Conditional Dawson-Watanabe Processes and
Fleming-Viot Processes

by

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Abstract

A class of time-inhomogeneous Fleming-Viot processes are introduced. Their laws are shown to be the laws of the normalized Dawson-Watanabe process, conditioned on the total mass process. This is motivated by, and gives another derivation of, a recent result of Etheridge and March which identifies the Fleming-Viot process as the Dawson-Watanabe process conditioned to have total mass one.

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There has been interest recently in establishing connections between the Dawson-Watanabe and Fleming-Viot superprocesses (eg. Konno-Shiga (1988), Etheridge-March (1990)). Sometimes results are more readily derived for one class of processes but one would like to be able to infer them for the other with minimal effort. Tribe (1989) used the Konno-Shiga (1988) results to analyze the Dawson-Watanabe superprocess near extinction.

Etheridge and March (1990) showed that the Fleming-Viot superprocesses is the Dawson-Watanabe superprocess, conditioned to have total mass one. Our goal is to generalize this pretty result via a different approach.

Let E be a locally compact space with a countable base and one-point compactification $E_\infty = E \cup \{\infty\}$. Its Borel σ-field is $\mathcal{E}$ and $b\mathcal{E}$ denotes the class of bounded $\mathcal{E}$-measurable functions from E to $\mathbf{R}$. $M_F(E)$ and $M_1(E)$ denote the spaces of finite measures, and probability measures, respectively, with the topologies of weak convergence. If $\phi \in b\mathcal{E}$ and $\nu \in M_F(E)$, $\nu(\phi)$ denotes $\int \phi(x)d\nu(x)$.

Let $\Omega = C([0,\infty), M_F(E))$ and $\hat{\Omega} = C([0,\infty), M_1(E))$ (compact-open topologies) and let $\mathcal{F}$ and $\hat{\mathcal{F}}$ be their respective Borel σ-fields. $X_t(w) = w(t)$ and $\hat{X}_t(\hat{w}) = \hat{w}(t)$ denote the coordinate mappings on Ω and $\hat{\Omega}$, respectively, and $\mathcal{F}_t^0 = \sigma(X_s : s \le t)$, $\hat{\mathcal{F}}_t^0 = \sigma(\hat{X}_s : s \le t)$, $\mathcal{F}_t = \mathcal{F}_{t+}^0$, $\hat{\mathcal{F}}_t = \hat{\mathcal{F}}_{t+}^0$.

Let (Y_t, P_y) be an E-valued conservative Feller process with generator A defined on $D(A) \subset C_o(E)$ (continuous functions on E, vanishing at ∞). Recall that this means T_t, the semigroup of Y, is strongly continuous on $C_o(E)$. For each $\sigma^2 > 0$ and $m \in M_F(E)$

there is a unique probability $\mathbb{P}_m$ on $(\Omega, \mathcal{F})$ such that

$$\forall \phi \in D(A) \qquad M_t(\phi) = X_t(\phi) - m(\phi) - \int_0^t X_s(A\phi)ds \quad \text{is an}$$

(DW_m) $\qquad (\mathcal{F}_t)$ – martingale starting at 0 and such that

$$< M(\phi) >_t = \sigma^2 \int_0^t X_s(\phi^2)ds.$$

For each $\sigma^2 > 0$ and $m \in M_1(E)$ there is a unique probability on $(\hat{\Omega}, \hat{\mathcal{F}})$ such that

$$\forall \phi \in D(A) \qquad \hat{M}_t(\phi) = \hat{X}_t(\phi) - m(\phi) - \int_0^t \hat{X}_s(A\phi)ds \quad \text{is an}$$

(FV_m) $\qquad (\hat{\mathcal{F}}_t)$ – martingale starting at 0 and such that

$$< \hat{M}(\phi) >_t = \sigma^2 \int_0^t \hat{X}_s(\phi^2) - \hat{X}_s(\phi)^2 ds.$$

See Ethier-Kurtz (1986, Ch. 9.4, 10.4) or Roelly-Coppoletta (1986) for the above results. $\mathbb{P}_m$ and $\hat{\mathbb{P}}_m$ are the laws of the A-Dawson-Watanabe and A-Fleming-Viot superprocesses, respectively. σ^2 is usually assumed to be one, unless otherwise indicated, and hence is suppressed in our notation..

Remark 1. (DW_m) and (FV_m) extend to $\phi \equiv 1$ by taking limits through a sequence $\{\phi_n\} \subset D(A)$ such that $\phi_n \to 1$ and $A\phi_n \to 0$, both in the bounded pointwise sense. For example, let $\phi_n(x) = \int_0^1 T_t f_n(x)dt$ where $f_n(x) = 1 - e^{-nd(x,\infty)}$ and d is a bounded metric on E_∞. The extension will also hold for $(FV_{m,f})$ described below and will be used without further comment.

If $T > 0$, let $(\hat{\Omega}_{T-}, \hat{\mathcal{F}}_{T-}) = (C([0,T), M_1(E))$, Borel sets) and let $(\hat{\Omega}_T, \hat{\mathcal{F}}_T)$ denote the same space with $[0,T]$ in place of $[0,T)$. $(\Omega_T, \mathcal{F}_T)$ and $(\Omega_{T-}, \mathcal{F}_{T-})$ denote the same spaces with $M_F(E)$ in place of $M_1(E)$. (We are abusing the $\mathcal{F}_T$ notation slightly here.)

