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Infinite dimensional stochastic differential equations of Ornstein–Uhlenbeck type

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Abstract

We consider the operator

$$\mathscr{L}f(x) = \frac{1}{2}\sum_{i,j=1}^{\infty} a_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) - \sum_{i=1}^{\infty} \lambda_i x_i b_i(x)\frac{\partial f}{\partial x_i}(x).$$

We prove existence and uniqueness of solutions to the martingale problem for this operator under appropriate conditions on the a_{ij}, b_i, and λ_i. The process corresponding to $\mathscr{L}$ solves an infinite dimensional stochastic differential equation similar to that for the infinite dimensional Ornstein–Uhlenbeck process.

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1. Introduction

Let λ_i be a sequence of positive reals tending to infinity, let σ_{ij} and b_i be functions defined on a suitable Hilbert space which satisfy certain continuity and non-degeneracy conditions, and let W_t^i be a sequence of independent one-dimensional Brownian motions. In this paper we consider the countable system of stochastic differential equations

$$\mathrm{d}X_t^i = \sum_{j=1}^{\infty} \sigma_{ij}(X_t)\,\mathrm{d}W_t^i - \lambda_i b_i(X_t) X_t^i\,\mathrm{d}t, \quad i = 1, 2, \ldots, \tag{1.1}$$

and investigate sufficient conditions for weak existence and weak uniqueness to hold. Note that when the σ_{ij} and b_i are constant, we have the stochastic differential equations characterizing the infinite-dimensional Ornstein–Uhlenbeck process.

We approach the weak existence and uniqueness of (1.1) by means of the martingale problem for the corresponding operator

$$\mathscr{L}f(x) = \frac{1}{2}\sum_{i,j=1}^{\infty} a_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) - \sum_{i=1}^{\infty} \lambda_i x_i b_i(x)\frac{\partial f}{\partial x_i}(x) \tag{1.2}$$

operating on a suitable class of functions, where $a_{ij}(x) = \sum_{k=1}^{\infty} \sigma_{ik}(x)\sigma_{jk}(x)$. Our main theorem says that if the a_{ij} are nondegenerate and bounded, the b_i are bounded above and below, and the a_{ij} and b_i satisfy appropriate Hölder continuity conditions, then existence and uniqueness hold for the martingale problem for $\mathscr{L}$; see Theorem 5.7 for a precise statement.

There has been considerable interest in infinite dimensional operators whose coefficients are only Hölder continuous. For perturbations of the Laplacian, see Cannarsa and Da Prato [6], where Schauder estimates are proved using interpolation theory and then applied to Poisson's equation in infinite dimensions with Hölder continuous coefficients (see also [14]).

Similar techniques have been used to study operators of the form (1.2). In finite dimensions see [17–19,12]. For the infinite dimensional case see [7–11,14,23]. Common to all of these papers is the use of interpolation theory to obtain the necessary Schauder estimates. In functional analytic terms, the system of equations (1.1) is a special case of the equation

$$\mathrm{d}X_t = (b(X_t)X_t + F(X_t))\,\mathrm{d}t + \sqrt{a(X_t)}\,\mathrm{d}W_t, \tag{1.3}$$

where a is a mapping from a Hilbert space H to the space of bounded nonnegative self-adjoint linear operators on H, b is a mapping from H to the nonnegative self-adjoint linear operators on H (not necessarily bounded), F is a bounded operator on H, and $b(x)x$ represents the composition of operators. Previous work on (1.3) has concentrated on the following cases: where a is constant, b is Lipschitz continuous, and $F \equiv 0$; where a and b are constant and F is bounded; and where F is bounded, b is constant and a is a perturbation of a constant operator by means of a Hölder continuous nonnegative self-adjoint operator. We also mention the paper [13] where weak solutions to (1.3) are considered. In our paper we consider Eq. (1.3) with the a and b satisfying certain Hölder conditions and $F \equiv 0$. There would be no difficulty introducing bounded $F(X_t)\,\mathrm{d}t$ terms, but we chose not to do so.

The paper most closely related to this one is that of Zambotti [23]. Our results complement those of [23] as each has its own advantages. We were able to remove the restriction that the a_{ij}'s be given by means of a perturbation by a bounded nonnegative operator which in turn facilitates localization, but at the expense of working with respect to a fixed basis and hence imposing summability conditions involving the off-diagonal a_{ij}. See Remark 5.10 for a further discussion in light of a couple of examples and our explicit hypotheses for Theorem 5.7.

There are also martingale problems for infinite dimensional operators with Hölder continuous coefficients that arise from the fields of superprocesses and stochastic partial differential equations (SPDE). See [20] for a detailed introduction to these. We mention [15], where superprocesses in the Fleming–Viot setting are considered, and [4], where uniqueness of a martingale problem for superprocesses on countable Markov chains with interactive branching is shown to hold. These latter results motivated the present approach as the weighted Hölder spaces used there for our perturbation bounds coincide with the function spaces S^α used here (see Section 2), at least in the finite-dimensional setting (see [1]).

Consider the one dimensional SPDE

$$\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(x,t) + A(u)\,\mathrm{d}\dot{W}, \tag{1.5}$$

where $\dot{W}$ is space-time white noise. If one sets

$$X_t^j = \int_0^{2\pi} \mathrm{e}^{\mathrm{i}jx} u(x,t)\,\mathrm{d}x, \quad j = 0, \pm 1, \pm 2, \ldots,$$

then the collection $\{X^i\}_{i=-\infty}^{\infty}$ can be shown to solve system (1.1) with $\lambda_i = i^2$, the b_i constant, and the a_{ij} defined in an explicit way in terms of A. Our original interest in the problem solved in this paper was to understand (1.5) when the coefficients A were bounded above and below but were only Hölder continuous as a function of u. The results in this paper do not apply to (1.5) and we hope to return to this in the future.

The main novelties of our paper are the following.

(1) *C^α estimates (i.e., Schauder estimates) for the infinite dimensional Ornstein–Uhlenbeck process*. These were already known (see [14]), but we point out that in contrast to using interpolation theory, our derivation is quite elementary and relies on a simple real variable lemma together with some semigroup manipulations.
(2) *Localization*. We use perturbation theory along the lines of Stroock–Varadhan to establish uniqueness of the martingale problem when the coefficients are sufficiently close to constant. We then perform a localization procedure to establish our main result. In infinite dimensions localization is much more involved, and this argument represents an important feature of this work.
(3) *A larger class of perturbations*. Unlike much of the previous work cited above, we do not require that the perturbation of the second order term be bounded by an operator that is nonnegative. The price we pay is that we require additional conditions on the off-diagonal a_{ij}'s.

After some definitions and preliminaries in Section 2, we establish the needed Schauder estimates in Section 3. Section 4 contains the proof of existence and Section 5 the uniqueness. Section 5 also contains some specific examples where our main result applies. This includes coefficients a_{ij} which depend on a finite number of local coordinates near (i,j) in a Hölder manner.

We use the letter c with or without subscripts for finite positive constants whose value is unimportant and which may vary from proposition to proposition. α will denote a real number between 0 and 1.

2. Preliminaries

We use the following notation. If H is a separable Hilbert space and $f : H \to \mathbb{R}$, $D_w f(x)$ is the directional derivative of f at $x \in H$ in the direction w; we do not require w to be a unit vector. The inner product in H is denoted by $\langle \cdot, \cdot \rangle$, and $|\cdot|$ denotes the norm generated by this inner product. $C_b = C_b(H)$ is the collection of $\mathbb{R}$-valued bounded continuous functions on H with the usual supremum norm. Let C_b^2 be the set of functions in C_b for which the first and second order partials are also in C_b. For $\alpha \in (0,1)$, set

$$|f|_{C^\alpha} = \sup_{x \in H, h \neq 0} \frac{|f(x+h) - f(x)|}{|h|^\alpha}$$

and let C^α be the set of functions in C_b for which $\|f\|_{C^\alpha} = \|f\|_{C_b} + |f|_{C^\alpha}$ is finite.

Let $V : \mathscr{D}(V) \to H$ be a (densely defined) self-adjoint nonnegative definite operator such that

$$V^{-1} \text{ is a trace class operator on } H. \tag{2.1}$$

Then there is a complete orthonormal system $\{\varepsilon_n : n \in \mathbb{N}\}$ of eigenvectors of V^{-1} with corresponding eigenvalues λ_n^{-1}, $\lambda_n > 0$, satisfying

$$\sum_{n=1}^{\infty} \lambda_n^{-1} < \infty, \quad \lambda_n \uparrow \infty, \quad V\varepsilon_n = \lambda_n \varepsilon_n$$

(see, e.g. Section 120 in [21]). Let $Q_t = \mathrm{e}^{-tV}$ be the semigroup of contraction operators on H with generator $-V$. If $w \in H$, let $w_n = \langle w, \varepsilon_n \rangle$ and we will write $D_i f$ and $D_{ij} f$ for $D_{\varepsilon_i} f$ and $D_{\varepsilon_i} D_{\varepsilon_j} f$, respectively.

Assume $a : H \to L(H,H)$ is a mapping from H to the space of bounded self-adjoint operators on H and $b : H \to L(\mathscr{D}(V), H)$ is a mapping from H to self-adjoint nonnegative definite operators on $\mathscr{D}(V)$ such that $\{\varepsilon_n\}$ are eigenvectors of $b(x)$ for all $x \in H$. If $a_{ij}(x) = \langle \varepsilon_i, a(x)\varepsilon_j \rangle$ and $b(x)(\varepsilon_i) = \lambda_i b_i(x) \varepsilon_i$, we assume that for some $\gamma > 0$

$$\gamma^{-1}|z|^2 \geqslant \sum_{i,j} a_{ij}(x) z_i z_j \geqslant \gamma |z|^2, \quad x, z \in H,$$

$$\gamma^{-1} \geqslant b_i(x) \geqslant \gamma, \quad x \in H, \ i \in \mathbb{N}. \tag{2.2}$$

We consider the martingale problem for the operator $\mathscr{L}$ which, with respect to the coordinates $\langle x, \varepsilon_i \rangle$, is defined by

$$\mathscr{L}f(x) = \frac{1}{2} \sum_{i,j=1}^{\infty} a_{ij}(x) D_{ij} f(x) - \sum_{i=1}^{\infty} \lambda_i x_i b_i(x) D_i f(x). \tag{2.3}$$

Let $\mathcal{T}$ be the class of functions in C_b^2 that depend on only finitely many coordinates and $\mathcal{T}_0$ be the set of functions in $\mathcal{T}$ with compact support. More precisely, $f \in \mathcal{T}$ if there exists n and $f_n \in C_b^2(\mathbb{R}^n)$ such that $f(x_1, \ldots, x_n, \ldots) = f_n(x_1, \ldots, x_n)$ for each point $(x_1, x_2, \ldots)$ and $f \in \mathcal{T}_0$ if, in addition, f_n has compact support. Let X_t denote the coordinate maps on the space $C([0, \infty), H)$ of continuous H-valued paths. We say that a probability measure $\mathbb{P}$ on $C([0, \infty), H)$ is a solution to the martingale problem for $\mathcal{L}$ started at x_0 if $\mathbb{P}(X_0 = x_0) = 1$ and $f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)\,\mathrm{d}s$ is a martingale for each $f \in \mathcal{T}$.

