Ito's formula for a pair of interacting super-Brownian motions was derived, we can easily verify the following approximate Tanaka formula for $\phi : \mathcal{Z} \to \mathbb{R}$ bounded:

$$\begin{aligned} \mathbf{X}_t^N(G_N^\alpha \phi) &= \mathbf{X}_0^N(G_N^\alpha \phi) - \gamma_1 \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G_N^\alpha \phi(x_1, x_2) X_s^{2,N}(dx_2)\, L^{1,N}(ds, dx_1) \\ &\quad + \gamma_2 \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G_N^\alpha \phi(x_1, x_2) X_s^{1,N}(dx_1)\, L^{2,N}(ds, dx_2) \\ &\quad + \alpha \int_0^t \mathbf{X}_s^N(G_N^\alpha \phi)\,ds \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G_N^\alpha \phi(x_1, x_2) \left(X_s^{1,N}(dx_1)\, M^{2,N}(ds, dx_2) + X_s^{2,N}(dx_2)\, M^{1,N}(ds, dx_1)\right) \\ &\quad - L_t^N(\phi), \end{aligned} \tag{4.5}$$

where $M^{i,N},\ i=1,2$, are the martingale measures in $\mathbf{M}^{-\gamma_1,\gamma_2,A}_{X_0^1,X_0^2}$. Let $(X^1,X^2,A,A,A)\in C_{\mathcal{M}_F{}^5}$ be an arbitrary limit point of $(X^{1,N},X^{2,N},L^{1,N},L^{2,N},L^N)$ (they exist by Proposition 3.1)), and to simplify the notation we assume

$$(X^{1,N},X^{2,N},L^{1,N},L^{2,N},L^N)\Rightarrow(X^1,X^2,A,A,A),$$

as $N\to\infty$. Moreover, throughout this section, by the Skorohod representation theorem, we may assume that

$$(X^{1,N},X^{2,N},L^{1,N},L^{2,N},L^N)\to(X^1,X^2,A,A,A),\ \text{ in } (D_{\mathcal{M}_F{}^5}),\ P-a.s. \tag{4.6}$$

Recall that $\mathrm{p}_t(\cdot)$ is transition density of the Brownian motion B with generator $\frac{\sigma^2}{2}\Delta$. Let Π_x be the law of B with $B_0=x$ and denote its semigroup by P_t. If $\phi:\mathbb{R}^d\to\mathbb{R}$ is Borel and bounded define

$$\begin{aligned} G^\alpha\phi(x_1\,,x_2) &\equiv \lim_{\epsilon\downarrow 0}\Pi_{x_1}\times\Pi_{x_2}\left[\int_0^\infty e^{-\alpha s}\mathrm{p}_\epsilon(B_s^1-B_s^2)\phi\left(\frac{B_s^1+B_s^2}{2}\right)ds\right] \\ &= \int_0^\infty e^{-\alpha s}p_{2s}(x_1-x_2)P_{s/2}\phi\Big(\frac{x_1+x_2}{2}\Big)\,ds. \end{aligned}$$

A change of variables shows this agrees with the definition of $G^\alpha\phi$ in Section 5 of Barlow-Evans and Perkins (1991) and so is finite (and the above limit exists) for all $x_1\neq x_2$, and all (x_1,x_2) if $d=1$. For $\phi\equiv 1$, $G^\alpha 1(x_1,x_2)$ is bounded by $c(1+\log^+(1/|x_1-x_2|))$ if $d=2$ and $c|x_1-x^2|^{-1}$ if $d=3$ (see (5.5)–(5.7) of Barlow, Evans and Perkins (1991)).

In this section we intend to prove the following proposition.

Proposition 4.1 *Let (X^1,X^2,A) be an arbitrary limiting point described above. Then for $\phi\in C_b(\mathbb{R}^d)$,*

$$\begin{aligned} \mathbf{X}_t(G^\alpha\phi) &= \mathbf{X}_0(G^\alpha\phi)-\gamma_1\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}G^\alpha\phi(x_1,x_2)X_s^2(dx_2)\,A(ds,dx_1) \\ &\quad +\gamma_2\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}G^\alpha\phi(x_1,x_2)X_s^1(dx_1)\,A(ds,dx_2) \\ &\quad +\alpha\int_0^t\mathbf{X}_s(G^\alpha\phi)\,ds+\tilde{M}_t(\phi)-A_t(\phi), \end{aligned} \tag{4.7}$$

where $\tilde{M}_t(\phi)$ is a continuous $\mathcal{F}_t^{\mathbf{X},A}$-local martingale.

To verify the proposition we will establish the convergence of all the terms in (4.5) through a series of lemmas.

If $\mu\in\mathcal{M}_F(\mathcal{Z}_N)$, $\mu*p^N(dx)$ denotes the convolution measure on $\mathcal{Z}_N$. The proof of the following lemma is trivial and hence is omitted.

Lemma 4.2 *If $\mu\in\mathcal{M}_F(\mathcal{Z}_N)$ then*

$$\sup_x \mu*p^N(\mathbf{B}(x,r))\leq\varrho_\delta(\mu)r^{(2\wedge d)-\delta},\ \ \forall r\in[0,1].$$

Now let us formulate a number of helpful results whose proofs are deferred to Section 8.

Lemma 4.3 *For* $0 < \epsilon < 1$,

$$\sup_{x_1} \int_{\mathcal{Z}_N} \psi_N^\epsilon(x\,,x_1)\mu^N(dx) \quad\leq\quad c\hat{\varrho}_\delta^N(\mu^N)\epsilon^{1-l_d-\tilde{\delta}}\,,\ \ \forall N \geq \epsilon^{-2}\,. \tag{4.8}$$

Lemma 4.4 *If* $0 < \eta < 2/7$, *then for all* $0 < \epsilon < 1/2$,

$$\begin{aligned} &E\left[\int_0^t \int_{\mathcal{Z}_N^2} \psi_N^\epsilon(x_1,x)\bar{X}_s^{2,N}(dx_1)\bar{L}^{1,N}(ds,dx)\,ds|\bar{X}^{1,N}\right] \\ &\quad\leq\ \ \bar{R}_N(t)\hat{\varrho}_\delta^N(\bar{X}_0^{2,N})^2\epsilon^{\eta(1-l_d-3\tilde{\delta})}t^{1-l_d-\tilde{\delta}}\,,\ \ \forall t>0,\ N\geq\epsilon^{-2}. \end{aligned}$$

Define

$$q^N(x) = 1_{\mathbf{0}}(x) + p^N(x). \tag{4.9}$$

Lemma 4.5 *If* $0 < \eta < 2/7$, *then for all* $0 < \epsilon < 1/2$,

$$\begin{aligned} &\int_{\mathbb{R}^d} E\left[\int_{\mathbb{R}^d} \psi_N^\epsilon(x,x_1)\bar{X}_t^{2,N}(dx_1)\int_{\mathbb{R}^d} G_N^\alpha\mathbf{1}(x\,,x_2)\bar{X}_t^{2,N}(dx_2)|\bar{X}^{1,N}\right]\left(\bar{X}_t^{1,N} * q^N\right)(dx) \\ &\quad\leq\ \ \bar{R}_N(t)[\hat{\varrho}_\delta^N(\bar{X}_0^{2,N})^2+1]\epsilon^{\eta(1-l_d-3\tilde{\delta})},\ \ \forall t\geq 0,\ N\geq\epsilon^{-2}. \end{aligned}$$

Now, for any $\hat{\delta} > 0$ define

$$\begin{aligned} G_N^{\alpha,\hat{\delta}}\phi(x_1\,,x_2) &\equiv \Pi_{x_1}^N \times \Pi_{x_2}^N\left[\int_{\hat{\delta}}^\infty e^{-\alpha s}K_N p^N(B_s^{1,N}-B_s^{2,N})\phi((B_s^{1,N}+B_s^{2,N})/2)\,ds\right], \\ G^{\alpha,\hat{\delta}}\phi(x_1\,,x_2) &\equiv \int_{\hat{\delta}}^\infty e^{-\alpha s}p_{2s}(x_1-x_2)P_{s/2}\phi\Big(\frac{x_1+x_2}{2}\Big)\,ds. \end{aligned}$$

Unlike $G^\alpha\phi$, $G^{\alpha,\hat{\delta}}\phi$ is bounded on $\mathbb{R}^{2d}$ for bounded Borel $\phi : \mathbb{R}^d \to \mathbb{R}$.

Lemma 4.6 **(a)** *For any* $\phi \in C_b(\mathbb{R}^d)$

$$G_N^\alpha\phi(\cdot,\cdot) \to G^\alpha\phi(\cdot,\cdot),\ \ \text{as } N\to\infty, \tag{4.10}$$

uniformly on the compact subsets of $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x_1\,x_2) : x_1 = x_2\}$.
(b) *For any* $\phi \in C_b(\mathbb{R}^d), \hat{\delta} > 0$,

$$G_N^{\alpha,\hat{\delta}}\phi(\cdot,\cdot) \to G^{\alpha,\hat{\delta}}\phi(\cdot,\cdot),\ \ \text{as } N\to\infty, \tag{4.11}$$

uniformly on the compact subsets of $\mathbb{R}^d \times \mathbb{R}^d$.

Proof: Let $\varepsilon_0 \in (0,1)$. The key step will be to show

$$\lim_{\varepsilon\downarrow 0}\ \sup_{N,|x_1-x_2|\geq\varepsilon_0} \int_0^\varepsilon N^{d/2}\Pi_{x_1}^N\times\Pi_{x_2}^N(|B_s^{1,N}-B_s^{2,N}|\leq N^{-1/2})\,ds = 0. \tag{4.12}$$

Once (4.12) is established we see that the contribution to $G_N^\alpha\phi$ from times $s\le\varepsilon$ is small uniformly for $|x_1-x_2|\ge\varepsilon_0$ and N. A straightforward application of the continuous time local central limit theorem (it is easy to check that (5.2) in Durrett (2004) works in continuous time) and Donsker's Theorem shows that uniformly in $x_1,\ x_2$ in compacts,

$$\begin{aligned}
&\lim_{N\to\infty} G_N^{\alpha,\varepsilon}\phi(x_1,x_2)\\
&\quad=\lim_{N\to\infty}\int_\varepsilon^\infty 2^{-d}N^{d/2}\Pi_{x_1}^N\times\Pi_{x_2}^N\Big(1(|B_s^{1,N}-B_s^{2,N}|\le N^{-1/2})\phi\Big(\frac{B_s^{1,N}+B_s^{2,N}}{2}\Big)\Big)e^{-\alpha s}\,ds\\
&\quad=\int_\varepsilon^\infty p_{2s}(x_1-x_2)P_{s/2}\phi\Big(\frac{x_1+x_2}{2}\Big)e^{-\alpha s}\,ds=G^{\alpha,\varepsilon}\phi(x_1,x_2).
\end{aligned}$$

This immediately implies (b), and, together with (4.12), also gives (a).

It remains to establish (4.12). Assume $|x|\equiv|x_1-x_2|\ge\varepsilon_0$ and let $\{S_j\}$ be as in Lemma 7.1. Then for $N^{-1/2}<\varepsilon_0$, use Lemma 7.1 to obtain

$$\begin{aligned}
&\int_0^\varepsilon N^{d/2}\Pi_{x_1}^N\times\Pi_{x_2}^N(|B_s^{1,N}-B_s^{2,N}|\le N^{-1/2})\,ds\\
&\quad=\int_0^\varepsilon N^{d/2-1}\frac{e^{-2Ns}}{2}\sum_{j=1}^\infty\frac{(2Ns)^j}{j!}P\Big(\frac{S_j}{\sqrt{j}}\in\frac{\sqrt{N}}{\sqrt{j}}x+[-j^{-1/2},j^{-1/2}]^d\Big)2N\,ds\\
&\quad\le\sum_{j=1}^\infty N^{d/2-1}C\exp\{-c((N/j)|x|^2\wedge\sqrt{N}|x|)\}j^{-d/2}\int_0^{2N\varepsilon}e^{-u}\frac{u^j}{j!}\,du. \qquad (4.13)
\end{aligned}$$

Now use Stirling's formula to conclude that for $j\ge1$ and $\varepsilon'=2e\varepsilon$,

$$\int_0^{2N\varepsilon}e^{-u}\frac{u^j}{j!}\,du\le\frac{(2N\varepsilon)^j}{j!}\le\frac{c_0}{\sqrt{j}}\Big(\frac{e2N\varepsilon}{j}\Big)^j\le c_0\Big(\frac{N\varepsilon'}{j}\Big)^j,$$

and so conclude

$$\int_0^{2N\varepsilon}e^{-u}\frac{u^j}{j!}\,du\le c_0\min\Big(1,\Big(\frac{N\varepsilon'}{j}\Big)^j\Big).$$

Use this to bound (4.13) by

$$\begin{aligned}
&C'\Big[N^{d/2-1}e^{-c\sqrt{N}\varepsilon_0}\sum_{1\le j\le\sqrt{N}\varepsilon_0}j^{-d/2}+N^{d/2-1}\sum_{j\ge2\varepsilon'N}2^{-j}\\
&\qquad+N^{-1}\sum_{\sqrt{N}\varepsilon_0<j<2\varepsilon'N}((j+1)/N)^{-d/2}\exp\{-c\varepsilon_0^2/(j/N)\}\Big]\\
&\le C'\Big[N^{d/2}e^{-c\sqrt{N}\varepsilon_0}+N^{d/2-1}2^{-2\varepsilon'N}+\int_0^{2\varepsilon'}u^{-d/2}\exp\{-c\varepsilon_0^2/u\}\,du.\Big]
\end{aligned}$$

Choose $\varepsilon=\varepsilon(\varepsilon_0)$ such that the right-hand side is at most ε_0 for $N\ge N_0(\varepsilon_0)$. By making ε smaller still we can handle the finitely many values of $N\le N_0$ and hence prove (4.12). ∎

Lemma 4.7 *For any $\phi \in C_{b,+}(\mathbb{R}^d)$ and $T>0$,*

$$\sup_{t\le T}\int_{\mathbb{R}^d\times\mathbb{R}^d} G^\alpha\phi(x_1,x_2)\,X_t^1(dx_1)\,X_t^2(dx_2)<\infty,\quad P-\text{a.s.}, \tag{4.14}$$

and

$$\begin{aligned}\sup_{t\le T}\bigg|\int_{\mathbb{R}^d\times\mathbb{R}^d} & G_N^\alpha\phi(x_1,x_2)\,X_t^{1,N}(dx_1)\,X_t^{2,N}(dx_2)\\ &-\int_{\mathbb{R}^d\times\mathbb{R}^d} G^\alpha\phi(x_1,x_2)\,X_t^1(dx_1)\,X_t^2(dx_2)\bigg|\to 0,\end{aligned} \tag{4.15}$$

in probability, as $N\to\infty$. Finally

$$t\to\int_{\mathbb{R}^d\times\mathbb{R}^d} G^\alpha\phi(x_1,x_2)\,X_t^1(dx_1)\,X_t^2(dx_2)\ \textit{is a.s. continuous.} \tag{4.16}$$

Proof First let us prove (4.15). By Proposition 2.4, Lemma 4.3, Proposition 3.2, Lemma 3.12 and Proposition 2.2, for any $\delta_1,\delta_2>0$, there exists $\epsilon_*>0$, such that

$$P\left(\sup_{t\le T}\int_{\mathbb{R}^d\times\mathbb{R}^d} G_N^\alpha\phi(x_1,x_2)1(|x_1-x_2|\le 2\epsilon)\,X_t^{1,N}(dx_1)\,X_t^{2,N}(dx_2)>\delta_1\right)\le\delta_2, \tag{4.17}$$

for any $\epsilon\le\epsilon_*$ and $N\ge\epsilon_*^{-2}$.

As in the previous section, for $1/2>\epsilon>0$, $f_\epsilon\in[0,1]$ is a C^∞ function such that

$$f_\epsilon(x)\ =\ \begin{cases}1, & \text{if } |x|\le\epsilon,\\ 0 & \text{if } |x|>2\epsilon.\end{cases} \tag{4.18}$$

By Lemma 4.6(b) and the convergence

$$(X^{1,N},X^{2,N})\to(X^1,X^2),\ \text{ in } D_{\mathcal{M}_F{}^2},\ P-\text{a.s.}$$

with $(X^1,X^2)\in C_{\mathcal{M}_F{}^2}$, we get

$$\begin{aligned}&P\left(\sup_{t\le T}\int_{\mathbb{R}^d\times\mathbb{R}^d} G^{\alpha,\hat\delta}\phi(x_1,x_2)f_\epsilon(x_1-x_2)\,X_t^1(dx_1)\,X_t^2(dx_2)>\delta_1\right)\\ &\quad\le\ \lim_{N\to\infty}P\left(\sup_{t\le T}\int_{\mathbb{R}^d\times\mathbb{R}^d} G_N^{\alpha,\hat\delta}\phi(x_1,x_2)f_\epsilon(x_1-x_2)\,X_t^{1,N}(dx_1)\,X_t^{2,N}(dx_2)>\delta_1\right)\\ &\quad\le\ \lim_{N\to\infty}P\left(\sup_{t\le T}\int_{\mathbb{R}^d\times\mathbb{R}^d} G_N^{\alpha}\phi(x_1,x_2)f_\epsilon(x_1-x_2)\,X_t^{1,N}(dx_1)\,X_t^{2,N}(dx_2)>\delta_1\right)\\ &\quad\le\ \delta_2.\end{aligned}$$

Since $\hat\delta>0$ was arbitrary we can take $\hat\delta\downarrow 0$ in the above to get

$$P\left(\sup_{t\le T}\int_{\mathbb{R}^d\times\mathbb{R}^d} G^{\alpha}\phi(x_1,x_2)f_\epsilon(x_1-x_2)\,X_t^1(dx_1)\,X_t^2(dx_2)>\delta_1\right)\ \le\ \delta_2\,. \tag{4.19}$$

Now

$$
\begin{aligned}
& P\left(\sup_{t\le T}\left|\int_{\mathbb{R}^d\times\mathbb{R}^d} G_N^\alpha\phi(x_1,x_2)\,X_t^{1,N}(dx_1)\,X_t^{2,N}(dx_2)\right.\right.\\
&\qquad \left.\left. - \int_{\mathbb{R}^d\times\mathbb{R}^d} G^\alpha\phi(x_1,x_2)\,X_t^1(dx_1)\,X_t^2(dx_2)\right| \ge 3\delta_1\right)\\
\le\ & P\left(\sup_{t\le T}\left|\int_{\mathbb{R}^d\times\mathbb{R}^d} G_N^\alpha\phi(x_1,x_2)(1-f_\epsilon(x_1-x_2))\,X_t^{1,N}(dx_1)\,X_t^{2,N}(dx_2)\right.\right.\\
&\qquad \left.\left. - \int_{\mathbb{R}^d\times\mathbb{R}^d} G^\alpha\phi(x_1,x_2)(1-f_\epsilon(x_1-x_2))\,X_t^1(dx_1)\,X_t^2(dx_2)\right| \ge \delta_1\right)\\
&\qquad + P\left(\sup_{t\le T}\int_{\mathbb{R}^d\times\mathbb{R}^d} G_N^\alpha\phi(x_1,x_2)f_\epsilon(x_1-x_2)\,X_t^{1,N}(dx_1)\,X_t^{2,N}(dx_2) \ge \delta_1\right)\\
&\qquad + P\left(\sup_{t\le T}\int_{\mathbb{R}^d\times\mathbb{R}^d} G^\alpha\phi(x_1,x_2)f_\epsilon(x_1-x_2)\,X_t^1(dx_1)\,X_t^2(dx_2) \ge \delta_1\right).
\end{aligned}
$$

Now let $N\to\infty$. Apply Lemma 4.6(a), convergence of $X^{1,N}, X^{2,N}$ and (4.17), (4.19) to get

$$
\begin{aligned}
& \limsup_{N\to\infty} P\left(\sup_{t\le T}\left|\int_{\mathbb{R}^d\times\mathbb{R}^d} G_N^\alpha\phi(x_1,x_2)\,X_t^{1,N}(dx_1)\,X_t^{2,N}(dx_2)\right.\right.\\
&\qquad \left.\left. - \int_{\mathbb{R}^d\times\mathbb{R}^d} G^\alpha\phi(x_1,x_2)\,X_t^{1,N}(dx_1)\,X_t^{2,N}(dx_2)\right| \ge 3\delta_1\right)\\
\le\ & 2\delta_2\,,
\end{aligned}
$$

and since $\delta_1\,,\delta_2>0$ were arbitrary (4.15) follows. Now (4.14) follows immediately from (4.15) and (4.17).

Weak continuity of $t\to X_t^i$ and the fact that $(1-f_\epsilon(x_1-x_2))G^\alpha\phi(x_1,x_2)$ is bounded and continuous imply the continuity of

$$
t\to\int_{\mathbb{R}^{2d}} G^\alpha\phi(x_1,x_2)(1-f_\epsilon(x_1-x_2))X_t^1(dx_1)X_t^2(dx_2)
$$

for any $\epsilon>0$. (4.16) now follows from (4.19). ∎

Lemma 4.8 *For any $\phi\in C_{b,+}(\mathbb{R}^d)$ and $T>0$,*

$$
\int_0^T\int_{\mathbb{R}^d\times\mathbb{R}^d} G^\alpha\phi(x_1,x_2)\,X_s^2(dx_2)\,A(ds,dx_1)<\infty,\quad P-\text{a.s.},\tag{4.20}
$$

and

$$
\begin{aligned}
& \sup_{t\le T}\left|\int_0^t\int_{\mathbb{R}^d\times\mathbb{R}^d} G_N^\alpha\phi(x_1,x_2)\,X_s^{2,N}(dx_2)\,L^{1,N}(ds,dx_1)\right.\\
&\qquad \left. - \int_0^t\int_{\mathbb{R}^d\times\mathbb{R}^d} G^\alpha\phi(x_1,x_2)\,X_s^2(dx_2)\,A(ds,dx_1)\right| \to 0,
\end{aligned}\tag{4.21}
$$

in probability, as $N\to\infty$.

Proof By Lemma 4.4, Assumption 2.5 and Proposition 2.2, for any $\delta_1\,,\delta_2>0$, there exists $\epsilon^*>0$, such that

$$(4.22)\; P\left(\sup_{t\le T}\int_0^t\int_{\mathbb{R}^d\times\mathbb{R}^d} G_N^\alpha\phi(x_1,x_2)1(|x_1-x_2|\le 2\epsilon)\,X_s^{2,N}(dx_2)\,L^{1,N}(ds,dx_1)>\delta_1\right)\le\delta_2,$$

for any $\epsilon\le\epsilon_*,\ N\ge\epsilon_*^{-2}$. Then, arguing as in the derivation of (4.19) in Lemma 4.7, we get

$$(4.23)\qquad P\left(\int_0^T\int_{\mathbb{R}^d\times\mathbb{R}^d} G^\alpha\phi(x_1,x_2)f_\epsilon(x_1-x_2)\,X_s^2(dx_2)\,A(ds,dx_1)>\delta_1\right)\ \le\ \delta_2\,,$$

and hence, again as in Lemma 4.7,

$$\begin{aligned}
&\limsup_{N\to\infty} P\left(\sup_{t\le T}\left|\int_0^t\int_{\mathbb{R}^d\times\mathbb{R}^d} G_N^\alpha\phi(x_1,x_2)\,X_s^{2,N}(dx_2)\,L^{1,N}(ds,dx_1)\right.\right.\\
&\qquad\left.\left.-\int_0^t\int_{\mathbb{R}^d\times\mathbb{R}^d} G^\alpha\phi(x_1,x_2)\,X_s^2(dx_2)\,A(ds,dx_1)\right|\ge 3\delta_1\right)\\
&\le\ 2\delta_2\,.
\end{aligned}$$

Since $\delta_1\,,\delta_2$ were arbitrary we are done, as in Lemma 4.7.