Each of these spaces is given the compact-open topology. If $\mathbb{P}$ is a probability on $(\Omega, \mathcal{F})$ (or $(\hat{\Omega}, \hat{\mathcal{F}})$), $\mathbb{P}|_{T-}$ is defined on $(\Omega_{T-}, \mathcal{F}_{T-})$ (or on $(\hat{\Omega}_{T-}, \hat{\mathcal{F}}_{T-})$) by $\mathbb{P}|_{T-}(A) = \mathbb{P}(X|_{[0,T)} \in A)$ (or use $\hat{X}$ in place of X). Similarly one defines $\mathbb{P}|_T$.

Here then is a slight restatement of the result of Etheridge and March (1990).

<u>Theorem A.</u> (Etheridge-March (1990)). Assume $m_n \to m$ in $M_F(E)$ where $m(E) = 1$. Let $\epsilon_n \downarrow 0$ and $T_n \to T$, where $T_n \in (0, \infty)$ and $T \in (0, \infty]$. Then

$$\mathbb{P}_{m_n}|_{T-}(\,\cdot \mid \sup_{t \le T_n} |X_t(1) - 1| < \epsilon_n) \xrightarrow{w} \hat{\mathbb{P}}_m|_{T-} \quad \text{on} \quad (\Omega_{T-}, \mathcal{F}_{T-}).$$

The best way to understand this result is to recall the "particle pictures" of these two processes.

Consider a system of K_N particles which follow independent copies of Y on $[0, 1/N]$ and then at $t = 1/N$ independently produce offspring according to a law ν with mean one and variance one. The offspring then follow independent copies of Y on $[1/N, 2/N]$ and this pattern of alternating branching and spatial motions continues. If $X_N(t)(A)$ is N^{-1} times the number of particles in A at time t and $\mathbb{P}_N$ is the law of X_N on $D([0, \infty), M_F(E))$ then $X_N(0) \to m$ in $M_F(E)$ implies $\mathbb{P}_N \xrightarrow{w} \mathbb{P}_m$.

Now consider a system of N particles which follow independent copies of Y on $[0, 1/N]$. At $t = 1/N$ these N particles produce a vector of offspring in $\mathbb{Z}_+^{\mathbb{N}}$ distributed as a multinomial random vector with N trials and $p_1 = \ldots = p_N = 1/N$. This pattern of alternating spatial motions and "multinomial branching" continues. $\hat{X}_N(t)$ denotes the empirical probability distribution of the N particles at time t and $\hat{\mathbb{P}}_N$ is the law of $\hat{X}_N$ on $D([0, \infty), M_1(E))$. If $\hat{X}_N(0) \to m$ in $M_1(E)$ then $\hat{\mathbb{P}}_N \xrightarrow{w} \hat{\mathbb{P}}_m$.

These results are minor modifications of results in Ethier-Kurtz (1986, Ch. 9.4, 10.4).

If $\{X_i : i \le N\}$ are Poisson (1) and $S_N = \sum_{i=1}^{N} X_i$, then an easy calculation shows that

$P((X_1 ... X_N) \in \cdot | S_N = N)$ is multinomial with N trials and $p_1 = ... = p_N = \frac{1}{N}$.

This shows that if we take ν to be Poisson (1) in the above construction of $\mathbb{P}_m$, then

$$\mathbb{P}_N(\cdot | X_N(t) = 1 \text{ for } t \le T)|_T = \hat{\mathbb{P}}_N|_T.$$

Letting $N \to \infty$ suggests (but does not prove) the result of Etheridge and March. Our original proof of our main result (Theorem 3 below) used this particle picture. The proof given below has sacrificed intuition for brevity.

Let

$$C_+ = \{f : [0,\infty) \to [0,\infty) : f \text{ continuous}, \ \exists t_f \in (0,\infty] \text{ such that}$$

$$f(t) > 0 \text{ if } t \in [0, t_f) \text{ and } f(t) = 0 \text{ if } t \ge t_f\}$$

with the compact-open topology. If $A \subset C_+$ and $T > 0$ let $A|_{T-} = \{f|_{[0,T)} : f \in A\}$ and $A|_T = \{f|_{[0,T]} : f \in A\}$.

Theorem 2.

(a) If $f \in C_+$ and $m \in M_1(E)$, there is a unique probability $\hat{\mathbb{P}}_{m,f}$ on $(\hat{\Omega}, \hat{\mathcal{F}})$ such that under $\hat{\mathbb{P}}_{m,f}$:

$$\forall \phi \in D(A) \ \ \hat{M}_t(\phi) = \hat{X}_t(\phi) - m(\phi) - \int_0^t \hat{X}_s(A\phi)ds, \ t < t_f,$$

$(FV_{m,f})$ is an $(\hat{\mathcal{F}}_t)$ – martingale starting at 0 and such that

$$< \hat{M}(\phi) >_t = \int_0^t (\hat{X}_s(\phi^2) - \hat{X}_s(\phi)^2) f(s)^{-1} ds \ \ \forall t < t_f.$$

$$\hat{X}_t = \hat{X}_{t_f} \text{ for all } t \ge t_f.$$

(b) If $(m_n, f_n|_{[0,T)}) \to (m, f|_{[0,T)})$ in $M_1(E) \times C_+|_{T-}$ where $T \leq t_f$, then $\hat{\mathbb{P}}_{m_n,f_n}|_{T-} \xrightarrow{w} \hat{\mathbb{P}}_{m,f}|_{T-}$ on $(\hat{\Omega}_{T-}, \hat{\mathcal{F}}_{T-})$. In particular if $t_f = \infty$, $\hat{\mathbb{P}}_{m_n,f_n} \xrightarrow{w} \hat{\mathbb{P}}_{m,f}$.