The connection between systems of stochastic differential equations and martingale problems continues to hold in infinite dimensions; see, for example, [16, pp. 166–168]. We will use this fact without further mention.

There are different possible martingale problems depending on what class of functions we choose as test functions. Since existence is the easier part for the martingale problem (see Theorem 4.2) and uniqueness is the more difficult part, we will get a stronger and more useful theorem if we have a smaller class of test functions. The collection $\mathcal{T}$ is a reasonably small class. When $a(x) \equiv a^0$ and $b(x) \equiv V$ are constant functions, the process associated with $\mathcal{L}$ is the well-known H-valued Ornstein–Uhlenbeck process. We briefly recall the definition; see Section 5 of [1] for details. Let $(W_t, t \geqslant 0)$ be the cylindrical Brownian motion on H with covariance a. Let $\mathcal{F}_t$ be the right continuous filtration generated by W. Consider the stochastic differential equation

$$\mathrm{d}X_t = \mathrm{d}W_t - VX_t\,\mathrm{d}t. \tag{2.4}$$

There is a pathwise unique solution to (2.4) whose laws $\{\mathbb{P}^x, x \in H\}$ define a unique homogeneous strong Markov process on the space of continuous H-valued paths (see, e.g. Section 5.2 of [16]). $\{X_t, t \geqslant 0\}$ is an H-valued Gaussian process satisfying

$$\mathbb{E}(\langle X_t, h \rangle) = \langle X_0, Q_t h \rangle \quad \text{for all } h \in H, \tag{2.5}$$

and

$$\mathrm{Cov}(\langle X_t, g \rangle \langle X_t, h \rangle) = \int_0^t \langle Q_{t-s}h, aQ_{t-s}g \rangle\,\mathrm{d}s. \tag{2.6}$$

The law of X started at x solves the martingale problem for

$$\mathcal{L}_0 f(x) = \frac{1}{2}\sum_{i,j=1}^{\infty} a_{ij}^0 D_{ij} f(x) - \sum_{i=1}^{\infty} \lambda_i x_i D_i f(x). \tag{2.7}$$

We let $P_t f(x) = \mathbb{E}^x f(X_t)$ be the semigroup corresponding to $\mathcal{L}_0$, and $R_\lambda = \int_0^\infty \mathrm{e}^{-\lambda s} P_s\,\mathrm{d}s$ be the corresponding resolvent. We define the semigroup norm $\|\cdot\|_{S^\alpha}$ for $\alpha \in (0, 1)$ by

$$|f|_{S^\alpha} = \sup_{t>0} t^{-\alpha/2} \|P_t f - f\|_{C_b} \tag{2.8}$$

and

$$\|f\|_{S^\alpha} = \|f\|_{C_b} + |f|_{S^\alpha}.$$

Let S^α denote the space of measurable functions on H for which this norm is finite.

For $x \in H$ and $\beta \in (0, 1)$ define $|x|_\beta = \sup_k |\langle x, \varepsilon_k \rangle| \lambda_k^{\beta/2}$ and

$$H_\beta = \{x \in H : |x|_\beta < \infty\}. \tag{2.9}$$

3. Estimates

We start with the following real variable lemma.

Lemma 3.1. *Let $A>0, B>0$. Assume $K : C_b(H) \to C_b(H)$ is a bounded linear operator such that*

$$\|Kf\|_{C_b} \leqslant A\|f\|_{C_b}, \quad f \in C_b(H), \tag{3.1}$$

and there exists $v \in H$ such that

$$\|Kf\|_{C_b} \leqslant B\|D_v f\|_{C_b}, \tag{3.2}$$

for all f such that $D_v f \in C_b(H)$. Then for each $\alpha \in (0,1)$ there is a constant $c_1 = c_1(\alpha)$ such that

$$\|Kf\|_{C_b} \leqslant c_1 |v|^\alpha |f|_{C^\alpha} B^\alpha A^{1-\alpha} \quad \text{for all } f \in C^\alpha.$$

Proof. Assume (3.1) and (3.2), the latter for some $v \in H$. Let $\{p_t : t \geqslant 0\}$ be the standard Brownian density on $\mathbb{R}$. If $f \in C^\alpha$, set

$$p_\varepsilon * f(x) = \int_{\mathbb{R}} f(x+zv) p_\varepsilon(z)\,\mathrm{d}z, \quad x \in H.$$

Since a change of variables shows that

$$p_\varepsilon * f(x+hv) - p_\varepsilon * f(x) = \int_{\mathbb{R}} f(x+zv)p_\varepsilon(z-h)\,\mathrm{d}z - \int_{\mathbb{R}} f(x+zv)p_\varepsilon(z)\,\mathrm{d}z,$$

it follows that

$$D_v(p_\varepsilon * f)(x) = -\int f(x+zv)p'_\varepsilon(z)\,\mathrm{d}z;$$

this is in $C_b(H)$ and

$$\begin{aligned}
|D_v(p_\varepsilon * f)(x)| &= \left| -\int f(x+zv)p'_\varepsilon(z)\,\mathrm{d}z \right| \\
&= \left| \int (f(x+zv) - f(x))p'_\varepsilon(z)\,\mathrm{d}z \right| \\
&\leqslant |f|_{C^\alpha} |v|^\alpha \int |z|^\alpha \frac{|z|}{\varepsilon} p_\varepsilon(z)\,\mathrm{d}z \\
&= c_2 |f|_{C^\alpha} |v|^\alpha \varepsilon^{(\alpha-1)/2},
\end{aligned}$$

where $c_2 = \int |z|^{\alpha+1} p_1(z)\,\mathrm{d}z$. We therefore obtain from (3.2) that

$$\|K(p_\varepsilon * f)\|_{C_b} \leqslant c_2 B |f|_{C^\alpha} |v|^\alpha \varepsilon^{(\alpha-1)/2}. \tag{3.3}$$

Next note that

$$\begin{aligned}
|p_\varepsilon * f(x) - f(x)| &\leqslant \int |f(x+zv) - f(x)| p_\varepsilon(z)\,\mathrm{d}z \\
&\leqslant |f|_{C^\alpha} |v|^\alpha \int |z|^\alpha p_\varepsilon(z)\,\mathrm{d}z \\
&= c_3 |f|_{C^\alpha} |v|^\alpha \varepsilon^{\alpha/2},
\end{aligned}$$

where $c_3 = \int |z|^\alpha p_1(z)\,dz$. By (3.1)

$$\|K(p_\varepsilon * f - f)\|_{C_b} \leqslant c_3 A |f|_{C^\alpha} |v|^\alpha \varepsilon^{\alpha/2}. \tag{3.4}$$

Let $c_4 = c_2 \vee c_3$ and $\varepsilon = B^2/A^2$. Combining (3.3) and (3.4) we have

$$\begin{aligned}\|Kf\|_{C_b} &\leqslant c_4 |f|_{C^\alpha} |v|^\alpha \varepsilon^{\alpha/2}[A + B\varepsilon^{-1/2}]\\ &= 2c_4 |f|_{C^\alpha} |v|^\alpha B^\alpha A^{1-\alpha}. \qquad \square\end{aligned}$$

Set

$$h(u) = \begin{cases} (2u)/(e^{2u} - 1), & u \neq 0;\\ 1, & u = 0,\end{cases}$$

and

$$|w|_t = \left(\sum_i w_i^2 h(\lambda_i t)\right)^{1/2} \leqslant |w|.$$

Recall

$$Q_t w = \sum_{i=1}^\infty e^{-\lambda_i t} w_i e_i.$$

We have the following by Propositions 5.1 and 5.2 of [1]:

Proposition 3.2. (a) *For all $w \in H$, $f \in C_b(H)$, and $t > 0$, $D_w P_t f \in C_b(H)$ and*

$$\|D_w P_t f\|_{C_b} \leqslant \frac{|w|_t \|f\|_{C_b}}{\sqrt{\gamma t}}. \tag{3.5}$$

(b) *If $t \geqslant 0$, $w \in H$, and $f : H \to \mathbb{R}$ is in $C_b(H)$ such that $D_{Q_t w} f \in C_b(H)$, then*

$$D_w P_t f(x) = P_t(D_{Q_t w} f)(x), \quad x \in H.$$

In particular,

$$\|D_w P_t f\|_{C_b} \leqslant \|D_{Q_t w} f\|_{C_b}. \tag{3.6}$$

We now prove:

Corollary 3.3. *Let $f \in C^\alpha$, $u, w \in H$. Then for all $t > 0$, $D_w P_t f$ and $D_u D_w P_t f$ are in $C_b(H)$ and there exists a constant $c_1 = c_1(\alpha, \gamma)$ independent of t such that*

$$\|D_w P_t f\|_{C_b} \leqslant c_1 |w|_t |f|_{C^\alpha} t^{(\alpha-1)/2} \leqslant c_1 |w|\, |f|_{C^\alpha} t^{(\alpha-1)/2} \tag{3.7}$$

and

$$\begin{aligned}\|D_u D_w P_t f\|_{C_b} &\leqslant c_1 |Q_{t/2} u|_{t/2} |w|_{t/2} |f|_{C^\alpha} t^{\frac{\alpha}{2}-1} \leqslant c_1 |u|_{t/2} |w|_{t/2} |f|_{C^\alpha} t^{\frac{\alpha}{2}-1}\\ &\leqslant c_1 |u| |w| |f|_{C^\alpha} t^{\frac{\alpha}{2}-1}.\end{aligned} \tag{3.8}$$

Proof. That $D_w P_t f$ is in $C_b(H)$ is immediate from Proposition 3.2(a). By (3.5) and (3.6) we may apply Lemma 3.1 to $K = D_w P_t$ with $v = Q_t w$, $A = |w|_t (\gamma t)^{-1/2}$ and $B = 1$ to conclude for $f \in C^\alpha$

$$\begin{aligned}\|D_w P_t f\|_{C_b} &\leqslant c_2 |Q_t w|^\alpha |f|_{C^\alpha} |w|_t^{1-\alpha} (\gamma t)^{-(1-\alpha)/2} \\ &\leqslant c_2 \gamma^{(\alpha-1)/2} |w|_t |f|_{C^\alpha} t^{(\alpha-1)/2}. \end{aligned} \tag{3.9}$$

This gives (3.7).

