NON-UNIQUENESS FOR NON-NEGATIVE SOLUTIONS OF PARABOLIC STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

K. BURDZY, C. MUELLER, AND E.A. PERKINS

Dedicated to Don Burkholder.

Abstract. Pathwise non-uniqueness is established for non-negative solutions of the parabolic stochastic pde

$$\frac{\partial X}{\partial t} = \frac{\Delta}{2} X + X^p \dot{W} + \psi, \ X_0 \equiv 0$$

where $\dot{W}$ is a white noise, $\psi \geq 0$ is smooth, compactly supported and non-trivial, and $0 < p < 1/2$. We further show that any solution spends positive time at the 0 function.

1. Introduction

Let $\sigma : \mathbb{R} \to \mathbb{R}$ be p-Hölder continuous (so $|\sigma(x) - \sigma(y)| \leq K|x-y|^p$), let $\psi \in C_c^1(\mathbb{R})$ (the space of C^1 functions on $\mathbb{R}$ with compact support), and consider the parabolic stochastic partial differential equation

$$\frac{\partial X}{\partial t}(t,x) = \frac{\Delta}{2} X(t,x) + \sigma(X(t,x))\dot{W}(t,x) + \psi. \tag{1.1}$$

Here $\dot{W}$ is a space-time white noise on $\mathbb{R}_+ \times \mathbb{R}$. If σ is Lipschitz continuous, pathwise uniqueness of solutions to (1.1) is classical (see, e.g., [Wal86]). Particular cases of (1.1) for non-Lipschitz σ arise in equations modeling populations undergoing migration (leading to the Laplacian) and critical reproduction or resampling (leading to the white noise term). For example if $\sigma(X) = \sqrt{X}$ and $X \geq 0$, we have the equation for the density of one-dimensional super-Brownian motion with immigration ψ (see Section III.4 of [P01]). If $\sigma(X) = \sqrt{X(1-X)}$, $\psi = 0$ and $X \in [0,1]$ we get the equation for

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the density of the stepping stone model on the line [Shi88]. In both cases pathwise uniqueness of solutions remains open while uniqueness in law is obtained by (different) duality arguments (see the above references). The duality arguments are highly non-robust and fail, for example if $\sigma(x,X)=\sqrt{f(x,X)X}$, which models a critically branching population with branching rate at site x in state X is $f(x,X)$. This is one reason that there is interest in proving pathwise uniqueness in (1.1) under Hölder continuous conditions on σ, corresponding to the classical results of [YW71] for one-dimensional SDE's with Hölder 1/2-continuous diffusion coefficients.

In [MP10] pathwise uniqueness for (1.1) is proved if $p>3/4$ and in [MMP11] pathwise uniqueness and uniqueness in law are shown to fail in (1.1) when $\sigma(X)=|X|^p$ for $1/2\le p<3/4$. Here a non-zero solution to (1.1) is constructed for zero initial conditions and the signed nature of the solution is critical. In the examples cited above the solutions of interest are non-negative and so it is natural to ask whether the results in [MP10] can be improved if there is only one point (say $u=0$) where $\sigma(u)$ fails to be Lipschitz, and we are only interested in non-negative solutions. Finding weaker conditions which imply pathwise uniqueness of non-negative solutions in this setting is a topic of ongoing research. In this paper we give counterexamples to pathwise uniqueness of non-negative solutions in the admittedly easier setting where $p<1/2$. Even here, however, we will find there are new issues which arise in our infinite dimensional setting. Our methods will also allow us to extend the nonuniqueness result in [MMP11] mentioned above to $0<p<1/2$.

We assume $\dot W$ is a white noise on the filtered probability space $(\Omega,\mathcal{F},\mathcal{F}_t,P)$, where $\mathcal{F}_t$ satisfies the usual hypotheses. This means $W_t(\phi)$ is an $\mathcal{F}_t$-Brownian motion with variance $\|\phi\|_2^2$ for each $\phi\in L^2(\mathbb{R},dx)$ and $W_t(\phi_1)$ and $W_t(\phi_2)$ are independent if $\langle\phi_1,\phi_2\rangle\equiv\int\phi_1(x)\phi_2(x)dx=0$. A stochastic process $X:\Omega\times\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ which is $\mathcal{F}_t-$previsible $\times$ Borel measurable will be called a solution to the stochastic heat equation (1.1) with initial condition $X_0:\mathbb{R}\to\mathbb{R}$ if for each $\phi\in C_c^\infty(\mathbb{R})$,

$$\begin{aligned}\langle X_t,\phi\rangle =&\langle X_0,\phi\rangle+\int_0^t\left\langle X_s,\frac{\Delta}{2}\phi\right\rangle ds\\&+\int_0^t\int\sigma(X(s,x))\phi(x)W(ds,dx)+t\langle\phi,\psi\rangle \text{ for all } t\ge 0 \ a.s.\end{aligned}$$

(The existence of all the integrals is of course part of the definition.) It is convenient to use the space $C_{rap}(\mathbb{R})$ of rapidly decreasing continuous functions on $\mathbb{R}$ as a state space for our solutions. To describe this space, for $f\in C(\mathbb{R})$ (the continuous functions on $\mathbb{R}$) let

$$|f|_\lambda=\sup_{x\in\mathbb{R}}e^{\lambda|x|}|f(x)|,$$

and set

$$C_{rap} = \{f \in C(\mathbb{R}) : |f|_\lambda < \infty \ \forall \lambda > 0\},$$
$$C_{tem} = \{f \in C(\mathbb{R}) : |f|_\lambda < \infty \ \forall \lambda < 0\}.$$

Equip C_{rap} with the complete metric

$$d(f,g) = \sum_{k=1}^{\infty} 2^{-k}(\|f-g\|_k \wedge 1),$$

and C_{tem} is given the complete metric

$$d_{tem}(f,g) = \sum_{k=1}^{\infty} 2^{-k}(\|f-g\|_{-1/k} \wedge 1).$$

Let C^+_{rap} be the subspace of non-negative functions in C_{rap}, which is a Polish space. Our primary interest is in the smaller space C^+_{rap} resulting in stronger non-uniqueness results.

A C^+_{rap}-valued solution to (1.1) is a solution X such that $t \to X(t,\cdot)$ is in $C(\mathbb{R}_+, C^+_{rap})$, the space of continuous C^+_{rap}-valued paths for all ω. In general if E is a Polish space we give $C(\mathbb{R}_+, E)$ the topology of uniform convergence on compact sets.

The following result is proved just as in Theorem 2.5 of [Shi94].

Theorem 1. *(Weak Existence of Solutions). Assume $\psi \geq 0$ and the p-Hölder continuous function σ satisfies $\sigma(0) = 0$. If $X_0 \in C^+_{rap}$, there exists a filtered space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with a white noise $\dot{W}$ and a C^+_{rap}-valued solution of* (1.1).