∎

Lemma 4.9 *For any $\phi\in C_{b,+}(\mathbb{R}^d)$, $T>0$,*

$$(4.24)\qquad \int_0^T\int_{\mathbb{R}^d\times\mathbb{R}^d} G^\alpha\phi(x_1,x_2)\,X_s^1(dx_1)\,A(ds,dx_2)<\infty,\quad P-\text{a.s.},$$

and

$$\begin{aligned}
(4.25)\qquad &\sup_{t\le T}\left|\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G_N^\alpha\phi(x_1,x_2)X_s^{1,N}(dx_1)\,L^{2,N}(ds,dx_2)\right.\\
&\qquad\left.-\int_0^t\int_{\mathbb{R}^d\times\mathbb{R}^d} G^\alpha\phi(x_1,x_2)\,X_s^1(dx_1)\,A(ds,dx_2)\right|\to 0,
\end{aligned}$$

in probability, as $N\to\infty$.

Proof Let $f_\epsilon \in [0,1]$ be a continuous function on $\mathbb{R}^d$ satisfying (4.18). Then

$$
\begin{aligned}
&\sup_{t\le T}\Big|\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G_N^\alpha\phi(x_1,x_2)X_s^{1,N}(dx_1)\,L^{2,N}(ds,dx_2)\\
&\qquad -\int_0^t\int_{\mathbb{R}^d\times\mathbb{R}^d} G^\alpha\phi(x_1,x_2)\,X_s^1(dx_1)\,A(ds,dx_2)\Big|\\
&\le\ \sup_{t\le T}\Big|\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G_N^\alpha\phi(x_1,x_2)(1-f_\epsilon(x_1-x_2))X_s^{1,N}(dx_1)\,L^{2,N}(ds,dx_2)\\
&\qquad -\int_0^t\int_{\mathbb{R}^d\times\mathbb{R}^d} G^\alpha\phi(x_1,x_2)(1-f_\epsilon(x_1-x_2))\,X_s^1(dx_1)\,A(ds,dx_2)\Big|\\
&\qquad +\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G_N^\alpha\phi(x_1,x_2)f_\epsilon(x_1-x_2)X_s^{1,N}(dx_1)\,L^{2,N}(ds,dx_2)\\
&\qquad +\int_0^T\int_{\mathbb{R}^d\times\mathbb{R}^d} G^\alpha\phi(x_1,x_2)f_\epsilon(x_1-x_2)\,X_s^1(dx_1)\,A(ds,dx_2) \qquad (4.26)\\
&\equiv\ I^{1,N,\epsilon}+I^{2,N,\epsilon}+I^{3,\epsilon}
\end{aligned}
$$

With Lemma 4.6(a) at hand, it is easy to check that for any compact $K\subset\mathbb{R}^d$

$$\sup_{(x_1\,x_2)\in K\times K}|G_N^\alpha\phi(x_1,x_2)(1-f_\epsilon(x_1-x_2))-G^\alpha\phi(x_1,x_2)(1-f_\epsilon(x_1-x_2))|\to 0.$$

Therefore by the convergence

$$(X^{1,N},L^{2,N})\to(X^1,A),\ \text{ in } D_{\mathcal{M}_F{}^2}\,,\ P-\text{a.s.}$$

with $(X^1,A)\in C_{\mathcal{M}_F{}^2}$ and uniform boundedness of $G^\alpha\phi$ away from the diagonal, we easily get

$$I^{1,N,\epsilon}\to 0,\ \ P-\text{a.s.},\ \forall\epsilon>0. \tag{4.27}$$

By Lemma 4.3, Proposition 2.2, and Proposition 2.4 we get that for all $N>\epsilon^{-2}$

$$
\begin{aligned}
I^{2,N,\epsilon}\ &\le\ \bar{L}_T^{2,N}(1)\left\{\sup_{x_2,s\le T}\int_{\mathbb{R}^d} G_N^\alpha\phi(x_1,x_2)1(|x_1-z|\le 2\epsilon)X_s^{1,N}(dx_1)\right\}\\
&\le\ c\bar{L}_T^{2,N}(1)R_N\epsilon^{1-l_d-\tilde\delta}\quad \text{recall (2.20)}\\
&=\ c\bar{L}_T^{2,N}(1)R_N\epsilon^{1-l_d-\tilde\delta}.
\end{aligned}
$$

Hence by Lemma 3.11, for any $\delta_1\,,\delta_2>0$ there exists ϵ^* such that

$$P\left(I^{2,N,\epsilon}>\delta_1\right)\le\delta_2\,,\ \ \forall N\ge\epsilon_*^{-2}\,,\epsilon\le\epsilon_*. \tag{4.28}$$

Arguing as in the derivation of (4.19) in Lemma 4.7, we get

$$P\left(I^{3,\epsilon}>\delta_1\right)\ \le\ \delta_2\,,\ \ \forall\epsilon\le\epsilon_*. \tag{4.29}$$

Now combine (4.27), (4.28), (4.29) to complete the proof.

■

Lemma 4.10 *For any $\phi \in C_{b,+}(\mathbb{R}^d)$ and $T > 0$,*

$$\int_0^T \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} G^\alpha \phi(x_1, x_2)\, X_s^2(dx_2) \right)^2 X_s^1(dx_1)\, ds < \infty, \quad P-\text{a.s.}, \tag{4.30}$$

and for $\hat{p}^N(z) = p^N(z)$ or $\hat{p}^N(z) = 1_{\mathbf{0}}(z)$ we have

$$\begin{aligned} \sup_{t\le T} \bigg| &\int_0^t \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} G_N^\alpha \phi(x_1, x_2)\, X_s^{2,N}(dx_2) \right)^2 \left(X_s^{1,N} * \hat{p}^N \right)(dx_1)\, ds \\ &- \int_0^t \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} G^\alpha \phi(x_1, x_2)\, X_s^2(dx_2) \right)^2 X_s^1(dx_1)\, ds \bigg| \to 0, \end{aligned} \tag{4.31}$$

in probability, as $N \to \infty$.

Proof Let f_ϵ be continuous function of $\mathbb{R}^d$ given by (4.18). Recall that

$$\psi_N^\epsilon(x_1\,, x_2) = G_N^\alpha \mathbf{1}(x_1\,, x_2) 1(|x_1 - x_2| \le \epsilon)$$

and let $\psi^\epsilon(x_1\,, x_2) = G^\alpha \mathbf{1}(x_1\,, x_2) 1(|x_1 - x_2| \le \epsilon)$. Then by simple algebra

$$\begin{aligned} &\sup_{t\le T} \bigg| \int_0^t \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} G_N^\alpha \phi(x_1, x_2)\, X_s^{2,N}(dx_2) \right)^2 \left(X_s^{1,N} * \hat{p}^N \right)(dx_1)\, ds \\ &\qquad - \int_0^t \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} G^\alpha \phi(x_1, x_2)\, X_s^2(dx_2) \right)^2 X_s^1(dx_1)\, ds \bigg| \\ &\le \sup_{t\le T} \bigg| \int_0^t \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} G_N^\alpha \phi(x_1, x_2)(1 - f_\epsilon(x_1 - x_2))\, X_s^{2,N}(dx_2) \right)^2 \left(X_s^{1,N} * \hat{p}^N \right)(dx_1)\, ds \\ &\qquad - \int_0^t \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} G^\alpha \phi(x_1, x_2)(1 - f_\epsilon(x_1 - x_2))\, X_s^2(dx_2) \right)^2 X_s^1(dx_1)\, ds \bigg| \\ &\qquad + 3\, \|\phi\|_\infty^2 \int_0^T \int_{\mathbb{R}^{3d}} \psi_N^\epsilon(x_1, x_2) X_s^{2,N}(dx_2) G_N^\alpha \mathbf{1}(x_1\,, x_2')\, X_s^{2,N}(dx_2') \left(X_s^{1,N} * \hat{p}^N \right)(dx_1)\, ds \\ &\qquad + 3\, \|\phi\|_\infty^2 \int_0^T \int_{\mathbb{R}^{3d}} \psi^\epsilon(x_1, x_2) X_s^2(dx_2) G^\alpha \mathbf{1}(x_1\,, x_2')\, X_s^2(dx_2')\, X_s^1(dx_1)\, ds \\ &\equiv I^{1,N,\epsilon} + I^{2,N,\epsilon} + I^{3,\epsilon}. \end{aligned}$$

Therefore by Lemma 4.6(a) and the convergence

$$(X^{1,N}, X^{2,N}) \to (X^1, X^2), \ \text{ in } D_{\mathcal{M}_F{}^2}\,, \ P-\text{a.s.}$$

with $(X^1, X^2) \in C_{\mathcal{M}_F{}^2}$, as in the previous proof we get

$$I^{1,N,\epsilon} \to 0, \ \ P-\text{a.s.}\ \forall \epsilon > 0. \tag{4.32}$$

By Lemma 4.5, Assumption 2.5 and Proposition 2.2, for any $\delta_1\,, \delta_2 > 0$ there exists ϵ^* such that

$$P\left(I^{2,N,\epsilon} > \delta_1 \right) \le \delta_2\,, \ \ \forall N \ge \epsilon_*^{-2}\,, \epsilon \le \epsilon_*. \tag{4.33}$$

Then arguing as in the derivation of (4.19) in Lemma 4.7 we get

$$P\left(I^{3,\epsilon} > \delta_1\right) \;\leq\; \delta_2\,, \;\; \forall \epsilon \leq \epsilon_*. \tag{4.34}$$

Now combine (4.32), (4.33), and (4.34) to complete the proof.

■

Lemma 4.11 *For any $\phi \in C_{b,+}(\mathbb{R}^d)$ and $T > 0$,*

$$\int_0^T \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} G^\alpha \phi(x_1, x_2)\, X_s^1(dx_1) \right)^2 X_s^2(dx_2)\, ds < \infty, \quad P-\text{a.s.}, \tag{4.35}$$

and for $\hat{p}^N(z) = p^N(z)$ or $\hat{p}^N(z) = 1_{\mathbf{0}}(z)$,

$$\begin{aligned}\sup_{t\leq T} \Bigg| &\int_0^t \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} G_N^\alpha \phi(x_1, x_2)\, X_s^{1,N}(dx_1) \right)^2 \left(X_s^{2,N} * \hat{p}^N\right)(dx_2)\, ds \\ &- \int_0^t \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} G^\alpha \phi(x_1, x_2)\, X_s^1(dx_1) \right)^2 X_s^2(dx_2)\, ds \Bigg| \to 0,\end{aligned} \tag{4.36}$$

in probability, as $N \to \infty$.

Proof The proof goes along the same lines as of Lemma 4.10, with the only difference being that we use Lemmas 4.3 and 4.2 instead of Lemma 4.5.

■

Before we formulate the next lemma, let us introduce the following notation for the martingales in the approximate Tanaka formula (4.5):

$$\tilde{M}_t^N(\phi) \;\equiv\; \sum_{i=1}^2 \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G_N^\alpha \phi(x_1, x_2) X_s^{3-i,N}(dx_1) M^{i,N}(ds, dx_2), \;\; i = 1, 2,\; \phi \in C_b(\mathbb{R}^d). \tag{4.37}$$

Lemma 4.12 *For any $\phi \in C_b(\mathbb{R}^d)$ there is a continuous $\mathcal{F}_t^{\mathbf{X},A}$-local martingale $\tilde{M}(\phi)$ such that for any $T > 0$,*

$$\sup_{t\leq T} \left| \tilde{M}_t^N(\phi) - \tilde{M}_t(\phi) \right| \to 0,$$

in probability, as $N \to \infty$.

Proof Lemmas 4.7, 4.8 and 4.9 show that all the terms in (4.5), except perhaps $\tilde{M}_t^N(\phi)$, converge in probability, uniformly for t in compact time sets, to an a.s. continuous limit as $N \to \infty$. Hence there is an a.s. continuous $\mathcal{F}_t^{\mathbf{X},A}$-adapted process $\tilde{M}_t(\phi)$ such that $\sup_{t\leq T} |\tilde{M}_t^N(\phi) - \tilde{M}_t(\phi)| \to 0$ in probability as $N \to \infty$.

(4.2) and Lemma 7.2 below imply

$$\begin{aligned}
|G_N^\alpha\phi(x_1,x_2)| &\le |\phi|_\infty 2^{-d}\int_0^\infty e^{-\alpha s}N^{d/2}P(B_{2s}^N\in x_2-x_1+[-N^{-1/2}N^{-1/2}]^d)\,ds\\
&\le c\|\phi\|_\infty\int_0^\infty e^{-\alpha s}\Big(\frac{N}{Ns+1}\Big)^{d/2}ds\\
&\le c\|\phi\|_\infty N^{1/2}.
\end{aligned}$$

Therefore

$$|\Delta\tilde M_s^N(\phi)|\le N^{-1}\sup_{x_2}\sum_{i=1}^2\Big|\int G_N^\alpha\phi(x_1,x_2)X_s^{i,N}(dx_1)\Big|\le c\|\phi\|_\infty N^{-1/2}\sum_{i=1}^2 X_s^{i,N}(1).$$

In view of $|\Delta X_s^{i,N}(1)|\le N^{-1}$, and Proposition 3.1, we see that if

$$T_n^N=\inf\{s:|\tilde M_s^N(\phi)|+\sum_{i=1}^2 X_s^{i,N}(1)\ge n\},$$

then

(4.38) $|\tilde M^N_{s\wedge T_n^N}|$ is uniformly bounded and as $n\to\infty$ T_n^N is large in probability, unformly in N.

Therefore $\tilde M^N_{\cdot\wedge T_n^N}$ is a uniformly bounded continuous $\mathcal{F}_t^{\mathbf{X}^N}$-local martingale and from this and (4.38) it is easy and standard to check that $\tilde M(\phi)$ is a continuous $\mathcal{F}_t^{\mathbf{X},A}$-local martingale. ∎

Proof of Proposition 4.1 Immediate from the approximate Tanaka formula (4.5), the Lemmas 4.7, 4.8, 4.9, 4.12 and convergence of L^N to A.

∎

5 Proofs of Theorems 1.1, 1.2 and 1.5

Lemma 5.1 *Let (X^1,X^2,A) be any limit point of $(X^{1,N},X^{2,N},L^{2,N})$. Then the collision local time $L(X^1,X^2)$ exists and for any $\phi\in C_b(\mathbb{R}^d)$,*

$$\begin{aligned}
\mathbf{X}_t(G^\alpha\phi) &= \mathbf{X}_0(G^\alpha\phi)-\gamma_1\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}G^\alpha\phi(x_1,x_2)X_s^2(dx_2)\,A(ds,dx_1)\\
&\quad+\gamma_2\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}G^\alpha\phi(x_1,x_2)X_s^1(dx_1)\,A(ds,dx_2)\\
&\quad+\alpha\int_0^t\mathbf{X}_s(G^\alpha\phi)\,ds-L_t(X^1,X^2)(\phi)\\
&\quad+\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}G^\alpha\phi(x_1,x_2)\left(X_s^1(dx_1)M^2(ds,dx_2)+X_s^2(dx_1)M^1(ds,dx_2)\right),
\end{aligned}\tag{5.1}$$

where the stochastic integral term is a continuous local martingale with quadratic variation

$$\int_0^t \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} G^\alpha \phi(x_1, x_2) X_s^2(dx_1) \right)^2 X^1(dx_2)\, ds$$
$$+ \int_0^t \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} G^\alpha \phi(x_1, x_2) X_s^1(dx_1) \right)^2 X^2(dx_2)\, ds.$$

Proof Define

$$G_\epsilon^\alpha \phi(x_1\,, x_2) = \Pi_{x_1} \times \Pi_{x_2} \left[\int_0^\infty e^{-\alpha s} \mathrm{p}_\epsilon(B_s^1 - B_s^2) \phi \left(\frac{B_s^1 + B_s^2}{2} \right) ds \right].$$

As in Section 5 of Barlow, Evans and Perkins (1991), $\mathbf{M}_{X_0^1, X_0^2}^{\gamma_1, \gamma_2, A}$ implies

$$\begin{aligned} \mathbf{X}_t(G_\epsilon^\alpha \phi) \quad &= \quad \mathbf{X}_0(G_\epsilon^\alpha \phi) - \gamma_1 \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G_\epsilon^\alpha \phi(x_1, x_2) X_s^2(dx_2)\, A(ds, dx_1) \\ &\quad + \gamma_2 \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G_\epsilon^\alpha \phi(x_1, x_2) X_s^1(dx_1)\, A(ds, dx_2) + \alpha \int_0^t \mathbf{X}_s(G_\epsilon^\alpha \phi)\, ds \\ &\quad \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G_\epsilon^\alpha \phi(x_1, x_2) \left(X_s^1(dx_1) M^2(ds, dx_2) + X_s^2(dx_1) M^1(ds, dx_2) \right) \\ &\quad - \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathrm{p}_\epsilon(x_1 - x_2) \phi((x_1 + x_2)/2) X_s^1(dx_1) X^2(dx_2)\, ds. \end{aligned} \tag{5.2}$$

Now apply Lemmas 4.7, 4.8, 4.9, 4.10, 4.11 (trivially dropping the non-negativity hypothesis on ϕ), and argue as in Section 5 of Barlow, Evans, Perkins (1991), essentially using dominated convergence, to show that all the terms (5.2) (except possibly the last one) converge in probability to the corresponding terms of (5.1), as $\epsilon \downarrow 0$. Hence the last term in (5.2), $L_t^\epsilon(X^1, X^2)(\phi)$, converges in probability to say $L_t(\phi)$ for each $\phi \in C_b(\mathbb{R}^d)$. This gives (5.1) with $L_t(\phi)$ in place of $L_t(X^1, X^2)(\phi)$. As each term in (5.1) is a.s. continuous in t (use (4.16) for the left-hand side) the same is true of $t \to L_t(\phi)$. This implies uniform convergence in probability for $t \leq T$ of $L_t^\epsilon(X^1, X^2)$ to $L_t(\phi)$ for each $\phi \in C_b(\mathbb{R}^d)$. It is now easy to construct L as a random non-decreasing continuous $\mathcal{M}_F$-valued process, using a countable convergence determining class and hence we see that by definition $L_t = L_t(X^1, X^2)$.

■

Proof of Theorem 1.1 In view of Proposition 3.1, it only remains to show that $L_t(X^1, X^2)$ exists and equals A_t. This, however now follows from Lemma 5.1, Proposition 4.1, and the uniqueness of the decomposition of the continuous semimartingale $\mathbf{X}_t(G^\alpha \phi)$.

■

Remark 5.2 *Since $A_t = L_t(X^1, X^2)$, Lemma 5.1 immediately gives us the following form of*

Tanaka's formula for (X^1, X^2):

$$\begin{aligned}\mathbf{X}_t(G^\alpha\phi) &= \mathbf{X}_0(G^\alpha\phi) - \gamma_1\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G^\alpha\phi(x_1,x_2)X_s^2(dx_2)\,L(ds,dx_1)\\ &\quad+\gamma_2\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G^\alpha\phi(x_1,z)X_s^1(dx_1)\,L(ds,dx_2)\\ &\quad+\alpha\int_0^t \mathbf{X}_s(G^\alpha\phi)\,ds - L_t(X^1,X^2)(\phi)\\ &\quad+\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G^\alpha\phi(x_1,x_2)\left(X_s^1(dx_1)M^2(ds,dx_2)+X_s^2(dx_1)M^1(ds,dx_2)\right),\end{aligned}$$

where the stochastic integral term is a continuous local martingale with quadratic variation

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} G^\alpha\phi(x_1,x_2)X_s^2(dx_1)\right)^2 X^1(dx_2)\,ds\\ &+\int_0^t\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} G^\alpha\phi(x_1,x_2)X_s^1(dx_1)\right)^2 X^2(dx_2)\,ds.\end{aligned}$$

Proof of Theorem 1.2 As mentioned in the Introduction, the case of $\gamma_2 \le 0$ is proved in Theorem 4.9 of Evans and Perkins (1994) and so we assume $\gamma_2 > 0$. Let X_0^i satisfy **UB**, $i = 1, 2$, and assume P is a law on $C_{\mathcal{M}_F{}^2}$ under which the canonical variables (X^1, X^2) satisfy $\mathbf{MP}^{0,\gamma_2}_{X_0^1,X_0^2}$. Then X^1 is a super-Brownian motion with branching rate 2, variance parameter σ^2, and law P^{X^1}, say. By Theorem 4.7 of Barlow, Evans and Perkins (1991),

$$\mathcal{H}_\delta \equiv \sup_{t\ge 0}\rho_\delta(X_t^1) < \infty \quad \forall\delta>0 \;\; P^{X^1} - \text{a.s.} \tag{5.3}$$

Let $q_\eta : \mathbb{R}_+ \to \mathbb{R}_+$ be a C^∞ function with support in $[0,\eta]$ such that $\int q_\eta(u)du = 1$. For $\varepsilon > 0$ we will choose an appropriate $\eta = \eta(\varepsilon) \le \varepsilon$ below and so may define

$$h_\varepsilon(X^1,s,x) = \gamma_2\int_0^\infty q_\eta(u-s)p_\varepsilon * X_u^1(x)\,du \equiv p_\varepsilon * X_s^{\varepsilon,1}(x).$$

Then (5.3) implies

$$\sup_{t\ge0,\varepsilon>0}\rho_\delta(X_t^{\varepsilon,1}) \le \gamma_2\mathcal{H}_\delta < \infty \quad \forall\delta>0 \;\; a.s. \tag{5.4}$$

Let $E_{r,x}$ denote expectation with respect to a Brownian motion B beginning at x at time r and let $P_{r,t}f(x) = E_{r,x}(f(B_t))$ for $r \le t$. It is understood that B is independent of (X^1, X^2). Define

$$\ell_t^\varepsilon(B,X^1) = \int_0^t h_\varepsilon(X^1,s,B_s)\,ds,$$

where the integrand is understood to be 0 for $s < r$ under $P_{r,x}$. The additional smoothing in time in our definition of h_ε does force some minor changes, but it is easy enough to modify the arguments in Theorems 4.1 and 4.7 of Evans and Perkins (1994), using (5.4), to see there is a Borel $\ell : C_{\mathbb{R}^d} \times C_{\mathcal{M}_F} \to C_{\mathbb{R}_+}$ such that

$$\text{(5.5)} \qquad \lim_{\varepsilon \downarrow 0} \sup_{r \geq 0, x \in \mathbb{R}^d} E_{r,x}\Big(\sup_{t \geq r} |\ell^\varepsilon_t(B, X^1) - \ell_t(B, X^1)|^2\Big) = 0 \quad P^{X^1} - a.s.,$$

and for a.a. X^1, $t \to \ell_t(B, X^1)$ is an increasing continuous additive functional of B. It is easy to use (5.4) and (5.5) to see that for a.a. X^1, ℓ_t is an admissible continuous additive functional in the sense of Dynkin and Kuznetsov (see (K_1) in Kuznetsov (1994)). Let

$$C_\ell = \{\phi \in C_b(\mathbb{R}^d) : \phi \text{ as a finite limit at } \infty\},$$

and C^+_ℓ be the set of non-negative functions in C_ℓ. Let $\phi : C_{\mathcal{M}_F} \to C^+_\ell$ be Borel and let $V_{r,t} = V^{X^1}_{r,t}\phi$ be the unique continuous C^+_ℓ-valued solution of

$$\text{(5.6)} \qquad V_{r,t}(x) = P_{r,t}\phi(x) + E_{r,x}\Big(\int_r^t V_{s,t}(B_s)\ell(B, X_1)(ds) - V_{s,t}(B_s)^2\, ds\Big),\ r \leq t.$$

The existence and uniqueness of such solutions is implicit in Kuznetsov (1994) and may be shown by making minor modifications in classical fixed point arguments (e.g. in Theorems 6.1.2 and 6.1.4 of Pazy (1983)), using the L^2 bounds on ℓ_t from (5.5) and the negative quadratic term in (5.6) to see that explosions cannot occur. The construction shows $V^{X^1}_{r,t}$ is Borel in X^1–we will not comment on such measurability issues in what follows. Theorems 1 and 2 of Kuznetsov (1994) and the aforementioned a.s. admissibility of $\ell(B, X^1)$ give the existence of a unique right continuous measure-valued Markov process X such that

$$\text{(5.7)} \qquad E_{X_0}(e^{-X_t(\phi)}) = e^{-X_0(V_{0,t}\phi)}, \quad \phi \in C^+_\ell.$$

If $\mathcal{P}_{X_0|X^1}$ is the associated law on path space we will show

$$\text{(5.8)} \qquad P(X^2 \in \cdot | X^1) = \mathcal{P}_{X^2_0|X^1}(\cdot)$$

and hence establish uniqueness of solutions to $\mathbf{MP}^{0,\gamma_2}_{X^1_0, X^2_0}$.