By Proposition 3.2, $D_w D_u P_t f = D_w P_{t/2} D_{Q_{t/2} u} P_{t/2} f$, and the latter is seen to be in $C_b(H)$ by invoking Proposition 3.2(a) twice. Using (3.5) and then (3.9) we have

$$\begin{aligned}\|D_w D_u P_t f\|_{C_b} &= \|D_w P_{t/2} D_{Q_{t/2} u} P_{t/2} f\|_{C_b} \\ &\leqslant |w|_{t/2} (\gamma t/2)^{-1/2} \|D_{Q_{t/2} u} P_{t/2} f\|_{C_b} \\ &\leqslant |w|_{t/2} (\gamma t/2)^{-1/2} c_2 \gamma^{(\alpha-1)/2} |Q_{t/2} u|_{t/2} |f|_{C^\alpha} (t/2)^{(\alpha-1)/2}.\end{aligned}$$

This gives (3.8). □

Remark 3.4. We often will use the fact that there exists c_1 such that

$$\|f\|_{C^\alpha} \leqslant c_1 \|f\|_{S^\alpha}. \tag{3.10}$$

This is (5.20) of [1].

Corollary 3.5. *There exists $c_1 = c_1(\alpha, \gamma)$ such that for all $\lambda > 0$, $f \in C^\alpha$, $i \leqslant j$, we have $D_i R_\lambda f, D_{ij} R_\lambda f \in C_b$, and*

$$\|D_i R_\lambda f\|_{C_b} \leqslant c_1 (\lambda + \lambda_i)^{-(\alpha+1)/2} |f|_{C^\alpha}, \tag{3.11}$$

$$\|D_{ij} R_\lambda f\|_{C_b} \leqslant c_1 (\lambda + \lambda_j)^{-\alpha/2} |f|_{C^\alpha}, \tag{3.12}$$

$$\|D_i R_\lambda f\|_{C^\alpha} \leqslant c_1 (\lambda + \lambda_i)^{-1/2} \|f\|_{C^\alpha}, \tag{3.13}$$

$$\|D_{ij} R_\lambda f\|_{C^\alpha} \leqslant c_1 \|f\|_{C^\alpha}. \tag{3.14}$$

Proof. Corollary 3.3 is exactly the same as Proposition 5.4 in [1], but with the S^α norms replaced by C^α norms. We may therefore follow the proofs of Theorem 5.6 and Corollary 5.7 in [1] and then use (3.10) to obtain our result. However, the proofs in [1] can be streamlined, so for the sake of clarity and completeness we give a more straightforward proof.

From (3.7) and (3.8) we may differentiate under the time integral and conclude that the first and second order partial derivatives of $R_\lambda f$ are continuous. To derive (3.12), note first that by (3.8),

$$\begin{aligned}\|D_{ij} P_t f\|_{C_b} &= \|D_{ji} P_t f\|_{C_b} \leqslant c_2 |Q_{t/2} \varepsilon_j| |\varepsilon_i| |f|_{C^\alpha} t^{\frac{\alpha}{2}-1} \\ &= c_2 \mathrm{e}^{-\lambda_j t/2} |f|_{C^\alpha} t^{\frac{\alpha}{2}-1}.\end{aligned} \tag{3.15}$$

Multiplying by $\mathrm{e}^{-\lambda t}$ and integrating over t from 0 to ∞ yields (3.12).

Next we turn to (3.14). Recall the definition of the S^α norm from (2.8). In view of (3.10) it suffices to show

$$\|D_{ij} R_\lambda f\|_{S^\alpha} \leqslant c_3 \|f\|_{C^\alpha}.$$

Since

$$\|P_t D_{ij} R_\lambda f - D_{ij} R_\lambda f\|_{C_b} \leqslant 2\|D_{ij} R_\lambda f\|_{C_b} \leqslant c_1 |f|_{C^\alpha} (\lambda + \lambda_j)^{-\alpha/2}$$

by (3.12), we need only consider $t \leqslant (\lambda + \lambda_j)^{-1}$.

Use Proposition 3.2(b) to write

$$\begin{aligned} P_t D_{ij} R_\lambda f - D_{ij} R_\lambda f &= [\mathrm{e}^{-\lambda_i t} \mathrm{e}^{-\lambda_j t} D_{ij} P_t R_\lambda f - D_{ij} P_t R_\lambda f] \\ &\quad + [D_{ij} P_t R_\lambda f - D_{ij} R_\lambda f]. \end{aligned} \tag{3.16}$$

Recalling that $\lambda_i \leqslant \lambda_j$, we see that the first term is bounded in absolute value by

$$\begin{aligned} c_4 (\lambda_j t)^{\alpha/2} \|D_{ij} P_t R_\lambda f\|_{C_b} &\leqslant c_5 t^{\alpha/2} \int_0^\infty \lambda_j^{\alpha/2} \mathrm{e}^{-\lambda s} \|D_{ij} P_{t+s} f\|_{C_b} \, \mathrm{d}s \\ &\leqslant c_5 t^{\alpha/2} |f|_{C^\alpha}, \end{aligned}$$

using (3.15).

The second term in (3.16) is equal, by the semigroup property, to

$$\begin{aligned} &\int_0^\infty \mathrm{e}^{-\lambda s} D_{ij} P_{t+s} f \, \mathrm{d}s - \int_0^\infty \mathrm{e}^{-\lambda s} D_{ij} P_s f \, \mathrm{d}s \\ &\quad = (\mathrm{e}^{\lambda t} - 1) \int_0^\infty \mathrm{e}^{-\lambda s} D_{ij} P_s f \, \mathrm{d}s - \mathrm{e}^{\lambda t} \int_0^t \mathrm{e}^{-\lambda s} D_{ij} P_s f \, \mathrm{d}s. \end{aligned}$$

Since $\lambda t \leqslant 1$, then $\mathrm{e}^{\lambda t} - 1 \leqslant c_6 (\lambda t)^{\alpha/2}$ and the bound for the second term in (3.16) now follows by using (3.15) to bound the above integrals, and recalling again that $\lambda t \leqslant 1$.

The proofs of (3.11) and (3.13) are similar but simpler, and are left to the reader (or refer to [1]). □

4. Existence

Before discussing existence, we first need the following tightness result.

Lemma 4.1. *Suppose Y is a real-valued solution of*

$$Y_t = y_0 + M_t - \lambda \int_0^t Y_r \, \mathrm{d}r, \tag{4.2}$$

where M_t is a martingale such that for some c_1,

$$\langle M \rangle_t - \langle M \rangle_s \leqslant c_1 (t - s), \quad s \leqslant t. \tag{4.3}$$

Let $T > 0$, $\varepsilon \in (0, 1)$. Let $Z_t = \int_0^t \mathrm{e}^{-\lambda(t-s)} \, \mathrm{d}M_s$. Then $Z_t = Y_t - \mathrm{e}^{-\lambda t} y_0$ and for each $q > \varepsilon^{-1}$, there exists a constant $c_2 = c_2(\varepsilon, q, T)$ such that for all $\delta \in (0, 1]$,

$$\mathbb{E}\left[\sup_{s,t \leqslant T, |t-s| \leqslant \delta} |Z_t - Z_s|^{2q} \right] \leqslant c_2(\varepsilon, q, T) \frac{\delta^{\varepsilon q - 1}}{\lambda^{(1-\varepsilon)q}}. \tag{4.4}$$

Proof. Some elementary stochastic calculus shows that

$$Y_t = \mathrm{e}^{-\lambda t} y_0 + \int_0^t \mathrm{e}^{-\lambda(t-s)} \, \mathrm{d}M_s,$$

which proves the first assertion about Z.

Fix $s_0 < t_0 \leqslant T$. Let

$$K_t = [\mathrm{e}^{-\lambda(t_0-s_0)} - 1]\mathrm{e}^{-\lambda s_0} \int_0^t \mathrm{e}^{\lambda r}\,\mathrm{d}M_r$$

and

$$L_t = \mathrm{e}^{-\lambda t_0} \int_{s_0}^t \mathrm{e}^{\lambda r}\,\mathrm{d}M_r.$$

Note

$$Z_{t_0} - Z_{s_0} = K_{s_0} + L_{t_0}.$$

Then

$$\begin{aligned}\langle K\rangle_{s_0} &= [\mathrm{e}^{-\lambda(t_0-s_0)} - 1]^2\mathrm{e}^{-2\lambda s_0} \int_0^{s_0} \mathrm{e}^{2\lambda r}\,\mathrm{d}\langle M\rangle_r \\ &\leqslant c_3[\mathrm{e}^{-\lambda(t_0-s_0)} - 1]^2\mathrm{e}^{-2\lambda s_0}\frac{\mathrm{e}^{2\lambda s_0}-1}{2\lambda} \\ &\leqslant c_3[\mathrm{e}^{-\lambda(t_0-s_0)} - 1]^2\lambda^{-1} \\ &\leqslant c_3\frac{(1\wedge\lambda(t_0-s_0))}{\lambda}.\end{aligned}$$

Considering the cases $\lambda(t_0 - s_0) > 1$ and $\leqslant 1$ separately, we see that for any $\varepsilon \in (0,1)$ this is less than

$$c_4(\varepsilon)\frac{(t_0-s_0)^\varepsilon}{\lambda^{1-\varepsilon}}.$$

Now applying the Burkholder–Davis–Gundy inequalities, we see that

$$\mathbb{E}|K_{s_0}|^{2q} \leqslant c_5(\varepsilon,q)\frac{(t_0-s_0)^{\varepsilon q}}{\lambda^{(1-\varepsilon)q}}, \quad q>1. \tag{4.5}$$

Similarly,

$$\begin{aligned}\langle L\rangle_{t_0} &\leqslant c_6\frac{1-\mathrm{e}^{-2\lambda(t_0-s_0)}}{2\lambda} \\ &\leqslant c_6(\lambda^{-1}\wedge(t_0-s_0)) \\ &= c_6\frac{(1\wedge\lambda(t_0-s_0))}{\lambda}.\end{aligned}$$

This leads to

$$\mathbb{E}|L_{t_0}|^{2q} \leqslant c_7(\varepsilon,q)\frac{(t_0-s_0)^{\varepsilon q}}{\lambda^{(1-\varepsilon)q}}, \quad q>1. \tag{4.6}$$

Combining (4.5) and (4.6) we get

$$\mathbb{E}|Z_{t_0} - Z_{s_0}|^{2q} \leqslant c_8(\varepsilon,q)\frac{|t_0-s_0|^{\varepsilon q}}{\lambda^{(1-\varepsilon)q}}.$$

It is standard to obtain (4.4) from this; cf. the proof of Theorem I.3.11 in [2]. $\square$

Recall the definition of H_β from (2.9).