Proof. Our conditions on σ imply the hypothesis on a in Theorem 2.5 of [Shi94], however that reference assumes $\psi(x,X)$ satisfies $\psi(x,X) \leq c|X|$. The proof, however, extends easily to our simpler setting of $\psi(x) \geq 0$. $\square$

Here is our main result on non-uniqueness. The proof is given in Section 3. Recall that $\psi \in C^1_c(\mathbb{R})$.

Theorem 2. *Consider* (1.1) *with $\sigma(X) = |X|^p$ for $p \in (0, 1/2)$ and $\psi \geq 0$ with $\int \psi(x)\,dx > 0$. There is a filtered space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ carrying a white noise $\dot{W}$ and two C^+_{rap}-valued solutions to* (1.1) *with initial conditions $X^1_0 = X^2_0 = 0$ such that $P(X^1 \neq X^2) > 0$. That is, pathwise uniqueness fails for non-negative solutions to* (1.1) *for σ, ψ as above.*

Remarks. 1. The state of affairs in Theorem 2 for $\psi = 0$ but X_0 non-zero remains unresolved. We expect the solutions to still be pathwise non-unique. The methods used to prove the above theorem do show pathwise uniqueness and uniqueness in law fail if $\psi = X_0 = 0$ and we drop the non-negativity condition on solutions. Namely, one can construct a non-zero solution to the resulting equation. We will not prove this as stronger results (described above) will be shown in [MMP11] using different methods.

2. Uniqueness in law holds for non-negative solutions to (1.1) for ψ, σ as above and general initial condition $X_0 \in C^+_{rap}$ but now with $1 \geq p \geq 1/2$. This may be proved as in [My98] where the case $\psi = 0$ is treated; for $p = 1/2$ this is of course the well-known uniqueness of super-Brownian motion with immigration ψ. We do not know if uniqueness in law fails for $p < 1/2$. The presence of a drift will play an important role in the proof of Theorem 2.

3. A key technique in this paper is to consider the total mass $M_t = \langle X_t, 1\rangle$, and then apply Theorem 4 which, given the Hölder continuity of $X(t,x)$, shows the brackets process $[M]$ to be bounded below by the integral of a power of M. This in turn allows one to apply comparison arguments with one dimensional diffusions.

In Section 4 below we prove that in the corresponding stochastic ordinary differential equation, although pathwise uniqueness again fails, uniqueness in law does hold. Of course the SDE is now one-dimensional so on one hand this is not surprising. On the other hand, the manner in which uniqueness in law holds is a bit surprising as the SDE picks out a particular boundary behaviour which has the solution spending positive time at 0 (see Section 4). This leads naturally to the following property for all solutions to the SPDE in Theorem 2.

Theorem 3. *Assume σ and ψ are as in Theorem 2. Let X be any C^+_{rap}-valued solution to* (1.1) *with $X_0 = 0$. Then*

$$\int_0^t 1(X(s,x) \equiv 0 \ \forall x)\, ds > 0 \textit{ for all } t > 0 \textit{ a.s.}$$

The proof will be given in Section 5 below. Let

$$b = \langle \psi, 1\rangle > 0.$$

We note that the above result fails for $p = 1/2$ since in that case $Y_t = 4\langle X_t, 1\rangle$ is a Bessel squared process of parameter $4b$ satisfying an ordinary sde of the form

$$dY_t = 2\sqrt{Y_t}dB_t + 4bdt.$$

Such solutions spend zero time at 0 (see for example, the analysis in Section V.48 of [RW].)

Finally we state the non-uniqueness result which complements that in [MMP11] in the much easier regime of $p < 1/2$. The solutions here will be signed.

Theorem 4. *If $0 < p < 1/2$ there is a C_{rap}-valued solution X to*

$$\frac{\partial X}{\partial t}(t,x) = \frac{\Delta X}{2}(t,x) + |X(t,x)|^p \dot{W}, \quad X(0) \equiv 0, \tag{1.2}$$

so that $P(X \not\equiv 0) > 0$. In particular uniqueness in law and pathwise uniqueness fail in (1.2).

Although the construction in [MMP11] for $1/2 \le p < 3/4$ is more delicate, it is a bit awkward to extend the reasoning to $p < 1/2$ and so we prefer to present the result here. The proof of Theorem 4 is simpler than that of Theorem 2 in that we can focus on a single process rather than a pair of solutions. The two proofs are similar in that approximate solutions are found by an excursion construction and the key ingredient required for the SPDE setting is Theorem 5 below. Hence we only give a brief sketch of the proof of Theorem 4 at the end of Section 3.

2. A Real Analysis Lemma

Theorem 5. *If $0 < \alpha, \beta < 1$ and $C > 0$, there is a constant $K_5(\beta, C) > 0$ such that if $f : \mathbb{R} \to \mathbb{R}_+$ satisfies*

$$|f(x) - f(y)| \le C|x-y|^\beta, \tag{2.1}$$

then

$$\int f^\alpha \, dx \ge K_5 \Big(\int f \, dx \Big)^{(\alpha\beta+1)/(\beta+1)}.$$

Proof. First we use a scaling argument to reduce to the case

$$\int f(x) dx = 1$$

for which we would have to prove

$$\int f^\alpha \ge K_4.$$

Indeed, if we take $b > 0$ and let

$$g(x) = b^{-\beta} f(bx)$$

then

$$|g(x) - g(y)| = b^{-\beta} |f(bx) - f(by)| \le C|x-y|^\beta$$

by the conditions of Theorem 5. Then setting $y = bx$, we get

$$\int g(x) dx = \int b^{-\beta} f(bx) dx = b^{-(\beta+1)} \int f(y) dy = 1$$

provided

$$b = \left(\int f(y) dy \right)^{\frac{1}{\beta+1}}.$$

So g satisfies the conditions of Theorem 5 with $\int g = 1$, and if we could show that

$$\int g^\alpha(x) dx \ge K_4$$

it would follow, substituting for b, that

$$\int f^\alpha(x) dx = b^{\alpha\beta+1} \int g^\alpha(x) dx \ge K_4 \left(\int f(x) dx \right)^{\frac{\alpha\beta+1}{\beta+1}}$$

as required.