The proof of the following lemma is similar to its discrete counterpart, Proposition 3.5(b), and so is omitted ((5.4) is used in place of Proposition 2.4).

Lemma 5.3

$$\sup_{r \leq T, x \in \mathbb{R}^d} \sup_{\varepsilon > 0} E_{r,x}\Big(e^{\lambda \ell^\varepsilon_T}\Big) < \infty \quad \forall T,\ \lambda > 0 \quad P^{X^1} - a.s.$$

Let $P^{\varepsilon, X^1}_{r,t} f(x) = E_{r,x}\Big(f(B_t)e^{\ell^\varepsilon_t}\Big)$ and $P^{X^1}_{r,t} f(x) = E_{r,x}\Big(f(B_t)e^{\ell_t}\Big)$. By Lemma 5.3 these are well-defined inhomogeneous semigroups for bounded Borel functions $f : \mathbb{R}^d \to \mathbb{R}$. It follows from (5.5), Lemma 5.3, and a dominated convergence argument that for each $t > 0$, and bounded Borel f,

$$\text{(5.9)} \qquad \begin{aligned} \sup_{r \leq t, x \in \mathbb{R}^d} |P^{\varepsilon, X^1}_{r,t} f(x) - P^{X^1}_{r,t} f(x)| &\leq \sup_{r \leq t, x \in \mathbb{R}^d} \|f\|_\infty E_{r,x}\Big(|e^{\ell^\varepsilon_t} - e^{\ell_t}|\Big) \\ &\to 0 \text{ as } \varepsilon \downarrow 0\ P^{X^1} - a.s. \end{aligned}$$

Let $D(\Delta/2)$ be the domain of the generator of B acting on the Banach space C_ℓ and $D(\Delta/2)_+$ be the nonnegative functions in this domain. Assume now that $\phi : C_{\mathcal{M}_F} \to D(\Delta/2)_+$ is Borel. Let $V^\varepsilon_{r,t} = V^{\varepsilon,X^1}_{r,t}$, be the unique continuous C^+_ℓ-valued solution of

$$V^\varepsilon_{r,t}(x) = P_{r,t}\phi(x) + E_{r,x}\Big(\int_r^t V^\varepsilon_{s,t}(B_s)\ell^\varepsilon(B, X_1)(ds) - V^\varepsilon_{s,t}(B_s)^2\,ds\Big),\ r \le t. \tag{5.10}$$

We claim $V^\varepsilon_{r,t}$ also satisfies

$$V^\varepsilon_{r,t}(x) = P^{\varepsilon,X^1}_{r,t}\phi(x) - \int_r^t P^{\varepsilon,X^1}_{r,s}((V^\varepsilon_{s,t})^2)(x)\,ds. \tag{5.11}$$

Theorem 6.1.5 of Pazy (1983) shows that $V^\varepsilon_{r,t} \in D(\Delta/2)$ for $r \le t$, is continuously differentiable in $r < t$ as a C_ℓ-valued map, and satisfies

$$\frac{\partial V^\varepsilon_{r,t}}{\partial r}(x) = -\Big(\frac{\sigma^2\Delta}{2} + h_\varepsilon(r,x)\Big)V^\varepsilon_{r,t}(x) + (V^\varepsilon_{r,t}(x))^2,\ r < t,\quad V^\varepsilon_{t,t} = \phi. \tag{5.12}$$

Then (5.11) is just the mild form of (5.12) and follows as in section 4.2 of Pazy (1983). Note here that h_ε is continuously differentiable in s thanks to the convolution with q_η and so Theorem 6.1.5 of Pazy (1983) does apply.

We next show that

$$\lim_{\varepsilon\downarrow 0}\sup_{r\le t}\|V^\varepsilon_{r,t}\phi - V_{r,t}\phi\|_\infty = 0 \quad P^{X^1} - a.s. \tag{5.13}$$

First use (5.11) and Lemma 5.3 to see that for each $t > 0$,

$$\begin{aligned}\sup_{r\le t, x\in\mathbb{R}^d,\varepsilon>0} V^\varepsilon_{r,t}(x) &\le \sup_{r\le t,x\in\mathbb{R}^d,\varepsilon>0} P^{\varepsilon,X^1}_{r,t}\phi(x)\\ &\le \|\phi(X^1)\|_\infty \sup_{r\le t,x\in\mathbb{R}^d,\varepsilon>0} E_{r,x}(e^{\ell^\varepsilon_t}) = c_{\phi,t}(X^1) < \infty \ \ P^{X^1} - a.s.\end{aligned} \tag{5.14}$$

Using (5.11) again, we see that

$$\begin{aligned}\|V^\varepsilon_{r,t} - V^{\varepsilon'}_{r,t}\|_\infty &\le \|P^{\varepsilon,X^1}_{r,t}\phi - P^{\varepsilon',X^1}_{r,t}\phi\|_\infty + \int_r^t \|(P^{\varepsilon,X^1}_{r,s} - P^{\varepsilon',X^1}_{r,s})((V^\varepsilon_{s,t})^2)\|_\infty\,ds\\ &\qquad + \int_r^t \|P^{\varepsilon',X^1}_{r,s}((V^\varepsilon_{s,t} + V^{\varepsilon'}_{s,t})(V^\varepsilon_{s,t} - V^{\varepsilon'}_{s,t}))\|_\infty\,ds\\ &\equiv T^{\varepsilon,\varepsilon'}_1 + T^{\varepsilon,\varepsilon'}_2 + T^{\varepsilon,\varepsilon'}_3.\end{aligned} \tag{5.15}$$

(5.14) shows that

$$\begin{aligned}T^{\varepsilon,\varepsilon'}_2 &\le \int_r^t \sup_x E_{r,x}(|e^{\ell^\varepsilon_s} - e^{\ell^{\varepsilon'}_s}|)c_{\phi,t}(X^1)^2\\ &\to 0 \text{ uniformly in } r \le t \text{ as } \varepsilon,\varepsilon' \downarrow 0 \ \ P^{X^1} - a.s. \quad \text{(by (5.5) and Lemma 5.3).}\end{aligned}$$

(5.9) implies $T_1^{\varepsilon,\varepsilon'} \to 0$ uniformly in $r \le t$ as $\varepsilon, \varepsilon' \downarrow 0$ and (5.14) also implies

$$T_3^{\varepsilon,\varepsilon'} \le 2c_{\phi,t}(X^1) \int_0^t \|V_{s,t}^{\varepsilon} - V_{s,t}^{\varepsilon'}\|_\infty \, ds.$$

The above bounds and a simple Gronwall argument now show that $V_{r,t}^{\varepsilon}(x)$ converges uniformly in $x \in \mathbb{R}^d$, and $r \le t$ to a continuous C_ℓ^+-valued map as $\varepsilon \downarrow 0$. It is now easy to let $\varepsilon \downarrow 0$ in (5.10) and use (5.5) to see that this limit is $V_{r,t}$, the unique solution to (5.6). This completes the derivation of (5.13).

As usual $\mathcal{F}_t^{X^i}$ is the canonical right-continuous filtration generated by X^i. Consider also the enlarged filtration $\bar{\mathcal{F}}_t = \mathcal{F}_\infty^{X^1} \times \mathcal{F}_t^{X^2}$. Argue as in the proof of Theorem 4.9 of Evans and Perkins (1994), using the predictable representation property of X^1, to see that for $\phi \in D(\Delta/2)$, $M_t^2(\phi)$ is a continuous $\bar{\mathcal{F}}_t$-local martingale such that $\langle M^2(\phi)\rangle_t = \int_0^t X_s^2(2\phi^2)\, ds$. The usual extension of the orthogonal martingale measure M^2 now shows that if $f : \mathbb{R}_+ \times \Omega \times \mathbb{R}^d \to \mathbb{R}$ is $\mathcal{P}(\bar{\mathcal{F}}.) \times$ Borel-measurable (here $\mathcal{P}(\bar{\mathcal{F}}.)$ is the $\bar{\mathcal{F}}_t$-predictable σ-field), such that $\int_0^t X_s^2(f_s^2)\, ds < \infty \quad \forall t > 0$ a.s., then

$$M_t^2(f) \equiv \int_0^t \int f(s,\omega,x) M^2(ds,dx) \ \text{ is a well-defined continuous } (\bar{\mathcal{F}}_t) - \text{local martingale}$$

$$\text{such that } \langle M(f)\rangle_t = \int_0^t X_s^2(2f_s^2)\, ds.$$

It is easy to extend the martingale problem for X^2 to bounded $f : [0,t] \times \mathbb{R}^d \to \mathbb{R}$ such that $f_s \in C_b^2(\mathbb{R}^d)$ for $s \le t$, and $\frac{\partial f_s}{\partial s}$, $(\Delta/2) f_s \in C_b([0,t] \times \mathbb{R}^d)$, (e.g. argue as in Proposition II.5.7 of Perkins (2002)). For such an f one has

$$\text{(5.16)}\quad X_u^2(f_u) \;=\; X_0^2(f_0) + M_u^2(f) + \int_0^u X_s^2\Big(\frac{\sigma^2}{2}\Delta f_s + \frac{\partial f_s}{\partial s}\Big)\, ds + \gamma_2 L_u(\bar{X}^1, \bar{X}^2)\,(f)\,, \quad u \le t.$$

Next we claim (5.16) holds for $f(s,x,X^1)$ where $f : [0,t] \times \mathbb{R}^d \times C_{\mathcal{M}_F} \to \mathbb{R}$ is a Borel map such that for P^{X^1}-a.a. X^1,

$$\text{(5.17)}\qquad f(s,\cdot,X^1) \in D(\Delta/2) \text{ for all } s \le t,\ (\Delta/2) f_s,\ \frac{\partial f_s}{\partial s} \in C_b([0,t] \times \mathbb{R}^d),$$
$$\sup_{s \le t, x \in \mathbb{R}^d} |f(s,x,X^1)| + \Big|\frac{\Delta}{2} f(s,x,X^1)\Big| + \Big|\frac{\partial f}{\partial s}(s,x,X^1)\Big| = C(X^1) < \infty.$$

To see this, for $f : [0,t] \times \mathbb{R}^d \times C_{\mathcal{M}_F} \to \mathbb{R}$ bounded, introduce

$$f^\delta(s,x,X^1) = \int_0^\infty p_\delta * f_{u,X^1}(x) q_\delta(u-s)\, du.$$

Note that if $f_n \to f$ in the bounded pointwise sense where the bound may depend on X^1, then $\frac{\partial f_n^\delta}{\partial u} \to \frac{\partial f^\delta}{\partial u}$ and $\frac{\Delta}{2} f_n^\delta \to \frac{\Delta}{2} f^\delta$ in the same sense as $n \to \infty$. By starting with $f(u,x,X^1) = f_1(u,x) f_2(X^1)$ where f_i are bounded and Borel, and using a monotone class argument we obtain (5.16) for f_δ where f is any Borel map on $\mathbb{R}_+ \times \mathbb{R}^d \times C_{\mathcal{M}_F}$ with $\sup_{s \le t, x} |f(s,x,X^1)| < \infty$ for each

X^1. If f is as in (5.17) then it is easy to let $\delta \downarrow 0$ to obtain (5.16) for f. Note we are using the extension of the martingale measure M^2 to the larger filtration $\bar{\mathcal{F}}_t$ in these arguments.

Now recall $\phi : C_{\mathcal{M}_F} \to D(\Delta/2)_+$ (Borel) and $V^\varepsilon_{r,t}$ is the unique solution to (5.10). Recall from (5.12) that $V^\varepsilon_{r,t}$ is a classical solution of the non-linear pde and in particular (5.17) is valid for $f(r,x) = V^\varepsilon_{r,t}(x)$. Therefore we may use (5.12) in (5.16) to get

$$(5.18)\quad \begin{aligned} X^2_s(V^\varepsilon_{s,t}) &= X^2_0(V^\varepsilon_{0,t}) + \int_0^s \int V^\varepsilon_{r,t}(x) M^2(dr,dx) + \int_0^s X^2_r((V^\varepsilon_{r,t})^2)\,dr \\ &\quad + \int_0^s \int V^\varepsilon_{r,t}(x)[\gamma_2 L(X^1,X^2)(dr,dx) - h_\varepsilon(X^1,r,x)X^2_r(dx)dr],\ s \le t. \end{aligned}$$

We claim the last term in (5.18) approaches 0 uniformly in $s \le t$ P-a.s. as $\varepsilon = \varepsilon_k \downarrow 0$ for an appropriate choice of $\eta_k = \eta(\varepsilon_k)$ in the definition of h_ε. The definition of collision local time allows us to select $\varepsilon_k \downarrow 0$ so that $L^{\varepsilon_k}(X^1,X^2) \to L(X^1,X^2)$ in $\mathcal{M}_F(\mathbb{R}_+ \times \mathbb{R}^d)$ a.s. Note that

$$(5.19)\quad \begin{aligned} &\Big|\gamma_2 \int_0^s \int\int V_{r,t}\Big(\frac{x_1+x_2}{2}\Big) p_{\varepsilon_k}(x_1-x_2) X^1_r(dx_1)X^2_r(dx_2)\,dr \\ &\quad - \int_0^s \int V^{\varepsilon_k}_{r,t}(x_2) h_{\varepsilon_k}(X^1,r,x_2)X^2_r(dx_2)\,dr\Big| \\ &\le \gamma_2 \int_0^s \int\int |V_{r,t}\Big(\frac{x_1+x_2}{2}\Big) - V^{\varepsilon_k}_{r,t}(x_2)| p_{\varepsilon_k}(x_1-x_2) X^1_r(dx_1)X^2_r(dx_2)\,dr \\ &\quad + \gamma_2 \Big|\int_0^s \int \Big[P_{\varepsilon_k} * X^1_r(x_2) - \int_0^\infty q_{\eta(\varepsilon_k)}(u-r) P_{\varepsilon_k} * X^1_u(x_2)\,du\Big] V^{\varepsilon_k}_{r,t}(x_2) X^2_r(dx_2)\,dr\Big| \\ &= I^k_1(s) + I^k_2(s). \end{aligned}$$

Let $\delta_0 > 0$. By (5.13) and the uniform continuity of $(r,x) \to V_{r,t}(x)$ there is a $k_0 = k_0(X^1) \in \mathbb{N}$ a.s. such that

$$\sup_{r \le t} |V_{r,t}\Big(\frac{x_1+x_2}{2}\Big) - V^{\varepsilon_k}_{r,t}(x_2)| < \delta_0 \text{ for } k > k_0 \text{ and } |x_1 - x_2| < \varepsilon_{k_0}.$$

By considering $|x_1 - x_2| < \varepsilon_{k_0}$ and $|x_1 - x_2| \ge \varepsilon_{k_0}$ separately one can easily show there is a $k_1 = k_1(X^1)$ so that

$$\sup_{s \le t} I^k_1(s) < \delta_0 \quad \text{if } k > k_1.$$

Next use the upper bound in (5.14) and the continuity of $u \to p_{\varepsilon_k} * X^1_u(x_2)$ to choose η so that $\eta_k = \eta(\varepsilon_k) \downarrow 0$ fast enough so that

$$\sup_{s \le t} I^k_2(s) \to 0 \quad P\text{-a.s. as } k \to \infty.$$

The above bounds show the lefthand side of (5.19) converges to 0 a.s. as $k \to \infty$. The a.s. convergence of $L^{\varepsilon_k}(X^1,X^2)$ to $L(X^1,X^2)$ therefore shows that

$$\lim_{k\to\infty} \sup_{s \le t} \Big|\gamma_2 \int_0^s \int V_{r,t}(x) L(X^1,X^2)(dr,dx) - \int_0^s \int V^{\varepsilon_k}_{r,t}(x) h_{\varepsilon_k}(X^1,r,x) X^2_r(dx)dr\Big| = 0 \quad \text{a.s.}$$

The uniform convergence in (5.13) now shows that the last term in (5.18) approaches 0 uniformly in $s \le t$ P-a.s. as $\varepsilon = \varepsilon_k \to 0$. (5.13) also allows us to let $\varepsilon_k \downarrow 0$ in (5.18), taking a further subsequence perhaps to handle the martingale term, and conclude

$$X_s^2(V_{s,t}) = X_0^2(V_{0,t}) + \int_0^s \int V_{r,t}(x) M^2(dr,dx) + \int_0^s X_r^2((V_{r,t})^2)\,dr \quad \text{for all } s \le t \text{ a.s.}$$

Now apply Ito's lemma to conclude

$$e^{-X_s^2(V_{s,t})} = e^{-X_0^2(V_{0,t})} - \int_0^s \int e^{-X_r^2(V_{r,t})} V_{r,t}(x) M^2(dr,dx), \quad s \le t.$$

The stochastic integral is a bounded $\bar{\mathcal{F}}_t$-local martingale and therefore is an $\bar{\mathcal{F}}_t$-martingale. This proves for $t_1 < t$,

$$E(e^{-X_t^2(\phi)}|\bar{\mathcal{F}}_{t_1}) = e^{X_{t_1}^2(V_{t_1,t}\phi)} \text{ for any Borel } \phi : C_{\mathcal{M}_F} \to D(\Delta/2)_+.$$

Now $V_{t_1,t}\phi : C_{\mathcal{M}_F} \to D(\Delta/2)_+$ is also Borel and so if $\phi_1 : C_{\mathcal{M}_F} \to D(\Delta/2)_+$ is Borel, we can also conclude

$$\begin{aligned} E(e^{-X_t^2(\phi)-X_{t_1}^2(\phi_1)}|\bar{\mathcal{F}}_0) &= E(e^{-X_{t_1}^2((V_{t_1,t}\phi)+\phi_1)}|\bar{\mathcal{F}}_0) \\ &= \exp(-X_0^2(V_{0,t_1}((V_{t_1,t}\phi)+\phi_1))). \end{aligned}$$

This uniquely identifies the joint distribution of $(X_{t_1}^2, X_t^2)$ conditional on X^1. Iterating the above a finite number of times, we have identified the finite-dimensional distributions of X^2 conditional on X^1, and in fact have shown that conditional on X^1, X^2 has the law of the measure-valued process considered by Dynkin and Kuznetsov in (5.7). ∎

Proof of Theorem 1.5. Let (X^1, X^2) be a solution to $\mathbf{M}_{X_0^1,X_0^2}^{-\gamma_1,\gamma_2}$ and let P denote its law on the canonical space of $\mathcal{M}_F{}^2$-valued paths. Conditionally on X, let Y denote a super-Brownian motion with immigration $\gamma_1 L_t(X^1, X^2)$ (see Theorem 1.1 of Barlow, Evans and Perkins (1991)) constructed perhaps on a larger probability space. This means for $\phi \in C_b^2(\mathbb{R}^2)$,

$$Y_t(\phi) = \gamma_1 L_t(X^1, X^2)(\phi) + M_t^Y(\phi) + \int_0^t Y_s(\frac{\sigma^2\Delta\phi}{2})\,ds,$$

where

$$\langle M^Y(\phi)\rangle_t = \int_0^t Y_s(2\phi^2)\,ds,$$

and M^Y is orthogonal with respect to the M^{X^i}, $i = 1, 2$. All these martingales are martingales with respect to a common filtration. Then it is easy to check that $\bar{X}^1 = X^1 + Y$ satisfies the martingale problem characterizing super-Brownian motion starting at X_0^1, i.e., is as in the first component in $\mathbf{M}_{X_0^1,X_0^2}^{0,0}$. Therefore there is jointly continuous function, $\bar{u}^1(t,x)$, with compact support such that $\bar{X}_t^1(dx) = \bar{u}^1(t,x)dx$ (see, e.g., Theorem III.4.2 and Corollary III.1.7 of Perkins (2002)) and so there is a bounded function on compact support, $u^1(t,x)$, so that $X_t^1(dx) = u^1(t,x)dx$ by the domination $X^1 \le \bar{X}^1$. Let $\phi \in C_b(\mathbb{R}^d)$. Then Lebesgue's differentiation theorem implies that

$$\lim_{\delta\to 0} \int p_\delta(x_1 - x_2)\phi(\frac{x_1+x_2}{2})u^1(s,x_1)dx_1 = \phi(x_2)u^1(s,x_2) \text{ for Lebesgue a.a. } (s,x_2)\ a.s.$$

Moreover the approximating integrals are uniformly bounded by $\|\phi\|_\infty \|u^1\|_\infty$ and so by Dominated Convergence one gets from the definition of $L(X^1, X^2)$ that

$$L_t(X^1, X^2)(\phi) = \int_0^t \int \phi(x_2) u^1(s, x_2) X_s^2(dx_2)\, ds.$$

Evans and Perkins (1994) (Theorem 3.9) used Dawson's Girsanov theorem to show there is a unique in law solution to $\mathbf{M}^{-\gamma_1,0}_{X_0^1,X_0^2}$ in our one-dimensional setting. If $P^{-\gamma_1,0}$ denotes this unique law on the canonical path space of measures, then they also showed

$$P^{-\gamma_1,0} << P_{X_0^1} \times P_{X_0^2}, \tag{5.20}$$

the product measure of two super-Brownian motions with diffusion parameter σ^2 and branching rate 2. Our boundedness of u^1 shows that

$$\int_0^t \int u^1(s,x)^2 X_s^2(dx) < \infty \text{ for all } t > 0 \ P - a.s. \text{ and } P^{-\gamma_1,0} - a.s.$$

The latter is a special case of our argument when $\gamma_2 = 0$. This allows us to apply Dawson's Girsanov theorem (see Theorem IV. 1.6 (a) of Perkins (2002)) to conclude that

$$\frac{dP}{dP^{-\gamma_1,0}}\Big|_{\mathcal{F}_t} = \exp\Big\{\int_0^t \int u^1(s,x)/2\, M^{X^2}(ds,dx) - \frac{1}{8}\int_0^t \int u^1(s,x)^2 X_s^2(dx) ds\Big\}.$$

Here M^{X^2} is the martingale measure associated with X_2 and u^1 is the density of X^1, both under $P^{-\gamma_1,0}$. Although the Girsanov theorem quoted above considered absolute continuity with respect to $P_{X_0^1} \times P_{X_0^2}$, the same proof gives the above result. This proves uniqueness of P and, together with (5.20) shows that P is absolutely continuous with respect to $P_{X_0^1} \times P_{X_0^2}$. This gives the required properties of the densities of X^i as they are well-known for super-Brownian motion (see Theorem III.4.2 of Perkins (2002)).