Theorem 4.2. *Assume* $a_{ij}: H \to \mathbb{R}$ *is continuous for all* i,j, b_i *is continuous for all* i, (2.2) *holds, and for some* $p>1$ *and positive constant* c_1

$$\lambda_k \geqslant c_1 k^p, \quad k \geqslant 1. \tag{4.7}$$

Then for every $x_0 \in H$, *there is a solution* $\mathbb{P}$ *to the martingale problem for* $\mathscr{L}$ *starting at* x_0. *Moreover if* $\beta \in (0,1)$, *then any such solution has* $\sup_{\varepsilon \leqslant t \leqslant \varepsilon^{-1}} |X_t|_\beta < \infty$ *for all* ε $\mathbb{P}$*-a.s. If in addition* $x_0 \in H_\beta$ *for some* $\beta \in (0,1)$, *then any solution* $\mathbb{P}$ *to the martingale problem for* $\mathscr{L}$ *starting at* x_0 *will satisfy*

$$\sup_{t \leqslant T} |X_t|_\beta < \infty \quad \textit{for all } T>0, \ \mathbb{P}-a.s. \tag{4.8}$$

Proof. This argument is standard and follows by making some minor modifications to the existence result in Section 5.2 of [16]. We give a sketch and leave the details to the reader. Fix x_0 in H. Using the finite dimensional existence result, we may construct a solution $X_t^n = (X_t^{n,k} : k \in \mathbb{N})$ of

$$X_t^{n,k} = x_0(k) + 1_{(k \leqslant n)} \left[-\int_0^t \lambda_k X_s^{n,k} b_k(X_s^n)\,\mathrm{d}s + \sum_{j=1}^n \int_0^t \sigma_{k,j}^n(X_s^n)\,\mathrm{d}W_s^j \right].$$

Here $\{W^j\}$ is a sequence of independent one-dimensional standard Brownian motions and $\sigma^n(x)$ is a symmetric positive definite square root of $(a_{ij}(x))_{i,j \leqslant n}$ which is continuous in $x \in H$ (see Lemma 5.2.1 of [22]). Then $X_t^n = \sum_{k=1}^n X_t^{n,k} \varepsilon_k$ has paths in $C([0,\infty), H)$ and we next verify this sequence of processes is relatively compact in this space. Once one has relative compactness, it is routine to use the continuity of the a_{ij} and b_i on H to show that any weak limit point of $\{X^n\}$ will be a solution to the martingale problem for $\mathscr{L}$ starting at x_0.

By our assumptions on b_k, each b_k is bounded above by γ^{-1} and below by γ. We perform a time change on $X_t^{n,k}$: let $A_t^{n,k} = \int_0^t b_k(X_s^n)\,\mathrm{d}s$, let $\tau_t^{n,k}$ be the inverse of $A_t^{n,k}$, and let $Y_t^{n,k} = X_{\tau_t^{n,k}}^{n,k}$. Then $Y_t^{n,k}$ solves the stochastic differential equation

$$Y_t^{n,k} = x_0(k) + 1_{(k \leqslant n)} \left[-\int_0^t \lambda_k Y_s^{n,k}\,\mathrm{d}s + M_t^{n,k} \right],$$

where $M_t^{n,k}$ is a martingale satisfying $|\langle M^{n,k}\rangle_t - \langle M^{n,k}\rangle_s| \leqslant c_2|t-s|$, and c_2 is a constant not depending on n or k.

We may use stochastic calculus to write

$$Y_t^{n,k} = x^{n,k}(t) + Z_t^{n,k},$$

where

$$x^{n,k}(t) = [1_{(k \leqslant n)} \mathrm{e}^{-\lambda_k t} + 1_{(k>n)}]x_0(k)$$

and

$$Z_t^{n,k} = 1_{(k \leqslant n)} \int_0^t \mathrm{e}^{-\lambda_k(t-s)}\,\mathrm{d}M_s^{n,k}.$$

Let $T>0$ and $s\leqslant t\leqslant T$. Choose $\varepsilon\in(0,1-\frac{1}{p})$ and $q>2/\varepsilon$. By Lemma 4.1 we have for $k\leqslant n$ and any $\delta\in(0,\gamma]$,

$$\mathbb{E}\left[\sup_{u,v\leqslant\gamma^{-1}T,|u-v|\leqslant\delta\gamma^{-1}}|Z_v^{n,k}-Z_u^{n,k}|^{2q}\right]\leqslant c_2(\varepsilon,q,\gamma^{-1}T)\gamma^{-\varepsilon q+1}\frac{\delta^{\varepsilon q-1}}{\lambda_k^{(1-\varepsilon)q}}.$$

Hence, undoing the time change tells us that

$$\mathbb{E}\left[\sup_{s,t\leqslant T,|s-t|\leqslant\delta}|\widetilde{X}_t^{n,k}-\widetilde{X}_s^{n,k}|^{2q}\right]\leqslant 1_{(k\leqslant n)}c_3(\varepsilon,q,\gamma,T)\frac{\delta^{\varepsilon q-1}}{\lambda_k^{(1-\varepsilon)q}},$$

where

$$\widetilde{X}_t^{n,k}=1_{(k\leqslant n)}(X_t^{n,k}-\mathrm{e}^{-\lambda_k\int_0^t b_k(X_r^n)\,\mathrm{d}r}x_0(k))+1_{(k>n)}x_0(k),$$

so that $\tilde{X}^{n,k}_{\tau_t^{n,k}}=Z_t^{n,k}$. Now for $0\leqslant s,t\leqslant T$ and $|t-s|\leqslant\gamma$,

$$\begin{aligned}(\mathbb{E}|\widetilde{X}_t^n-\widetilde{X}_s^n|^{2q})^{1/q}&=\||\widetilde{X}_t^n-\widetilde{X}_s^n|^2\|_q=\left\|\sum_k|\widetilde{X}_t^{n,k}-\widetilde{X}_s^{n,k}|^2\right\|_q\\&\leqslant\sum_k\||\widetilde{X}_t^{n,k}-\widetilde{X}_s^{n,k}|^2\|_q=\sum_k(\mathbb{E}|\widetilde{X}_t^{n,k}-\widetilde{X}_s^{n,k}|^{2q})^{1/q}\\&\leqslant c_3(\varepsilon,q,\gamma,T)^{1/q}\sum_k\frac{|t-s|^{\varepsilon-1/q}}{\lambda_k^{1-\varepsilon}},\end{aligned}$$

where $\|\cdot\|_q$ is the usual $L^q(\mathbb{P})$ norm.

By our choice of ε this is bounded by $c_4(\varepsilon,q,\gamma,T)|t-s|^{\varepsilon/2}$, and hence

$$\sup_n\mathbb{E}|\widetilde{X}_t^n-\widetilde{X}_s^n|^{2q}\leqslant c_4^q|t-s|^{\varepsilon q/2},\quad s,t\leqslant T,\ \ |s-t|\leqslant\gamma.$$

It is well known ([5]) that this implies the relative compactness of $\widetilde{X}^n$ in $C(\mathbb{R}_+,H)$.

We may write

$$X_t^n=\widetilde{X}_t^n-U^n(t),\tag{4.9}$$

where

$$U^n(t)=\sum_{k=1}^n\mathrm{e}^{-\lambda_k\int_0^t b_k(X_r^n)\,\mathrm{d}r}x_0(k)\varepsilon_k.$$

If $s<t$, then

$$\begin{aligned}|U^n(t)-U^n(s)|^2&=\sum_{k=1}^n\left[\mathrm{e}^{-\lambda_k\int_0^t b_k(X_r^n)\,\mathrm{d}r}-\mathrm{e}^{-\lambda_k\int_0^s b_k(X_r^n)\,\mathrm{d}r}\right]^2x_0(k)^2\\&\leqslant\sum_{k=1}^n((\lambda_k^2\gamma^{-2}|t-s|^2)\wedge 1)x_0(k)^2\end{aligned}$$

$$\leqslant \sum_{k=1}^{\infty} 1_{(\lambda_k \leqslant \gamma|t-s|^{-1})} \lambda_k^2 x_0(k)^2 \gamma^{-2} |t-s|^2 + \sum_{k=1}^{\infty} 1_{(\lambda_k > \gamma|t-s|^{-1})} x_0(k)^2. \tag{4.10}$$

Fix $\varepsilon > 0$. First choose N so that $\sum_{k=N}^{\infty} x_0(k)^2 < \varepsilon$, and then $\delta > 0$ so that

$$\sum_{k=1}^{\infty} 1_{(\lambda_k > \gamma\delta^{-1})} x_0(k)^2 < \varepsilon$$

and

$$\sum_{k=1}^{N} \lambda_k^2 x_0(k)^2 \gamma^{-2} \delta^2 < \varepsilon.$$

If $0 < t - s < \delta$, then use the above bounds in (4.10) to conclude that

$$\begin{aligned} |U^n(t) - U^n(s)|^2 &\leqslant \sum_{k=1}^{N} \lambda_k^2 x_0(k)^2 \gamma^{-2} \delta^2 + \sum_{k=N}^{\infty} x_0(k)^2 \\ &\quad + \sum_{k=1}^{\infty} 1_{(\lambda_k > \gamma\delta^{-1})} x_0(k)^2 \\ &< 3\varepsilon. \end{aligned}$$

This and the fact that $U^n(0) \to x_0$ in H prove that $\{U^n\}$ is relatively compact in $C(\mathbb{R}_+, H)$. The relative compactness of $\{X^n\}$ now follows from (4.9).

