Kiyosi Itô, Inaugural Recipient of the Gauss Prize 2006
by Ed Perkins

Kyosi Itô, now 91 and one of the founders of modern probability, became the first recipient of the Gauss prize for applications of mathematics at the 2006 ICM in Madrid.

While working in relative isolation (to a degree this may be said of all probabilists at the time) at the Statistical Bureau of the Government of Japan, in 1942 Professor Itô published an article [I2] in a mathematics journal of Osaka University. It laid the foundations for much of the field for the next 40 years and contained the groundbreaking ideas for which he was awarded the Gauss Prize.

The relationship between the applied and pure ends of the mathematical spectrum has always been a symbiotic one and Itô's career is a wonderful demonstration of this. In [I4] he notes that as a student he loved both the beauty of pure mathematics but also the interactions between mathematics and mechanics and his introduction to probability was through statistical mechanics. One of his earliest papers (1943) was on turbulence. His groundbreaking work on stochastic integration, stochastic differential equations and stochastic calculus was carried out as a purely mathematical pathwise realization of the ideas of Feller and Kolmogorov. It has formed the basis of a vast range of stochastic models in filtering theory, population genetics, mathematical finance and statistical physics. In Itô's words [Fu]:

Because my own research on stochastic analysis is in pure mathematics, the fact that my work has been chosen for the the Gauss Prize for applications of mathematics is truly unexpected and deeply gratifying. I hope therefore to share this great honor and joy with my family, teachers, colleagues, and students in mathematics, as well as with all those who took my work in stochastic analysis and extended it to areas far beyond my imagination.

Prior to 1930, Bachelier (1900) proposed Brownian motion as a model for fluctuations of the stock market, Einstein (1905) won a Nobel prize for his work which used Brownian motion to give experimental confirmation of the atomic theory, and Wiener (1923) gave a mathematically rigorous construction of Brownian motion. (Having to be absolutely certain of every line, we mathematicians place last as usual, but are able to verify that Einstein did earn his Nobel.) Nonetheless, most probabilists would likely pick 1933 as the birth of modern probability. It is the year Kolmogorov's treatise on the foundations of the subject appeared. Itô studied at the University of Tokyo (1935–38) and read Kolmogorov [K], Feller [Fe], Lévy [L], and Doob [D]. Probability as a modern mathematical discipline was in its infancy. His thesis supervisor at Tokyo was Shokichi Iyanaga. At a ceremony held at Kyoto U. in Sept. 2006 Prof. Itô wrote:

Today my only regret is that I was not able to share the news of the Gauss prize with my teacher and mentor, Professor Shokichi Iyanaga, who passed away this June at the age of 100. As a mathematics student at the University of Tokyo in the 1930's, I would not have been able to continue my research interests in probability theory, had it not been for his kind and constant encouragements. Professor Iyanaga taught all of his students to pursue their own interests, whether or not these were popular at the time, and whether or not they had any visible potential practical applications.

As a beginning researcher I had the pleasure of speaking to Prof. Itô at a 1983 Conference at Baton Rouge when Prof. Itô was lecturing on his recent work on SDE's in

infinite-dimensional spaces. It was summer in Louisiana and we were staying in unair-conditioned dormitory rooms. Many of the senior mathematicians had opted for hotels (as I would certainly do these days myself) but Itô stayed in the dorm rooms and in the evenings would discuss mathematics with many of the perspiring younger participants. When I mentioned that I had studied at U. Illinois and had taken courses on potential theory and measure theory from Prof. Doob, he immediately told me how Doob's works on continuous parameter stochastic processes were pivotal in his decision to study probability. Kolmogorov had constructed Markov transition functions as the fundamental solutions of the forward equation associated with a second order differential operator (described here in one spatial dimension) $\mathcal{L} = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx}$. He would then use his general existence theorem to construct a Markov process associated with this transition kernel. Itô felt much of this work was fine analysis but somehow the probability was secondary (the title suggests as much). Lévy offered up something more exciting in a direct manipulation with the sample paths of a process with stationary and independent increments (now called Lévy processes). The Fourier transforms at a fixed time of such processes were characterized by Lévy and Khinchin, but it is a formula which only comes to life with Lévy's pathwise construction of the stochastic process as a "Poisson sum of compensated jumps".[1] Lévy's pathwise approach had excited Prof. Itô–pathwise arguments are what draw many of us to the subject. Many of Lévy's arguments were non-rigorous, however (although almost always correct). In [I4] Itô writes "I had a hard time following Lèvy's argument because of his unique intrinsic description". It was only when he read Doob's works on modifications of continuous parameter processes that Itô realized how to carry out Lévy's construction rigorously. The construction was written in [I1], formed the basis of his thesis, and is now called the Lévy-Itô construction. This pathwise perspective has stayed with Itô throughout his career.