Remark. If $f(s) = \sigma^{-2}$ is constant clearly $\hat{\mathbb{P}}_{m,f}$ is just the unique solution $\hat{\mathbb{P}}_m$ of (FV_m).

The proof of Theorem 2 is easy (although some tedious calculations make it a little long), and given at the end of this work. $\hat{\mathbb{P}}_{m,f}$ will be constructed by making a deterministic time change of a Fleming-Viot process whose underlying Markov process, Y, is time-inhomogeneous.

If $m \in M_F(E) - \{0\}$, let $\bar{m}(A) = m(A)/m(E)$. If $t_X(w) = \inf\{u : X_u(E) = 0\}(w \in \Omega)$, then Tribe (1989) showed that $\lim_{t \uparrow t_X} \bar{X}_t$ exists $\mathbb{P}_m$–a.s. Hence we may $\mathbb{P}_m$-a.s. extend $\{\bar{X}_t : t < t_X\}$ to a continuous $M_1(E)$-valued process on $[0, \infty)$ by setting $\bar{X}_t = \bar{X}_{t_X-}$ for $t \geq t_X$. In fact Tribe's result will follow from our arguments but this is not surprising as we will borrow some of his methods.

Let $Q_y \in M_1(C_+)$ denote the law of the unique solution of

$$Z_t = y + \int_0^t \sqrt{Z_s} dB_s$$

(B a standard Brownian motion). If follows from (DW_m) with $\phi = 1$ that

$\mathbb{P}_m(X.(1) \in A) = Q_{m(1)}(A)$.

Theorem 3. If $m \in M_F(E) - \{0\}$, then

$$\mathbb{P}^m(\bar{X} \in A | X.(1) = f) = \hat{\mathbb{P}}_{\bar{m},f}(A) \quad Q_{m(1)} - \text{a.a.f} \quad \forall A \in \hat{\mathcal{F}}.$$

Hence $\hat{\mathbb{P}}_{\bar{m},f}(\cdot)$ is a regular conditional distribution for $\bar{X}$ on $(\Omega, \mathcal{F}, \mathbb{P}_m)$ given $X.(1) = f$.

Proof. If $M_t(\phi)$ is as in (DW_m), $\phi \in D(A)$, $T_n = \inf\{t : X_t(1) \leq n^{-1}\}$, and

(1) $\bar{M}^n_t(\phi) = \int_0^t 1(s \leq T_n) X_s(1)^{-1} dM_s(\phi) - \int_0^t 1(s \leq T_n) X_s(\phi) X_s(1)^{-2} dM_s(1)$, then

Itô's Lemma implies

$$\bar{X}_{t\wedge T_n}(\phi) = \bar{m}(\phi) + \int_0^t 1(s \le T_n)\bar{X}_s(A\phi)ds + \bar{M}_t^n(\phi). \tag{2}$$

(2) implies that

$$\sup_{t\le K, n\in\mathbb{N}} |\bar{M}_t^n(\phi)| \le 2\|\phi\|_\infty + K\|A\phi\|_\infty. \tag{3}$$

Since $\{\bar{M}_t^n(\phi) : n \in \mathbb{N}\}$ is a martingale in n (t fixed) by (1), it converges a.s. as $n \to \infty$ for each $t \ge 0$ by the Martingale Convergence Theorem and (3). A simple application of the L^2-maximal inequality shows that the convergence is uniform for t in compacts a.s. (by perhaps passing to a subsequence). Hence the limit, $\bar{M}_t(\phi)$, is a continuous martingale which clearly satisfies

$$\bar{M}_t^n(\phi) = \bar{M}_{t\wedge T_n}(\phi) \quad \forall t \ge 0 \ a.s. \tag{4}$$

$$\sup_{t\le K} |\bar{M}_t(\phi)| \le 2\|\phi\|_\infty + K\|A\phi\|_\infty \ a.s. \tag{5}$$

We now may let $n \to \infty$ in (2) to see

$$\bar{X}_t(\phi) = \bar{m}(\phi) + \int_0^t 1(s < t_X)\bar{X}_s(A\phi)ds + \bar{M}_t(\phi) \ \ \forall t \ge 0 \text{ a.s. } \forall \phi \in D(A). \tag{6}$$

Let $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(X_s(1) : s \ge 0)$. We claim $\bar{M}_t(\phi)$ is a $(\mathcal{G}_t)$-martingale ($\phi \in D(A)$ fixed). Let $s < t$ and let F be a bounded $\sigma(X.(1))$-measurable random variable. The predictable representation theorem of Jacod and Yor (see Yor (1978, Thm. 3) and recall $Q^{m(1)}$ is the law of $X.(1)$) shows that

$$F = \mathbb{P}_m(F) + \int_0^\infty f(s,w)dX_s(1) \tag{7}$$

for some $\sigma(X_s(1) : s \leq t)$-predictable f. Therefore

$$\mathbb{P}_m((\bar{M}_{t\wedge T_n}(\phi) - \bar{M}_{s\wedge T_n}(\phi))F|\mathcal{F}_s)$$

$$= \mathbb{P}_m((\bar{M}_t^n(\phi) - \bar{M}_s^n(\phi))\int_0^\infty f(u)dM_u(1)|\mathcal{F}_s) \quad \text{(by (4) and (7))}$$