Assume now $\mathbb{P}$ is any solution to the martingale problem for $\mathscr{L}$ starting at $x_0 \in H$ and let X_t^i denote $\langle X_t, \varepsilon_i \rangle$. Fix $\beta \in (0,1)$ and $T > 1$. Choose $\varepsilon \in (0, 1-\beta)$. Using a time change argument as above but now with no parameter n and $\delta = 1$, we may deduce for any $q > 1/\varepsilon$ and $k \in \mathbb{N}$

$$\mathbb{P}\left(\sup_{t \leqslant T} |X_t^k - \mathrm{e}^{-\lambda_k \int_0^t b_k(X_s)\,\mathrm{d}s} x_0(k)| > \lambda_k^{-\beta/2} \right) \leqslant c_5(\varepsilon, q, T/\gamma) \lambda_k^{\beta q - q(1-\varepsilon)}.$$

The right-hand side is summable over k by our choice of ε and (4.7). The Borel–Cantelli lemma therefore implies that

$$\sup_{t \leqslant T} |X_t^k - \mathrm{e}^{-\lambda_k \int_0^t b_k(X_s)\,\mathrm{d}s} x_0(k)| \leqslant \lambda_k^{-\beta/2} \quad \text{for } k \text{ large enough, a.s.} \tag{4.11}$$

If $x_0 \in H_\beta$, this implies that with probability 1, for large enough k,

$$\sup_{t \leqslant T} |X_t^k| \lambda_k^{\beta/2} \leqslant 1 + x_0(k) \lambda_k^{\beta/2} \leqslant 1 + |x_0|_\beta,$$

and hence

$$\sup_{t \leqslant T} |X_t|_\beta < \infty \quad \text{a.s.}$$

For general $x_0 \in H$, (4.11) implies

$$\sup_{T^{-1}\leqslant t\leqslant T} |X_t^k|\lambda_k^{\beta/2} \leqslant 1 + \mathrm{e}^{-\lambda_k \gamma T^{-1}} \lambda_k^{\beta/2} |x_0| \leqslant c_6(\gamma, T, \beta, x_0) \quad \text{for large enough } k, \text{ a.s.}$$

This implies $\sup_{T^{-1}\leqslant t\leqslant T} |X_t|_\beta < \infty$ a.s. and so completes the proof. $\square$

5. Uniqueness

We continue to assume that (a_{ij}) and (b_i) are as in Section 2 and in particular will satisfy (2.2). Let $y_0 \in H$ and let $\mathbb{P}$ be any solution to the martingale problem for $\mathscr{L}$ started at y_0. For any bounded function f define

$$S_\lambda f = \mathbb{E} \int_0^\infty \mathrm{e}^{-\lambda s} f(X_s)\,\mathrm{d}s.$$

Fix $z_0 \in H$ and define

$$\mathscr{L}_0 f(x) = \frac{1}{2} \sum_{i,j=1}^\infty a_{ij}(z_0) D_{ij} f(x) - \sum_i^\infty \lambda_i x_i b_i(z_0) D_i f(x). \tag{5.1}$$

Set $\mathscr{B} = \mathscr{L} - \mathscr{L}_0$ and let R_λ be the resolvent for $\mathscr{L}_0$ as in Section 2.

To make this agree with the definition of $\mathscr{L}_0$ in Section 2 we must replace λ_i by $\widehat{\lambda}_i = b_i(z_0)\lambda_i$ and set $a_{ij}^0 = a_{ij}(z_0)$. As $\gamma \leqslant b_i(z_0) \leqslant \gamma^{-1}$, and the constants in Corollary 3.5 may depend on γ, we see that the bounds in Corollary 3.5 involving the original λ_i remain valid for R_λ. We also will use the other results in Section 3 with $\widehat{\lambda}_i$ in place of λ_i without further comment. In addition, if we simultaneously replace b_i by $\widehat{b}_i = b_i/b_i(z_0)$, then

$$\mathscr{L} f(x) = \frac{1}{2} \sum_{i,j=1}^\infty a_{ij}(x) D_{ij} f(x) - \sum_{i=1}^\infty \widehat{\lambda}_i x_i \widehat{b}_i(x) D_i f(x),$$

$$\mathscr{L}_0 f(x) = \frac{1}{2} \sum_{i,j=1}^\infty a_{ij}(z_0) D_{ij} f(x) - \sum_{i=1}^\infty \widehat{\lambda}_i x_i D_i f(x),$$

and

$$\widehat{b}_i(z_0) = 1 \quad \text{for all } i.$$

In Propositions 5.1 and 5.2 we will simply assume $b_i(z_0) = 1$ for all i without loss of generality, it being understood that the above substitutions are being made. In each case it is easy to check that the hypotheses on (b_i, λ_i) carry over to $(\widehat{b}_i, \widehat{\lambda}_i)$ and as the conclusions only involve $\mathscr{L}$, $\mathscr{L}_0$, R_λ, and our solution X, which remain unaltered by these substitutions, this reduction is valid.

Let

$$\eta = \sup_x \sum_{i,j=1}^\infty |a_{ij}(x) - a_{ij}(z_0)|. \tag{5.2}$$

Set

$$B_i(x) = x_i(b_i(x) - 1).$$

As before, α will denote a parameter in $(0, 1)$.

Proposition 5.1. *Assume*

$$\sum_{i\leqslant j} |a_{ij}|_{C^\alpha}\lambda_j^{-\alpha/2} < \infty, \tag{5.3}$$

$$\sum_i \lambda_i^{1/2}\|B_i\|_{C_b} < \infty, \tag{5.4}$$

and

$$\sum_i \lambda_i^{(1-\alpha)/2}|B_i|_{C^\alpha} < \infty. \tag{5.5}$$

There exists $c_1(\lambda) \to 0$ as $\lambda \to \infty$ and $c_2 = c_2(\alpha, \gamma)$ such that for all $f \in C^\alpha$, we have $\mathscr{B}R_\lambda f \in C^\alpha$ and

$$\|\mathscr{B}R_\lambda f\|_{C^\alpha} \leqslant (c_1(\lambda) + c_2\eta)\|f\|_{C^\alpha}.$$

Proof. We have

$$\begin{aligned}|\mathscr{B}R_\lambda f(x)| &\leqslant \sum_{i,j} |a_{ij}(x) - a_{ij}(z_0)||D_{ij}R_\lambda f(x)| \\ &\quad + \sum_i \lambda_i|x_i||b_i(x) - 1||D_iR_\lambda f(x)| \\ &\leqslant \eta c_3|f|_{C^\alpha} + c_4(\lambda)|f|_{C^\alpha},\end{aligned} \tag{5.6}$$

where $c_4(\lambda) \to 0$ as $\lambda \to \infty$ by (5.4) and (3.11). In particular, the series defining $\mathscr{B}R_\lambda f$ is absolutely uniformly convergent.

Let $\widehat{a}_{ij}(x) = a_{ij}(x) - a_{ij}(z_0)$. If $h \in H$, then

$$\begin{aligned}|\mathscr{B}R_\lambda f(x+h) - \mathscr{B}R_\lambda f(x)| &= \Bigg|\sum_{i,j}[\widehat{a}_{ij}(x+h)D_{ij}R_\lambda f(x+h) - \widehat{a}_{ij}(x)D_{ij}R_\lambda f(x)] \\ &\qquad + \sum_i \lambda_i[B_i(x+h)D_iR_\lambda f(x+h) - B_i(x)D_iR_\lambda f(x)]\Bigg| \\ &\leqslant \left|\sum_{i,j}\widehat{a}_{ij}(x+h)(D_{ij}R_\lambda f(x+h) - D_{ij}R_\lambda f(x))\right| \\ &\quad + \left|\sum_{i,j}(\widehat{a}_{ij}(x+h) - \widehat{a}_{ij}(x))D_{ij}R_\lambda f(x)\right| \\ &\quad + \left|\sum_i \lambda_iB_i(x+h)(D_iR_\lambda f(x+h) - D_iR_\lambda f(x))\right| \\ &\quad + \left|\sum_i \lambda_i(B_i(x+h) - B_i(x))D_iR_\lambda f(x)\right| \\ &= S_1 + S_2 + S_3 + S_4.\end{aligned} \tag{5.7}$$

Use (3.14) to see that

$$\begin{aligned} S_1 &\leqslant c_5 \sum_{i,j} |\widehat{a}_{ij}(x+h)||f|_{C^\alpha}|h|^\alpha \\ &\leqslant c_6 \eta |f|_{C^\alpha}|h|^\alpha. \end{aligned} \tag{5.8}$$

By (3.12)

$$\begin{aligned} S_2 &\leqslant \sum_{i,j} |a_{ij}(x+h) - a_{ij}(x)|\,|D_{ij}R_\lambda f(x)| \\ &\leqslant c_7 \sum_{i\leqslant j} |a_{ij}|_{C^\alpha}|h|^\alpha(\lambda+\lambda_j)^{-\alpha/2}|f|_{C^\alpha} \\ &\leqslant c_8(\lambda)|f|_{C^\alpha}|h|^\alpha, \end{aligned} \tag{5.9}$$

where (5.3) and dominated convergence imply $\lim_{\lambda\to\infty} c_8(\lambda) = 0$. By (3.13)

$$S_3 \leqslant c_9 \sum_i \lambda_i |B_i(x+h)|(\lambda+\lambda_i)^{-1/2}|f|_{C^\alpha}|h|^\alpha \leqslant c_{10}(\lambda)|f|_{C^\alpha}|h|^\alpha, \tag{5.10}$$

where $c_{10}(\lambda) \to 0$ as $\lambda \to \infty$ by (5.4) and dominated convergence. By (3.11)

$$S_4 \leqslant c_{11} \sum_i \lambda_i |B_i|_{C^\alpha}(\lambda+\lambda_i)^{-(1+\alpha)/2}|f|_{C^\alpha}|h|^\alpha \leqslant c_{12}(\lambda)|f|_{C^\alpha}|h|^\alpha, \tag{5.11}$$

where again $c_{12}(\lambda) \to 0$ as $\lambda \to \infty$ by (5.5). Combining (5.8)–(5.11) yields

$$|\mathscr{B}R_\lambda f|_{C^\alpha} \leqslant [c_{13}(\lambda) + c_{14}\eta]|f|_{C^\alpha}.$$

This and (5.6) complete the proof. □

Let C^α_n denote those functions in C^α which only depend on the first n coordinates. Note that $\mathscr{T}_0 \subset \bigcup_n C^\alpha_n$. Note also that $S_\lambda f$ is a real number while $R_\lambda f$ is a function.