Now we concentrate on proving $\int f^\alpha \geq K_4$ assuming that $\int f = 1$ and assuming the Hölder condition (2.1) on f. Let $M = \sup_x f(x)$, and note the conclusion is obvious if $M = \infty$ so assume it is finite. If $M < 1$, then since $0 < \alpha < 1$, we have

$$\int f^\alpha(x)dx \geq \int f(x)dx = 1.$$

On the other hand, if $M \geq 1$, then the Hölder condition on f implies that $f \geq \frac{1}{2}$ on an interval I whose length is bounded below by a constant $L > 0$ depending only on C, β. So in this case, too, we conclude

$$\int f^\alpha(x)dx \geq \frac{L}{2^\alpha} \geq \frac{L}{2},$$

and Theorem 5 is proved. □

3. Proof of Theorem 2

If $\psi \in C_c^1(\mathbb{R})$, $\psi \geq 0$, $b = \int \psi dx > 0$ and $0 < p < 1/2$, we want to construct distinct solutions X, Y to

$$\frac{\partial X}{\partial t}(t,x) = \frac{\Delta X}{2}(t,x) + (X(t,x))^p \dot{W}(t,x) + \psi(x),\ \ X \geq 0, X_0 = 0. \tag{3.1}$$

Let C_b^k denote the space of bounded C^k functions on $\mathbb{R}$ with bounded jth order partials for all $j \leq k$, and set $C_b = C_b^0$. The standard Brownian semigroup is denoted by $(P_t, t \geq 0)$ and $p_t(\cdot)$ is the Brownian density.

Here is an overview of the proof. We will proceed by constructing approximate solutions $(X^\varepsilon, Y^\varepsilon)$ to (3.1) and then let (X, Y) be an appropriate weak limit point of $(X^{\varepsilon_n}, Y^{\varepsilon_n})$. These approximate solutions will satisfy $X^\varepsilon \geq Y^\varepsilon \geq 0$ and $Y^\varepsilon \geq X^\varepsilon \geq 0$, respectively, on alternating excursions away from 0 by $M = \langle X^\varepsilon, 1\rangle \vee \langle Y^\varepsilon, 1\rangle$. M will equal $2b\varepsilon$ at the effective start of each excursion. We then calculate an upper bound on the probability that M will hit 1 on a given excursion (see (3.43) below) and a lower bound on $\bar{D}^\varepsilon = |\langle X^\varepsilon, 1\rangle - \langle Y^\varepsilon, 1\rangle|$ hitting an appropriate $x_0 \in (0,1)$ during each excursion (see (3.58) below). Theorem 5 is used in the proof of the first bound (see (3.36) below). These bounds will then show there is positive probability (independent of ε) of $\bar{D}^\varepsilon$ hitting x_0 before M hits 1. The result follows by taking weak limits as $\varepsilon_n \downarrow 0$. The use of Theorem 5 will mean the above upper bound is valid only up to a stopping time V_k^ε which will be large with high probability. This necessitates a "padding out" of the above excursions after this stopping time, and this technical step unfortunately complicates the construction.

Fix $\varepsilon > 0$ and define $(X^\varepsilon, Y^\varepsilon)$, $D^\varepsilon = |X^\varepsilon - Y^\varepsilon|$, the white noise $\dot{W}$ and a sequence of stopping times inductively on j as follows. Let $T_0^\varepsilon = 0$, $U_j^\varepsilon =$

$T_j^\varepsilon+\varepsilon$, and assume $X^\varepsilon_{T^\varepsilon_{2j}}=Y^\varepsilon_{T^\varepsilon_{2j}}\equiv 0$ on $\{T^\varepsilon_{2j}<\infty\}$. Assuming $\{T^\varepsilon_{2j}<\infty\}$, on $[T^\varepsilon_{2j},U^\varepsilon_{2j}]$ define

$$Y^\varepsilon(t,x)\equiv 0,\ \text{and}\ D^\varepsilon(t,\cdot)=X^\varepsilon(t,\cdot)=2\int_0^{t-T^\varepsilon_{2j}}P_s\psi(\cdot)ds\in C^+_{rap}. \tag{3.2}$$

That is,

$$\frac{\partial X^\varepsilon}{\partial t}=\frac{\Delta}{2}X^\varepsilon+2\psi\quad\text{for } T^\varepsilon_{2j}\le t\le U^\varepsilon_{2j}. \tag{3.3}$$

Next let $t\to(Y^\varepsilon_{U^\varepsilon_{2j}+t},D^\varepsilon_{U^\varepsilon_{2j}+t})$ in $C(\mathbb{R}_+,C^+_{rap})^2$ solve the following SPDE for $t\ge U^\varepsilon_{2j}$,

$$\begin{aligned}\frac{\partial Y^\varepsilon}{\partial t}&=\frac{\Delta}{2}Y^\varepsilon+\psi+(Y^\varepsilon)^p\dot W,\ Y^\varepsilon_{U^\varepsilon_{2j}}\equiv 0,\\ \frac{\partial D^\varepsilon}{\partial t}&=\frac{\Delta}{2}D^\varepsilon+[(Y^\varepsilon+D^\varepsilon)^p-(Y^\varepsilon)^p]\dot W,D^\varepsilon_{U^\varepsilon_{2j}}(x)=2\int_0^\varepsilon P_s\psi(x)ds.\end{aligned} \tag{3.4}$$

The existence of such a solution on some filtered space carrying a white noise follows as in Theorem 2.5 of [Shi94]. To be careful here one has to construct an appropriate conditional probability given $\mathcal{F}_{U^\varepsilon_{2j}}$ and so inductively construct our white noise along with $(Y^\varepsilon,D^\varepsilon)$. Set $X^\varepsilon_t=Y^\varepsilon_t+D^\varepsilon_t$ for $U^\varepsilon_{2j}\le t\le T^\varepsilon_{2j+1}$, where

$$T^\varepsilon_{2j+1}=\inf\{t\ge U^\varepsilon_{2j}:\langle X^\varepsilon_t,1\rangle=0\}\ (\inf\emptyset=\infty),$$

and also restrict the above definition of $(Y^\varepsilon,D^\varepsilon)$ to $[U^\varepsilon_{2j},T^\varepsilon_{2j+1}]$. Therefore on $[U^\varepsilon_{2j},T^\varepsilon_{2j+1}]$,

$$\begin{aligned}&\frac{\partial Y^\varepsilon}{\partial t}=\frac{\Delta}{2}Y^\varepsilon+\psi+(Y^\varepsilon)^p\dot W,\ Y^\varepsilon_{U^\varepsilon_{2j}}\equiv 0,\\ &\frac{\partial X^\varepsilon}{\partial t}=\frac{\Delta}{2}X^\varepsilon+\psi+(X^\varepsilon)^p\dot W,\ X^\varepsilon_{U^\varepsilon_{2j}}(x)=2\int_0^\varepsilon P_s\psi(x)ds,\\ &D^\varepsilon=|X^\varepsilon-Y^\varepsilon|=X^\varepsilon-Y^\varepsilon\ge 0,\\ &X^\varepsilon,Y^\varepsilon\text{ are continuous and } C^+_{rap}\text{-valued.}\end{aligned} \tag{3.5}$$

Note that $X^\varepsilon_{T^\varepsilon_{2j+1}}=Y^\varepsilon_{T^\varepsilon_{2j+1}}=0$. The precise meaning of the above formulas for $\partial Y^\varepsilon/\partial t$ and $\partial X^\varepsilon/\partial t$ is that equality holds after multiplying by $\phi\in C_c^\infty$ and integrating over $\mathbb{R}$ and over any time interval in $[U^\varepsilon_{2j},T^\varepsilon_{2j+1}]$.