■

6 Proof of the Concentration Inequality–Proposition 2.4

As we will be proving the concentration inequality for the ordinary rescaled critical branching random walk, $\bar{X}^{1,N}$, in order to simplify the notation we will write X^N for $\bar{X}^{1,N}$, and write ξ^N, or just ξ, for $\bar{\xi}^{1,N}$. Dependence of the expectation on the initial measure $X_0^N = \bar{X}_0^{1,N}$ will be denoted by $E_{X_0^N}$. $\{P_u^N, u \geq 0\}$ continues to denote the semigroup of our rescaled continuous time random walk B^N.

Notation. If $\psi : \mathcal{Z}_N \to \mathbb{R}$, let $P^N\psi(x) = \sum_y p_N(y-x)\psi(y)$ and let

$$R\psi(x) = R^N\psi(x) = \sum_{k=0}^{\infty} 2^{-k}(P^N)^k \psi(x).$$

To bound the mass in a fixed small ball we will need good exponential bounds. Here is a general exponential bound whose proof is patterned after an analogous bound for super-Brownian motion (see e.g. Lemma III.3.6 of Perkins (2002)). The discrete setting does complicate the proof a bit.

Proposition 6.1 *Let $f:\mathcal{Z}_N\to\mathbb{R}_+$ be bounded and define*

$$\bar f^N(u)=\|P^N_u f\|_\infty,\quad \bar I^N f(t)=\int_0^t \bar f^N(u)\,du.$$

If $t>0$ satisfies

$$\bar I^N f(t)\le \frac{1}{14}\exp(-4\|f\|_\infty/N), \tag{6.1}$$

then

$$E_{X_0^N}\Big(\exp(X_t^N(f))\Big)\le \exp\Big(2X_0^N(P_t^N Rf)\Big). \tag{6.2}$$

Proof Assume $\phi:[0,t]\times\mathcal{Z}_N\to\mathbb{R}_+$ is such that ϕ and $\dot\phi\in C_b([0,t]\times\mathcal{Z}_N)$. Then $\mathbf{M}^{N,0,0}_{X_0^{1,N},X_0^{2,N}}$ and (2.9) imply

$$\begin{aligned}X_t^N(\phi_t) &= X_0^N(\phi_0)+\int_0^t X_s^N(\Delta^N\phi_s+\dot\phi_s)\,ds\\ &+ \sum_x\int_0^t\int\frac{1}{N}\phi(s,x)1(u\le\xi_{s-}(x))[\hat\Lambda_x^{1,+}(ds,du)-\hat\Lambda_x^{1,-}(ds,du)]\\ &+ \sum_{x,y}\int_0^t\int\frac{1}{N}[\phi(s,x)-\phi(s,y)]1(u\le\xi_{s-}(y))\hat\Lambda^{1,m}_{x,y}(ds,du).\end{aligned}\tag{6.3}$$

A short calculation using Ito's lemma for Poisson point processes (see p. 66 in Ikeda and Watanabe (1981)) shows that

$$\begin{aligned}&\exp\Big(X_t^N(\phi_t)\Big)-\exp\Big(X_0^N(\phi_0)\Big)\\ &\quad=\int_0^t\exp\Big(X_s^N(\phi_s)\Big)[X_s^N(\Delta^N\phi_s+\dot\phi_s)+d_s^N]\,ds+\tilde M_t^N,\end{aligned}\tag{6.4}$$

where

$$\begin{aligned}d_s^N &= N\sum_x\xi_s(x)\Big[\exp\Big(\frac{1}{N}\phi(s,x)\Big)+\exp\Big(-\frac{1}{N}\phi(s,x)\Big)-2\Big]\\ &\quad+N\sum_y\xi_s(y)\sum_x p_N(y-x)\Big[\exp\Big(\frac{1}{N}(\phi(s,x)-\phi(s,y)\Big)-\frac{1}{N}(\phi(s,x)-\phi(s,y))-1\Big],\end{aligned}$$

and $\tilde M^N$ is a locally bounded local martingale. In fact

$$\begin{aligned}\tilde M_t^N &= \sum_x\int_0^t\int\exp\Big(X_{s-}^N(\phi_s)+\frac{1}{N}1(u\le\xi_{s-}(x))\phi(s,x)\Big)-\exp\Big(X_{s-}^N(\phi_s)\Big)\hat\Lambda_x^{1,+}(ds,du)\\ &\quad+\sum_x\int_0^t\int\exp\Big(X_{s-}^N(\phi_s)-\frac{1}{N}1(u\le\xi_{s-}(x))\phi(s,x)\Big)-\exp\Big(X_{s-}^N(\phi_s)\Big)\hat\Lambda_x^{1,-}(ds,du)\\ &\quad+\sum_{x,y}\int_0^t\int\exp\Big(X_{s-}^N(\phi_s)+\frac{1}{N}1(u\le\xi_{s-}(y))(\phi(s,x)-\phi(s,y))\Big)-\exp\Big(X_{s-}^N(\phi_s)\Big)\hat\Lambda^{1,m}_{x,y}(ds,du).\end{aligned}$$

Assume now that for some $c_0 > 0$,

$$\|\phi\|_\infty \le 4c_0 N, \tag{6.5}$$

and note that if $|w| \le 4c_0$, then

$$0 \le e^w - 1 - w = w^2 \sum_{k=0}^\infty \frac{w^k}{(k+2)!} \le w^2 \frac{e^{4c_0}}{2}. \tag{6.6}$$

Now use (6.6) with $w = \phi(s,x)/N \in [0, 4c_0]$ or $w = \frac{\phi(s,x)-\phi(s,y)}{N} \in [-4c_0, 4c_0]$ to see that

$$\begin{aligned} d_s^N &\le e^{4c_0} X_s^N(\phi_s^2) + \frac{e^{4c_0}}{2}\frac{1}{N}\sum_y \xi_s(y)\sum_x p_N(y-x)(\phi(s,x)-\phi(s,y))^2 \\ &\le e^{4c_0} X_s^N(\phi_s^2) + \frac{e^{4c_0}}{2} X_s^N\Big(\phi_s^2 + P^N(\phi_s^2)\Big) \\ &= e^{4c_0}\frac{3}{2} X_s^N(\phi_s^2) + \frac{e^{4c_0}}{2} X_s^N(P^N(\phi_s^2)). \end{aligned} \tag{6.7}$$

Now assume $t > 0$ satisfies (6.1), let $c_1 = 7\exp(4\|f\|_\infty/N)$ and define $\kappa(u) = (1 - c_1 \bar{I}^N f(t-u))^{-1}$ for $u \in [0,t]$. Introduce

$$\phi(u,x) = P_{t-u}^N Rf(x)\kappa(u), \quad u \le t, \quad x \in \mathcal{Z}_N.$$

As convoluting P^N with P_t^N amounts to running B^N until the first jump after time t, one readily sees that these operators commute and hence so do R and P_t^N. Therefore

$$|P_u^N(Rf)(x)| = |R(P_u^N f)(x)| \le 2\bar{f}^N(u), \quad u \le t. \tag{6.8}$$

We also have

$$\begin{aligned} P^N(P_u^N Rf)(x) &= P^N R(P_u^N f)(x) = \sum_{k=0}^\infty 2^{-k}(P^N)^{k+1}(P_u^N f)(x) \\ &\le 2R(P_u^N f)(x) = 2P_u^N(Rf)(x). \end{aligned} \tag{6.9}$$

By (6.1) and (6.8), for $u \le t$,

$$|\phi(u,x)| \le 2\bar{f}^N(t-u)\kappa(u) \le 4\bar{f}^N(t-u) \le \frac{4\|f\|_\infty}{N} N,$$

and so (6.5) holds with $c_0 = \frac{\|f\|_\infty}{N}$. Clearly $\phi, \dot\phi \in C_b([0,t]\times \mathcal{Z}_N)$ and so (6.7) is valid with this choice of c_0. We therefore have

$$\begin{aligned}
&X_s^N(\dot\phi_s + \Delta^N\phi_s) + d_s^N \\
&= X_s^N(P_{t-s}^N Rf)\dot\kappa_s + d_s^N \\
&= -X_s^N(P_{t-s}^N Rf)\kappa_s^2 c_1 \bar f^N(t-s) + d_s^N \\
&\le \kappa_s^2 X_s^N\Big(\frac{e^{4c_0}3}{2}\big(P_{t-s}^N Rf\big)^2 + \frac{e^{4c_0}}{2}P^N\big((P_{t-s}^N Rf)^2\big) - c_1\bar f^N(t-s)P_{t-s}^N Rf\Big) \quad \text{(by (6.7))} \\
&\le \kappa_s^2 \bar f^N(t-s) X_s^N\Big(3e^{4c_0}\big(P_{t-s}^N Rf\big) + e^{4c_0}P^N\big(P_{t-s}^N Rf\big) - c_1 P_{t-s}^N Rf\Big) \quad \text{(by (6.8))} \\
&\le \kappa_s^2 \bar f^N(t-s)\Big[5e^{4c_0} - c_1\Big] X_s^N\big(P_{t-s}^N Rf\big) \quad \text{(by (6.9))} \\
&\le 0,
\end{aligned}$$

the last by the definition of c_1. Now return to (6.4) with the above choice of ϕ. By choosing stopping times $T_k^N \uparrow \infty$ as $k\to\infty$ such that $E(\tilde M^N_{t\wedge T_k^N}) = 0$ and using Fatou's lemma we get from the above that

$$\begin{aligned}
E\Big(\exp\big(X_t^N(f)\big)\Big) &\le E\Big(\exp\big(X_t^N(Rf)\big)\Big) \\
&\le \exp\Big[\frac{X_0^N(P_t^N Rf)}{1 - c_1 \bar I^N f(t)}\Big] \\
&\le \exp\Big(2X_0^N(P_t^N Rf)\Big) \quad \text{(by (6.1))}.
\end{aligned}$$

■

We now specialize the above to obtain exponential bounds on the mass in a ball of radius r. In fact we will use this bound for the ball in a torus and so present the result in a more general framework. Lemma 7.3 below will show that the key hypothesis, (6.10) below, is satisfied in this context.

Corollary 6.2 *Let $c_{6.10} \ge 1$, $T>0$ and $\delta \in (0, 2\wedge d)$. There is an $r_0 = r_0(c_{6.10}, \delta, T) \in (0,1]$ such that for any N, $r \in [N^{-1/2}, r_0]$ and any $C \subset \mathbb{R}^d$ satisfying*

$$\sup_x \Pi_x^N(B_u^N \in C) \le c_{6.10} r^d (1 + u^{-d/2}) \quad \forall u \in (0,T], \tag{6.10}$$

then for any $c>0$,

$$E_{X_0^N}\Big[\exp\big(r^{\delta - 2\wedge d} X_t^N(C)\big)\Big] \le e^{2c},$$

for all $0\le t\le T$ satisfying

$$X_0^N(P_t^N R 1_C) \le c r^{2\wedge d - \delta}. \tag{6.11}$$

Proof We want to apply Proposition 6.1 with $f = r^{\delta - 2\wedge d}1_C$, where $N^{-1/2}\le r\le 1$ and $C\subset \mathbb{R}^d$ satisfies (6.10). Note that

$$\|f\|_\infty \le r^{\delta - 2\wedge d} \le N^{\frac{-\delta}{2} + \frac{2\wedge d}{2}} \le N, \tag{6.12}$$

and by (6.10), for $t \leq T$

$$\begin{aligned}\bar{I}^N f(t) &\leq r^{\delta-2\wedge d} c_{6.10}\Big[\int_0^t r^d\,du + \int_0^t (r^d u^{-d/2}) \wedge 1\,du\Big] \\ &\leq r^{\delta-2\wedge d} c_{6.10}\Big[r^d t + \int_0^{r^2} du + r^d \int_{r^2}^t u^{-d/2}\,du\,1(t > r^2)\Big].\end{aligned}$$

If

$$\phi_d(r,t) = 1 + t + [\log^+(t/r^2)1(d=2)],$$

then a simple calculation leads to

$$\bar{I}^N f(t) \leq 3c_{6.10} r^\delta \phi_d(r,t) \quad t \leq T.$$

In view of (6.12), condition (6.1) in Proposition 6.1 will hold for all $t \leq T$ if $r \leq r_0(c_{6.10}, \delta, T)$ for some $r_0 > 0$. Therefore Proposition 6.1 now implies the required result. ∎

Corollary 6.3 *If $0 < \theta \leq N$ and $t \leq e^{-4}(14\theta)^{-1}$, then*

$$E_{X_0^N}(\exp(\theta X_t^N(1)) \leq \exp(4\theta X_0^N(1)).$$

Proof Take $f \equiv \theta$ (θ as above) in Proposition 6.1. Note that condition (6.1) holds iff $t \leq \exp(-4\theta/N)(14\theta)^{-1}$ and so for $\theta \leq N$ is implied by our bound on t. As $P_t^N Rf = 2\theta$, Proposition 6.1 gives the result. ∎

Remark 6.4 *The above corollary is of course well known as $X_t^N(1) = Z_{Nt}/N$, where $\{Z_u, u \geq 0\}$ is a rate 1 continuous time Galton-Watson branching process starting with $NX_0^N(1)$ particles and undergoing critical binary branching. It is easy to show, e.g. by deriving a simple non-linear o.d.e in t for $E(u^{Z_t}|Z_0 = 1)$, (see (9.1) in Ch. V of Harris (1963)) that for $\theta > 0$ and $t < [N(e^{\theta/N} - 1)]^{-1}$, or $\theta \leq 0$ and all $t \geq 0$,*

$$E_{X_0^N}\Big[\exp(\theta X_t^N(1)\Big] = \Big[1 + \frac{e^{\theta/N} - 1}{1 - tN(e^{\theta/N} - 1)}\Big]^{NX_0^N(1)}. \tag{6.13}$$

The above exponential bounds will allow us to easily obtain the required concentration inequality on a mesh of times and spatial locations. To interpolate between these space-time points we will need a uniform modulus of continuity for the individuals making up our branching random walk. For this it will be convenient to explicitly label these individuals by multi-indices

$$\beta \in I \equiv \cup_{n=0}^\infty \{0, 1, \ldots, |\xi_0| - 1\} \times \{0,1\}^n.$$

If $\beta = (\beta_0, \ldots, \beta_n) \in I$, let $\pi\beta = (\beta_0, \ldots, \beta_{n-1})$ if $n \geq 1$ be the parent index of β, set $\pi\beta = \emptyset$ if $n = 0$, and let $\beta|i = (\beta_0, \ldots, \beta_i)$ if $0 \leq i \leq n \equiv |\beta|$. Define $\beta|(-1) = \emptyset$.

Let $\{\tau^\beta : \beta \in I\}$, $\{b^\beta : \beta \in I\}$ be two independent collections of i.i.d. random variables with τ^β exponentially distributed with rate $2N$ and $P(b^\beta = 0) = P(b^\beta = 2) = 1/2$. Let $T^\emptyset = 0$ and define

$$T^\beta = \sum_{0 \leq i \leq |\beta|} \tau^{\beta|i} 1\Big(\prod_{j=1}^{i-1} b^{\beta|j} > 0\Big).$$

We will think of $[T^{\pi\beta}, T^\beta)$ as the lifetime of particle β, so that $T^\beta = T^{\pi\beta}$ (iff $b^{\beta|j} = 0$ for some $j < |\beta|$) means particle β never existed, while b^β is the number of offspring of particle β. Write $\beta \sim t$ iff particle β is alive at time t, i.e., iff $T^{\pi\beta} \leq t < T^\beta$. Let $\{B_0^k : 0 \leq k < |\xi_0|\}$ be points in $\mathcal{Z}_N$ satisfying $\sum_k 1(B_0^k = x) = \xi_0(x)$ for all $x \in \mathcal{Z}_N$. Now condition on $\{T^\beta : \beta \in I\}$. Let $\{B_s^\beta - B_{T^{\pi\beta}}^\beta : s \in [T^{\pi\beta}, T^\beta)\}_{T^{\pi\beta} < T^\beta}$ be a collection of independent copies of B^N, starting at 0. Formally they may be defined inductively with $B_{T^{\pi\beta}}^\beta$ chosen to be $B_{T^{\pi\beta}-}^{\pi\beta}$. Such labelling schemes may be found in Ch. 8 of Walsh (1986) or Ch. II of Perkins (2002).

If $\hat{\xi}(x) = \sum_{\beta\sim t} 1(B_t^\beta = x)$, then $(\xi_t, t \geq 0)$ and $(\hat{\xi}_t, t \geq 0)$ are identical in law. One can see this by noting $\{\hat{\xi}_t : t \geq 0\}$ is an S_F-valued Markov process with the same jump rates and initial condition as $\{\xi_t : t \geq 0\}$. Formally one can work with the associated empirical processes $\hat{X}_t^N = \frac{1}{N}\sum_x \hat{\xi}_t(x)$ and X_t^N and calculate their respective generators as in Section 9.4 of Ethier and Kurtz (1986). The generator of the former can be found by arguing as in Section II.4 of Perkins (2002).

Alternatively, one can in fact define the above branching particle system from our original Poisson equations (2.2) since one can use the uniform variables in our driving Poisson point processes to trace back ancestries. We briefly outline the construction. We begin by labeling the $|\xi_0|$ initial particles as above with multi-indices $0, \dots, |\xi_0| - 1 \in I$ and assigning each particle at each site x an integer $1 \leq k \leq \xi_0(x)$ that we call its level. Since there are only finitely many particles we can explain how these labels and levels propagate forwards in time at the jump times in (2.2). If at time t there is an "arrival" in $\Lambda_x^{1,+}$ with height $u \in [k-1, k)$, where $k \leq \xi_{t-}(x)$, then the particle at level k at site x branches. If β is the label of this branching particle at time $t-$ then two new particles are created at x with labels $\beta 0$ at level k and $\beta 1$ at level $\xi_t(x) = \xi_{t-}(x) + 1$. ($\beta$ no longer labels a particle at time t.) All other particles keep their current labels and levels. If at time t there is an "death" in $\Lambda_x^{1,-}$ with height $u \in [k-1, k)$, where $k \leq \xi_{t-}(x)$, then the particle at level k at site x dies. It is removed from the population (it's label β no longer labels a particle at time t), and all particles at x with levels greater than k have their levels decreased by 1, and keep their current labels at time t. If at time t there is a "migration" from y to x in $\Lambda_{x,y}^{1,m}$ with height $u \in [k-1, k)$, where $k \leq \xi_{t-}(y)$, then the particle at level k at site y migrates to x where its new level is $\xi_t(x) = \xi_{t-}(x) + 1$. The particles at y with levels greater than k have their levels reduced by 1. The migrating particle keeps its label β, so that this label now refers to particle $\xi_t(x)$ at site x. All other particles keep their current labels and levels. At this point we have inductively defined a multi-index $\alpha(t, x, k) \in I$ which labels the particle with level $k \leq \xi_t(x)$ at site $x \in \mathcal{Z}_N$ and time $t \geq 0$. Here we use α to denote this random function as β will denote the independent I-valued variable. As the birth of a label coincides with a branching event and its death must coincide with either a branching event (it has 2 children) or a death event (it has 0 children) it is clear the set of times at which β is alive, $A_\beta = \{t \geq 0 : \exists x, k \text{ s.t. } \alpha(t, x, k) = \beta\}$ is a left semi-closed interval $[U^\beta, T^\beta)$ (possibly empty) with $U^\beta = T^{\pi\beta}$, where again $T^\emptyset = 0$. Also we see that $T^\beta - T^{\pi\beta}$ is exponential with rate $2N$ (the time to a birth or death event of the appropriate height corresponding to the level labelled by β). The independence properties of the Poisson point processes allow one to show that the collection of these exponential increments are independent. They are only indexed by those β which have a positive lifespan but the collection can be padded out with iid exponentials to reconstruct the indendent τ^β's described above. The same reasoning applies to reconstruct the b^β's. For $t \in [T^{\pi\beta}, T^\beta)$ define B_t^β and ℓ_t^β by

$$\alpha(t, B_t^\beta, \ell_t^\beta) = \beta.$$

Then B^β starts at $B^{\pi\beta}_{T^{\pi\beta}-}$ and jumps according to migration events of the appropriate height. Therefore conditional on the $\{T^{\beta'}, b^{\beta'}\}$ it is a copy of B^N. The independence property of the Poisson point processes $\Lambda^{1,m}_{x,y}$ show they are (conditionally on the birth and death times) independent.

We have shown that the collection of labelled branching particles constructed from (2.2) are identical in law to those described above and used for example in Ch. II of Perkins (2002). We have been a bit terse here as the precise details are a bit tedious and in fact one can proceed without this intrinsic labelling as noted above since starting from the labelled system one can reconstruct the process of interest, X^N.

However you prefer to proceed, it will be convenient to extend the definition of B^β_s to all $s \in [0, T^\beta)$ by following the paths of β's ancestors, i.e.,

$$B^\beta_s = B^{\beta|i}_s \quad \text{if } T^{\beta|i-1} \le s < T^{\beta|i}, \quad i \le |\beta|.$$

Then for $\beta \sim t$, $s \to B^\beta_s$ is a copy of B^N on $[0, T^\beta)$ as it is a concatenation of independent random walks with matching endpoints.

Here is the modulus of continuity we will need.

Proposition 6.5 *Let $\varepsilon \in (0, 1/2)$. There is a random variable $\delta_{N,\varepsilon}$ s.t. for some $c_i(\varepsilon) > 0$, $i = 1, 2$,*

$$P(\delta_{N,\varepsilon} < 2^{-n}) \le c_1(\varepsilon) \exp\Big(-c_2(\varepsilon) 2^{n\varepsilon}\Big) \text{ for all } 2^{-n} \ge N^{-1}, \tag{6.14}$$

and whenever $\delta_{N,\varepsilon} \ge 2^{-n} \ge N^{-1}$, for $n \in \mathbb{N}$,

$$|B^\beta_t - B^\beta_{j2^{-n}}| \le 2^{-n(\frac{1}{2}-\varepsilon)} \quad \text{for any } j \in \{0, 1, \dots, n2^n\},\ t \in [j2^{-n}, (j+1)2^{-n}] \text{ and } \beta \sim t. \tag{6.15}$$

Proof In Section 4 of Dawson, Iscoe and Perkins (1989) a more precise result is proved for a system of branching Brownian motions in discrete time. Our approach is the same and so we will only point out the changes needed in the argument. Let $h(r) = r^{\frac{1}{2}-\varepsilon}$.