$$= \mathbb{P}_m((\int_s^t 1(u \leq T_n)X_u(1)^{-1}dM_u(\phi) - \int_s^t 1(u \leq T_n)X_u(\phi)X_u(1)^{-2}dM_u(1))\int_s^t f(u)dM_u(1)|\mathcal{F}_s)$$

$$= \mathbb{P}_m(\int_s^t 1(u \leq T_n)(X_u(\phi)X_u(1)^{-1} - X_u(\phi)X_u(1)^{-1})f(u)du|\mathcal{F}_s)$$

$$= 0.$$

Let $n \to \infty$ in the above and use (4,5) to see that $\mathbb{P}_m((\bar{M}_t(\phi) - \bar{M}_s(\phi))F|\mathcal{F}_s) = 0$ and hence $\bar{M}_t(\phi)$ is a $(\mathcal{G}_t)$-martingale.

It follows from (4) and $\bar{M}_{t\wedge t_X}(\phi) = \bar{M}_t(\phi)$ that

$$(8) \qquad <\bar{M}(\phi)>_t = \int_0^t 1(s < t_X)(\bar{X}_s(\phi^2) - \bar{X}_s(\phi)^2)X_s(1)^{-1}ds \quad \mathbb{P}_m - \text{a.s.}$$

Let $\{\mathbb{P}(A|f) : A \in \hat{\mathcal{F}}, f \in C_+\}$ be a regular conditional probability for $\bar{X}$ given $X.(1) = f(\cdot)$ (under $\mathbb{P}_m$). If $\phi \in D(A)$ and $f \in C_+$, define $\hat{M}_t^f(\phi) = \hat{M}_t^f(\phi)(\hat{X})$ on $(\hat{\Omega}, \hat{\mathcal{F}})$ by the first equation in $(FV_{\bar{m},f})$ for $t < t_f$ and set $\hat{M}_t^f(\phi) = \hat{M}_{t_f-}^f(\phi)$ for $t \geq t_f$. Note that (6) and $\bar{M}_t(\phi) = \bar{M}_{t\wedge t_X}(\phi)$ imply

$$(9) \qquad \bar{M}_t(\phi) = \hat{M}_t^{X.(1)}(\phi)(\bar{X}) \;\; \forall t \geq 0 \;\; \mathbb{P}_m - \text{a.s.} \; \forall \phi \in D(A).$$

If $G \in b\hat{\mathcal{F}}_s^0$ and $s < t$, then the $(\mathcal{G}_t)$-martingale property of $\bar{M}_t(\phi)$ shows that

$$\mathbb{P}_m((\bar{M}_t(\phi) - \bar{M}_s(\phi))G(\bar{X})|X.(1)) = 0 \quad \mathbb{P}_m - \text{a.s.}$$

and hence, by (9),

$$\mathbb{P}((\hat{M}_t^f(\phi) - \hat{M}_s^f(\phi))G|f) = 0 \quad Q^{m(1)} - \text{a.a.f.}$$

Consider the null set, Λ, of f's off which the above holds for all rational $s < t$ and all G in C_s, a countable set in $b\hat{\mathcal{F}}_s^0$ whose bounded pointwise closure is $b\hat{\mathcal{F}}_s^0$. By working on Λ^c and taking limits in both s and G one easily shows that

(10) $\quad \{\hat{M}_t^f(\overset{\emptyset}{\cancel{u}}) : t \geq 0\}$ is an $(\hat{\mathcal{F}}_t)$ – martingale under $\mathbb{P}(\cdot|f)$ for $Q^{m(1)}$ – a.a.f.

If $t_n^f = \inf\{u : f(u) \leq 1/n\}$, then (8) implies that

(11) $\hat{M}^f_{t\wedge t_n^f}(\phi)^2 - \int_0^t 1(s < t_n^f)(\hat{X}_s(\phi^2) - \hat{X}_s(\phi)^2)f(s)^{-1}ds$ is an $(\hat{\mathcal{F}}_t)$-martingale under $\mathbb{P}(\cdot|f)$ $\forall n \in \mathbb{N}$ for $Q^{m(1)}$–a.a.f.

Now consider a countable core, D_0, for A and fix f outside a $Q^{m(1)}$-null set so that (10) and (11) hold for all $\phi \in D_0$. Take uniform limits in $(\phi, A\phi)$ (recall the definition of $\hat{M}_t^f$) to see that (10) and (11) hold for all ϕ in $D(A)$ for $Q^{m(1)}$-a.a.f. Therefore $\mathbb{P}(\cdot|f)$ solves $(FV_{\bar{m},f})$ for $Q^{m(1)}$-a.a.f. and so $\mathbb{P}(\cdot|f) = \hat{\mathbb{P}}_{\bar{m},f}(\cdot)$ for $Q^{m(1)}$-a.a.f. by Theorem 2.∎

Corollary 4. Let $\{m_n\} \subset M_F(E) - \{0\}$ satisfy $\bar{m}_n \to m$ in $M_1(E)$. Assume $\{A_n\}$ is a sequence of Borel subsets of C_+, $f \in C_+$ and $T \in (0, t_f]$ satisfy:

(12) $Q^{m_n(1)}(A_n) > 0 \;\; \forall n$

(13) $\sup\{|g(t) - f(t)| : g \in A_n, t \leq S\} \to 0$ as $n \to \infty \;\; \forall S < T$.