Much of the above work was written while Itô was at the Statistical Bureau (1939-43). M. Fukushima indicated to me that government agencies and private companies would then often hire bright young students as a future investment and let them pursue their own interests as well. Itô writes [I4] "I was given sufficient time for my own study thanks to the kindness of Mr. T. Kawashima, Head of the Bureau". This kindness in wartime Japan also led to the seminal work [I2] which shaped what many of us have been doing during our entire careers. As described in the Introduction and Foreword in [I4], his goal here was to construct the diffusion processes of Kolmogorov [K] by a direct manipulation of the sample paths in the spirit of Lèvy. (Feller [Fe] had extended Kolmogorov's construction to allow

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[1] The idea in [L] and [I1] is to construct the process by adding the sum of its jumps–for compound Poisson processes (jumps occur at a finite rate according to some common distribution) this works fine. In general, small jumps occur at an infinite rate so one must truncate them at ϵ and take a limit. The delicate issue here is that, in general, the sample paths have unbounded variation and the sum of jumps only converges conditionally. To get convergence in the above scheme one must subtract off the mean of the jumps to get a difference of two terms, each approaching infinity, which converges to the limiting value of the process. To do this at all times simultaneously one introduces a sea of potential jumps indexed by size and time at which they occur–the space-time Poisson point process of jumps.

jump kernels and it was in this general setting Itô was working but these extensions were published later in his 1951 AMS Memoir.) His observation was that instantaneously at x Kolmogorov's processes were performing Brownian motion with drift $b(x)$ and standard deviation $\sigma(x) = \sqrt{a(x)}$. Kolmogorov carries this out at the level of probability distributions of the underlying random motion; Itô did it directly at the level of sample paths by building the random process $X(t)$ as the solution of the stochastic differential equation

$$(SDE) \qquad\qquad dX(t) = \sigma(X(t))dB(t) + b(X(t))dt.$$

Here B is a Brownian motion and so has sample paths of unbounded variation on every interval of positive length almost surely. So, to make sense of the first term in (SDE) Itô had to interpret the integral, not sample path by sample path, but rather process by process. One puts in a stochastic *process* $f(s, \cdot)$ (i.e., a measurable function of the ever-suppressed ω) satisfying certain integrability and measurability conditions and one produces a stochastic *process* $\int_0^t f(s, \cdot)dB(s)$ satisfying certain integrability and measurability conditions. To get a useful integral with a dominated convergence theorem, one must restrict the class of integrands $f(s, \omega)$ to those that depend on past information, say of the Brownian motion B. It is a condition which is satisfied for the integrand arising in (SDE). Having made sense of (SDE) Itô then connected it up with Kolmogorov's process (and the operator $\mathcal{L}$) through the formula that bears his name

$$df(X(t)) = f'(X(t))dX(t) + \frac{1}{2}f''(X(t))\sigma^2(X(t))dt$$
$$(IF) \qquad\qquad\qquad = f'(X(t))\sigma(X(t))dB(t) + \mathcal{L}f(X(t))dt.$$

The intuition is clear enough with hindsight. B has infinite variation but has quadratic variation on $[0, t]$ equal to t in a certain sense. One must use a second order Taylor expansion for $X(t)$ locally and sum over the increments to derive (IF). Itô's integral had a precursor–as Stroock and Varadhan put it, "Itô is the Lebesgue of this branch of integration theory; Paley and Wiener were its Riemann." Paley and Wiener (1934) had constructed the stochastic integral for square integrable *deterministic* integrands which would not help one in interpreting (SDE). [2]

Like many significant breakthroughs, Itô's construction and associated analysis was ahead of its time. Itô writes [I4]: "I do not know anyone who read this paper [I2] thoroughly when it appeared except my friend G. Maruyama." Maruyama told Fukushima that he read it repeatedly under the light of a gatekeeper's box while in a military camp where he had been drafted. Itô moved to Nagoya U. (1943-52) and submitted an extended version of his work (in English) on stochastic d.e.'s including jumps to the Memoirs of the AMS. Itô credits Doob with assisting with the publication in the US. Itô's original work was done without the continuous parameter martingale theory which Doob was just developing at the same time on the other side of the ocean. It is the approach which greatly simplifies