In place of the Brownian tail estimates we use the following exponential bound for our continuous time random walks which may be proved using classical methods:

$$\Pi^N_0(|B^N(2^{-n})| > h(2^{-n})) \le 3d \exp\Big(-c_{6.16} 2^{n\varepsilon}\Big) \text{ for } 2^{-n} \ge N^{-1} \text{ and some positive } c_{6.16}(\varepsilon). \tag{6.16}$$

We write $\beta' > \beta$ iff β' is a descendent of β, i.e., if $\beta = \beta'|i$ for some $i \le |\beta'|$. The next ingredient is a probability bound on

$$\Gamma_n(j2^{-n}) = \text{card}\{\beta \sim j2^{-n} : \exists \beta' > \beta \text{ s.t. } \beta' \sim (j+1)2^{-n}\},$$

the number of ancestors at time $j2^{-n}$ of the population at $(j+1)2^{-n}$. The probability of a particular individual having ancestors at time 2^{-n} in the future is $(1 + N2^{-n})^{-1}$ (use (6.13) with $\theta = -\infty$). From this and easy binomial probability estimates one gets

$$P(\exists j \le n2^n \text{ s. t. } \Gamma_n(j2^{-n}) \ge [e^2 X^N_{j2^{-n}}(1) + 1]2^n) \le n2^n \exp(-2^n). \tag{6.17}$$

Define $n_1 = n_1(N)$ by $2^{-n_1} \ge \frac{1}{N} > 2^{-n_1-1}$ and for $1 \le n \le n_1$ define

$$A_n = \Big\{ \max_{1 \le j \le n2^n} \max_{\beta \in \Gamma_n(j2^{-n})} |B^\beta_{j2^{-n}} - B^\beta_{(j-1)2^{-n}}| > h(2^{-n}) \Big\}.$$

Set $A_0 = \Omega$. Next introduce

$$A^*_{n_1} = \Big\{ \max_{j \le n_1 2^{n_1}} \sup_{t \in [j2^{-n_1},(j+1)2^{-n_1}], \beta \sim t} |B^\beta_t - B^\beta_{j2^{-n_1}}| > h(2^{-n}) \Big\}.$$

Set $\delta_{N,\varepsilon} = 2^{-n}$ iff $\omega \notin A^*_{n_1} \cup (\cup_{n'=n}^{n_1} A_{n'})$ and $\omega \in A_{n-1}$ for $n \in \mathbb{N}$. If $\omega \in A^*_{n_1}$, set $\delta_{N,\varepsilon} = 1$. A standard binomial expansion argument (as in Dawson, Iscoe and Perkins (1989)) shows that (6.15) holds with some universal constant c in front of the bound on the righthand side. The latter is easily removed by adjusting ε. It follows easily from (6.16) and (6.17) that $P(A_n) \le c_1 \exp\Big(-c_2 2^{n\varepsilon}\Big)$.

Therefore to show (6.14), and so complete the proof it remains to show

$$P(A^*_{n_1}) \le c_1 \exp\Big(-c_2 2^{n_1 \varepsilon}\Big). \tag{6.18}$$

This bound is the only novel feature of this argument. It arises due to our continuous time setting. Let $0 \le j \le n_1 2^{n_1}$ and condition on $N_j = NX^N_{j2^{-n_1}}(1)$. For each of these N_j particles run it until its first branching event. The particle is allowed to give birth iff it splits into two before time $(j+1)2^{-n_1}$. If it splits after this time or if it dies at the branching event, it is killed. Note that particles will split into two with probability

$$q = \frac{1}{2}\int_0^{2^{-n_1}} e^{-2Ns} 2N ds = \frac{1}{2}(1 - e^{-2N2^{-n_1}}) \le \frac{1}{2}(1 - e^{-4}) < \frac{1}{2},$$

the last inequality by our definition of n_1. Let Z_1 be the size of the population after this one generation. Now repeat the above branching mechanism allowing a split only if it occurs before the next time interval of length 2^{-n_1} and continue this branching mechanism until the population becomes extinct. In this way we get a subcritical discrete time Galton-Watson branching process with mean offspring size $\mu = 2q < 1$. Let T_j denote the extinction time of this process. Then $P(T_j > m) \le E(Z_m) = N_j \mu^m$ and so, integrating out the conditioning, we have

$$P(\max_{j \le n_1 2^{n_1}} T_j > m) \le n_1 2^{n_1} N X^N_0(1) \mu^m. \tag{6.19}$$

In keeping track of this branching process until T_j we may in fact be tracing ancestral lineages beyond the interval $[j2^{-n_1}, (j+1)2^{-n_1}]$, which is of primary interest to us, but any branching events in this interval will certainly be included in our Galton Watson branching process as they must occur within time 2^{-n_1} of their parent's branching time. The resulting key observation is therefore that T_j is an upper bound for the largest number of binary splits in $[j2^{-n_1}, (j+1)2^{-n_1}]$ over any line of descent of the entire population. Since the offspring can move at most $1/\sqrt{N}$ from the location of its parent at each branching event, $T_j/\sqrt{N}$ gives a (crude) upper bound on the maximum displacement of any particle in the population over the above time interval. It therefore follows from (6.19) that

$$\begin{aligned} P\Big(\max_{j \le n_1 2^{n_1}} \sup_{t \in [j2^{-n_1},(j+1)2^{-n_1}], \beta \sim t} |B^\beta_t - B^\beta_{j2^{-n_1}}| > mN^{-1/2}\Big) &\le n_1 2^{n_1} N X^N_0(1) \mu^m \\ &\le n_1 2^{2n_1+1} X^N_0(1) \mu^m. \end{aligned}$$

Now take m to be the integer part of $\sqrt{N} h(2^{-n_1})$ in the above. A simple calculation (recall Assumption 2.5) now gives (6.18) and so completes the proof. ∎

Proof of Proposition 2.4. Let $\varepsilon \in (0, 1/2)$ and $T \in \mathbb{N}$. We use $n_0 = n_0(\varepsilon, T)$ to denote a lower bound on n which may increase in the course of the proof. Let $h(r) = r^{\frac{1}{2}-\varepsilon}$ (as above) and $\phi(r) = r^{2\wedge d-\varepsilon}$ for $r > 0$. Following Barlow, Evans and Perkins (1991), let $B_n(y) = B(y, 3h(2^{-n}))$, $\mathcal{B}_n = \{B_n(y) : y \in 2^{-n}\mathbb{Z}^d\}$ and for $n \geq n_0$ define a class $\mathcal{C}_n$ of subsets of $\mathbb{R}^d$ such that from some constants $c_{6.22}$ and $c_{6.23}$, depending only on $\{X_0^N\}$ and ε,

$$\forall B \in \mathcal{B}_n \ \exists C \in \mathcal{C}_n \text{ s. t. } B \subset C \tag{6.20}$$

$$\forall C \in \mathcal{C}_n \ \exists B \in \mathcal{B}_n \text{ s.t. } C \subset \tilde{B} \equiv \cup_{k\in\mathbb{Z}^d} k + B \tag{6.21}$$

$$\forall C \in \mathcal{C}_n,\ N \in \mathbb{N},\quad X_0^N(P_{j2^{-n}}R1_C) \leq c_{6.22}\phi(h(2^{-n})) \ \text{ for } j \in \{0, \ldots, T2^n\} \tag{6.22}$$

$$\text{card}(\mathcal{C}_n) \leq c_{6.23}T2^{c(d)n}. \tag{6.23}$$

To construct $\mathcal{C}_n$ note first that as in Lemma 4.2 for $B \in \mathcal{B}_n$,

$$\begin{aligned} \mu_t^N(B) \equiv X_0^N(P_t^N R1_B) &\leq 2\sup_x X_0^N(B(x, 3h(2^{-n}))) \\ &\leq c_0(\{X_0^N\}, \varepsilon)\phi(h(2^{-n})). \end{aligned} \tag{6.24}$$

Here we have used Assumption 2.5. For a fixed $y \in (2^{-n}\mathbb{Z}^d) \cap [0, 1)^d$ let C be a union of balls in $\mathcal{B}_n$ of the form $k + B_n(y)$, $k \in \mathbb{Z}^d$, where such balls are added until $\max_{0\leq j\leq T2^n} \mu^N_{j2^{-n}}(C) > \phi(h(2^{-n}))$. It follows from (6.24) that for each $0 \leq j \leq 2^nT$, $\mu_{j2^{-n}}(C) \leq \phi(h(2^{-n})) + c_0\phi(h(2^{-n}))$ and so (6.22) holds with $c_{6.22} = 1 + c_0$. Continue the above process with new integer translates of $B_n(y)$ until every such translate is contained in a unique C in $\mathcal{C}_n$. If n_0 is chosen so that $6h(2^{-n_0}) < 1$, then these C's will be disjoint and all but one will satisfy $\max_{0\leq j\leq 2^nT} \mu_{j2^{-n}}(C) > \phi(h(2^{-n}))$. Therefore

$$\begin{aligned} \text{number of } C\text{'s constructed from } B_n(y) &\leq \sum_{j=0}^{2^nT} \mu_{j2^{-n}}(\tilde{B}_n(y))\phi(h(2^{-n}))^{-1} + 1 \\ &\leq X_0^N(1)T2^n\phi(h(2^{-n}))^{-1} + 1. \end{aligned} \tag{6.25}$$

Now repeat the above for each $y \in (2^{-n}\mathbb{Z}^d) \cap [0, 1)^d$. Then (6.25) gives (6.23), and (6.20) and (6.21) are clear from the construction.

Let $0 \leq t \leq T$, $x \in \mathbb{R}^d$ and assume $\frac{1}{N} \leq 2^{-n} \leq \delta_N$, where $\delta_N = \delta_{N,\varepsilon}$ is as in Proposition 6.5. Choose $0 \leq j \leq 2^nT$ and $y \in 2^{-n}\mathbb{Z}^d$ so that $t \in [j2^{-n}, (j + 1)2^{-n})$ and $|y - x| \leq 2^{-n} \leq h(2^{-n})$, respectively. Let $\beta \sim t$ and $B_t^\beta \in B(x, h(2^{-n})$. Then Proposition 6.5 implies

$$\begin{aligned} |B^\beta_{j2^{-n}} - y| &\leq |B^\beta_{j2^{-n}} - B_t^\beta| + |B_t^\beta - x| + |x - y| \\ &\leq 3h(2^{-n}), \end{aligned}$$

and so,

$$B^\beta_{j2^{-n}} \in B_n(y) \subset C \quad \text{for some } C \in \mathcal{C}_n, \tag{6.26}$$

where the last inclusion holds by (6.20). For $C \in \mathcal{C}_n$, let

$$X_t^{N,j2^{-n},C}(A) = \frac{1}{N}\sum_{\beta\sim t} 1(B^\beta_{j2^{-n}} \in C, B^\beta_t \in A), \quad t \geq j2^{-n}$$

denote the time t distribution of descendents of particles which were in C at time $j2^{-n}$. Use (6.26) to see that $X_t^N(B(x,h(2^{-n}))) \leq X_t^{N,j2^{-n},C}(1)$, and therefore

$$\sup_{0\leq t\leq T, x\in\mathbb{R}^d} X_t^N(B(x,h(2^{-n}))) \leq \max_{j\leq 2^nT, C\in\mathcal{C}_n}\sup_{t\leq 2^{-n}} X^{N,j2^{-n},C}_{t+j2^{-n}}(1). \tag{6.27}$$

The process $t \to X^{N,j2^{-n},C}_{t+j2^{-n}}(1)$ evolves like a continuous time critical Galton-Watson branching process–conditional on $\mathcal{F}_{j2^{-n}}$ it has law $P_{X^N_{j2^{-n}}|_C}$–and in particular is a martingale. If $\theta_n, \lambda_n > 0$ we may therefore apply the weak L^1 inequality to the submartingale $\exp(\theta_n X^{N,j2^{-n},C}_{t+j2^{-n}}(1))$ to see that (6.27) implies

(6.28)
$$P\Big(\sup_{t\leq T, x\in\mathbb{R}^d} X_t^N(B(x,h(2^{-n}))) > \lambda_n,\ 2^{-n} < \delta_N\Big) \leq \sum_{j\leq 2^nT}\sum_{C\in\mathcal{C}_n} e^{-\theta_n\lambda_n} E\Big(E_{X^N_{j2^{-n}}|_C}(\exp(\theta_n X^N_{2^{-n}}(1)))\Big).$$

Let $\theta_n = 2^{n\left(\frac{d\wedge 2}{2}-3\varepsilon\right)}$ and $\lambda_n = 2^{-n\left(\frac{d\wedge 2}{2}-4\varepsilon\right)}$. For $n \geq n_0$ we may assume $2^{-n} \leq e^{-4}(14\theta_n)^{-1}$, and as we clearly have $\theta_n \leq 2^n \leq N$, Corollary 6.3 implies

$$E_{X^N_{j2^{-n}}|_C}(\exp(\theta_n X^N_{2^{-n}}(1))) \leq \exp(4\theta_n X^N_{j2^{-n}}(C)). \tag{6.29}$$

Next we apply Corollary 6.2 with $C \in \mathcal{C}_n$ as above, $\delta = \varepsilon$, and $r = h(2^{-n}) \in [N^{-1/2}, h(2^{-n_0})]$. Lemma 7.3 below and (6.21) imply hypothesis (6.10) of Corollary 6.2 and (6.22) implies

$$X_0^N(P^N_{j2^{-n}}R1_c) \leq c_{6.22}\phi(r) = c_{6.22}r^{d\wedge 2-\varepsilon} \quad \forall 0 \leq j2^{-n} \leq T.$$

Note that for $n \geq n_0$,

$$r^{\varepsilon - d\wedge 2} = 2^{-n(\frac{1}{2}-\varepsilon)(\varepsilon - d\wedge 2)} \geq 2^{n\left(\frac{d\wedge 2}{2}-\varepsilon(d\wedge 2)\right)} \geq 4\theta_n.$$

Therefore Corollary 6.2 shows that for $n \geq n_0(\varepsilon, T)$,

$$E\Big(\exp(4\theta_n X^N_{j2^{-n}}(C))\Big) \leq e^{2c_{6.22}} \quad \forall 0 \leq j2^{-n} \leq T.$$

Use this in (6.29) and apply the resulting inequality and (6.23) in (6.28) to see that for $n \geq n_0(\varepsilon, T)$,

$$\begin{aligned}
&P\Big(\sup_{t\leq T, x\in\mathbb{R}^d} X_t^N(B(x,h(2^{-n}))) \geq 2^{-n(\frac{d\wedge 2}{2}-4\varepsilon)}\Big)\\
&\leq (2^nT+1)\, c_{6.23}T\exp(c(d)n)\exp(-2^{n\varepsilon})e^{2c_{6.22}} + P(\delta_N \leq 2^{-n})\\
&\leq T^2 c(\{X_0^N\},\varepsilon)\exp(c'(d)n - 2^{n\varepsilon}) + c_1(\varepsilon)\exp(-c_2(\varepsilon)2^{n\varepsilon}).
\end{aligned}$$

We have used (6.14) from Proposition 6.5 in the last line. If $r_0(\varepsilon,T)=h(2^{-n_0})$, the above bound is valid for (recall $n\geq n_0$ and $2^{-n}\geq N^{-1}$)

$$h(2^{-n})\in[N^{-\frac{1}{2}+\varepsilon},r_0(\varepsilon,T)].$$

A calculation shows that $2^{-n(1-4\varepsilon)}\leq h(2^{-n})^{d\wedge 2-8\varepsilon}2^{-n\varepsilon}$. Therefore elementary Borel-Cantelli and interpolation arguments (in r) show that if

$$\mathcal{H}_{\delta,N}(\varepsilon,T,r_0)=\sup_{0\leq t\leq T,x\in\mathbb{R}^d}\ \sup_{N^{-\frac{1}{2}+\varepsilon}\leq r\leq r_0}\frac{X_t^N(B(x,r))}{r^{d\wedge 2-\delta}},$$

then for any $\eta>0$ there is an $M=M(\eta,\epsilon,T)$ such that

$$\sup_N P(\mathcal{H}_{8\varepsilon,N}(\varepsilon,T,r_0(\varepsilon))\geq M)\leq\eta.$$

Here we are also using

$$P(\sup_{0\leq t\leq T}X_t^N(1)\geq K)\leq X_0^N(1)/K\leq C/K$$

by the weak L^1 inequality. The latter also allows us to replace $r_0(\varepsilon)$ by 1 in the above bound. If we set $r'=r^{\frac{1}{1-2\varepsilon}}\leq r$ so that $r\geq N^{-\frac{1}{2}+\varepsilon}$ iff $r'\geq N^{-1/2}$, one can easily show

$$\sup_{0\leq t\leq T,x\in\mathbb{R}^d}\ \sup_{\frac{1}{\sqrt{N}}\leq r'\leq 1}\frac{X_t^N(B(x,r'))}{r'^{d\wedge 2-10\varepsilon}}\leq\sup_{0\leq t\leq T,x\in\mathbb{R}^d}\ \sup_{N^{-\frac{1}{2}+\varepsilon}\leq r\leq 1}\frac{X_t^N(B(x,r))}{r'^{d\wedge 2-8\varepsilon}},$$

that is $\mathcal{H}_{10\varepsilon,N}(0,T,1)\leq\mathcal{H}_{8\varepsilon,N}(\varepsilon,T,1)$. Therefore we have shown that for each $T\in\mathbb{N}$ and $\varepsilon>0$, $\mathcal{H}_{10\varepsilon,N}(0,T,1)=\sup_{0\leq t\leq T}\rho^N_{10\varepsilon}(X_t^N)$ is bounded in probability uniformly in N. The fact that $X^N_\cdot$ has a finite lifetime which is bounded in probability uniformly in N (e.g. let $\theta\to-\infty$ in (6.13)) now lets us take $T=\infty$ and so complete the proof of Proposition 2.4. ∎

7 Proof of Proposition 3.5 and Lemma 3.7

In this section we will verify some useful moment bounds on the positive colicin models which, through the domination in Proposition 2.2, will be important for the proof of our theorems. As usual, $|x|$ is the L^∞ norm of x.

Lemma 7.1 *Let*

$$S_n=\sum_{i=1}^n X_i,$$

where $\{X_i\,,i\geq 1\}$ are i.i.d. random variables, distributed uniformly on $(\mathbb{Z}^d/M)\cap[0,1]^d$. Then there exist $c,C>0$ such that

$$P\left(\frac{S_n}{\sqrt{n}}\in x+[-r,r]^d\right)\leq Ce^{-c(|x|^2\wedge(|x|\sqrt{n}))}r^d, \tag{7.1}$$

for all $(x,r)\in\mathbb{R}^d\times[n^{-1/2},1]$.

Proof It suffices to bound $P(S_n/\sqrt{n} = x)$ for $x \in \mathbb{Z}^d/M\sqrt{n}$ by the right-hand side with $r = n^{-1/2}$ for then we can add up the points in the cube to get the result. For this, note if $|x| \le 2$ we can ignore the exponentials on the right-hand side, and for $|x| > 2$ we have $|x'| > |x|/2$ for all $x' \in x + [-r,r]^d$ and any $r \le 1$. The local central limit theorem implies

$$\sup_z P(S_n = z) \le Cn^{-d/2}. \tag{7.2}$$

Suppose now that $n = 2m$ is even. For $1 \le i \le d$,

$$P(|S_n^i| \ge an, S_n = y) \le 2P(|S_m^i| \ge am, S_n = y) \tag{7.3}$$

since for $|S_n^i| \ge an$ we must have $|S_m^i| \ge am$ or $|S_n^i - S_m^i| \ge am$ and the increments of S_k^i conditional on $S_n = y$ are exchangeable. A standard large deviations result (see Section 1.9 in Durrett (2004)) implies

$$P(|S_m^i| \ge am) \le e^{-\gamma(a)m}$$

where $\gamma(a) = \sup_{\theta>0} -\theta a + \phi(\theta)$ and $\phi(\theta) = E\exp(\theta X_i)$. γ is convex with $\gamma(0) = 0$, $\gamma'(0) = 0$, and $\gamma''(0) > 0$, so $\gamma(a) \ge c(a^2 \wedge a)$.

Let $y = x\sqrt{n}$ and choose i so that $|y_i| = |y|$. Taking $a = |x|/\sqrt{n}$ it follows that

$$P(|S_m^i| \ge am) \le Ce^{-c(|x|^2 \wedge |x|\sqrt{n})},$$

so using (7.2) and (7.3) we have

$$\begin{aligned} P(S_n = y) = P(|S_n^i| \ge an,\ S_n = y) &\le 2P(|S_m^i| \ge am, S_n = y) \\ &\le E[1(|S_m^i| \ge am)P(S_m = y - S_m(\omega))] \\ &\le Ce^{-c(|x|^2 \wedge |x|\sqrt{n})}n^{-d/2}. \end{aligned}$$

The result for odd times $n = 2m+1$ follows easily from the observation that if $S_n = x$ then $S_{n-1} = y$ where y is a neighbor of x and on the step n the walk must jump from y to x. ∎

Using the local central limit theorem for the continuous time random walk to estimate the probability of being at a given point leads easily to:

Lemma 7.2 *There is a constant c such that for all $s > 0$*

$$\sup_x \Pi_{\mathbf{0}}^N \left(B_s^N \in x + [-r,r]^d\right) \le \frac{cr^d N^{d/2}}{(Ns+1)^{d/2}}, \quad \forall r \ge N^{-1/2}. \tag{7.4}$$

and in particular for $r = 1/\sqrt{N}$ and any $N \ge 1$ we have

$$\sup_x \Pi_{\mathbf{0}}^N \left(B_s^N \in x + \left[-1/\sqrt{N}, 1/\sqrt{N}\right]^d\right) \le \frac{c}{(Ns+1)^{d/2}}. \tag{7.5}$$

We also used the following version of Lemma 7.2 for the torus in the proof of the concentration inequality in the previous section. This time we give the details.

Notation. If $x \in \mathbb{R}^d$, $r > 0$, let $C(x,r) = \cup_{k\in\mathbb{Z}^d} B(x+k,r)$.