Then $\mathbb{P}_{m_n}(\bar{X} \in \cdot | X.(1) \in A_n)|_{T-} \xrightarrow{w} \hat{\mathbb{P}}_{m,f}|_{T-}$ on $(\hat{\Omega}_{T-}, \hat{\mathcal{F}}_{T-})$.

Proof. Let $\phi : \hat{\Omega}_{T-} \to \mathbf{R}$ be bounded and continuous. Then

$$
\begin{aligned}
&|\mathbb{P}_{m_n}(\phi(\bar{X})|X.(1) \in A_n) - \hat{\mathbb{P}}_{m,f}(\phi)| \\
&= \left|\int_{A_n} \hat{\mathbb{P}}_{\bar{m}_n,g}(\phi) - \hat{\mathbb{P}}_{m,f}(\phi) dQ^{m_n(1)} Q^{m_n(1)}(A_n)^{-1}\right| \quad \text{(Theorem 3)} \\
&\leq \sup_{g \in A_n} |\hat{\mathbb{P}}_{\bar{m}_n,g}(\phi) - \hat{\mathbb{P}}_{m,f}(\phi)|
\end{aligned}
$$

which approaches 0 as $n \to \infty$ by (13) and Theorem 2(b).∎

Corollary 5. Let $\{m_n\}$ and m be as in Corollary 4. Let $f \in C_+$, $T \in (0, t_f]$, $T_n \to T$ $(T_n < \infty)$, $\epsilon_n \downarrow 0$ and assume $|m_n(1) - f(0)| < \epsilon_n$. Then

(a) $\mathbb{P}_{m_n}(\bar{X} \in \cdot | \sup_{t \le T_n} |X_t(1) - f(t)| < \epsilon_n)|_{T-} \xrightarrow{w} \hat{P}_{m,f}|_{T-}$ on $(\hat{\Omega}_{T-}, \hat{\mathcal{F}}_{T-})$

(b) $\mathbb{P}_{m_n}(X/f \in \cdot | \sup_{t \le T_n} |X_t(1) - f(t)| < \epsilon_n)|_{T-} \xrightarrow{w} \hat{\mathbb{P}}_{m,f}|_{T-}$ on $(\Omega_{T-}, \mathcal{F}_{T-})$.

Proof.

(a) follows from Corollary 4 with

$$A_n = \{g \in C_+ : \sup_{t \le T_n} |g(t) - f(t)| < \epsilon_n\}.$$

(b) Note that for $S < T$ and large n, if $\delta_n = \epsilon_n(\inf_{t \le S} f(t)(f(t) - \epsilon_n))^{-1}$, then

$$\mathbb{P}_{m_n}(\sup_{t \le S} |X_t(1)^{-1} - f(t)^{-1}| < \delta_n | A_n) = 1$$

(A_n as above). Since $\delta_n \to 0$, (b) is immediate from (a).∎

Clearly Theorem A is just (b) of the above result with $f(t) \equiv 1$.

Let $\phi(r) = r^2 \log\log 1/r$ $(r < e^{-1})$ and let $\phi - m(A)$ denote the Hausdorff ϕ-measure of $A \subset \mathbf{R}^d$. Let $S(\nu)$ denote the closed support of $\nu \in M_F(\mathbf{R}^d)$. If $\mathbb{P}_m$ denotes the law of super-Brownian motion then for $d \ge 3$ there is a universal constant $K_d \in (0, \infty)$ such that

$$X_t(A) = K_d \phi - m(A \cap S(X_t)) \quad \forall A \in B(\mathbf{R}^d)\ \mathbb{P}_m - \text{a.s. } \forall t > 0 \tag{14}$$

(see Dawson-Perkins (1990, Theorem 5.2)). Theorem 3 therefore shows that for all $t > 0$ and $Q^{m(1)}$-a.a.f,

$$\hat{X}_t(A) = f(t)^{-1} K_d \phi - m(A \cap S(\hat{X}_t)) \quad \forall A \in B(\mathbf{R}^d)\ \hat{\mathbb{P}}_{m,f} - \text{a.s.} \tag{15}$$

It would of course be nice to know if (15) holds for a particular f such as 1. It seems likely that (14) together with the result of Konno-Shiga (1988) and a version of the 0-1 law used to prove (14) will give (14) for the $\Delta/2$-Fleming-Viot superprocess but with another unknown constant K'_d. I have no idea on how one could prove or disprove $K_d = K'_d$.

We now return to the proof of Theorem 2.

If $g : [0,\infty) \to (0,\infty)$ is continuous, let $G(u) = \int_0^u g(s)ds$. For $(t_o, y) \in [0,\infty) \times E \equiv \vec{E}$ define $P^g_{(t_o,y)}$ on the Borel sets of $D([0,\infty), \vec{E})$ by

$$P^g_{(t_o,y)}(A) = P_y(\vec{Y}^g \in A)$$

where $\vec{Y}^g(t) = (t_o + t, Y(G(t_o + t) - G(t_o)))$. Let $P_{(t_o,y)} = P^1_{(t_o,y)}$ and let $\vec{Y}(t)$ denote the canonical process on $D([0,\infty), \vec{E})$).

<u>Proposition 6.</u> $(\vec{Y}, P^g_{(t_o,y)})$ is an $\vec{E}$-valued Feller process, that is, it is a strong Markov process with a strongly continuous semigroup, $\vec{T}^g_t$, on $\mathcal{C}_o(\vec{E})$.