Now assume $T^\varepsilon_{2j+1}<\infty$ and construct $(X^\varepsilon,Y^\varepsilon)$ and $D^\varepsilon=|X^\varepsilon-Y^\varepsilon|=Y^\varepsilon-X^\varepsilon$ as above but with the roles of X and Y reversed. This means that on $[T^\varepsilon_{2j+1},U^\varepsilon_{2j+1}]=[T^\varepsilon_{2j+1},T^\varepsilon_{2j+1}+\varepsilon]$,

$$D^\varepsilon(t,\cdot)=Y^\varepsilon(t,\cdot)=2\int_0^{t-T^\varepsilon_{2j+1}}P_s\psi(\cdot)\,ds\in C^+_{rap}(\mathbb{R})\text{ and }X^\varepsilon(t,\cdot)\equiv 0, \tag{3.6}$$

and so

$$\frac{\partial Y^\varepsilon}{\partial t}=\frac{\Delta}{2}Y^\varepsilon+2\psi, \tag{3.7}$$

and on $[U^\varepsilon_{2j+1}, T^\varepsilon_{2j+2}]$,

$$
\begin{aligned}
&\frac{\partial X^\varepsilon}{\partial t} = \frac{\Delta}{2} X^\varepsilon + \psi + (X^\varepsilon)^p \dot W,\ X^\varepsilon_{U^\varepsilon_{2j+1}} \equiv 0,\ X^\varepsilon \ge 0,\\
&\frac{\partial Y^\varepsilon}{\partial t} = \frac{\Delta}{2} Y^\varepsilon + \psi + (Y^\varepsilon)^p \dot W,\ Y^\varepsilon_{U^\varepsilon_{2j+1}} = 2\int_0^\varepsilon P_s\psi(x)\,ds,\ Y^\varepsilon \ge 0,\\
&D^\varepsilon = |X^\varepsilon - Y^\varepsilon| = Y^\varepsilon - X^\varepsilon \ge 0\\
&X^\varepsilon, Y^\varepsilon \text{ are continuous and } C^+_{rap}\text{-valued.}
\end{aligned}
\tag{3.8}
$$

Here, as before, we have

$$T^\varepsilon_{2j+2} = \inf\{t \ge U^\varepsilon_{2j+1} : \langle Y^\varepsilon_t, 1\rangle = 0\}.$$

Clearly $X^\varepsilon_{T^\varepsilon_{2j+2}} = Y^\varepsilon_{T^\varepsilon_{2j+2}} \equiv 0$ on $\{T^\varepsilon_{2j+2} < \infty\}$ and $T^\varepsilon_j \uparrow \infty$ and so our inductive construction of $(X^\varepsilon, Y^\varepsilon)$ is complete. It is also clear from the construction that if $X_j(t) = X^\varepsilon_{(T^\varepsilon_j + t)\wedge T^\varepsilon_{j+1}}$ and similarly for Y_j, then we may assume

$$
\begin{aligned}
&P((X_{2j}, Y_{2j}, T^\varepsilon_{2j+1} - T^\varepsilon_{2j}) \in \cdot | \mathcal{F}_{T^\varepsilon_{2j}})\\
&\quad = P((X_0, Y_0, T^\varepsilon_1) \in \cdot) \ \text{ a.s. on } \{T^\varepsilon_{2j} < \infty\},
\end{aligned}
\tag{3.9}
$$

and

$$
\begin{aligned}
&P((Y_{2j+1}, X_{2j+1}, T^\varepsilon_{2j+2} - T^\varepsilon_{2j+1}) \in \cdot | \mathcal{F}_{T^\varepsilon_{2j+1}})\\
&\quad = P((X_0, Y_0, T^\varepsilon_1) \in \cdot) \ \text{ a.s. on } \{T^\varepsilon_{2j+1} < \infty\}.
\end{aligned}
\tag{3.10}
$$

Define $J_\varepsilon = \bigcup_{j=1}^\infty [U^\varepsilon_{j-1}, T^\varepsilon_j]$,

$$A^\varepsilon_1(t,x) = \sum_{j=0}^\infty (-1)^j \int_0^t 1_{[T^\varepsilon_j, U^\varepsilon_j]}(s)\psi(x)ds,$$

and $A^\varepsilon_2(t,x) = -A^\varepsilon_1(t,x)$. Combine (3.3), (3.5), (3.8) and the fact that on $[T^\varepsilon_{2j+1}, U^\varepsilon_{2j+1}]$ we have $X^\varepsilon_t = 0 = \frac{\Delta}{2} X^\varepsilon_t + \psi - \psi$ to see that for a test function $\phi \in C^\infty_c(\mathbb{R})$,

$$
\begin{aligned}
\langle X^\varepsilon_t, \phi\rangle &= \langle A^\varepsilon_1(t), \phi\rangle + \int_0^t \left(\langle X^\varepsilon_s, \frac{\Delta}{2}\phi\rangle + \langle \psi, \phi\rangle\right) ds\\
&\quad + \int_0^t \int 1_{J_\varepsilon}(s)\phi(x) X^\varepsilon(s,x)^p dW(s,x),\\
X^\varepsilon_\cdot &\in C(\mathbb{R}_+, C^+_{rap}).
\end{aligned}
\tag{3.11}
$$

Similar reasoning gives

$$\langle Y_t^\varepsilon,\phi\rangle = \langle A_2^\varepsilon(t),\phi\rangle + \int_0^t \Big(\langle Y_s^\varepsilon,\frac{\Delta}{2}\phi\rangle + \langle\psi,\phi\rangle\Big)\,ds + \int_0^t\int 1_{J_\varepsilon}(s)\phi(x)Y^\varepsilon(s,x)^p dW(s,x),$$
$$Y_\cdot^\varepsilon \in C(\mathbb{R}_+, C_{rap}^+). \tag{3.12}$$

Since $U_j^\varepsilon = T_j^\varepsilon+\varepsilon$, the alternating summation in the definition of A_1^ε implies that

$$\sup_t |A_i^\varepsilon(t,x)| \le \varepsilon\psi(x). \tag{3.13}$$

It follows from (3.2) and (3.6) (recall that $b=\int\psi(x)dx$) that

$$X^\varepsilon(t,x)1_{J_\varepsilon^c}(t) \le 2\int_0^\varepsilon P_s\psi(x)\,ds \le 4\varepsilon^{1/2}b.$$

Therefore for any $T>0$ and ϕ as above

$$E\Big(\Big[\int_0^T\int 1_{J_\varepsilon^c}(s)(X^\varepsilon(s,x))^p\phi(x)\,dW(s,x)\Big]^2\Big) \le (4\varepsilon^{1/2}b)^{2p}T\|\phi\|_2^2. \tag{3.14}$$

By identifying the white noise $\dot W$ with associated Brownian sheet, we may view W as a stochastic process with sample paths in $C(\mathbb{R}_+, C_{tem}(\mathbb{R}))$. Using bounds in Section 6 of [Shi94] (see especially the pth moment bounds in the proofs of Theorems 2.2 and 2.5 there) it is straightforward to verify that for $\varepsilon_n \downarrow 0$, $\{(X^{\varepsilon_n}, Y^{\varepsilon_n}, W) : n\in\mathbb{N}\}$ is tight in $C(\mathbb{R}_+, (C_{rap}^+)^2\times C_{tem})$. Some of the required bounds are in fact derived in the proof of Lemma 6 below. By (3.13), (3.14) and their analogues for Y^ε, one sees from (3.11) and (3.12) that for any limit point (X,Y,W), X and Y are C_{rap}^+-valued solutions of (3.1) with respect to the common $\dot W$. It remains to show that X and Y are distinct.