Proposition 5.2. *If* $f \in \bigcup_n C^\alpha_n$, *then*

$$S_\lambda f = R_\lambda f(y_0) + S_\lambda \mathscr{B} R_\lambda f. \tag{5.12}$$

Proof. Fix $z_0 \in H$. Suppose $h \in \mathscr{T}$. Since $h(X_t) - h(X_0) - \int_0^t \mathscr{L}h(X_s)\,\mathrm{d}s$ is a martingale, taking expectations we have

$$\mathbb{E}h(X_t) - h(y_0) = \mathbb{E}\int_0^t \mathscr{L}h(X_s)\,\mathrm{d}s.$$

Multiplying by $\mathrm{e}^{-\lambda t}$ and integrating over t from 0 to ∞, we obtain

$$\begin{aligned} S_\lambda h - \frac{1}{\lambda}h(y_0) &= \mathbb{E}\int_0^\infty \mathrm{e}^{-\lambda t}\int_0^t \mathscr{L}h(X_s)\,\mathrm{d}s\,\mathrm{d}t \\ &= \frac{1}{\lambda}\mathbb{E}\int_0^\infty \mathrm{e}^{-\lambda s}\mathscr{L}h(X_s)\,\mathrm{d}s = \frac{1}{\lambda}S_\lambda \mathscr{L}h. \end{aligned}$$

This can be rewritten as

$$\lambda S_\lambda h - S_\lambda \mathscr{L}_0 h = h(y_0) + S_\lambda \mathscr{B}h. \tag{5.13}$$

Define

$$\mathscr{L}_0^n f(x) = \sum_{i,j=1}^{n} a_{ij}(z_0) D_{ij} f(x) - \sum_{i=1}^{n} \lambda_i x_i D_i f(x).$$

Let R_λ^n be the corresponding resolvent. The corresponding process is an n-dimensional Ornstein–Uhlenbeck process which starting from x at time t is Gaussian with mean vector $(x_i e^{-\lambda_i t})_{i \leqslant n}$ and covariance matrix $C_{ij}(t) = a_{ij}(z_0)(1 - e^{-(\lambda_i + \lambda_j)t})(\lambda_i + \lambda_j)^{-1}$. These parameters are independent of n and the distribution coincides with the law of the first n coordinates (with respect to ε_i) of the process with resolvent R_λ.

Now take $f \in C_n^\alpha$ and let $h(x) = R_\lambda f(x) = R_\lambda^n f(x_1, \ldots, x_n)$. (Here we abuse our notation slightly by having f also denote its dependence on the first n variables.) By Corollary 3.5 and (3.10), $h \in \mathscr{T}$. Moreover, $\mathscr{L}_0 h = \mathscr{L}_0^n R_\lambda^n f = \lambda R_\lambda^n f - f = \lambda R_\lambda f - f$. The second equality is standard since on functions in C_b^2, $\mathscr{L}_0^n$ coincides with the generator of the finite-dimensional diffusion. Now substitute this into (5.13) to derive (5.12). $\square$

To iterate (5.12) we will need to extend it to $f \in C^\alpha$ by an approximation argument. Recall $\widehat{\lambda}_i = b_i(z_0)\lambda_i$.

Notation. Write $f_n \xrightarrow{\text{bp}} f$ if $\{f_n\}$ converges to f pointwise and boundedly.

Lemma 5.3. (a) *If $f \in C^\alpha$, then $pR_p f \xrightarrow{\text{bp}} f$ as $p \to \infty$ and*

$$\sup_{p>0} \|pR_p f\|_{C^\alpha} \leqslant \|f\|_{C^\alpha}.$$

(b) *For $p>0$ there is a $c_1(p)$ such that for any bounded measurable $f : H \to \mathbb{R}$, $R_p f \in C^\alpha$ and $\|pR_p f\|_{C^\alpha} \leqslant c_1(p)\|f\|_{C_b}$.*

Proof. (a) Note if $f \in C^\alpha$, then

$$\|pR_p f\|_{C_b} \leqslant \int_0^\infty p e^{-pt} \|P_t f\|_{C_b} \, dt \leqslant \|f\|_{C_b}$$

and

$$pR_p f(x) - f(x) = \int_0^\infty p e^{-pt} (P_t f(x) - f(x)) \, dt \to 0$$

because $P_t f(x) \xrightarrow{\text{bp}} f(x)$ as $t \to 0$.

Let X_t be the solution to (2.4) (so that X has resolvents (R_λ)) and let $X_t^i = \langle X_t, \varepsilon_i \rangle \varepsilon_i$. Then X_t^i satisfies

$$X_t^i = X_0^i + M_t^i - \widehat{\lambda}_i \int_0^t X_s^i \, ds, \tag{5.14}$$

where M_t^i is a one-dimensional Brownian motion with $\mathrm{Cov}(M_t^i, M_s^i) = a_{ii}(s \wedge t)$. Let $X_t^{x_i,i}$ denote the solution to (5.14) when $X_0^i = x_i$. Then

$$X_t^{x_i+h_i,i} - X_t^{x_i,i} = h_i - \widehat{\lambda}_i \int_0^t (X_s^{x_i+h_i,i} - X_s^{x_i,i}) \, ds,$$

and so

$$X_t^{x_i+h_i,i} - X_t^{x,i} = \mathrm{e}^{-\widehat{\lambda}_i t} h_i \varepsilon_i.$$

Hence, if X_t^x is defined by $\langle X_t^x, \varepsilon_i\rangle = X_t^{x_i,i}$,

$$|X_t^{x+h} - X_t^x| = \left|\sum h_i^2 \mathrm{e}^{-2\widehat{\lambda}_i t}\right|^{1/2} \leqslant |h|.$$

Therefore

$$|P_t f(x+h) - P_t f(x)| \leqslant |f|_{C^\alpha} \mathbb{E}(|X_t^{x+h} - X_t^x|^\alpha) \leqslant |f|_{C^\alpha} |h|^\alpha,$$

and so

$$|pR_p f(x+h) - pR_p f(x)| \leqslant \int_0^\infty p\mathrm{e}^{-pt} |P_t f(x+h) - P_t f(x)|\,\mathrm{d}t \leqslant |f|_{C^\alpha} |h|^\alpha,$$

i.e., $|pR_p f|_{C^\alpha} \leqslant |f|_{C^\alpha}$. This proves (a).

(b) As we mentioned above, for any bounded measurable f, $\|pR_p f\|_{C_b} \leqslant \|f\|_{C_b}$. We also have

$$\begin{aligned} P_s pR_p f - pR_p f &= \int_0^\infty p\mathrm{e}^{-pt}[P_{s+t} f - P_t f]\,\mathrm{d}t \\ &= (\mathrm{e}^{ps} - 1)\int_0^\infty p\mathrm{e}^{-pt} P_t f\,\mathrm{d}t - \mathrm{e}^{ps}\int_0^s p\mathrm{e}^{-pt} P_t f\,\mathrm{d}t. \end{aligned}$$

The right-hand side is bounded by

$$2(\mathrm{e}^{ps} - 1)\|f\|_{C_b}.$$

This in turn is bounded by $c_2(p)s^{\alpha/2}$ for $0 \leqslant s \leqslant 1$. Also,

$$\|P_s pR_p f - pR_p f\|_{C_b} \leqslant 2\|f\|_{C_b} \leqslant 2s^{\alpha/2}\|f\|_{C_b} \quad \text{for } s \geqslant 1.$$

Hence $\|pR_p f\|_{S^\alpha} \leqslant c_3(p)\|f\|_{C_b}$. Our conclusion follows by (3.10), which holds for the $\{\widehat{\lambda}_i\}$ just as it did for $\{\lambda_i\}$. $\square$

Lemma 5.4. *Suppose* $f_n \xrightarrow{\text{bp}} 0$ *where* $\sup_n \|f_n\|_{C^\alpha} < \infty$. *Then*

$$D_{ij} R_\lambda f_n \xrightarrow{\text{bp}} 0 \quad \textit{and} \quad D_i R_\lambda f_n \xrightarrow{\text{bp}} 0 \quad \textit{as } n \to \infty \textit{ for all } i,j.$$

Proof. We focus on the second order derivatives as the proof for the first order derivatives is simpler. We know from Corollary 3.3 that $D_{ij} R_\lambda f_n$ is uniformly bounded in C^α norm, so in particular, it is uniformly bounded in C_b norm and we need only establish the pointwise convergence. We have from (3.8) that

$$\|D_{ij} P_t f_n\|_{C_b} \leqslant c_1 \|f_n\|_{C^\alpha} t^{\alpha/2-1}. \tag{5.15}$$

From Proposition 3.2, we have

$$D_{ij} P_t f_n = D_i P_{t/2} D_{Q_{t/2}\varepsilon_j} P_{t/2} f_n. \tag{5.16}$$

Fix $t>0$ and $w \in H$. The proof of Proposition 5.2 in [1] shows there exist random variables $R(t,w)$ and Y_t such that

$$D_w P_t f(x) = \mathbb{E}[f(Q_t x + Y_t) R(t,w)], \quad f \in C_b(H),$$

and

$$\mathbb{E}[R(t,w)^2] \leqslant \frac{|w|^2}{\gamma t}.$$

Therefore

$$h_n(j,t,x) \equiv D_{Q_{t/2}e_j} P_{t/2} f_n(x) = \mathbb{E}(f_n(Q_{t/2}x + Y_{t/2}) R(t/2, Q_{t/2}\varepsilon_j)) \xrightarrow{\text{bp}} 0$$

by dominated convergence. Moreover Cauchy–Schwarz implies

$$\|h_n(j,t)\|_{C_b} \leqslant (\gamma t)^{-1/2} \sup_m \|f_m\|_{C_b}.$$

Repeating the above reasoning and using (5.16) we have

$$D_{ij} P_t f_n(x) = D_i P_{t/2} h_n(x) = \mathbb{E}(h_n(Q_{t/2}x + Y_{t/2}) R(t/2, \varepsilon_i)) \xrightarrow{\text{bp}} 0$$

and

$$\|D_{ij} P_t f_n\|_{C_b} \leqslant (\gamma t)^{-1} \sup_m \|f_m\|_{C_b}. \tag{5.17}$$

Fix $\varepsilon>0$. Write

$$|D_{ij} R_\lambda f_n(x)| \leqslant \left| \int_0^\varepsilon e^{-\lambda t} D_{ij} P_t f_n(x)\, dt \right| + \left| \int_\varepsilon^\infty e^{-\lambda t} D_{ij} P_t f_n(x)\, dt \right|;$$

by dominated convergence and (5.17) the second term tends to 0, while (5.15) shows the first term is bounded by

$$\int_0^\varepsilon c_2 \|f_n\|_{C^\alpha} t^{\alpha/2-1}\, dt \leqslant c_3 \left(\sup_m \|f_m\|_{C^\alpha} \right) \varepsilon^{\alpha/2}.$$

Therefore

$$\limsup_{n\to\infty} |D_{ij} R_\lambda f_n(x)| \leqslant c_4 \left(\sup_m \|f_m\|_{C^\alpha} \right) \varepsilon^{\alpha/2}.$$

Since ε is arbitrary,

$$\limsup_{n\to\infty} |D_{ij} R_\lambda f_n(x)| = 0. \qquad \square$$

Proposition 5.5. *Assume* (5.4). *If $f \in C^\alpha$, then*

$$S_\lambda f = R_\lambda f(y_0) + S_\lambda \mathscr{B} R_\lambda f. \tag{5.18}$$

Proof. We know $f_p = f - pR_p f \xrightarrow{\text{bp}} 0$ as $p \to \infty$ by Lemma 5.3. This lemma also shows $\|f_p\|_{C^\alpha} \leqslant 2\|f\|_{C^\alpha}$, and therefore we may use Lemma 5.4, the finiteness of η, (5.4) (in fact a weaker condition suffices here), and dominated convergence to conclude

$$\mathscr{B}R_\lambda f_p(x) = \sum_{i,j}(a_{ij}(x) - a_{ij}(z_0))D_{ij}(R_\lambda f_p)(x) + \sum_i \lambda_i x_i(b_i(x) - b_i(z_0))D_i(R_\lambda f_p)(x) \xrightarrow{\text{bp}} 0 \quad \text{as } p \to \infty.$$

Here we also use the bounds $\|D_{ij}R_\lambda f_p\|_{C_b} \leqslant c\|f\|_{C^\alpha}$ and $\|D_i R_\lambda f_p\|_{C_b} \leqslant c\lambda_i^{-1/2}\|f\|_{C^\alpha}$ from (3.11), (3.12) and Lemma 5.3(a). By using dominated convergence it is now easy to take limits through the resolvents to see that to prove (5.18) it suffices to fix $p>0$ and verify it for $f = pR_p h$ where $h \in C^\alpha$. Fix such an h.