Lemma 7.3 *There is a constant C such that*

$$\sup_{x\in\mathbb{R}^d} \Pi_0^N(B_u^N \in C(x,r)) \le Cr^d(1+u^{-d/2}) \quad \text{for all } u > 0,\ r \in [N^{-1/2}, 1].$$

Proof Note first it suffices to consider $x \in [-1,1]^d$ and

$$(7.6) \qquad N^{-1/2} \le r \le \sqrt{u/2}.$$

If $\{S_n\}$ is as in Lemma 7.1, we may define $B_u^N = S_{\tau_u^N}/\sqrt{N}$, where τ^N is an independent rate N Poisson process. We have

$$(7.7) \quad \Pi_0^N(B_u^N \in C(x,r)) \le P(\tau_u^N \notin [Nu/2, 2Nu]) + \sum_{Nu/2 \le j \le 2Nu} P(\tau_u^N = j) \sum_{k \in \mathbb{Z}^d} P\Big(\frac{S_j}{\sqrt{j}} \in B\Big((x+k)\sqrt{\frac{N}{j}}, r\sqrt{\frac{N}{j}}\Big)\Big).$$

Standard exponential bounds imply

$$(7.8) \qquad P(\tau_u^N \notin [Nu/2, 2Nu]) \le e^{-cNu}$$

for some $c > 0$. By (7.6), we have $j^{-1/2} \le r\sqrt{\frac{N}{j}} \le 1$ for $Nu/2 \le j \le 2Nu$ and so we may use Lemma 7.1 to bound the series on the right-hand side of (7.7) by

$$\begin{aligned}
\sum_{Nu/2 \le j \le 2Nu} \quad & P(\tau_u^N = j) \sum_{k \in \mathbb{Z}^d} C \exp\Big(-c\Big(\frac{|x+k|^2 N}{j} \wedge |x+k|\sqrt{N}\Big)\Big)(N/j)^{d/2} r^d \\
&\le 2^{d/2} C \sum_{k \in \mathbb{Z}^d} \exp\Big(-c\Big(\frac{|x+k|^2}{2u} \wedge |x+k|\sqrt{N}\Big)\Big) u^{-d/2} r^d \\
&\le C \sum_{k \in \mathbb{Z}^d} \Big(\exp\Big(\frac{-c|x+k|^2}{u}\Big) + \exp(-c|x+k|)\Big) u^{-d/2} r^d \\
&\le C(u^{d/2} + 1) u^{-d/2} r^d.
\end{aligned}$$

In the last line we have carried out an elementary calculation (recall $|x| \le 1$) and as usual are changing constants from line to line. Use the above and (7.8) in (7.7) to get

$$\begin{aligned}
\sup_x \Pi_0^N(B_u^N \in C(x,r)) &\le e^{-cNu} + C(1 + u^{-d/2}) r^d \\
&\le C(Nu)^{-d/2} + C(1 + u^{-d/2}) r^d \\
&\le C r^d u^{-d/2} + C(1 + u^{-d/2}) r^d,
\end{aligned}$$

since $r \ge N^{-1/2}$. The result follows. ∎

Eliminating times near 0 and small $|x|$ we can improve Lemma 7.2.

Lemma 7.4 *Let $0 < \delta^* < 1/2, 0 < \delta' < 1$. Then for any $a \ge 0$, there exists $c = c(a, \delta, \delta^*)) > 0$ such that*

$$\Pi_{\mathbf{0}}^N \left(B_s^N \in x + [-r,r]^d\right) \le c r^d (s^a \wedge 1),$$

for all $r \ge N^{-1/2}$, $s \ge N^{\delta'-1}$, and $|x| \ge s^{1/2-\delta^*}$.

Proof As before, it suffices to consider $r = N^{-1/2}$. Let τ_s^N = number of jumps of B^N up to time s. Then, as in 7.8, there exists $c > 0$ such that

$$\Pi_{\mathbf{0}}^N\left(\tau_s^N > 2Ns \text{ or } \tau_s^N < Ns/2\right) \le e^{-csN}. \tag{7.9}$$

Now for $Ns/2 \le j \le 2Ns$ and $r \ge N^{-1/2}$, Lemma 7.1 implies

$$P\left(S_j/\sqrt{j} \in \left(x + [-N^{-1/2}, N^{-1/2}]^d\right)\frac{\sqrt{N}}{\sqrt{j}}\right) \le C\exp\left(-c\left(\frac{|x|^2 N}{j} \wedge \frac{|x|\sqrt{N}}{\sqrt{j}}\right)\right) j^{-d/2}.$$

Using $2Ns \ge j \ge Ns/2$, $|x| \ge s^{1/2-\delta^*}$, and calculus the above is at most

$$\begin{aligned}
&C\exp(-c(|x|^2/s \wedge |x|/\sqrt{s}))N^{-d/2}s^{-d/2}\\
&\le C[\exp(-cs^{-\delta^*}) + \exp(-cs^{-2\delta^*})]N^{-d/2}s^{-d/2}\\
&\le C(a,\delta^*)(s^a \wedge 1)N^{-d/2}
\end{aligned}$$

To combine this with (7.9) to get the result, we note that $(s^a \wedge 1)^{-1}e^{-csN/2}$ is decreasing in s, so if $s \ge N^{\delta'-1}$, then

$$\frac{e^{-csN/2}}{s^a \wedge 1} \le \frac{e^{-cN^{\delta'}/2}}{N^{a(\delta'-1)}} \le C(\delta', a).$$

Using $e^{-cN^{\delta'}/2} \le c(\delta')N^{-d/2}$ we see that the right-hand side of (7.9) is also bounded by

$$C(\delta', a)(s^a \wedge 1)N^{-d/2}$$

and the proof is completed by combining the above bounds. ∎

Lemma 7.5 *For any $0 < \delta^* < 1/2$, let*

$$Q_N = \left\{(s,x) : |x| \ge s^{1/2-\delta^*} \vee N^{-1/2+\delta^*}\right\}.$$

Let $I_N(s,x) = N^{d/2}\,\Pi_{\mathbf{0}}^N\left(\left|B_s^N - x\right| \le 1/\sqrt{N}\right)$. Then there exists a constant $c_{7.10} = c_{7.10}(\delta^*)$ such that

$$\sup_{N\ge 1}\ \sup_{(s,x)\in Q_N} I_N(s,x) \le c_{7.10} \tag{7.10}$$

Proof Fix $0 < \delta' < \delta^*$. If $s \ge N^{\delta'-1}$ then the result follows from Lemma 7.4. For $s \le N^{\delta'-1}$ and $|x| \ge N^{-1/2+\delta^*}$, $|B_s^N - x| \le 1/\sqrt{N}$ implies $|B_s^N| > N^{-1/2+\delta^*}/2$. Therefore for $p \ge 2$, a simple martingale inequality implies

$$\begin{aligned}
I_N(s,x) &\le N^{d/2}\Pi_{\mathbf{0}}^N\left(\left|B_s^N\right| > N^{-1/2+\delta^*}/2\right) \le c_p N^{d/2}N^{p(1/2-\delta^*)}\Pi_{\mathbf{0}}^N\left[\left|B_s^N\right|^p\right]\\
&\le c_p N^{d/2}N^{p(1/2-\delta^*)}s^{p/2} \le c_p N^{d/2+p(\delta'/2-\delta^*)} \le c_p
\end{aligned}$$

if p is large enough so that $d/2 + p(\delta'/2 - \delta^*) < 0$. ∎

Recall that in (3.5) we fixed the constants $\delta, \tilde{\delta}$ such that $0 < \delta < \tilde{\delta} < 1/6$.

Lemma 7.6 *There exists $c_{7.11} = c_{7.11}(\delta, \tilde{\delta})$ such that for all $s > 0$, $z' \in \mathcal{Z}_N$ and $\mu^N \in \mathcal{M}_F(\mathcal{Z}_N)$*

$$N^{d/2}\int_{\mathbb{R}^d} \sup_{|z_1|\le 1/\sqrt{N}} \Pi_{\mathbf{0}}^N\left(\left|B_s^N + z' + z_1 - y\right| \le 1/\sqrt{N}\right)\mu^N(dy) \le c_{7.11}\,\hat{\varrho}_\delta^N(\mu^N)s^{-l_d-\tilde{\delta}}. \tag{7.11}$$

Proof Recall $l_d = (d/2-1)^+$. Let $s \geq 1$. Then from Lemma 7.2 we immediately get that

$$N^{d/2} \int_{\mathbb{R}^d} \sup_{|z_1| \leq 1/\sqrt{N}} \Pi_{\mathbf{0}}^N \left(\left| B_s^N + z' + z_1 - y \right| \leq 1/\sqrt{N} \right) \mu^N(dy)$$
$$\leq N^{d/2} \frac{c}{(Ns+1)^{d/2}} \mu^N(1) \leq c\mu^N(1) s^{-d/2} \leq c\mu^N(1) s^{-l_d-\tilde{\delta}}$$

and hence for $s \geq 1$ we are done.

Now let $0 < s < 1$ and fix $0 < \delta^* < 1/2$ such that $\delta/2 + \delta^*((2 \wedge d) - \delta) = \tilde{\delta}$. A little algebra converts the condition into

$$-d/2 + ((2 \wedge d) - \delta)(1/2 - \delta^*) = -l_d - \tilde{\delta}. \tag{7.12}$$

Define $A_N = \left\{ y : |z' - y| \leq (s^{1/2-\delta^*} \vee N^{-1/2+\delta^*}) + N^{-1/2} \right\}$. Let I_N^1 and I_N^2 be the contributions to (7.11) from the integrals over A_N and A_N^c, respectively. Let $s \geq 1/N$. Then we apply Lemma 7.2 to bound I_N^1 as follows:

$$\begin{aligned}
I_N^1 &\leq N^{d/2} \frac{c}{(Ns+1)^{d/2}} \int_{A_N} \mu^N(dy) \\
&\leq \hat{\varrho}_\delta^N(\mu^N) \frac{c}{(s+1/N)^{d/2}} \left((s^{1/2-\delta^*} \vee N^{-1/2+\delta^*}) + N^{-1/2} \right)^{2\wedge d-\delta} \\
&\leq c\hat{\varrho}_\delta^N(\mu^N) \frac{c}{(s)^{d/2}} \left(2s^{1/2-\delta^*} \right)^{2\wedge d-\delta} \leq c\hat{\varrho}_\delta^N(\mu^N) s^{-l_d-\tilde{\delta}},
\end{aligned}$$

where (7.12) is used in the last line. If $s \leq 1/N$ we get (using (7.12) again)

$$\begin{aligned}
I_N^1 &\leq N^{d/2} \mu^N \left(\mathbf{B}(z', N^{-1/2+\delta^*} + N^{-1/2}) \right) \leq c\hat{\varrho}_\delta^N(\mu^N) N^{d/2} N^{(-1/2+\delta^*)2\wedge d-\delta} \\
&= c\hat{\varrho}_\delta^N(\mu^N) N^{l_d+\tilde{\delta}} \leq c\hat{\varrho}_\delta^N(\mu^N) s^{-l_d-\tilde{\delta}}.
\end{aligned}$$

As for the second term I_N^2 we have the following. Note that for any $y \in A_N^c$ and $|z_1| \leq 1/\sqrt{N}$ we have,

$$\left| z' + z_1 - y \right| \geq s^{1/2-\delta^*} \vee N^{-1/2+\delta^*}. \tag{7.13}$$

Then by (7.13) and Lemma 7.5 we get

$$I_N^2 \leq c_{7.10} \mu^N(1) \leq c_{7.10} s^{-l_d-\tilde{\delta}} \mu^N(1),$$

where the last inequality is trivial since $s < 1$. Summing our bounds on I_N^1 and I_N^2 gives the required inequality for $0 < s < 1$ and the proof is complete. ∎

Return now the the proof of Proposition 3.5. Let $\phi : \mathcal{Z} \to [0, \infty)$. By our choice of $\tilde{\delta}$,

$$\tilde{l}_d \equiv 1 - l_d - \tilde{\delta} > 0.$$

From (2.19), Lemma 7.6, and the definition of $\mathcal{H}_{\delta,N}$ we get the following bounds by using the Markov property of B^N:

$$
\begin{aligned}
P^{g_N}_{s,t}(\phi)(x_1) \;\le\;& \Pi^N_{s,x_1}\left[\phi(B^N_t)\right] \\
&+ \gamma_2^+ 2^{-d} \int_s^t \int_{\mathcal{Z}_N} N^{d/2} \sum_{|z|\le N^{-1/2}} \Pi^N_{s,x_1}\left(B^N_{s_1} = y_1 + z\right) \Pi^N_{s_1,y_1+z}\left[\phi(B^N_t)\right] \\
&\qquad \times \bar{X}^{1,N}_{s_1}(dy_1)\, ds_1 \\
&+ \sum_{n=2}^{\infty} 2^{-dn} (\gamma_2^+)^n c^n_{7.11} \left(\mathcal{H}_{\delta,N} + \sup_{s\le u\le t} \bar{X}^{1,N}_u(1)\right)^n \int_{\mathbb{R}^n_+} 1(s < s_1 < \ldots < s_n \le t) \\
&\times \prod_{i=1}^n (s_i - s_{i-1})^{-l_d - \tilde{\delta}} \sup_x \Pi^N_{s_n,x}\left[\phi(B^N_t)\right] ds_1 \ldots ds_n\,,
\end{aligned} \tag{7.14}
$$

and for $\mu^N \in \mathcal{M}_F(\mathcal{Z}_N)$,

$$
\begin{aligned}
&\int P^{g_N}_{s,t}(\phi)(x_1)\mu^N(dx_1) \le \int \Pi^N_{s,x_1}\left[\phi(B^N_t)\right]\mu^N(dx_1) \\
&+ \sum_{n=1}^{\infty} \hat{\varrho}^N_\delta(\mu^N) 2^{-dn} (\gamma_2^+)^n c^n_{7.11} \left(\mathcal{H}_{\delta,N} + \sup_{s\le u\le t} \bar{X}^{1,N}_u(1)\right)^{n-1} \int_{\mathbb{R}^n_+} 1(s < s_1 < \ldots < s_n \le t) \\
&\times \prod_{i=1}^n (s_i - s_{i-1})^{-l_d - \tilde{\delta}} \\
&\times \int \sup_{|z_n|\le N^{-1/2}} \Pi^N_{s_n,y_n+z_n}\left[\phi(B^N_t)\right] \bar{X}^{1,N}_{s_n}(dy_n)\, ds_1 \ldots ds_n\,.
\end{aligned} \tag{7.15}
$$

In the above integrals, $s_0 = 0$ and we have distributed the $\bar{X}^{1,N}_{s_i}$ integrations among the B^N increments in different manners in (7.14) and (7.15).

Define

$$
J_n(s_n) \;\equiv\; \int_{\mathbb{R}^{n-1}_+} 1(s_1 < \ldots < s_n) \prod_{i=1}^n (s_i - s_{i-1})^{-l_d - \tilde{\delta}}\, ds_1 \ldots ds_{n-1}\,, \quad n \ge 1.
$$

Some elementary calculations give

$$
J_n(s_n) \;=\; \frac{\Gamma(\tilde{l}_d)^n}{\Gamma(n\tilde{l}_d)} s_n^{(n-2)\tilde{l}_d + 1 - 2l_d - 2\tilde{\delta}}, \;\; \forall n \ge 1. \tag{7.16}
$$

Define

$$
R^N_{7.17}(t,\omega) \;\equiv\; \sum_{n=2}^{\infty} \frac{\left(R_N(\omega) 2^{-d} \gamma_2^+ c_{7.11} \Gamma(\tilde{l}_d)\right)^n t^{(n-2)\tilde{l}_d + 1 - 2l_d}}{\Gamma(n\tilde{l}_d)}, \tag{7.17}
$$

$$
R^N_{7.18}(t,\omega) \;\equiv\; \sum_{n=1}^{\infty} \frac{\left(2^{-d} \gamma_2^+ c_{7.11} \Gamma(\tilde{l}_d)\right)^n R_N(\omega)^{n-1} t^{(n-1)\tilde{l}_d}}{\Gamma(n\tilde{l}_d)}. \tag{7.18}
$$

Note that

$$R^N_{7.17}(t,\omega)+R^N_{7.18}(t,\omega)=\sup_{s\le t}(R^N_{7.17}(s,\omega)+R^N_{7.18}(s,\omega)) \quad < \quad \infty, \quad P-\text{a.s.} \tag{7.19}$$

where the last inequality follows since

$$\frac{C^n}{\Gamma(n\epsilon)}\to 0, \text{ as } n\to\infty,$$

for any $C>0$ and $\epsilon>0$ (recall that $d\le 3$, $\tilde{l}_d>0, 1-2l_d\ge 0$). Moreover, (2.21) immediately yields

(7.20) for each $t>0$, $R^N_{7.17}(t,\omega)+R^N_{7.18}(t,\omega)$ is bounded in probability uniformly in N (tight).

Now we are ready to give some bounds on expectations.

Lemma 7.7 *Let $\phi:\mathcal{Z}\to[0,\infty)$. Then, for $0\le s<t$,*

$$\begin{aligned}
&\Pi^N_{s,x_1}\left[\phi(B^N_t)\exp\left\{\int_s^t \gamma_2^+ g_N(\bar{X}^{1,N}_r, B^N_r)\,dr\right\}\right]\\
&\quad\le \Pi^N_{s,x_1}\left[\phi(B^N_t)\right]\\
&\qquad + \gamma_2^+2^{-d}\int_s^t\int_{\mathcal{Z}_N} N^{d/2}\sum_{|z|\le N^{-1/2}}\Pi^N_{s,x_1}\left(B^N_{s_1}=y_1+z\right)\Pi^N_{s_1,y_1+z}\left[\phi(B^N_t)\right]\bar{X}^{1,N}_{s_1}(dy_1)\,ds_1\\
&\qquad + R^N_{7.17}(t)\int_s^t (s_n-s)^{-2\tilde{\delta}}\sup_x \Pi^N_{s_n,x}\left[\phi(B^N_t)\right]\,ds_n\,.
\end{aligned}$$

Proof Immediate from (7.14) and (7.16).

■

Proof of Proposition 3.5

(a) Immediate from (7.15), (7.18) and (7.20) with $\bar{R}_N(t)=R^N_{7.18}(t)$.

(b) By Lemma 7.7, Lemma 7.6 and the definition of R_N we get

$$\begin{aligned}
&\Pi^N_{s,x_1}\left[\phi(B^N_t)\exp\left\{\int_s^t \gamma_2^+ g_N(\bar{X}^{1,N}_r, B^N_r)\,dr\right\}\right]\\
&\le \|\phi\|_\infty\left(1+\gamma_2^+2^{-d}\int_s^t\int_{\mathcal{Z}_N} N^{d/2}\Pi^N_{s,x_1}\left(|B^N_{s_1}-y_1|\le N^{-1/2}\right)\bar{X}^{1,N}_{s_1}(dy_1)\,ds_1\right.\\
&\qquad\left.+R^N_{7.17}(t)\int_s^t (s_n-s)^{-2\tilde{\delta}}\,ds_n\right)\\
&\le \|\phi\|_\infty\left(1+2^{-d}\gamma_2^+c_{7.11}R_N\int_s^t (s_1-s)^{-l_d-\tilde{\delta}}\,ds_1+R^N_{7.17}(t)t^{1-2\tilde{\delta}}\right)\\
&\le \|\phi\|_\infty\, c\left(1+R_N t^{1-l_d-\tilde{\delta}}+R^N_{7.17}(t)t^{1-2\tilde{\delta}}\right)\\
&= \|\phi\|_\infty\,\bar{R}_N(t),
\end{aligned}$$

and we are done.

■

Proof of Lemma 3.7

(a) Recall that our choice of parameter $\tilde\delta$ in (3.5) implies that $\tilde l_d = 1 - l_d - \tilde\delta > 0$. By Proposition 3.5(a) we get

$$
\begin{aligned}
&\int_{\mathcal{Z}_N} P^{g_N}_{0,t}\left(N^{d/2}1\left(|\cdot - y| \le 1/\sqrt{N}\right)\right)(x_1)\mu^N(dx_1)\\
&\quad\le \int_{\mathbb{R}^d} N^{d/2}\Pi^N_{0,x_1}\left(\left|B^N_t - y\right| \le 1/\sqrt{N}\right)\mu^N(dx_1)\\
&\qquad + \hat\varrho^N_\delta(\mu^N)\bar R_N(t)\int_0^t s_n^{-\tilde\delta - l_d}\\
&\qquad\qquad \times \int_{\mathbb{R}^d} N^{d/2}\sup_{|z_n|\le N^{-1/2}}\Pi^N_{y_n+z_n}\left(\left|B^N_{t-s_n} - y\right| \le N^{-1/2}\right)\bar X^{1,N}_{s_n}(dy_n)\,ds_n.
\end{aligned}
\tag{7.21}
$$

By Lemma 7.6 the first term in (7.21) is bounded by

$$
c_{7.11}\hat\varrho^N_\delta(\mu^N)t^{-\tilde\delta - l_d}
\tag{7.22}
$$

uniformly on x_1.

Now it is easy to check that

$$
\int_0^t s^{-2\tilde\delta}(t-s)^{-l_d-\tilde\delta}\,ds \le c(t), \quad \text{and } c(t) \text{ is bounded uniformly on the compacts.}
\tag{7.23}
$$

Apply (7.23) and Lemma 7.6, and recall that $\bar R_N(t)$ may change from line to line, to show that the second term in (7.21) is bounded by

$$
\begin{aligned}
&c(t)\hat\varrho^N_\delta(\mu^N)\bar R_N(t)R_N\int_0^t s_n^{-\tilde\delta - l_d}(t-s_n)^{-l_d-\tilde\delta}\,ds_n\\
&\quad\le \hat\varrho^N_\delta(\mu^N)\bar R_N(t)t^{1-2l_d-2\tilde\delta}\\
&\quad\le \hat\varrho^N_\delta(\mu^N)\bar R_N(t)t^{-\tilde\delta - l_d}
\end{aligned}
\tag{7.24}
$$

Now put together (7.22, (7.24) to get that (7.21) is bounded by

$$
\hat\varrho^N_\delta(\mu^N)\bar R_N(t)t^{-l_d-\tilde\delta}
$$

and we are done.

(b) Apply Lemma 7.7 with $\phi(x) = N^{d/2} \int_{\mathcal{Z}} 1(|x-y| \le N^{-1/2})\mu^N(dy)$ to get

$$
\begin{aligned}
&\int_{\mathcal{Z}_N} P_{0,t}^{g_N}\left(N^{d/2}1\left(|\cdot - y| \le 1/\sqrt{N}\right)\right)(x_1)\mu^N(dy) \\
&\quad = \; P_{0,t}^{g_N}\phi(x_1) \\
&\quad \le \; \int_{\mathbb{R}^d} N^{d/2}\Pi_{x_1}^N\left(\left|B_t^N - y\right| \le 1/\sqrt{N}\right)\mu^N(dy) \\
&\quad + \gamma_2^+ 2^{-d}\int_0^t \int_{\mathcal{Z}_N^2} N^{d/2}\sum_{|z|\le N^{-1/2}} \Pi_{x_1}^N\left(B_{s_1}^N = y_1 + z\right) N^{d/2}\Pi_{y_1+z}^N\left(\left|B_{t-s_1}^N - y\right| \le 1/\sqrt{N}\right) \\
&\qquad\qquad \times \bar{X}_{s_1}^{1,N}(dy_1)\mu^N(dy)\,ds_1 \\
&\quad + \bar{R}_N(t)\int_0^t s_n^{-2\tilde{\delta}} \sup_{x_1}\left\{\Pi_{x_1}^N\left(N^{d/2}\int_{\mathcal{Z}_N}\left|B_{t-s_n}^N - y\right| \le 1/\sqrt{N}\mu^N(dy)\right)\right\} ds_n.
\end{aligned}
\tag{7.25}
$$

By Lemma 7.6 the first term in (7.25) is bounded by

$$
c_{7.11}\hat{\varrho}_\delta^N(\mu^N)t^{-l_d-\tilde{\delta}} \tag{7.26}
$$

uniformly on x_1. Let us consider the second term in (7.25). First apply Lemma 7.6 to bound the integrand.

$$
\begin{aligned}
&\int_{\mathcal{Z}_N^2} N^{d/2}\sum_{|z|\le N^{-1/2}} \Pi_{x_1}^N\left(B_{s_1}^N = y_1 + z\right) N^{d/2}\Pi_{y_1+z}^N\left(\left|B_{t-s_1}^N - y\right| \le 1/\sqrt{N}\right)\bar{X}_{s_1}^{1,N}(dy_1)\mu^N(dy) \\
&\quad \le \; \left(\sup_{z_1}\int_{\mathcal{Z}_N} N^{d/2}\Pi_{z_1}^N\left(\left|B_{t-s_1}^N - y\right| \le 1/\sqrt{N}\right)\mu^N(dy)\right) \\
&\qquad \times \int_{\mathcal{Z}_N} N^{d/2}\sum_{|z|\le N^{-1/2}} \Pi_{x_1}^N\left(B_{s_1}^N = y_1 + z\right)\bar{X}_{s_1}^{1,N}(dy_1) \\
&\quad \le \; c\hat{\varrho}_\delta^N(\mu^N)R_N s_1^{-l_d-\tilde{\delta}}(t-s_1)^{-l_d-\tilde{\delta}}.
\end{aligned}
$$

Hence the second term in (7.25) is bounded by

$$
\begin{aligned}
\hat{\varrho}_\delta^N(\mu^N)\bar{R}_N(t)\int_0^t s_1^{-l_d-\tilde{\delta}}(t-s_1)^{-l_d-\tilde{\delta}}\,ds_1 \;&\le\; c\hat{\varrho}_\delta^N(\mu^N)\bar{R}_N(t)t^{1-2l_d-2\tilde{\delta}}. \\
&\le\; \hat{\varrho}_\delta^N(\mu^N)\bar{R}_N(t)t^{-l_d-\tilde{\delta}}
\end{aligned}
\tag{7.27}
$$

Now apply (7.23) and Lemma 7.6 to show that the third term in (7.25) is bounded by

$$
c(t)\hat{\rho}_\delta^N(\mu^N)\bar{R}_N(t). \tag{7.28}
$$

Now put together (7.26), (7.27), (7.28) to bound (7.25) by

$$
\hat{\varrho}_\delta^N(\mu^N)\bar{R}_N(t)t^{-l_d-\tilde{\delta}}
$$

and we are done.