We know $X^\varepsilon(t,\cdot)$ and $Y^\varepsilon(t,\cdot)$ will be locally Hölder continuous of index 1/4 but it will be convenient to have a slightly stronger statement. We note parenthetically that any other index of Hölder continuity for $X^\varepsilon(t,\cdot)$ and $Y^\varepsilon(t,\cdot)$ would yield the same range for p in Theorem 2, provided that the index were less than 1/2. Let

$$V_k^\varepsilon = \inf\{s\ge 0 : \exists x,x'\in\mathbb{R} \text{ such that } |X^\varepsilon(s,x)-X^\varepsilon(s,x')| + |Y^\varepsilon(s,x)-Y^\varepsilon(s,x')| > k|x-x'|^{1/4}\}.$$

We will show in Lemma 6 that $\lim_{k\to\infty}\sup_{0<\varepsilon\le 1} P(V_k^\varepsilon \le M) = 0$ for any $M\in\mathbb{N}$.

Note that on $[T_{2j}^\varepsilon, U_{2j}^\varepsilon]$, $Y^\varepsilon = 0$ and

$$|X^\varepsilon(t,x)-X^\varepsilon(t,x')| = 2\Big|\int p_{t-T_{2j}^\varepsilon}(z)(\psi(z+x)-\psi(z+x'))dz\Big| \le 2\|\psi'\|_\infty|x-x'|.$$

This implies that on the above interval for all real x, x',

$$|X^\varepsilon(t,x) - X^\varepsilon(t,x')| + |Y^\varepsilon(t,x) - Y^\varepsilon(t,x')| \le 4(\|\psi'\|_\infty \vee \|\psi\|_\infty)|x-x'|^{1/4},$$

where the inequality holds trivially for $|x - x'| > 1$ since the left side is at most $4\|\psi\|_\infty$. By symmetry it also holds on $[T^\varepsilon_{2j+1}, U^\varepsilon_{2j+1}]$. We may assume $k \ge 4(\|\psi'\|_\infty \vee \|\psi\|_\infty)$ and so the above implies

$$V^\varepsilon_k \in \bigcup_{j=0}^\infty (U^\varepsilon_j, T^\varepsilon_{j+1}] \cup \{\infty\}. \tag{3.15}$$

We fix a value of k which will be chosen sufficiently large below. We will now enlarge our probability space to include a pair of processes $(\bar X^\varepsilon_t, \bar Y^\varepsilon_t)$ which will equal $(\langle X^\varepsilon_t, 1\rangle, \langle Y^\varepsilon_t, 1\rangle)$ up to time V^ε_k and then switch to a pair of approximate solutions to a convenient SDE. Set $p' = \frac{p+2}{5} \in (0, \frac{1}{2})$ and $K(k) = K_5(1/4, k)$ where K_5 is as in Theorem 5. We may assume our $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ carries a standard $\mathcal{F}_t$-Brownian motion $(B_s : s \le V^\varepsilon_k)$, independent of $(X^\varepsilon, Y^\varepsilon, W)$.

To define a law Q_0 on $C(\mathbb{R}, \mathbb{R}^2_+) \times [0, \infty]$, first construct a solution $(\tilde Y^\varepsilon, \tilde D^\varepsilon, B)$ of

(3.16)

$$\begin{aligned}
\tilde Y^\varepsilon_t &= b\int_0^t 1(\varepsilon < s)\, ds + \int_0^t 1(\varepsilon < s)(\tilde Y^\varepsilon_s)^{p'} \sqrt{K(k)}\, dB_s,\ \tilde Y^\varepsilon \ge 0,\\
\tilde D^\varepsilon_t &= 2b(t \wedge \varepsilon) + \int_0^t 1(\varepsilon < s)[(\tilde Y^\varepsilon_s + \tilde D^\varepsilon_s)^{p'} - (\tilde Y^\varepsilon_s)^{p'}]\sqrt{K(k)}\, dB_s,\ \tilde D^\varepsilon \ge 0.
\end{aligned}$$

Such a weak solution may again be found by approximation by solutions of Lipschitz SDE's as in Theorems 2.5 and 2.6 of [Shi94] for the more complicated stochastic pde setting. Set $\tilde X^\varepsilon = \tilde Y^\varepsilon + \tilde D^\varepsilon$ and $\tilde T^\varepsilon_1 = \inf\{t : \tilde X^\varepsilon_t = 0\} \ge \varepsilon$, and define

$$Q_0(A) = P((\tilde X^\varepsilon_{\cdot \wedge \tilde T^\varepsilon_1}, \tilde Y^\varepsilon_{\cdot \wedge \tilde T^\varepsilon_1}, \tilde T^\varepsilon_1) \in A). \tag{3.17}$$

Next we enlarge our space to include $(\bar{X}^\varepsilon, \bar{Y}^\varepsilon)$ so that for finite $t \le \bar{T}_1^\varepsilon$ (this time is defined below),

$$\bar{Y}_t^\varepsilon = \langle Y^\varepsilon_{t\wedge V_k^\varepsilon}, 1\rangle + b(t - V_k^\varepsilon)^+ + \int_0^t 1(s > V_k^\varepsilon)(\bar{Y}_s^\varepsilon)^{p'}\sqrt{K(k)}\, dB_s,\ \bar{Y}^\varepsilon \ge 0, \tag{3.18}$$

$$\bar{D}_t^\varepsilon = \langle D^\varepsilon_{t\wedge V_k^\varepsilon}, 1\rangle + \int_0^t 1(s > V_k^\varepsilon)[(\bar{D}_s^\varepsilon + \bar{Y}_s^\varepsilon)^{p'} - (\bar{Y}_s^\varepsilon)^{p'}]\sqrt{K(k)}\, dB_s,\ \bar{D}^\varepsilon \ge 0, \tag{3.19}$$

$$\bar{X}_t^\varepsilon = \bar{Y}_t^\varepsilon + \bar{D}_t^\varepsilon = \langle X^\varepsilon_{t\wedge V_k^\varepsilon}, 1\rangle + b(t - V_k^\varepsilon)^+ + \int_0^t 1(s > V_k^\varepsilon)(\bar{X}_s^\varepsilon)^{p'}\sqrt{K(k)}\, dB_s, \tag{3.20}$$

$$\bar{T}_1^\varepsilon = \inf\{t \ge 0 : \bar{X}_t^\varepsilon = 0\} \le \infty. \tag{3.21}$$