Let $z_n(x) = \sum_{i=1}^n x_i\varepsilon_i + \sum_{i>n}(z_0)_i\varepsilon_i \to x$ as $n \to \infty$ and define $h_n(x) = h(z_n(x))$. Then $h_n \xrightarrow{\text{bp}} h$ since $h \in C^\alpha$. Recall the definition of R_p^n from the proof of Proposition 5.2; by the argument there, we see that the function $pR_p h_n(x) = pR_p^n h_n(x_1, \dots, x_n)$ depends only on $(x_1, \dots, x_n)$. By Lemma 5.3(b) $pR_p h_n \in C^\alpha$ and therefore is in C_n^α. Proposition 5.2 shows that (5.18) is valid with $f = R_p h_n$. Now $pR_p h_n \xrightarrow{\text{bp}} pR_p h$ as $n \to \infty$ and $\sup_n \|pR_p h_n\|_{C^\alpha} \leqslant c_1(p)$ by Lemma 5.3(b). Therefore, if $d_n = pR_p(h_n - h)$ we may use Lemma 5.4, Corollary 3.5, and dominated convergence, as before, to conclude

$$\mathscr{B}R_\lambda d_n(x) = \sum_{i,j}(a_{ij}(x) - a_{ij}(z_0))D_{ij}(R_\lambda d_n)(x) + \sum_i \lambda_i x_i(b_i(x) - b_i(z_0))D_i(R_\lambda d_n)(x) \xrightarrow{\text{bp}} 0 \quad \text{as } n \to \infty.$$

We may now let $n \to \infty$ in (5.18) with $f = pR_p h_n$ to derive (5.18) with $f = pR_p h$, as required. □

Theorem 5.6. *Assume* (2.2), *each a_{ij} and each b_i is continuous*, (4.7), (5.3), (5.4), *and* (5.5) *hold. There exists η_0, depending only on (α, γ), such that if $\eta \leqslant \eta_0$, then for any $y_0 \in H$ there is a unique solution to the martingale problem for $\mathscr{L}$ started at y_0.*

Proof. Existence follows from Theorem 4.2.

Let $\mathbb{P}$ be any solution to the martingale problem and define S_λ as above. Suppose $f \in C^\alpha$. Then by Proposition 5.5 we have

$$S_\lambda f = R_\lambda f(y_0) + S_\lambda \mathscr{B}R_\lambda f.$$

Using Proposition 5.1 we can iterate the above and obtain

$$S_\lambda f = R_\lambda\left(\sum_{i=0}^k (\mathscr{B}R_\lambda)^i\right)f(y_0) + S_\lambda(\mathscr{B}R_\lambda)^{k+1}f.$$

Provided $\eta_0 = \eta_0(\alpha, \gamma)$ is small enough, our hypothesis that $\eta \leqslant \eta_0$ and Proposition 5.1 imply that for $\lambda > \lambda_0(\alpha, \gamma, (a_{ij}), (b_i))$, the operator $\mathscr{B}R_\lambda$ is bounded on C^α with norm strictly less than $\frac{1}{2}$. Therefore $\sum_{i=k+1}^\infty (\mathscr{B}R_\lambda)^i f$ converges to 0 and $(\mathscr{B}R_\lambda)^{k+1}f$ also converges to 0, both in C^α norm, as $k \to \infty$. In particular, they converge to 0 in sup norm, so $R_\lambda(\sum_{i=k+1}^\infty (\mathscr{B}R_\lambda)^i)f(y_0)$ and $S_\lambda(\mathscr{B}R_\lambda)^{k+1}f$ both converge to 0 as $k \to \infty$. It follows that

$$S_\lambda f = R_\lambda\left(\sum_{i=0}^\infty (\mathscr{B}R_\lambda)^i\right)f(y_0).$$

This is true for any solution to the martingale problem, so S_λ is uniquely defined for large enough λ. Inverting the Laplace transform and using the continuity of $t \to \mathbb{E}f(X_t)$, we see that for every $f \in C^\alpha$, $\mathbb{E}f(X_t)$ has the same value for every solution to the martingale problem. It is not hard to see that $\mathcal{T}_0 \subset C^\alpha$ is dense with respect to the topology of bounded pointwise convergence in the set of all bounded functions. From here standard arguments (cf. [3, Section VI.3]) allow us to conclude the uniqueness of the martingale problem of $\mathcal{L}$ starting at y_0 as long as we have $\eta \leqslant \eta_0$. $\square$

Set

$$Q_{\beta,N} = \{x \in H : |x|_\beta \leqslant N\}.$$

Theorem 5.7. *Assume* (b_i) *and* (a_{ij}) *are as in Section* 2, *so that* (2.2) *holds. Assume also that* $\alpha, \beta \in (0,1)$ *satisfy*:

(a) *There exist* $p > 1$ *and* $c_1 > 0$ *such that* $\lambda_j \geqslant c_1 j^p$.
(b) $\sum_{i \leqslant j} |a_{ij}|_{C^\alpha} \lambda_j^{-\alpha/2} < \infty$.
(c) $\sum_j \lambda_j^{-\beta} < \infty$. (*For example, this holds if* $\beta > 1/p$.)
(d) *For all* $N > 0$, *for all* $\eta_0 > 0$, *and for all* $x_0 \in Q_{\beta,N}$ *there exists* $\delta > 0$ *such that if* $|x - x_0| < \delta$ *and* $x \in Q_{\beta,N}$, *then*

$$\sum_{i,j} |a_{ij}(x) - a_{ij}(x_0)| < \eta_0.$$

(e) $\sum_i \lambda_i^{1/2} |b_i|_{C^\alpha} < \infty$.

Then for all $y \in H_\beta$ *there exists a unique solution to the martingale problem for* $\mathcal{L}$ *starting at* y.

Remark. By Theorem 4.2, any solution to the martingale problem for $\mathcal{L}$ starting at $y \in H$ will immediately enter H_β and remain there a.s. for any $\beta \in (0,1)$. Hence the spaces H_β are natural state spaces for the martingale problem.

Proof. Fix $\beta \in (0,1)$ as in (c) and write Q_N for $Q_{\beta,N}$. Let $\mathbb{P}$ be a solution to the martingale problem for $\mathcal{L}$. By Theorem 4.2 we only need consider uniqueness. If $T_N = \inf\{t : X_t \notin Q_N\}$, then by Theorem 4.2 we see that $T_N \uparrow \infty$, a.s. and it suffices to show uniqueness for $\mathbb{P}(X_{\cdot \wedge T_N} \in \cdot)$. (c) implies Q_N is compact and so as in the proof of Theorem VI.4.2 of [3] it suffices to show:

(5.19) for all $x_0 \in Q_N$ there exist $r > 0$, $\widetilde{a}_{ij}$, and $\widetilde{b}_i$ such that $a_{ij} = \widetilde{a}_{ij}$ and $b_i = \widetilde{b}_i$ on $Q_N \cap \{x \in H : |x - x_0| < r\}$ and the martingale problem for $\widetilde{\mathcal{L}}$ starting at y has a unique solution for all $y \in Q_N$. Here $\widetilde{\mathcal{L}}$ is defined analogously to $\mathcal{L}$ but with a_{ij} and b_i replaced by $\widetilde{a}_{ij}$ and $\widetilde{b}_i$, respectively.

Fix $x_0 \in Q_N$, η_0 as in Theorem 5.6. Choose δ as in (d). We claim we can choose $1 \geqslant \delta_1 > 0$ depending on δ and N such that if $x \in Q_N$ and $\|x - x_0\|_\infty < \delta_1$, then $|x - x_0| < \delta$. Here $|x|_\infty = \sup_i |\langle x, e_i \rangle|$.

To prove the claim, note that $\|x-x_0\|_\infty \leqslant \delta_1$ implies that for any K_0

$$\sum_k (x^k - x_0^k)^2 \leqslant \sum_k \delta_1^2 \wedge (4N^2\lambda_k^{-\beta}) \leqslant K_0\delta_1^2 + 4N^2 \sum_{k>K_0} \lambda_k^{-\beta}.$$

So first choose K_0 such that the second term is less than $\delta^2/2$ and then set $\delta_1 = \delta/\sqrt{2K_0}$.

Now let $[p_j, q_j] = [x_0^j - \delta_1, x_0^j + \delta_1] \cap [-N\lambda_j^{-\beta/2}, N\lambda_j^{-\beta/2}]$ and note $p_j < q_j$ as $x_0 \in Q_N$. Let $\psi_j : \mathbb{R} \to \mathbb{R}$ be defined by

$$\psi_j(x) = \begin{cases} x & \text{if } p_j \leqslant x \leqslant q_j, \\ p_j & \text{if } x < p_j, \\ q_j & \text{if } x > q_j. \end{cases}$$

Define $\psi : H \to Q_N \cap \{x \in H : \|x - x_0\|_\infty < \delta_1\}$ by

$$\psi(x) = \sum_{j=1}^{\infty} \psi_j(\langle x, \mathrm{e}_j\rangle)\mathrm{e}_j.$$

As $\|\psi_j\|_\infty^2 \leqslant N^2\lambda_j^{-\beta}$, ψ is well defined by (c).

Take $r = \delta_1 \in (0,1]$ and set $\widetilde{a}_{ij}(x) = a_{ij}(\psi(x))$. If $|x - x_0| < r$ and $x \in Q_N$, then $\|x - x_0\|_\infty < r$ and therefore $\psi(x) = x$, which says that $\widetilde{a}_{ij}(x) = a_{ij}(x)$ for all i,j.