<u>Proof.</u> This is routine. For example, for the strong continuity of $\vec{T}^g_t f$ use the uniform continuity of G on compacts and the fact that $\lim_{t_0 \to \infty} \|f(t_o, \cdot)\|_\infty = 0$ for $f \in \mathcal{C}_o(\vec{E})$. ∎

<u>Notation.</u> $C_K(A, B)$ denotes the space of continuous functions from A to B with compact support and $C^1_K(A, B)$ denotes those functions in $C_K(A, B)$ which are continuously differentiable.

$$\mathcal{C}_o = \{\psi \in \mathcal{C}_o(\vec{E}) : \psi(t,x) = \sum_{i=1}^n \psi_i(t)\phi_i(x), \quad \phi_i \in D(A), \quad \psi_i \in C^1_K([0,\infty), \mathbf{R})\}$$

$$\mathcal{C} = \{\psi \in \mathcal{C}_o(\vec{E}) : \; t \mapsto \frac{\partial \psi}{\partial t} t, \; A\psi_t \text{ and } \psi_t \text{ are all functions in } \; C_K([0,\infty), \mathcal{C}_o(E))\}$$

Let $\vec{A}^g$ denote the generator of $\vec{Y}^g$ and write $\vec{A}$ for $\vec{A}^1$.

<u>Proposition 7.</u>

(a) $\mathcal{C}$ is a core for $\vec{A}^g$ and if $\psi \in \mathcal{C}$, then

$$\vec{A}^g \psi(t,x) = \frac{\partial \psi}{\partial t}(t,x) + g(t) A\psi_t(x).$$

(b) $\mathcal{C}_o$ is a core for $\vec{A}$.

Proof. The closure of $\mathcal{C}_o$ in $C_o(\vec{E})$ contains all functions $\psi(t,x)$ as in the definition of $\mathcal{C}_o$ but with $\phi_i \in C_o(E)$. Apply Stone-Weierstrass to this latter class to see that $\mathcal{C}_o$ (and hence also $\mathcal{C} \supset \mathcal{C}_o$) is dense in $C_o(\vec{E})$. Note that if $\psi(t,x) = \sum_{i=1}^{n} \psi_i(t)\phi_i(x) \in \mathcal{C}_o$, then

$$\vec{T}_t^g \psi(t_o, x) = \sum_{i=1}^{n} \psi_i(t + t_o) T_{G(t+t_o)-G(t_o)} \phi_i(x).$$

It is now easy to check that $\vec{T}_t^g : \mathcal{C}_o \to \mathcal{C}$. A theorem of S. Watanabe (Ethier-Kurtz (1986, Chapter 1, Prop. 3.3)) will show $\mathcal{C}$ is a core for $\vec{A}^g$ providing we prove $\mathcal{C} \subset D(\vec{A}^g)$. Moreover since $\vec{T}_t^1 : \mathcal{C}_o \to \mathcal{C}_o$, the same argument will also prove (b). A direct calculation shows that if $\psi \in \mathcal{C}$ then $\psi \in D(\vec{A}^g)$ and $\vec{A}^g\psi$ is given by the formula in (a) above. We omit the details.∎

Proof of Theorem 2. (a) Consider first the uniqueness of $\hat{\mathbb{P}}_{m,f}$. Claim it suffices to prove the result when $t_f = \infty$ and $T_f \equiv \int_0^{t_f} \frac{1}{f(s)} ds = \infty$. To prove this claim let $T_n \uparrow t_f$ $(T_n < t_f)$ and let $f_n(t) = f(t \wedge T_n)$. Note that $t_{f_n} = T_{f_n} = \infty$. Let $\hat{\mathbb{P}}$ be a solution of $(FV_{m,f})$. A solution, $\hat{\mathbb{P}}_n$, of (FV_{m,f_n}) may be constructed by letting $\hat{\mathbb{P}}_n = \hat{\mathbb{P}}$ on $\hat{\mathcal{F}}^0_{T_n}$ and setting the conditional distribution of $\{\hat{X}_{t+T_n}, t \geq 0\}$ given $\hat{\mathcal{F}}^0_{T_n}$ equal to $\mathbb{P}_{\hat{X}_{T_n}}$, the unique solution of $(FV_{\hat{X}_{T_n}})$ with $\sigma^2 = f(T_n)^{-1}$. The assumed uniqueness of $\hat{\mathbb{P}}_n$ establishes the uniqueness of $\hat{\mathbb{P}}|_{\hat{\mathcal{F}}^0_{T_n}}$. Since $\hat{X}_t = \hat{X}_{t_f}$ for all $t \geq t_f$ (if $t_f < \infty$), the uniqueness of $\hat{\mathbb{P}}$ on $(\hat{\Omega}, \hat{\mathcal{F}})$ is clear, and the claim is proved.

Assume now that $t_f = T_f = \infty$. Let $\hat{\mathbb{P}}$ be a solution of $(FV_{m,f})$. It is easy to extend $\{\hat{M}_t(\phi) : \phi \in D(A)\}$ to a martingale measure $\{\hat{M}_t(h) : h \in b\mathcal{E}\}$ such that $< \hat{M}(h) >_t = \int_0^t (\hat{X}_s(h^2) - \hat{X}_s(h)^2) f(s)^{-1} ds$ (all under $\hat{\mathbb{P}}$). $\hat{M}_t$ is then a worthy martingale measure (in the sense of Walsh (1986, Ch. 2)) and so we can extend $\hat{M}_t$ to integrands $h(t,x)$ which are bounded measurable functions on $\hat{E} = [0, \infty) \times E$. For such an h we have

$$(16) \qquad < \hat{M}(h) >_t = \int_0^t (\hat{X}_s(h_s^2) - \hat{X}_s(h_s)^2) f(s)^{-1} ds \ \ \hat{\mathbb{P}} - \text{a.s.}$$