Note that if $V_k^\varepsilon \ge T_1^\varepsilon$, then $\bar{X}_t^\varepsilon = \langle X_t^\varepsilon, 1\rangle$ for $t \le T_1^\varepsilon$ and so $\bar{T}_1^\varepsilon = T_1^\varepsilon$. Therefore, $\bar{T}_1^\varepsilon \wedge V_k^\varepsilon \le T_1^\varepsilon$. We conclude that $\bar{X}^\varepsilon_{t\wedge V_k^\varepsilon} = \bar{Y}^\varepsilon_{t\wedge V_k^\varepsilon} + \bar{D}^\varepsilon_{t\wedge V_k^\varepsilon}$ for $t \le \bar{T}_1^\varepsilon$, thus proving (3.20).

To carry out the above construction first build $(\bar{Y}^\varepsilon_{V_k^\varepsilon+t}, \bar{D}^\varepsilon_{V_k^\varepsilon+t}, B_{V_k^\varepsilon+t} - B_{V_k^\varepsilon})$ by approximation by solutions to SDE's with Lipschitz coefficients as in Theorem 2.5 of [Shi94]. This and a measurable selection argument (see Section 12.2 of [SV]) allows us to build the appropriate regular conditional probability

$$\begin{aligned} &Q^0{}_{\langle Y^\varepsilon_{V_k^\varepsilon},1\rangle, \langle D^\varepsilon_{V_k^\varepsilon},1\rangle}(\cdot) \\ &\qquad \equiv P(((\bar{Y}^\varepsilon_{V_k^\varepsilon+t}, \bar{D}^\varepsilon_{V_k^\varepsilon+t}, B(V_k^\varepsilon + t) - B(V_k^\varepsilon)), t \ge 0) \in \cdot | X^\varepsilon, Y^\varepsilon, W), \end{aligned}$$

where $\{Q^0_{y,d} : y, d \ge 0\}$ is a measurable family of laws on $C(\mathbb{R}_+, \mathbb{R}_+^2 \times \mathbb{R})$. This then allows us to construct $(\bar{X}^\varepsilon, \bar{Y}^\varepsilon, \bar{D}^\varepsilon)$ as above on an enlargement of our original space which we still denote $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. We also may now prescribe another measurable family of laws $\{Q_{y,x} : (y, x) \in C(\mathbb{R}_+, \mathbb{R}_+^2)\}$ on $C(\mathbb{R}_+, \mathbb{R}_+^2) \times [0, \infty]$ such that for each Borel A, w.p. 1,

$$Q_{\langle X^\varepsilon_{\cdot\wedge V_k^\varepsilon},1\rangle, \langle Y^\varepsilon_{\cdot\wedge V_k^\varepsilon},1\rangle}(A) = P((\bar{X}^\varepsilon_{\cdot\wedge \bar{T}_1^\varepsilon}, \bar{Y}^\varepsilon_{\cdot\wedge \bar{T}_1^\varepsilon}, \bar{T}_1^\varepsilon) \in A | X^\varepsilon, Y^\varepsilon, W). \tag{3.22}$$

Define

$$Q_1(A) = P((\bar{X}^\varepsilon_{\cdot\wedge \bar{T}_1^\varepsilon}, \bar{Y}^\varepsilon_{\cdot\wedge \bar{T}_1^\varepsilon}, \bar{T}_1^\varepsilon) \in A) = E\Big(Q_{\langle X^\varepsilon_{\cdot\wedge V_k^\varepsilon},1\rangle, \langle Y^\varepsilon_{\cdot\wedge V_k^\varepsilon},1\rangle}(A)\Big). \tag{3.23}$$

Next, inductively define $(\bar{X}_t^\varepsilon, \bar{Y}_t^\varepsilon)$, $t \in [\bar{T}_j^\varepsilon, \bar{T}_{j+1}^\varepsilon]$, and $\{\bar{T}_j^\varepsilon\}$ in a manner reminiscent of that for $(X^\varepsilon, Y^\varepsilon)$, and consistent with the above construction for $j = 0$ (set $\bar{T}_0^\varepsilon = 0$). Assume the construction up to $\bar{T}_{2j}^\varepsilon$ is such that

$$\bar{X}^\varepsilon_{\bar{T}_{2j}^\varepsilon} = \bar{Y}^\varepsilon_{\bar{T}_{2j}^\varepsilon} = 0 \text{ on } \{\bar{T}_{2j}^\varepsilon < \infty\}, \tag{3.24}$$

and

$$\bar{T}^{\varepsilon}_{2j} = T^{\varepsilon}_{2j} \text{ on } \{V^{\varepsilon}_k > \bar{T}^{\varepsilon}_{2j}\}. \tag{3.25}$$

Define

$$\bar{X}_j(t) = \bar{X}^{\varepsilon}_{(\bar{T}^{\varepsilon}_j + t)\wedge \bar{T}^{\varepsilon}_{j+1}}, \tag{3.26}$$

and similarly define $\bar{Y}_j$. On $\{\bar{T}^{\varepsilon}_{2j} < V^{\varepsilon}_k\}$ set

$$\begin{aligned} P((\bar{X}_{2j}, \bar{Y}_{2j}, \bar{T}^{\varepsilon}_{2j+1} - \bar{T}^{\varepsilon}_{2j}) \in \cdot | \mathcal{F}_{\bar{T}^{\varepsilon}_{2j}} \vee \sigma(X^{\varepsilon}, Y^{\varepsilon}, W)) \\ = Q_{\langle X^{\varepsilon}_{(T^{\varepsilon}_{2j}+\cdot)\wedge V^{\varepsilon}_k}, 1\rangle, \langle Y^{\varepsilon}_{(T^{\varepsilon}_{2j}+\cdot)\wedge V^{\varepsilon}_k}, 1\rangle}(\cdot). \end{aligned} \tag{3.27}$$

On $\{\infty > \bar{T}^{\varepsilon}_{2j} \geq V^{\varepsilon}_k\}$ set

$$P((\bar{X}_{2j}, \bar{Y}_{2j}, \bar{T}^{\varepsilon}_{2j+1} - \bar{T}^{\varepsilon}_{2j}) \in \cdot | \mathcal{F}_{\bar{T}^{\varepsilon}_{2j}} \vee \sigma(X^{\varepsilon}, Y^{\varepsilon}, W)) = Q_0(\cdot). \tag{3.28}$$