Define

$$\rho(u) = \begin{cases} u & \text{if } |u| < r, \\ (2r - |u|)u/r & \text{if } r \leqslant |u| < 2r, \\ 0 & \text{if } 2r \leqslant |u|, \end{cases}$$

and set $\widetilde{b}_i(x) = b_i(x_0 + \rho(x - x_0))$. If $|x - x_0| < r$, then $\rho(x - x_0) = x - x_0$ and so $\widetilde{b}_i(x) = b_i(x)$. Also $\widetilde{b}_i$ is clearly continuous as (e) implies that b_i is.

We now show that $\widetilde{a}_{ij}$ satisfies the hypotheses of Theorem 5.6. For any x

$$\sum_{i,j} |\widetilde{a}_{ij}(x) - \widetilde{a}_{ij}(x_0)| = \sum_{i,j} |a_{ij}(\psi(x)) - a_{ij}(x_0)|. \tag{5.20}$$

Since $\|\psi(x) - x_0\|_\infty \leqslant r$ and $\psi(x) \in Q_N$, it follows that $|\psi(x) - x_0| < \delta$. (d) now implies that the right-hand side of (5.20) is less than η_0. It remains only to check (5.3) for $\widetilde{a}_{ij}$. But

$$|\psi_j(x) - \psi_j(x + h_j)| \leqslant |h_j|,$$

and so

$$|\psi(x) - \psi(x + h)| \leqslant |h|.$$

Therefore

$$\begin{aligned} |\widetilde{a}_{ij}(x + h) - \widetilde{a}_{ij}(x)| &= |a_{ij}(\psi(x + h)) - a_{ij}(\psi(x))| \\ &\leqslant |a_{ij}|_{C^\alpha} |\psi(x + h) - \psi(x)|^\alpha \\ &\leqslant |a_{ij}|_{C^\alpha} |h|^\alpha, \end{aligned}$$

and so

$$|\widetilde{a}_{ij}|_{C^\alpha} \leqslant |a_{ij}|_{C^\alpha}.$$

Hence $\widetilde{a}_{ij}$ satisfies (5.3) because a_{ij} does.

If we set $B_i(x) = x_i(\widetilde{b}_i(x) - \widetilde{b}_i(x_0))$, it is easy to check that $B_i(x)$ is 0 for $|x - x_0| \geqslant 2r$, $\|B_i\|_\infty \leqslant c_2 |\widetilde{b}_i|_{C^\alpha} \leqslant c_2 |b_i|_{C^\alpha}$, and $|B_i|_{C^\alpha} \leqslant c_2 |\widetilde{b}_i|_{C^\alpha} \leqslant c_2 |b_i|_{C^\alpha}$, where c_1 may depend on x_0. Therefore (e) implies $(\widetilde{b}_i)$ satisfies (5.4) and (5.5).

We see then that Theorem 5.6 applies to $\widetilde{a}_{ij}$ and $\widetilde{b}_i$ and so (5.19) holds. □

Example 5.8. We discuss a class of examples where the $b_i = 1$ and the a_{ij} are zero unless i and j are sufficiently close together. Let $M \in \mathbb{N}$, $\alpha \in (0,1)$ and $S_M(i,j)$ be the subspace of H generated by $\{\varepsilon_k : |k - i| \vee |k - j| \leqslant M\}$. Also let $\Pi_{S_M(i,j)}$ be the projection operator onto $S_M(i,j)$. Assume that $a_{ji}(x) = a_{ij}(x) = \langle \varepsilon_i, a(x)\varepsilon_j \rangle$ satisfies (2.2) and depends only on coordinates corresponding to $S_M(i,j)$, that is,

$$a_{ij}(x) = a_{ij}(\Pi_{S_M(i,j)}x) \quad \text{for all } x \in H, \ \ i,j \in \mathbb{N}. \tag{5.21}$$

In particular, (5.21) implies a_{ij} is constant if $|i-j| > 2M$. Also suppose that

$$\sup_{i,j} |a_{ij}|_{C^\alpha} = c_1 < \infty. \tag{5.22}$$

Set $b_i(x) = 1$ for all i, x and also assume

$$\lambda_j \geqslant c_2 j^p \quad \text{for all } j \text{ for some } p > 1, \tag{5.23}$$

and $\beta \in (0,1)$ satisfies

$$\sum_{j=1}^{\infty} \lambda_j^{\frac{-\beta\alpha}{2}+\delta} < \infty \quad \text{for some } \delta > 0. \tag{5.24}$$

For example, (5.24) will hold if $p > 2$ and $\beta\alpha > 2/p$. We then claim that the hypotheses of Theorem 5.7 hold and so there is a unique solution to the martingale problem for $\mathscr{L}f(x) = \sum_{i,j} a_{ij}(x) D_{ij} f(x) - \sum_i \lambda_i x_i D_i f(x)$, starting at any $y \in H_\beta$.

We must check conditions (b)–(d) of Theorem 5.7. Note first that

$$|a_{ij}(x+h) - a_{ij}(x)| \leqslant 1_{(|i-j| \leqslant 2M)} |a_{ij}|_{C^\alpha} |h|^\alpha,$$

so that $|a_{ij}|_{C^\alpha} \leqslant 1_{(|i-j| \leqslant 2M)} c_3$ and hence by (5.24),

$$\sum_{i \leqslant j} |a_{ij}|_{C^\alpha} \lambda_j^{-\alpha/2} \leqslant (2M+1) c_5 \sum_j \lambda_j^{-\alpha/2} < \infty.$$

This proves (b), and (c) is immediate from (5.24). If $N > 0$, $x, x_0 \in Q_{\beta,N}$, then for small enough $\varepsilon > 0$,

$$\begin{aligned}
&\sum_{i,j} |a_{ij}(x) - a_{ij}(x_0)| \\
&\quad \leqslant 2 \sum_{i \leqslant j} |a_{ij}|_{C^\alpha} \left[\sum_k 1_{(|k-i| \vee |k-j| \leqslant M)} (x(k) - x_0(k))^2 \right]^{\alpha/2}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant 2|x-x_0|^{\varepsilon}\sum_i\sum_{j=1}^{i+2M}\left[\sum_k 1_{(|k-i|\leqslant M)}|x(k)-x_0(k)|^{2-(2\varepsilon/\alpha)}\right]^{\alpha/2}\\
&\leqslant |x-x_0|^{\varepsilon}c_4(M)\sum_{k=1}^{\infty}|x(k)-x_0(k)|^{\alpha-\varepsilon}\\
&\leqslant c_5(M)|x-x_0|^{\varepsilon}\sum_{k=1}^{\infty}(2N)^{\alpha-\varepsilon}\lambda_k^{\frac{-\beta}{2}(\alpha-\varepsilon)}\\
&\leqslant c(M,N)|x-x_0|^{\varepsilon}.
\end{aligned}
$$

We have used (5.22), $x, x_0 \in Q_{\beta,N}$ and (5.24) in the above. This proves (d), as required.

Example 5.9. We give a more specific realization of the previous example. Continue to assume $b_i = 1$ for all i, (5.23), and (5.24). Let $L, N \geqslant 1$ (we can take $N = 1$, for example) and for $k \geqslant 1$ let $I_k = \{(k-1)N+1, \ldots, kN\}$. For each k assume $a^{(k)} : \mathbb{R}^{2L+N} \to \mathscr{S}_N^+$, the space of symmetric positive definite $N \times N$ matrices. Assume for all k, for all $x \in \mathbb{R}^{2L+N}$, and for all $z \in \mathbb{R}^N$,

$$\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^{(k)}(x)z_i z_j \in [\gamma|z|^2, \gamma^{-1}|z|^2] \tag{5.25}$$

and

$$\sup_k \max_{1\leqslant i,j\leqslant N} |a_{ij}^{(k)}|_{C^{\alpha}} < \infty. \tag{5.26}$$

Now for $x \in H$, let $\pi_k x = (\langle x, \varepsilon_{((\ell+k-1)N-L)\vee 1}\rangle)_{\ell=1,\ldots,2L+N} \in \mathbb{R}^{2L+N}$ and define $a : H \to L(H,H)$ by

$$
\begin{aligned}
\langle a(x)\varepsilon_i, \varepsilon_j\rangle &= a_{ij}(x) = a_{ji}(x)\\
&= \begin{cases} a_{i-(k-1)N,j-(k-1)N}^{(k)}(\pi_k x) & \text{if } i,j \in I_k, k \geqslant 1,\\ 0 & \text{if } (i,j) \notin \bigcup_{k=1}^{\infty} I_k \times I_k. \end{cases}
\end{aligned}
$$

Then for all $x, z \in H$,

$$
\begin{aligned}
\sum_i\sum_j a_{ij}(x)z_i z_j &= \sum_{k=1}^{\infty}\sum_{i,j\in I_k} a_{ij}(x)z_i z_j\\
&= \sum_{k=1}^{\infty}\sum_{i,j=1}^{N} a_{ij}^{(k)}(\pi_k x)z_{(k-1)N+i}z_{(k-1)N+j}\\
&\in [\gamma|z|^2, \gamma^{-1}|z|^2]
\end{aligned}
$$

by (5.25), and so (2.2) holds. Note that if $i,j \in I_k$, then (using the notation of Example 5.8) $\mathscr{S}_{L+N}(i,j) \supset \{(k-1)N-L+1, \ldots, kN+L\}$, and so (5.21) with $M = L+N$ is immediate from the above definitions. Also (5.22) is implied by (5.26). The conditions of Example 5.8 therefore hold and so weak existence and uniqueness of solutions hold for the martingale problem for $\mathscr{L}$ with initial conditions in H_{β}.

Remark 5.10. The above examples demonstrate the novel features of our results. The fact that our perturbation need not be nonnegative facilitates the localization argument (see Remark 9 in [23] for comparison) and the presence of $\{\lambda_j^{-\alpha/2}\}$ in condition (b) of Theorem 5.7 means that the perturbation need not be Hölder in the trace class norm. The latter allows for the possibility of locally dependent Hölder coefficients with just bounded Hölder norms, something that seems not to be possible using other results in the literature. On the other hand [23] includes an SPDE example which our approach cannot handle in general unless, for example, the orthonormal basis in the equation diagonalizes the second derivative operator. This is because he has decoupled the conditions on the drift operator and noise term, while ours are interconnected. The latter leads to the double summation in conditions (b) and (d) of Theorem 5.7, as opposed to the trace class conditions in [23]. All of these approaches seem to still be a long way from resolving the weak uniqueness problem for the one-dimensional SPDE described in the introduction which leads to much larger perturbations.

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