∎

8 Proof of Lemmas 4.3, 4.4 and 4.5

We start with the

Proof of Lemma 4.3

$$
\begin{aligned}
\psi_N^\epsilon(x\,,x_1) &= 2^{-d}N^{d/2}\sum_{|z|\le N^{-1/2}}\int_0^\infty e^{-\alpha s}\mathrm{p}_{2s}^N(x_1-x-z)\,ds1\left(|x-x_1|\le\epsilon\right)\\
&\le \int_\epsilon^\infty e^{-\alpha s}N^{d/2}\sum_{|z|\le N^{-1/2}}\mathrm{p}_{2s}^N(x_1-x-z)\,ds1\left(|x-x_1|\le\epsilon\right)\\
&\quad + N^{d/2}\sum_{|z|\le N^{-1/2}}\int_0^\epsilon e^{-\alpha s}\mathrm{p}_{2s}^N(x_1-x-z)\,ds\\
&\equiv I_\epsilon^{1,N}(x,x_1,z_1)+I_\epsilon^{2,N}(x,x_1,z_1)\,.
\end{aligned}
$$

By Lemma 7.2 we get the following bound on $I_\epsilon^{1,N}$:

$$
\begin{aligned}
I_\epsilon^{1,N}(x,x_1,z_1) &\le c\int_\epsilon^\infty s^{-d/2}e^{-\alpha s}\,ds1\left(|x-x_1|\le\epsilon\right)\\
&\le ch_d(\epsilon)1\left(|x-x_1|\le\epsilon\right).
\end{aligned}
$$

Now use condition $\mathbf{UB}_N$ to get

$$
\begin{aligned}
\int_{\mathcal{Z}_N} I_\epsilon^{1,N}(x,x_1,z_1)\mu_0^N(dx) &\le ch_d(\epsilon)\int_{\mathcal{Z}_N}1\left(|x-x_1|\le\epsilon\right)\mu^N(dx)\\
&\le c\hat{\varrho}_\delta^N(\mu^N)h_d(\epsilon)\epsilon^{(2\wedge d)-\tilde{\delta}}\\
&\le c\hat{\varrho}_\delta^N(\mu^N)\epsilon^{1-\tilde{\delta}},\ \forall x_1\in\mathcal{Z}_N\,. \qquad (8.1)
\end{aligned}
$$

As for $I_\epsilon^{2,N}$, apply Lemma 7.6 to get

$$
\begin{aligned}
\int_{\mathcal{Z}_N} I_\epsilon^{2,N}(x,x_1)\mu^N(dx) &\le c_{7.11}\hat{\varrho}_\delta^N(\mu^N)\int_0^\epsilon s^{-l_d-\tilde{\delta}}\,ds\\
&\le c\hat{\varrho}_\delta^N(\mu^N)\epsilon^{1-l_d-\tilde{\delta}},\ \forall x_1\in\mathcal{Z}_N\,. \qquad (8.2)
\end{aligned}
$$

By combining (8.1) and (8.2) we are done. ∎

Recall from (4.3) that

$$
h_d(t) \equiv \begin{cases} 1, & \text{if } d=1,\\ 1+\ln_+(1/t), & \text{if } d=2,\\ t^{1-d/2}, & \text{if } d=3.\end{cases}
$$

Recall also that $\alpha>0$ if $d\le 2$, and $\alpha\ge 0$ if $d=3$.

Lemma 8.1 *There is a $c=c_\alpha>0$ such that*

$$
\Pi_{x_1}^N\left[G_N^\alpha\mathbf{1}(B_t^N\,,x_2)\right] \le ch_d(t)\ \forall t>0,\ \forall x_1\,,x_2\in\mathcal{Z}_N\,.
$$

Proof First apply Lemma 7.2 to get

$$\begin{aligned}\Pi_{x_1}^N\left[G_N^\alpha \mathbf{1}(B_t^N,x_2)\right] &= \Pi_{x_1}^N\left[N^{d/2}\sum_{|z|\le N^{-1/2}}\int_0^t e^{-\alpha s}\mathrm{p}_{2s}^N(B_t^N-x_2-z)\,ds\right]\\ &\quad+\Pi_{x_1}^N\left[N^{d/2}\sum_{|z|\le N^{-1/2}}\int_t^\infty e^{-\alpha s}\mathrm{p}_{2s}^N(B_t^N-x_2-z)\,ds\right]\\ &\le ct^{-d/2}\int_0^t e^{-\alpha s}\,ds+c\int_t^\infty e^{-\alpha s}s^{-d/2}\,ds\\ &\le c(t^{1-d/2}1(t\le 1)+t^{-d/2}\int_0^t e^{-\alpha s}\,ds1(t\ge 1))+ch_d(t)\,.\end{aligned}$$

Now a trivial calculation shows that

$$c(t^{1-d/2}1(t\le 1)+t^{-d/2}\int_0^t e^{-\alpha s}\,ds1(t\ge 1))\le ch_d(t)$$

and we are done.

■

Notation, assumptions. Until the end of this section we will make the following assumption on N,ϵ

$$0<\epsilon<1/2, N\ge \epsilon^{-2}\,. \tag{8.3}$$

We will typically reserve notation μ^N for measures in $\mathcal{M}_F(\mathcal{Z}_N)$ and $c(t)$ will denote a constant depending on $t\ge 0$ which is bounded on compacts. Also recall that $\delta,\tilde\delta$ satisfy (3.5).

Lemma 8.2 *If $0<\eta<(2/d)\wedge 1$, there exists a constant $c=c_\eta>0$ such that*

$$\Pi_{x_1}^N\left[\psi_N^\epsilon(B_t^N,x_2)\right] \le \begin{cases} ch_d(t), & \text{if } 0<t\le\epsilon^\eta\\ c\epsilon^{1-d\eta/2} & \text{otherwise,}\end{cases}$$

uniformly in $x_1,x_2\in\mathcal{Z}$, $N\ge\epsilon^{-2}$.

Proof First let us treat the case $t\le\epsilon^\eta$. By Lemma 8.1, for all $x_1,x_2\in\mathcal{Z}_N$,

$$\begin{aligned}\Pi_{x_1}^N\left[\psi_N^\epsilon(B_t^N,x_2)\right] &\le \Pi_{x_1}^N\left[G_N^\alpha\mathbf{1}(B_t^N,x_2)\right]\\ &\le ch_d(t)\,.\end{aligned} \tag{8.4}$$

Now let us turn to the case $t>\epsilon^\eta$. Then

$$\begin{aligned}\Pi_{x_1}^N\left[\psi_N^\epsilon(B_t^N,x_2)\right] &= \Pi_{x_1}^N\left[1\left(\left|B_t^N-x_2\right|\le\epsilon\right)N^{d/2}\sum_{|z|\le N^{-1/2}}\int_\epsilon^\infty e^{-\alpha s}\mathrm{p}_{2s}^N(B_t^N-x_2-z)\,ds\right]\\ &\quad+\Pi_{x_1}^N\left[1\left(\left|B_t^N-x_2\right|\le\epsilon\right)N^{d/2}\sum_{|z|\le N^{-1/2}}\int_0^\epsilon e^{-\alpha s}\mathrm{p}_{2s}^N(B_t^N-x_2-z)\,ds\right]\\ &\equiv I_1+I_2.\end{aligned}$$

Then for any x_1

$$
\begin{aligned}
I_1 &\le \sup_x \Pi^N_{x_1}\left[1\left(\left|B^N_t - x\right| \le \epsilon\right) N^{d/2} \sum_{|z|\le N^{-1/2}} \int_\epsilon^\infty e^{-\alpha s} \mathrm{p}^N_{2s}(B^N_t - x - z)\, ds\right] \\
&\le \sup_x \Pi^N_{x_1}\left[1\left(\left|B^N_t - x\right| \le \epsilon\right) c \int_\epsilon^\infty e^{-\alpha s} s^{-d/2}\, ds\right] \quad \text{(by (7.5))} \\
&\le c h_d(\epsilon)\epsilon^d t^{-d/2} \quad \text{(by Lemma 7.2)} \\
&\le c h_d(\epsilon)\epsilon^{d-\eta d/2} \quad (\text{since } t > \epsilon^\eta) \\
&\le c\epsilon^{1-\eta d/2},
\end{aligned}
$$

where the last inequality follows by the definition of h_d. Now apply again Lemma 7.2 and the assumption $t > \epsilon^\eta$ to get

$$
\begin{aligned}
I_2 &\le \Pi^N_{x_1}\left[N^{d/2} \sum_{|z|\le N^{-1/2}} \int_0^\epsilon e^{-\alpha s} \mathrm{p}^N_{2s}(B^N_t - x_2 - z)\, ds\right] \\
&\le c t^{-d/2} \int_0^\epsilon ds \\
&\le c\epsilon^{-\eta d/2}\epsilon
\end{aligned}
$$

and we are done. ∎

We will also use the trivial estimate

$$
h_d(t) \le c t^{-l_d - \tilde{\delta}}, \quad t \le 1. \tag{8.5}
$$

The proof of the following trivial lemma is omitted.

Lemma 8.3 *Let $f(t)$ be a function such that*

$$
f(t) \le \begin{cases} c h_d(t), & \text{if } 0 < t \le \epsilon^\eta \\ c\epsilon^{1-d\eta/2} & \text{otherwise.} \end{cases}
$$

Then for $0 < \eta < 1/2$ and some $c(\eta, t)$, bounded for t in compacts, we have

$$
\int_0^t s^{-2\tilde{\delta}} f(t-s)\, ds \le c(t)\epsilon^{\eta(1-l_d-3\tilde{\delta})}. \tag{8.6}
$$

The next several lemmas will give some bounds on the Green's functions G^α_N used in Section 4.

Lemma 8.4 *For any $0 < \eta < 2/7$ there exists $\{\bar{R}_N(t)\}_{t\ge 0}$ (possibly depending on η), such that for all $0 < \epsilon < 1/2$,*

$$
P^{g_N}_{s,t}\left(\psi^\epsilon_N(\cdot, x_2)\right)(x_1) \le \begin{cases} h_d(t-s)\bar{R}_N(t), & \text{if } 0 < t-s \le \epsilon^\eta, \\ \epsilon^{\eta(1-l_d-3\tilde{\delta})}\bar{R}_N(t) & \text{if } t-s > \epsilon^\eta, \end{cases}
$$

uniformly in x_1, x_2, $N \ge \epsilon^{-2}$.

Proof First, by Lemma 7.7 we get

$$
\begin{aligned}
P^{g_N}_{s,t}\left(\psi^\epsilon_N(\cdot,x_2)\right)(x_1) \;\le\;& \Pi^N_{s,x_1}\left[\psi^\epsilon_N(x_2\,,B^N_t)\right] \\
&+\gamma_2^+ 2^{-d}\int_s^t\int_{\mathcal{Z}_N} N^{d/2}\sum_{|z|\le N^{-1/2}} \Pi^N_{s,x_1}\left(B^N_{s_1}=y_1+z\right)\Pi^N_{s_1,y_1+z}\left[\psi^\epsilon_N(x_2\,,B^N_t)\right] \\
&\quad\times \bar X^{1,N}_{s_1}(dy_1)\,ds_1 \\
&+\bar R_N(t)\int_s^t (s_n-s)^{-2\tilde\delta}\sup_{x_1,x_2}\Pi^N_{x_1}\left[\psi^\epsilon_N(x_2\,,B^N_{t-s_n})\right]ds_n \\
\equiv\;& I^{1,N}(s,t)+I^{2,N}(s,t)+I^{3,N}(s,t).
\end{aligned}
\tag{8.7}
$$

Note that $I^{1,N}(s,t)=I^{1,N}(t-s)$, and hence, if $t-s\ge\epsilon^\eta$, then by Lemma 8.2, $I^{1,N}(s,t)$ in (8.7) is bounded by

$$
c\epsilon^{1-d\eta/2} \;\le\; c\epsilon^{\eta(1-l_d-3\tilde\delta)},\;\; \forall\epsilon\le 1,
\tag{8.8}
$$

where the inequality follows since $1-d\eta/2\ge\eta(1-l_d-3\tilde\delta)$ by our assumptions on $\eta,\tilde\delta$. To bound $I^{1,N}(s,t)$ for $t-s\le\epsilon^\eta$ use again Lemma 8.2 and hence obtain

$$
I^{1,N}(s,t) \;\le\; \begin{cases} ch_d(t-s), & \text{if } 0<t-s\le\epsilon^\eta,\\ c\epsilon^{\eta(1-l_d-3\tilde\delta)} & \text{if } t-s>\epsilon^\eta. \end{cases}
\tag{8.9}
$$

Next consider $I^{2,N}(s,t)$. First bound the integrand:

$$
\begin{aligned}
&\int_{\mathcal{Z}_N} N^{d/2}\sum_{|z|\le N^{-1/2}}\Pi^N_{s,x_1}\left(B^N_{s_1}=y_1+z\right)\Pi^N_{s_1,y_1+z}\left[\psi^\epsilon_N(x_2\,,B^N_t)\right]\bar X^{1,N}_{s_1}(dy_1) \\
&\le\left(\sup_{x\in\mathcal{Z}_N}\Pi^N_x\left[\psi^\epsilon_N(x_2\,,B^N_{t-s_1})\right]\right) \\
&\qquad\times\int_{\mathcal{Z}_N} N^{d/2}\sum_{|z|\le N^{-1/2}}\Pi^N_{x_1}\left(B^N_{s_1-s}=y_1+z\right)\bar X^{1,N}_{s_1}(dy_1) \\
&\le c_{7.11}R_N(s_1-s)^{-l_d-\tilde\delta}\sup_{x\in\mathcal{Z}_N}\Pi^N_x\left[\psi^\epsilon_N(x_2\,,B^N_{t-s_1})\right],\quad s\le s_1\le t,
\end{aligned}
\tag{8.10}
$$

where the last inequality follows by Lemma 7.6 and definition of R_N. Now for $t-s\le\epsilon^\eta$ we apply (8.10), (8.5), and Lemma 8.2 to get

$$
\begin{aligned}
I^{2,N}(t) \;\le\;& \gamma_2^+2^{-d}c_{7.11}R_N\int_s^t (s_1-s)^{-l_d-\tilde\delta}\sup_{x,x_2\in\mathcal{Z}_N}\Pi^N_x\left[\psi^\epsilon_N(x_2\,,B^N_{t-s_1})\right]ds_1 \\
\le\;& cR_N\int_s^t (s_1-s)^{-l_d-\tilde\delta}(t-s_1)^{-l_d-\tilde\delta}\,ds_1 \\
\le\;& cR_N(t-s)^{1-2l_d-2\tilde\delta}\int_0^1 u^{-l_d-\tilde\delta}(1-u)^{-l_d-\tilde\delta}\,du \\
\le\;& c(t)R_Nh_d(t-s),\;\; t-s\le\epsilon^\eta.
\end{aligned}
\tag{8.11}
$$

Now, for $t-s \geq \epsilon^\eta$ use (8.10) to get

$$
\begin{aligned}
I^{2,N}(s,t) &\leq \gamma_2^+ 2^{-d} c_{7.11} R_N \int_s^t (s_1-s)^{-l_d-\tilde\delta} \sup_{x\in\mathcal{Z}_N} \Pi_x^N \left[\psi_N^\epsilon(x_2\,, B_{t-s_1}^N)\right] ds_1 \\
&\leq cR_N \int_s^{t-\epsilon^{2\eta}} (s_1-s)^{-l_d-\tilde\delta} \sup_{x\in\mathcal{Z}_N} \Pi_x^N \left[\psi_N^\epsilon(x_2\,, B_{t-s_1}^N)\right] ds_1 \\
&\quad + cR_N \int_{t-\epsilon^{2\eta}}^t (s_1-s)^{-l_d-\tilde\delta} \sup_{x\in\mathcal{Z}_N} \Pi_x^N \left[\psi_N^\epsilon(x_2\,, B_{t-s_1}^N)\right] ds_1.
\end{aligned}
\tag{8.12}
$$

Let us estimate the first integral on the right hand side of (8.12). By Lemma 8.2 with $2\eta < 2/3$ in place of η, it is bounded by

$$
\begin{aligned}
& c\epsilon^{1-\eta d} \int_s^{t-\epsilon^{2\eta}} (s_1-s)^{-l_d-\tilde\delta}\, ds_1 \\
&= c(t)\epsilon^{1-\eta d} \\
&\leq c(t)\epsilon^{\eta(1-l_d-3\tilde\delta)}.
\end{aligned}
\tag{8.13}
$$

The last line follows as in (8.8) and uses $\eta < 2/7$.

Now, let us estimate the second integral on the right hand side of (8.12). Since $t-s \geq \epsilon^\eta$, we see that for $\epsilon < 1/2$, there is a constant $c = c(\eta)$ such that

$$
t-\epsilon^{2\eta} \geq s+\epsilon^\eta - \epsilon^{2\eta} \geq s + c\epsilon^\eta,
$$

and hence the second integral on the right hand side of (8.12) is bounded by

$$
\begin{aligned}
& c\epsilon^{\eta(-l_d-\tilde\delta)} \int_{t-\epsilon^{2\eta}}^t \sup_{x\in\mathcal{Z}_N} \Pi_x^N \left[\psi_N^\epsilon(x_2\,, B_{t-s_1}^N)\right] ds_1 \\
&\leq c\epsilon^{\eta(-l_d-\tilde\delta)}\epsilon^{2\eta(1-l_d-\tilde\delta)} \quad \text{(by (8.5) and Lemma 8.2).}
\end{aligned}
\tag{8.14}
$$

Since $l_d \leq 1/2$, the right-hand side of (8.14) is bounded by

$$
c\epsilon^{\eta(1-l_d-3\tilde\delta)}. \tag{8.15}
$$

Combining (8.11)—(8.15) we get

$$
I^{2,N}(s,t) \leq \begin{cases} c(t)R_N h_d(t-s), & \text{if } 0 < t-s \leq \epsilon^\eta, \\ c(t)R_N\epsilon^{\eta(1-l_d-3\tilde\delta)} & \text{if } t-s > \epsilon^\eta. \end{cases} \tag{8.16}
$$

For $I^{3,N}(t)$, we apply Lemmas 8.2 and 8.3 to get

$$
I^{3,N}(s,t) \leq \bar R_N(t)\epsilon^{\eta(1-l_d-3\tilde\delta)} \tag{8.17}
$$

Moreover, since $\epsilon^{\eta(1-l_d-3\tilde\delta)} \leq h_d(t-s)$ for $t-s \leq 1$, we obtain

$$
I^{3,N}(s,t) \leq \left(\epsilon^{\eta(1-l_d-3\tilde\delta)} 1(t-s\geq\epsilon^\eta) + h_d(t-s)1(t-s\leq\epsilon^\eta)\right)\bar R_N(t). \tag{8.18}
$$

Now put together (8.9), (8.16), (8.18) to bound (8.7) by

$$\left(\epsilon^{\eta(1-l_d-3\tilde{\delta})}1(t-s\geq\epsilon^{\eta})+h_d(t-s)1(t-s\leq\epsilon^{\eta})\right)\bar{R}_N(t), \tag{8.19}$$

and we are done.

∎

Lemma 8.5 *There is a $c=c_\alpha$ such that*

$$\sup_{x_1}\int_{\mathcal{Z}_N}G_N^{\alpha}\mathbf{1}(x\,,x_1)\mu^N(dx)\leq c\hat{\varrho}_\delta^N(\mu^N),\ \ \forall\mu^N\in\mathcal{M}_F(\mathcal{Z}_N),\ \forall N\in\mathbb{N}\,,$$

Proof By Lemma 7.2 we get

$$\begin{aligned}G_N^{\alpha}\mathbf{1}(x\,,x_1) &\leq N^{d/2}\sum_{|z|\leq N^{-1/2}}\int_0^1 e^{-\alpha s}\mathrm{p}_{2s}^N(x_1-x-z)\,ds\\ &\quad+c\int_1^{\infty}e^{-\alpha s}s^{-d/2}\,ds\\ &\equiv N^{d/2}\sum_{|z|\leq N^{-1/2}}\int_0^1 e^{-\alpha s}\mathrm{p}_{2s}^N(x_1-x-z)\,ds+c_{8.20}(\alpha),\end{aligned} \tag{8.20}$$

for all $x,x_1\in\mathcal{Z}_N$. Apply Lemma 7.6 to get

$$\begin{aligned}\int_{\mathcal{Z}_N}G_N^{\alpha}\mathbf{1}(x\,,x_1)\mu^N(dx) &\leq c_{7.11}\hat{\varrho}_\delta^N(\mu^N)\int_0^1 s^{-l_d-\tilde{\delta}}\,ds+c_{8.20}(\alpha)\mu^N(1)\\ &\leq c\hat{\varrho}_\delta^N(\mu^N)\,,\ \forall\, x_1\in\mathcal{Z}_N\,,\end{aligned}$$

and we are done.

∎

Lemma 8.6 *There is an $\bar{R}_N(t)$ so that*

$$\int_{\mathcal{Z}_N}P_{s,t}^{g_N}\left(G_N^{\alpha}\mathbf{1}(x\,,\cdot)\right)(x_1)\mu^N(dx)\ \ \leq\ \ c(t)\hat{\varrho}_\delta^N(\mu^N)\bar{R}_N(t)\,,$$

for all $t\geq s\geq 0,\ x_1\in\mathcal{Z}_N\,,\ \mu^N\in\mathcal{M}_F(\mathcal{Z}_N),$ and $N\in\mathbb{N}$.

Proof The result is immediate by Lemma 8.5 and Proposition 3.5(b).