An elementary stochastic calculus argument now shows that

$$(17) \qquad \hat{X}(\psi_t) = m(\psi_o) + \int_0^t \hat{X}_s(\vec{A}\psi_s) ds + \hat{M}_t(\psi) \ \ \forall t \geq 0 \ \ \hat{\mathbb{P}} - \text{a.s.}$$

first for $\psi \in \mathcal{C}_o$ and then for all $\psi \in D(\vec{A})$ by Proposition 7(b). Let $C_t = \int_0^t f(s)^{-1} ds$. The conditions $t_f = T_f = \infty$ show C is a continuous function from $[0, \infty)$ onto $[0, \infty)$ with a continuous inverse τ. Let $g(u) = f(\tau_u)$, $\tilde{\mathcal{F}}_u = \hat{\mathcal{F}}_{\tau_u}$ and $\tilde{X}(u) = \hat{X}(\tau_u)$. Let $\phi \in \mathcal{C}$ and set $\psi(t, x) = \phi(C_t, x)$. It is easy to check that $\psi \in \mathcal{C}$ and hence (17) holds (since $\mathcal{C} \subset D(\vec{A})$). Therefore if $\tilde{M}_u(\phi) = \hat{M}_{\tau_u}(\psi)$, then, using the expression for $\vec{A}\psi$ from Proposition 7, we have

$$\begin{aligned} \tilde{X}_t(\phi_t) = \hat{X}_{\tau_t}(\psi_{\tau_t}) &= m(\phi_o) + \int_0^{\tau_t} \hat{X}_s\Big(\frac{\partial \psi}{\partial s} + A\psi_s\Big) ds + \hat{M}_{\tau(t)}(\psi) \\ (18) \qquad &= m(\phi_o) + \int_0^{\tau_t} \hat{X}_s\Big(\frac{\partial \phi}{\partial s}(C_s, \cdot) f(s)^{-1} + A\phi_{C_s}\Big) ds + \tilde{M}_t(\phi) \\ &= m(\phi_o) + \int_0^t \tilde{X}_u(\vec{A}^g \phi_u) du + \tilde{M}_t(\phi), \end{aligned}$$

where we have changed variables and used Proposition 7(a). $\tilde{M}_t(\phi)$ is an $(\tilde{\mathcal{F}}_t)$-martingale with

$$\begin{aligned} < \tilde{M}(\phi) >_t &= \int_0^{\tau_t} (\hat{X}_s(\psi_s^2) - \hat{X}_s(\psi_s)^2) f(s)^{-1} ds \\ (19) \qquad &= \int_0^t \tilde{X}_u(\phi_u^2) - \tilde{X}_u(\phi_u)^2 du. \end{aligned}$$

Since $\mathcal{C}$ is a core for $\vec{A}^g$ by Proposition 7, (18) and (19) hold for all $\psi \in D(\vec{A}^g)$. This proves that $\hat{\mathbb{P}}(\delta. \times \tilde{X}. \in \cdot)$ solves $(FV_{\delta_o \times m})$ for $\vec{A}^g$ and hence $V_t^g = \delta_t \times \tilde{X}_t$ is the $\vec{A}^g - FV$ process starting at $\delta_o \times m$ (under $\hat{\mathbb{P}}$). Since

$$(20) \qquad \hat{X}_t(\phi) = V_{C_t}^g(\bar{\phi}), \ \ \bar{\phi}(t, x) = \phi(x)$$

we see that $\hat{\mathbb{P}}$ is unique.

Consider now the problem of existence of $\hat{\mathbb{P}}_{m,f}$. If $t_f = T_f = \infty$, then use (20) to define $\hat{\mathbb{P}}_{m,f}$ in terms of an $\vec{A}^g - FV$ process. It is then straightforward to check that $\hat{\mathbb{P}}_{m,f}$ satisfies $(FV_{m,f})$. Assume now that $t_f \wedge T_f < \infty$. Choose $T_n \uparrow t_f, T_n < t_f$. If $f_n(t) = f(t \wedge T_n)$ then clearly $t_{f_n} = T_{f_n} = \infty$ and so by the above there is a unique $\hat{\mathbb{P}}_{m,f_n}$ on $(\hat{\Omega}, \hat{\mathcal{F}})$ solving (FV_{m,f_n}). It is easy to use uniqueness to see that if $k \geq n$ then $\hat{\mathbb{P}}_{m,f_n} = \hat{\mathbb{P}}_{m,f_k}$ on $\hat{\mathcal{F}}^0_{T_n}$. Taking a projective limit of $\{\hat{\mathbb{P}}_{m,f_n} : n \in \mathbb{N}\}$, we may construct a probability $\hat{\mathbb{P}}_{m,f}$ on $(\hat{\Omega}, \hat{\mathcal{F}}_{t_f-})(\hat{\mathcal{F}}_{\infty-} = \hat{\mathcal{F}})$ such that all the conditions of $(FV_{m,f})$ hold except the last ($\hat{X}_t = \hat{X}_{t_f}$ for $t \geq t_f$). We may, and shall, assume that $t_f < \infty$ (or we are done). Then for $\phi \in D(A) \cup \{1\}$