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[2] It appears, however, that Itô did not know of this in his 1942 paper although he mentions in [I4] that he read it carefully after moving to Nagoya in 1943, and cites it in a 1944 paper on the integral where their influence can be seen.

and extends Itô's original construction (and lack thereof makes the original construction all the more impressive). Doob immediately appreciated what Itô had done and started to extend Itô's stochastic integral and calculus to general martingales, using, of course, martingale theory. Further extensions were carried out by Itô's former students H. Kunita and S.Watanabe (1967) and P.A. Meyer and the Strasbourg school, culminating in the most general theory of stochastic integration with respect to semimartingales–that is a theorem not an opinion. Stepping back a bit, it was really in the 60's that the area of stochastic integration and stochastic differential equations seemed to reignite. I learned the theory from two excellent sources which appeared then: Itô's Tata Institute Notes [I3] and H. McKean's deceivingly slim book, Stochastic Integrals [McK]. The latter is an effective way to learn stochastic analysis as each sentence is an exercise left for the reader (Itô describes it as "compact but full of interesting materials"). There is a tricky step in the definition of Itô's integral where one proves denseness of a certain class of integrands. It is easy if one assumes the modern notions of predictability or progressive measurability but Itô originally worked with a larger class of adapted and jointly measurable integrands and many of the (then) "modern" proofs seemed to get it wrong. Moving backward in time I discovered the correct and somewhat trickier proof in [I3].

Prof. Itô moved to Kyoto University in 1952 and spent 27 years there–half of which he spent abroad. It gave him the opportunity to work at many of the leading centres in the world (Princeton, Stanford, Cornell, Aarhus) and also bring his ideas to the rest of the world. It is hard to imagine any university today being so far-sighted and generous with such a valuable resource. It clearly enabled Itô to embrace some new and rich theories and in turn exert a positive influence on the development of probability in Japan as he would frequently return to Kyoto U. and lecture on new areas of probability. He arrived in Princeton in 1954 shortly after Feller had completed his analytical classification of one-dimensional diffusions (continuous strong Markov processes). Again the challenge of a pathwise construction of these processes was natural for Itô. He teamed up with Feller's then student, McKean, and 10 years later they completed *the book* on the subject [IM]. Itô describes a key interaction with Feller in [I5]:

"*McKean tried to explain to Feller my work on the stochastic differential equations ... It seemed to me that Feller did not fully understand its significance, but when I explained Lévy's local time to Feller, he immediately appreciated its relevance to the study of the one dimensional diffusion. Indeed Feller immediately gave us a conjecture ... which was eventually substantiated in my joint paper with McKean (1963).*"

Lévy's local time which Itô had read as a student in [L] turned out to be the critical ingredient in the Itô-McKean pathwise construction of general one-dimensional continuous strong Markov processes. It provides the means to a random change of time of a Brownian path which, together with a deterministic change of scale, gave the general result (those who have actually read [IM] will know this is a simplification but a justifiable one). Although the class of processes Itô constructed as solutions of stochastic differential equations form the central and canonical examples it is curious that they are barely mentioned in their book. Although they both start with a Brownian path, the two constructions are in fact completely different. For one-dimensional diffusions the time and scale change technique proved to be the most general and so it is natural that the Kyoto U. seminar in the later

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50's and early 60's would try to extend some of these ideas to general Markov processes. Feller's one-dimensional diffusions were symmetric Markov processes with respect to his speed measure which governed the rate of change of the aforementioned random time change. Beurling and Deny had already introduced the study of Dirichlet forms for general symmetric Markov processes. Masatoshi Fukushima, one of Itô's graduate students then, extended the above time-change ideas to the study of general symmetric Markov processes and with M. Silverstein became one of the two leaders in a subject of continuing importance. Martin boundary theory as developed by Doob and Dynkin as well as Wentzell's boundary conditions for d-dimensional diffusions were other topics of study. To this day there is no general classification of d-dimensional continuous Markov processes. A host of examples (eg. of Tom Salisbury (York U.) (1986)) suggest this question is not going to be so tractable. On the other hand the theory of d-dimensional Markov processes is still dominated by Itô's stochastic differential equations and extensions thereof. His methods of stochastic analysis run through most of the infinite-dimensional theories as well, where Itô has also done influential work. The multiparameter extension of Itô's stochastic integration pioneered by Renzo Cairoli and John Walsh (UBC) play an important role in the modern theory of stochastic partial differential equations. The theory of Dawson-Watanabe superprocesses developed by Itô's student Shinzo Watanabe and Don Dawson (Carleton) is one celebrated class of infinite-dimensional examples in which Itô's stochastic analysis plays a central role. He also played a more direct role. During Itô's tenure as a professor at Stanford (1961–64) he became interested in branching processes through proximity to Harris, Karlin and others. M. Fukushima recalls a resulting series of stimulating lectures on branching processes given by Itô during a trip back to Kyoto U. This and subsequent discussions led to the Ikeda-Nagasawa-Watanabe work on branching Markov processes and S. Watanabe's celebrated paper (1968) on continuous state branching processes. This paper has shaped many of my own research contributions.