∎

Lemma 8.7 *There is a $c_{8.7}$ such that if $\mu^N\in\mathcal{M}_F(\mathcal{Z}_N)$, then*

$$\begin{aligned}&\int_{\mathcal{Z}_N}\sup_{|z_1|\leq 1/\sqrt{N}}\Pi_{x_1+z_1}^N\left[G_N^{\alpha}\mathbf{1}(x\,,B_t^N)\right]\mu^N(dx_1)\\ &\quad\leq\ c_{8.7}\hat{\varrho}_\delta^N(\mu^N)\,,\ \forall t\geq 0,\ x\in\mathcal{Z}_N,\ N\in\mathbb{N}.\end{aligned}$$

Proof Use (8.20) to get

$$\Pi^N_{x_1+z_1}\left[G^\alpha_N \mathbf{1}(x\,,B^N_t)\right] \;\le\; N^{d/2}\Pi^N_{x_1+z_1}\left[\sum_{|z|\le N^{-1/2}}\int_0^1 e^{-\alpha s}\mathrm{p}^N_{2s}(B^N_t-x-z)\,ds\right] \tag{8.21}$$
$$+\,c_{8.20}(\alpha),$$

for all $t\ge 0, x, x_1, z_1\in\mathcal{Z}_N$. If $\mu^N\in\mathcal{M}_F(\mathcal{Z}_N)$, we have

$$\begin{aligned}
&N^{d/2}\int_{\mathcal{Z}_N}\sup_{|z_1|\le 1/\sqrt{N}}\Pi^N_{x_1+z_1}\left[\sum_{|z|\le N^{-1/2}}\int_0^1 e^{-\alpha s}\mathrm{p}^N_{2s}(B^N_t-x-z)\,ds\right]\mu^N(dx_1)\\
&\le \int_0^1 N^{d/2}\int_{\mathbb{R}^d}\sup_{|z_1|\le 1/\sqrt{N}}\Pi^N_{x_1+z_1}\left(\left|B^N_{t+2s}-x\right|\le 1/\sqrt{N}\right)\mu^N(dx_1)\,ds\\
&\le c_{8.22}\hat{\rho}_\delta(\mu^N),\quad \forall t\ge 0,\ x\in\mathcal{Z}_N,
\end{aligned}\tag{8.22}$$

where the last inequality follows by Lemma 7.6. This and (8.21) imply

$$\begin{aligned}
&\int_{\mathcal{Z}_N}\sup_{|z_1|\le 1/\sqrt{N}}\Pi^N_{x_1+z_1}\left[G^\alpha_N\mathbf{1}(x\,,B^N_t)\right]\mu^N(dx_1)\\
&\le c_{8.22}\hat{\rho}_\delta(\mu^N)+c_{8.20}(\alpha)\mu^N(1)\\
&\le c\hat{\varrho}^N_\delta(\mu^N)\,,\ \forall t\ge 0,\ x\in\mathcal{Z}_N\,,
\end{aligned}$$

and the proof is finished.

■

Lemma 8.8 *There is an $\bar{R}_N(t)$ such that*

$$\int_{\mathcal{Z}_N} P^{g_N}_{0,t}\left(G^\alpha_N\mathbf{1}(x\,,\cdot)\right)(x_1)\mu^N(dx_1) \;\le\; \hat{\varrho}^N_\delta(\mu^N)\bar{R}_N(t)\,,$$

for all $t\ge 0,\ x\in\mathcal{Z}_N\,,\mu^N\in\mathcal{M}_F(\mathcal{Z}_N),\ N\in\mathbb{N}$.

Proof First, by Lemma 7.7 we get

$$\begin{aligned}
&\int_{\mathcal{Z}_N} P^{g_N}_{0,t}\left(G^\alpha_N\mathbf{1}(x\,,\cdot)\right)(x_1)\mu^N(dx_1)\\
&\le \int_{\mathbb{R}^d}\Pi^N_{0,x_1}\left[G^\alpha_N\mathbf{1}(x\,,B^N_t)\right]\mu^N(dx_1)\\
&\quad+\gamma_2^+2^{-d}\int_0^t\int_{\mathcal{Z}_N^2}N^{d/2}\sum_{|z|\le N^{-1/2}}\Pi^N_{x_1}\left(B^N_{s_1}=y_1+z\right)\Pi^N_{y_1+z}\left[G^\alpha_N\mathbf{1}(x\,,B^N_{t-s_1})\right]\\
&\qquad\times\bar{X}^{1,N}_{s_1}(dy_1)\mu^N(dx_1)\,ds_1+\bar{R}_N(t)\mu^N(1)\int_0^t s_n^{-2\tilde{\delta}}\sup_{x_1\,,x}\Pi^N_{x_1}\left[G^\alpha_N\mathbf{1}(x\,,B^N_{t-s_n})\right]\,ds_n\\
&\equiv I^{1,N}+I^{2,N}+I^{3,N}.
\end{aligned}\tag{8.23}$$

By Lemma 8.7,

$$I^{1,N} \quad \leq \quad c(t)\hat{\varrho}_\delta^N(\mu^N). \tag{8.24}$$

Now, let us consider the second term in (8.23). By Lemma 8.7 and Lemma 7.6 we get

$$\begin{aligned}
I^{2,N} \quad &\leq \quad \gamma_2^+ 2^{-d} \int_0^t \int_{\mathcal{Z}_N} \left(\sup_{|z_1|\leq N^{-1/2}} \Pi^N_{y_1+z_1}\left[G^\alpha_N \mathbf{1}(x\,, B^N_{t-s_1})\right]\right) \\
&\qquad \times \left(\int_{\mathcal{Z}_N} N^{d/2} \sum_{|z|\leq N^{-1/2}} \Pi^N_{x_1}\left(B^N_{s_1} = y_1 + z\right)\mu^N(dx_1)\right) \bar{X}^{1,N}_{s_1}(dy_1)\,ds_1 \\
&\leq \quad c_{7.11}\hat{\varrho}_\delta^N(\mu^N)\int_0^t s_1^{-l_d-\tilde{\delta}} c_{8.7}\hat{\varrho}_\delta^N(\bar{X}^{1,N}_{s_1})\,ds_1 \\
&\leq \quad c\hat{\varrho}_\delta^N(\mu^N)R_N\int_0^t s_1^{-l_d-\tilde{\delta}}\,ds_1 \\
&\leq \quad c(t)\hat{\varrho}_\delta^N(\mu^N)R_N\,, \ \ \forall t\geq 0. \qquad (8.25)
\end{aligned}$$

Apply (7.23), (8.5) and Lemma 8.1 to get that

$$I^{3,N} \quad \leq \quad \bar{R}_N(t)\mu^N(1), \tag{8.26}$$

where we recall again that $\bar{R}_N(\cdot)$ may change from line to line. Now combine (8.24), (8.25), (8.26) to get that the expression in (8.23) is bounded by

$$\hat{\varrho}_\delta^N(\mu^N)\bar{R}_N(t), \tag{8.27}$$

and we are done.

∎

Lemma 8.9 *There is an $\bar{R}_N(\cdot)$ such that for all $\mu^N \in \mathcal{M}_F(\mathcal{Z}_N)$,*

$$\begin{aligned}
&\int_{\mathcal{Z}_N} P^{g_N}_{0,t}\left(\psi^\epsilon_N(\cdot, x)\right)(x_1)\mu^N(dx) \\
&\quad \leq \ \ \epsilon^{1-l_d-\tilde{\delta}}c(t)\hat{\varrho}_\delta^N(\mu^N)\bar{R}_N(t)\,,\ \forall t\geq 0,\ x_1\in\mathcal{Z}_N\,,\ N\geq\epsilon^{-2}\,.
\end{aligned} \tag{8.28}$$

Proof The result is immediate by Lemma 4.3, Proposition 3.5(b) and the symmetry of ψ^ϵ_N. ∎

We next need a bound on the convolution of $P^{g_N}_{s,t}$ with p^N. Let

$$\bar{g}_N(\bar{X}^{1,N}_s, x) = 2^{-d}N^{d/2}\bar{X}^{1,N}_s(\mathbf{B}(x, 2N^{-1/2})).$$

Clearly

$$P^{g_N}_{s,t}(\phi) \quad \leq \quad P^{\bar{g}_N}_{s,t}(\phi)\,,\ \ \forall s\leq t,\ \phi:\mathcal{Z}_N\to[0,\infty). \tag{8.29}$$

Lemma 8.10 *For any $\phi:\mathcal{Z}_N\to[0,\infty).$, and $s\leq t$*

$$\sum_{x_1\in\mathcal{Z}_N} P^{g_N}_{s,t}(\phi)(x_1+x)p^N(x_1) \quad \leq \quad \sum_{x_1\in\mathcal{Z}_N} P^{\bar{g}_N}_{s,t}(\phi(x_1+\cdot))(x)p^N(x_1)$$

Proof It is easy to see that

$$\begin{aligned}
&\sum_{x_1\in\mathcal{Z}_N} P^{g_N}_{s,t}(\phi)(x_1+x)p^N(x_1)\\
&\quad= \sum_{|x_1|\le N^{-1/2}} p^N(x_1)\Pi^N_x\left[\phi\left(B^N_{t-s}+x_1\right)\exp\left\{\int_0^{t-s}\gamma_2^+ g_N(\bar X^{1,N}_{r+s},B^N_r+x_1)\,dr\right\}\right]\\
&\quad\le \sum_{|x_1|\le N^{-1/2}} p^N(x_1)\Pi^N_x\left[\phi\left(B^N_{t-s}+x_1\right)\exp\left\{\int_0^{t-s}\gamma_2^+ \bar g_N(\bar X^{1,N}_{r+s},B^N_r)\,dr\right\}\right]\\
&\quad= \sum_{x_1\in\mathcal{Z}_N} P^{\bar g_N}_{s,t}(\phi(x_1+\cdot))(x)p^N(x_1).
\end{aligned}$$

■

Proof of Lemma 4.4 By Lemma 2.3 (see below) and (3.7),

$$\begin{aligned}
&(8.30)\quad E\left[\int_{\mathcal{Z}_N^3}\psi^\epsilon_N(x_1,x)1(|x_2-x|\le 1/\sqrt{N})N^{d/2}\bar X^{2,N}_t(dx_1)\bar X^{2,N}_t(dx_2)\bar X^{1,N}_t(dx)|\bar X^{1,N}\right]\\
&\le \int_{\mathcal{Z}_N^3} P^{g_N}_{0,t}(\psi^\epsilon_N(\cdot,x))(x_1)N^{d/2}P^{g_N}_{0,t}\left(1(|\cdot-x|\le 1/\sqrt{N})\right)(x_2)\\
&\qquad\times \bar X^{2,N}_0(dx_1)\bar X^{2,N}_0(dx_2)\bar X^{1,N}_t(dx)\\
&\quad+c\Big[1+\mathcal{H}_{\delta,N}N^{-1+\ell_d+\delta/2}\Big]E\Big[\int_0^t\left(\sup_{y\in\mathcal{Z}_N}P^{g_N}_{s,t}(\psi^\epsilon_N(\cdot,x))(y)\right)\\
&(8.31)\quad\times\int_{\mathcal{Z}_N^2}N^{d/2}P^{g_N}_{s,t}\left(1(|\cdot-x|\le 1/\sqrt{N})\right)(z)\bar X^{2,N}_s(dz)\,ds\,\bar X^{1,N}_t(dx)|\bar X^{1,N}\Big]\\
&\quad+cE\Big[\int_0^t\left(\sup_{y\in\mathcal{Z}_N}P^{g_N}_{s,t}(\psi^\epsilon_N(\cdot,x))(y)\right)\\
&\quad\times\int_{\mathcal{Z}_N^2}\left(\sum_{y\in\mathcal{Z}_N}N^{d/2}P^{g_N}_{s,t}\left(1(|\cdot-x|\le 1/\sqrt{N})\right)(y)p^N(z-y)\right)\bar X^{2,N}_s(dz)\,ds\,\bar X^{1,N}_t(dx)|\bar X^{1,N}\Big].
\end{aligned}$$

Here we have noted that if $\phi_1,\phi_2\ge 0$ in Lemma 2.3, then the third term on the right-hand side of (2.18) is at most

$$E\Big(\int_0^t\int_{\mathcal{Z}_N}\Big(\sum_y P^{g_N}_{s,t}\phi_1(y)P^{g_N}_{s,t}\phi_2(y)p_N(z-y)\Big)+P^{g_N}_{s,t}\phi_1(z)P^{g_N}_{s,t}\phi_2(z)\bar X^{2,N}_s(dz)ds|\bar X^{1,N}\Big).$$

The first term on the right hand side of (8.30) is bounded by

$$
\begin{aligned}
(8.32) \qquad & \int_{\mathcal{Z}_N^2} P_{0,t}^{g_N}\left(\psi_N^\epsilon(\cdot,x)\right)(x_1)\bar{X}_0^{2,N}(dx_1)\bar{X}_t^{1,N}(dx) \\
& \qquad \times\left\{\sup_x N^{d/2}P_{0,t}^{g_N}\left(1(|\cdot-x|\le 1/\sqrt{N})\right)(x_2)\bar{X}_0^{2,N}(dx_2)\right\} \\
& \le \epsilon^{1-l_d-\tilde{\delta}}\bar{X}_0^{2,N}(1)\hat{\varrho}_\delta^N(\bar{X}_0^{2,N})R_N\bar{R}_N(t)^2t^{-l_d-\tilde{\delta}} \\
& = \bar{R}_N(t)\epsilon^{1-l_d-\tilde{\delta}}\hat{\varrho}_\delta^N(\bar{X}_0^{2,N})^2t^{-l_d-\tilde{\delta}}
\end{aligned}
$$

where the first inequality follows by Lemmas 8.9 and 3.7(a), and the second is immediate consequence of the definition of $\hat{\varrho}_\delta^N(\bar{X}_0^{2,N})$.

Now we consider the second term on the right-hand side of (8.30). By Lemma 8.4, for $\eta\in(0,2/7)$, it is bounded by

$$
\begin{aligned}
(8.33) \qquad & c(1+R_N)\bar{R}_N(t)\int_{\mathcal{Z}_N}\int_0^t\left(\epsilon^{\eta(1-l_d-3\tilde{\delta})}1(t-s\ge\epsilon^\eta)+h_d(t-s)1(t-s\le\epsilon^\eta)\right) \\
& \qquad \times E\left[\int_{\mathcal{Z}_N}N^{d/2}P_{s,t}^{g_N}\left(1(|\cdot-x|\le 1/\sqrt{N})\right)(z)\bar{X}_s^{2,N}(dz)|\bar{X}^{1,N}\right]\,ds\,\bar{X}_t^{1,N}(dx).
\end{aligned}
$$

Now use (2.17) to evaluate the conditional expectation inside the integral with respect to s to get

$$
\begin{aligned}
(8.34) \qquad & E\left[\int_{\mathcal{Z}_N}N^{d/2}P_{s,t}^{g_N}\left(1(|\cdot-x|\le 1/\sqrt{N})\right)(z)\bar{X}_s^{2,N}(dz)|\bar{X}^{1,N}\right] \\
& = \int_{\mathcal{Z}_N}N^{d/2}P_{0,t}^{g_N}\left(1(|\cdot-x|\le 1/\sqrt{N})\right)(z)\bar{X}_0^{2,N}(dz) \\
& \le \bar{R}_N(t)\hat{\varrho}_\delta^N(\bar{X}_0^{2,N})t^{-l_d-\tilde{\delta}},\ \ \forall x\in\mathcal{Z}_N\,,
\end{aligned}
$$

where the last inequality follows by Lemma 3.7(a). This and (8.5) allows us to bound (8.33) by

$$
\begin{aligned}
(8.35) \qquad & c(1+R_N)\hat{\varrho}_\delta^N(\bar{X}_0^{2,N})\bar{X}_t^{1,N}(1)t^{-l_d-\tilde{\delta}}\bar{R}_N(t)^2\left(t\epsilon^{\eta(1-l_d-3\tilde{\delta})}+c\epsilon^{\eta(1-l_d-\tilde{\delta})}\right) \\
& = \bar{R}_N(t)\epsilon^{\eta(1-l_d-3\tilde{\delta})}t^{-l_d-\tilde{\delta}}\hat{\varrho}_\delta^N(\bar{X}_0^{2,N}).
\end{aligned}
$$

Now we consider the third term on the right-hand side of (8.30). We use Lemmas 8.4, 8.10 to show that it is bounded by

$$
\begin{aligned}
& \bar{R}_N(t)\int_0^t\left(\epsilon^{\eta(1-l_d-3\tilde{\delta})}1(t-s\ge\epsilon^\eta)+h_d(t-s)1(t-s\le\epsilon^\eta)\right) \\
(8.36) \qquad & \times\int_{\mathcal{Z}_N}E\left[\int_{\mathcal{Z}_N}\left(\sum_{y\in\mathcal{Z}_N}N^{d/2}P_{s,t}^{\bar{g}_N}\left(1(|\cdot+y-x|\le 1/\sqrt{N})\right)(z)p^N(y)\right)\right. \\
& \qquad \left.\times\bar{X}_s^{2,N}(dz)|\bar{X}^{1,N}\right]\,ds\,\bar{X}_t^{1,N}(dx).
\end{aligned}
$$

Now use (2.17), (8.29), Chapman-Kolmogorov and Lemma 3.7(a) to get that

$$\begin{aligned}
&\sum_{y\in\mathcal{Z}_N}\left(E\left[\int_{\mathcal{Z}_N} N^{d/2}P^{\bar g_N}_{s,t}\left(1(|\cdot+y-x|\le 1/\sqrt{N})\right)(z)\bar X^{2,N}_s(dz)|\bar X^{1,N}\right]\right)p^N(y)\\
&= \sum_{y\in\mathcal{Z}_N}\left(\int_{\mathcal{Z}_N} N^{d/2}P^{\bar g_N}_{0,t}\left(1(|\cdot+y-x|\le 1/\sqrt{N})\right)(z)\bar X^{2,N}_0(dz)\right)p^N(y)\\
&\le \hat\varrho^N_\delta(\bar X^{2,N}_0)\bar R_N(t)t^{-l_d-\tilde\delta},
\end{aligned}$$

where in the last inequality we applied Lemma 3.7(a) for the $P^{\bar g_N}$ semigroup.

This, (8.5) and (8.36) imply that the third term is bounded by

$$\bar R_N(t)\bar X^{1,N}_t(1)\hat\varrho^N_\delta(\bar X^{2,N}_0)t^{-l_d-\tilde\delta}\epsilon^{\eta(1-l_d-3\tilde\delta)}. \tag{8.37}$$

Combine (8.32) and (8.35), (8.37) to get the desired result. ∎

Proof of Lemma 4.5 It is easy to use Lemma 2.3 and (3.7), as in (8.30), to show that

$$\begin{aligned}
&\int_{\mathbb{R}^d} E\left[\int_{\mathbb{R}^d}\psi^\epsilon_N(x,x_1)\bar X^{2,N}_t(dx_1)\int_{\mathbb{R}^d} G^\alpha_N\mathbf{1}(x\,,x_2)\bar X^{2,N}_t(dx_2)|\bar X^{1,N}\right]\left(\bar X^{1,N}_t * q^N\right)(dx)\\
&\le \int_{\mathcal{Z}_N^3} P^{g_n}_{0,t}\left(\psi^\epsilon_N(x\,,\cdot)\right)(x_1)P^{g_n}_{0,t}\left(G^\alpha_N\mathbf{1}(x\,,\cdot)\right)(x_2)\bar X^{2,N}_0(dx_1)\bar X^{2,N}_0(dx_2)\left(\bar X^{1,N}_t * q^N\right)(dx)\\
&\quad + c\Big[1+\mathcal{H}_{\delta,N}N^{-1+\ell_d+\delta/2}\Big]E\left[\int_0^t\left(\sup_{z,x\in\mathcal{Z}_N} P^{g_n}_{s,t}\left(\psi^\epsilon_N(x\,,\cdot)\right)(z)\right)\right.\\
&\quad\left.\times\int_{\mathcal{Z}_N}\left(\sup_{z\in\mathcal{Z}_N}\int_{\mathcal{Z}_N} P^{g_n}_{s,t}\left(G^\alpha_N\mathbf{1}(x\,,\cdot)\right)(z)\left(\bar X^{1,N}_t * q^N\right)(dx)\right)\bar X^{2,N}_s(dz_1)\,ds|\bar X^{1,N}\right]\\
&\equiv I^{1,N}+I^{2,N}.
\end{aligned} \tag{8.38}$$

In the above the sup over z in the last integrand avoids the additional convolution with p^N we had in (8.30). First, apply Lemmas 8.8, 8.9, and 4.2 to get

$$\begin{aligned}
I^{1,N} &\le \hat\varrho^N_\delta(\bar X^{2,N}_0)\bar R_N(t)\int_{\mathcal{Z}_N^2} P^{g_n}_{0,t}\left(\psi^\epsilon_N(x\,,\cdot)\right)(x_1)\left(\bar X^{1,N}_t * q^N\right)(dx)\bar X^{2,N}_0(dx_1)\\
&\le \hat\varrho^N_\delta(\bar X^{2,N}_0)\bar R_N(t)\epsilon^{1-l_d-\tilde\delta}\hat\varrho^N_\delta(\bar X^{1,N}_t)\bar X^{2,N}_0(1)\\
&\le \hat\varrho^N_\delta(\bar X^{2,N}_0)^2\bar R_N(t)\epsilon^{1-l_d-\tilde\delta}.
\end{aligned} \tag{8.39}$$

Now, let us bound $I^{2,N}$. By Lemmas 8.4, 8.6, and 4.2 we get for $\eta \in (0, 2/7)$,

$$\begin{aligned}
I^{2,N} &\leq c(1+R_N)\bar{R}_N(t)\int_0^t \left(\epsilon^{\eta(1-l_d-3\tilde{\delta})}1(t-s\geq \epsilon^\eta) + h_d(t-s)1(t-s\leq \epsilon^\eta)\right) \\
&\quad \times \sup_{z\in\mathcal{Z}_N}\left\{\int_{\mathcal{Z}_N} P^{g_N}_{s,t}\left(G^\alpha_N \mathbf{1}(x\,,\cdot)\right)(z)\left(\bar{X}^{1,N}_t * q^N\right)(dx)\right\} \\
&\quad \times E\left[\bar{X}^{2,N}_s(1)|\bar{X}^{1,N}\right]\, ds \\
&\leq \hat{\varrho}^N_\delta(\bar{X}^{1,N}_t)\bar{R}_N(t) \\
&\quad \times \int_0^t \left(\epsilon^{\eta(1-l_d-3\tilde{\delta})}1(t-s\geq \epsilon^\eta) + h_d(t-s)1(t-s\leq \epsilon^\eta)\right) E\left[\bar{X}^{2,N}_s(1)|\bar{X}^{1,N}\right]\, ds.
\end{aligned}$$

Apply Corollary 3.6(b) to get

$$\begin{aligned}
I^{2,N} &\leq \hat{\varrho}^N_\delta(\bar{X}^{2,N}_t)\bar{R}_N(t)^2\bar{X}^{2,N}_0(1)\int_0^t \left(\epsilon^{\eta(1-l_d-3\tilde{\delta})}1(t-s\geq \epsilon^\eta) + h_d(t-s)1(t-s\leq \epsilon^\eta)\right) ds \\
&\leq \hat{\varrho}^N_\delta(\bar{X}^{2,N}_0)\bar{R}_N(t)\epsilon^{\eta(1-l_d-3\tilde{\delta})}, \qquad (8.40)
\end{aligned}$$

where (8.5) is used in the last line. Combine this with (8.39) to finish the proof.

■

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