$$\sup_{t<t_f} |\hat{M}_t(\phi)| \leq 2\|\phi\|_\infty + t_f\|A\phi\|_\infty$$

and so by the Martingale Convergence Theorem, $\lim_{t\uparrow t_f} \hat{M}_t(\phi)$ exists $\hat{\mathbb{P}}_{m,f}$—a.s. It now follows trivially from $(FV_{m,f})$ (on $[0, t_f)$) that $\lim_{t\uparrow t_f} \hat{X}_t(\phi)$ exists $\hat{\mathbb{P}}_{m,f}$-a.s. first for all $\phi \in D(A) \cup \{1\}$ and hence for all $\phi \in C(E_\infty)$ therefore $\lim_{t\uparrow t_f} \hat{X}_t = \hat{X}_{t_f-}$ exists in $M_1(E_\infty)$. It follows easily from $(FV_{m,f})$ that for $\phi \in C(E_\infty)$

$$\begin{aligned}
\hat{\mathbb{P}}_{m,f}(\hat{X}_{t_f-}(\phi)) &= \lim_{t\uparrow t_f} \hat{\mathbb{P}}_{m,f}(\hat{X}_t(\phi)) \\
&= \lim_{t\uparrow t_f} P_{m(1)}(\phi(Y_t)) \qquad (21)\\
&= P_{m(1)}(\phi(Y_{t_f-})).
\end{aligned}$$

This shows $\hat{X}_{t_f-}(\{\infty\}) = 0$ a.s. and hence $\hat{X}_{t_f-} \in M_1(E)$. Now extend $\hat{\mathbb{P}}_{m,f}$ to $\hat{\mathcal{F}}$ by requiring that $\hat{X}_t = \hat{X}_{t_f-}$ $\forall t \geq t_f$ $\hat{\mathbb{P}}_{m,f}$-a.s. Clearly $\hat{\mathbb{P}}_{m,f}$ solves $(FV_{m,f})$ and the proof of (a) is complete.

(b) Fix $S < T$. Since Ω_{T-} is equipped with the compact-open topology it suffices to prove

$$\hat{\mathbb{P}}_{m_n,f_n}|_S \to \hat{\mathbb{P}}_{m,f}|_S \text{ on } \hat{\Omega}_S. \tag{22}$$

If $\epsilon = \inf_{t \le S} f(t) (> 0)$ (recall $S < T \le t_f$) then $\inf_{t \le S} f_n(t) \ge \epsilon/2$ for $n \ge N$ and so for $\phi \in D(A)$ and $n \ge N$,

$$| < \hat{M}(\phi) >_t - < \hat{M}(\phi) >_s | \le \|\phi\|_\infty^2 \int_s^t f_n(u) du \le \|\phi\|_\infty^2 2\epsilon^{-1} |t-s| \; \forall s,t \le S \; \hat{\mathbb{P}}_{m_n,f_n}\text{–a.s.}$$

Standard arguments now give the tightness of $\{\hat{\mathbb{P}}_{m_n,f_n}|_S : n \in \mathbb{N}\}$ viewed as probabilities on $\hat{\Omega}_S^\infty = C([0,S], M_1(E_\infty))$ (see for example Thm. 2.3 of Roelly-Coppoletta (1986) but note she is implicitly working with the vague topology and hence we only get tightness on $\hat{\Omega}_S^\infty$, not $\hat{\Omega}_S$). To obtain tightness in $\hat{\Omega}_S$ introduce $h_p(x) = e^{-pd(x,\infty)}$ (d a bounded metric on E_∞), $g_p = \int_0^1 T_t h_p(\cdot) dt \in D(A)$ and note that $Ag_p = T_1 g_p - g_p$. Then by (FV_{m_n,f_n}) we have

$$\sup_{t \le S} \hat{X}_t(g_p) \le m_n(g_p) + \sup_{t \le S} |\hat{M}_t(g_p)| + \int_0^S \hat{X}_u(T_1 g_p) du \quad \hat{\mathbb{P}}_{m_n,f_n} - \text{a.s.}$$

Now it is easy to use $< \hat{M}(g_p) >_S \le \int_0^S \hat{X}_u(g_p^2) f_n(u)^{-1} du \;\; \hat{\mathbb{P}}_{m_n,f_n}$ – a.s. and the superprocess property (see (21)) to conclude that

$$\lim_{p \to \infty} \sup_n \hat{\mathbb{P}}_{m_n,f_n}(\sup_{t \le S} \hat{X}_t(g_p)) = 0.$$

Since $\lim_{x \to \infty} g_p(x) = 1$, this proves the compact containment property needed in order to conclude $\{\hat{\mathbb{P}}_{m_n,f_n}|_S : n \in \mathbb{N}\}$ are tight in $\hat{\Omega}_S$.

Let $\mathbb{P}$ be a limit point of the above sequence ($\mathbb{P} \in M_1(\hat{\Omega}_S)$). Since everything in sight is uniformly bounded it is clear that the two equations in $(FV_{m,f})$ are satisfied under $\mathbb{P}$

for $t \leq S$. Extend $\mathbb{P}$ to $(\hat{\Omega}, \hat{\mathcal{F}})$ by setting the conditional distribution of $\{\hat{X}_{t+S} : t \geq 0\}$ given $\hat{\mathcal{F}}_S^0$ equal to $\hat{\mathbb{P}}_{\hat{X}_S, g}$ where $g(t) = f(S+t)$. Then $\mathbb{P} = \hat{\mathbb{P}}_{m,f}$ and so $\mathbb{P}|_S = \hat{\mathbb{P}}_{m,f}|_S$. (22) follows and the proof is complete.∎

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