These definitions imply $\bar{T}^{\varepsilon}_{2j+1} = \inf\{t > \bar{T}^{\varepsilon}_{2j} : \bar{X}^{\varepsilon}_t = 0\}$ and that on our enlarged probability space, conditional on $\mathcal{F}_{\bar{T}^{\varepsilon}_{2j}}$ and on $\{V^{\varepsilon}_k > \bar{T}^{\varepsilon}_{2j}\}$, (3.18)-(3.20) hold for $t \in [\bar{T}^{\varepsilon}_{2j}, \bar{T}^{\varepsilon}_{2j+1}]$, while on $\{\infty > \bar{T}^{\varepsilon}_{2j} \geq V^{\varepsilon}_k\}$, for $t \in [\bar{T}^{\varepsilon}_{2j}, \bar{T}^{\varepsilon}_{2j+1}]$, (3.16), (3.17) and (3.28) give

$$\begin{aligned} \bar{Y}^{\varepsilon}_t &= b \int_0^t 1(\bar{T}^{\varepsilon}_{2j} + \varepsilon < s)\, ds \\ &\quad + \int_0^t 1(\bar{T}^{\varepsilon}_{2j} + \varepsilon < s)(\bar{Y}^{\varepsilon}_s)^{p'} \sqrt{K(k)}\, dB(s),\ \bar{Y}^{\varepsilon} \geq 0, \\ \bar{D}^{\varepsilon}_t &= 2b\big((t - \bar{T}^{\varepsilon}_{2j}) \wedge \varepsilon\big) \\ &\quad + \int_0^t 1(\bar{T}^{\varepsilon}_{2j} + \varepsilon < s)\big((\bar{Y}^{\varepsilon}_s + \bar{D}^{\varepsilon}_s)^{p'} - (\bar{Y}^{\varepsilon}_s)^{p'}\big) \sqrt{K(k)}\, dB_s,\ \bar{D}^{\varepsilon} \geq 0, \\ \bar{X}^{\varepsilon}_t &= 2b\big((t - \bar{T}^{\varepsilon}_{2j}) \wedge \varepsilon\big) + b \int_0^t 1(\bar{T}^{\varepsilon}_{2j} + \varepsilon < s)\, ds \\ &\qquad + \int_0^t 1(\bar{T}^{\varepsilon}_{2j} + \varepsilon < s)(\bar{X}^{\varepsilon}_s)^{p'} \sqrt{K(k)}\, dB(s). \end{aligned} \tag{3.29}$$

Now assume $\bar{T}^{\varepsilon}_{2j+1} < \infty$ and construct $(\bar{X}^{\varepsilon}_t, \bar{Y}^{\varepsilon}_t)$ and $\bar{D}^{\varepsilon}_t = |\bar{X}^{\varepsilon}_t - \bar{Y}^{\varepsilon}_t|$ for $t \in [\bar{T}^{\varepsilon}_{2j+1}, \bar{T}^{\varepsilon}_{2j+2}]$ as above but with the roles of $\bar{X}$ and $\bar{Y}$ reversed. This means that on $\{\bar{T}^{\varepsilon}_{2j+1} < V^{\varepsilon}_k\}$,

$$\begin{aligned} P((\bar{Y}_{2j+1}, \bar{X}_{2j+1}, \bar{T}^{\varepsilon}_{2j+2} - \bar{T}^{\varepsilon}_{2j+1}) \in \cdot | \mathcal{F}_{\bar{T}^{\varepsilon}_{2j+1}} \vee \sigma(X^{\varepsilon}, Y^{\varepsilon}, W)) \\ = Q_{\langle Y^{\varepsilon}_{(T^{\varepsilon}_{2j+1}+\cdot)\wedge V^{\varepsilon}_k}, 1\rangle, \langle X^{\varepsilon}_{(T^{\varepsilon}_{2j+1}+\cdot)\wedge V^{\varepsilon}_k}, 1\rangle}(\cdot), \end{aligned} \tag{3.30}$$

and on $\{\infty > \bar{T}^{\varepsilon}_{2j+1} \geq V^{\varepsilon}_k\}$, the above conditional probability is again Q_0. The apparent lack of symmetry in the definitions arises because we have also reversed the roles of X^{ε} and Y^{ε} on $[\bar{T}^{\varepsilon}_{2j+1}, \bar{T}^{\varepsilon}_{2j+2}]$. The above definition implies that $\bar{D}^\varepsilon_t = \bar{Y}^\varepsilon_t - \bar{X}^\varepsilon_t \geq 0$ on $[\bar{T}^\varepsilon_{2j+1}, \bar{T}^\varepsilon_{2j+2}]$, $\bar{T}^\varepsilon_{2j+2} = \inf\{t > \bar{T}^\varepsilon_{2j+1} : \bar{Y}^\varepsilon_t = 0\}$, and $\bar{X}^\varepsilon(\bar{T}^\varepsilon_{2j+2}) = \bar{Y}^\varepsilon(\bar{T}^\varepsilon_{2j+2}) = 0$ on $\{\bar{T}^\varepsilon_{2j+2} < \infty\}$.

It follows from (3.18), (3.20) (now with $t \in [\bar{T}^\varepsilon_{2j}, \bar{T}^\varepsilon_{2j+1}]$) and (3.25) that on $\{V^\varepsilon_k > \bar{T}^\varepsilon_{2j+1}\}$ we have $\bar{T}^\varepsilon_{2j+1} = T^\varepsilon_{2j+1}$. Symmetric reasoning shows that on $\{V^\varepsilon_k > \bar{T}^\varepsilon_{2j+2}\}$, $\bar{T}^\varepsilon_{2j+2} = T^\varepsilon_{2j+2}$. We have verified (3.24) and (3.25) for $j+1$. Since $\bar{T}^\varepsilon_{j+1} - \bar{T}^\varepsilon_j \geq \varepsilon$ (by (3.18),(3.20) and (3.29)), $\bar{T}^\varepsilon_j \uparrow \infty$ and our inductive definition is complete.