I will not try to make this an encyclopedic description of Itô's contributions, as that would take a book. Fortunately the book has been written [I4]. I should, however, mention Itô's groundbreaking work on excursion theory (1970) which decomposes a general Markov process into excursions from a recurrent point and the process up to the recurrent point. The complicating issue is that typically the process returns to the point instantaneously. The solution is an infinite dimensional version of the same Poisson point process ideas Itô used in his thesis to prove the Lévy-Itô decomposition of a Lévy process. Instead of individual jumps taking values in the line, the Poisson points being glued together are now excursion paths taking values in an appropriate path space. Itô also suggested the converse construction of building a Markov process up from its excursions and, under suitable conditions, this synthesis was carried out by Tom Salisbury (York U.) and Chris Rogers.

The Gauss Prize in Madrid seemed to be part of a general recognition that the field of probability had come of age as the work of Fields medallists Okounkov and Werner both highlight probability. Canadians played a role in the celebration as Okounkov's work with Rick Kenyon (UBC) played a significant role in his citation, and Don Dawson (Carleton) was on the Fields Medal Committee. (The mathematics community in Canada had already anticipated this emergence when they elected three probabilists to the current

CMS Executive.)

I have always felt fortunate to be part of a field where the leaders are so supportive of younger researchers. The kindness and interest Prof. Itô showed in my work on that hot Baton Rouge evening remains with me. His mentoring established a whole generation of probabilists in Japan–his students include such influential figures as T. Sirao, T. Hida, N. Ikeda, M. Nisio, S. Watanabe, H. Kunita, M. Fukushima and T. Yamada. Many more were active participants in Itô's Probability Seminar at Kyoto U. including M. Motoo, T. Watanabe, T. Ueno, H. Tanaka and K. Sato. Their students in turn are the current leaders in today's flourishing Japanese probability community, and collectively represent one of Prof. Itô's greatest legacies.

References

[D] J. L. Doob, Stochastic processes depending on a continuous parameter, *Trans. Amer. Math. Soc.* **42**, 107-140 (1937).

[Fe] W. Feller, Zur theorie der stochastischen prozesse (existenz und eindeutigkeits-sätze), *Math. Ann.* **113**, 113-160 (1936).

[Fo] H. Föllmer, On Kiyosi Itô's work and its impact, to appear, *Proc. 2006 ICM, Madrid* (2007).

[Fu] M. Fukushima, On the works of Kiyosi Itô and stochastic analysis, to appear in *Jap. J. Math.*(2007).

[I1] K. Itô, On stochastic processes (infinitely divisible laws of probability, *Japan. Jour. Math.* **XVIII**, 261-301 (1942).

[I2] K. Itô, Differential equations determining a Markov Process (in Japanese), *Journ. Pan-Japan Math. Coll.* No. 1077, 1352-1400 (1942); (in English) in *Kiyosi Itô, Selected Papers*, D. W. Stroock and S.R.S. Varadhan (ed.'s), 42-75 (1986).

[I3] K. Itô, *Lectures on Stochastic Processes*, Tata Institute of Fundamental Research, Bombay (1960).

[I4] K. Itô, *Kiyosi Itô, Selected Papers*, D. W. Stroock and S.R.S Varadhan (ed.'s), Springer-Verlag, 1986.

[I5] K. Itô, Memoirs of my research in stochastic analysis, in *Proc. Abel Symp. 2005, Stochastic Analysis and Applications–A Symposium in Honor of K. Itô*, 1-5, Springer (2006).

[IM] K. Itô and H. P. McKean Jr., *Diffusion Processes and Their Sample Paths*, Springer-Verlag, 1965.

[K] A. Kolmogorov, Über die analytischen methoden in wahrscheinlichkeitsrechnung, *Math. Ann.* **104**, 415-458 (1931).

[L] P. Lévy, *Thëorie de l'Addition des Variables Aléatoires*, Gauthier-Villars, Paris, 1937.

[McK] H. P. McKean Jr., *Stochastic Integrals*, Academic Press, NY and London, 1969.