The reasoning above to show $\bar{T}^\varepsilon_{2j+1} = T^\varepsilon_{2j+1}$ on $\{V^\varepsilon_k > \bar{T}^\varepsilon_{2j+1}\}$ and the obvious induction also shows that

$$\begin{aligned}&\bar{X}^\varepsilon_{t\wedge V^\varepsilon_k} = \langle X^\varepsilon_{t\wedge V^\varepsilon_k}, 1\rangle,\ \bar{Y}^\varepsilon_{t\wedge V^\varepsilon_k} = \langle Y^\varepsilon_{t\wedge V^\varepsilon_k}, 1\rangle,\\ &\bar{D}^\varepsilon_{t\wedge V^\varepsilon_k} = |\langle X^\varepsilon_{t\wedge V^\varepsilon_k}, 1\rangle - \langle Y^\varepsilon_{t\wedge V^\varepsilon_k}, 1\rangle| \quad \forall t \geq 0 \text{ a.s.}\end{aligned} \tag{3.31}$$

The following consequence of the above construction will be important for us:

$$\begin{aligned}&P\big((\bar{X}_{2j}, \bar{Y}_{2j}, \bar{T}^\varepsilon_{2j+1} - \bar{T}^\varepsilon_{2j}) \in \cdot \big| \mathcal{F}_{\bar{T}^\varepsilon_{2j}}\big)\\ &\qquad\qquad = Q_1(\cdot) \text{ a.s. on } \{\bar{T}^\varepsilon_{2j} < V^\varepsilon_k\},\\ &P\big((\bar{Y}_{2j+1}, \bar{X}_{2j+1}, \bar{T}^\varepsilon_{2j+2} - \bar{T}_{2j+1}) \in \cdot \big| \mathcal{F}_{\bar{T}^\varepsilon_{2j+1}}\big)\\ &\qquad\qquad = Q_1(\cdot) \text{ a.s. on } \{\bar{T}^\varepsilon_{2j+1} < V^\varepsilon_k\},\end{aligned} \tag{3.32}$$

and

$$\begin{aligned}&P\big((\bar{X}_{2j}, \bar{Y}_{2j}, \bar{T}^\varepsilon_{2j+1} - \bar{T}^\varepsilon_{2j}) \in \cdot \big| \mathcal{F}_{\bar{T}^\varepsilon_{2j}}\big)\\ &\qquad\qquad = Q_0(\cdot) \text{ a.s. on } \{V^\varepsilon_k \leq \bar{T}^\varepsilon_{2j}\},\\ &P\big((\bar{Y}_{2j+1}, \bar{X}_{2j+1}, \bar{T}^\varepsilon_{2j+2} - \bar{T}^\varepsilon_{2j+1}) \in \cdot \big| \mathcal{F}_{\bar{T}^\varepsilon_{2j+1}}\big)\\ &\qquad\qquad = Q_0(\cdot) \text{ a.s. on } \{V^\varepsilon_k \leq \bar{T}^\varepsilon_{2j+1}\}.\end{aligned} \tag{3.33}$$

Consider, for example, the first equality in (3.32). By (3.27) we have for a Borel set B and $A \in \mathcal{F}_{\bar{T}^\varepsilon_{2j}}$, $A \subset \{V^\varepsilon_k > \bar{T}^\varepsilon_{2j}\}$,

$$\begin{aligned}&P\big(\{(\bar{X}_{2j}, \bar{Y}_{2j}, \bar{T}^\varepsilon_{2j+1} - \bar{T}^\varepsilon_{2j}) \in B\} \cap A\big)\\ &\quad = E\Big(Q_{\langle X^\varepsilon_{(T^\varepsilon_{2j}+\cdot)\wedge V^\varepsilon_k}, 1\rangle, \langle Y^\varepsilon_{(T^\varepsilon_{2j}+\cdot)\wedge V^\varepsilon_k}, 1\rangle}(B) 1_A\Big)\\ &\quad = E\Big(E\Big(Q_{\langle X^\varepsilon_{(T^\varepsilon_{2j}+\cdot)\wedge V^\varepsilon_k}, 1\rangle, \langle Y^\varepsilon_{(T^\varepsilon_{2j}+\cdot)\wedge V^\varepsilon_k}, 1\rangle}(B) \Big| \mathcal{F}_{T^\varepsilon_{2j}\wedge V^\varepsilon_k}\Big) 1_A\Big).\end{aligned} \tag{3.34}$$

In the last line we used the fact that $V^\varepsilon_k > \bar{T}^\varepsilon_{2j} = T^\varepsilon_{2j}$ on A to see that $A \in \mathcal{F}_{T^\varepsilon_{2j}\wedge V^\varepsilon_k}$. Formula (3.31) shows that our construction of $(\bar{X}^\varepsilon, \bar{Y}^\varepsilon)$ has not increased the information in $\mathcal{F}_{T^\varepsilon_{2j}\wedge V^\varepsilon_k}$ so we may use (3.9). Applying (3.9) and the fact that $V^\varepsilon_k = T^\varepsilon_{2j} + V^\varepsilon_k \circ \theta_{T^\varepsilon_{2j}}$ on $\{V^\varepsilon_k > T^\varepsilon_{2j}\}$, where (θ_t) are the shift operators for $(X^\varepsilon, Y^\varepsilon)$, we conclude from (3.34) that the far left-hand side of

(3.34) equals

$$E\Big(Q_{\langle X^\varepsilon_{\cdot\wedge V^\varepsilon_k},1\rangle,\langle Y^\varepsilon_{\cdot\wedge V^\varepsilon_k},1\rangle}(B)\Big)1_A) = E\Big(Q_1(B)1_A\Big),$$

by (3.23). This gives the first equality in (3.32) and the second inequality holds by a symmetric argument. The proof of (3.33) is easier.

Our next goal is to show there is positive probability, independent of ε, of $\bar{D}^\varepsilon$ hitting some appropriately chosen $x_0 \in (0,1)$ before $\bar{X}^\varepsilon$ or $\bar{Y}^\varepsilon$ hits 1. By (3.32) and (3.33) the excursions of $\bar{X}^\varepsilon \vee \bar{Y}^\varepsilon$ away from 0 are governed by Q_0 or Q_1, depending on whether or not V^ε_k has occurred. Therefore we need to analyze these two laws.

Consider the more complex Q_1 first. Use (3.11), with $\phi = 1$, in (3.20) and the fact that $V^\varepsilon_k > \varepsilon$ (by (3.15)) to conclude that under Q_1, $\bar{X}^\varepsilon_t = 2b(t\wedge\varepsilon)$ for $t \le \varepsilon$ and for $0 \le t \le \bar{T}^\varepsilon_1 - \varepsilon$,

$$\begin{aligned}\bar{X}^\varepsilon_{t+\varepsilon} &= \varepsilon b + ((t+\varepsilon)\wedge V^\varepsilon_k)b + b\int_0^{t+\varepsilon} 1(s > V^\varepsilon_k)\,ds \\ &\quad + \int_\varepsilon^{t+\varepsilon}\int 1(s \le V^\varepsilon_k)X^\varepsilon(s,x)^p\,dW(s,x) \\ &\quad + \int_\varepsilon^{t+\varepsilon} 1(s > V^\varepsilon_k)(\bar{X}^\varepsilon_s)^{p'}\sqrt{K(k)}\,dB_s \\ &= 2\varepsilon b + tb + N_t, \end{aligned} \tag{3.35}$$

where N is a continuous $(\mathcal{F}_{t+\varepsilon})-$local martingale such that

$$\begin{aligned}\langle N\rangle_t &= \int_\varepsilon^{t+\varepsilon}\Big[1(s\le V^\varepsilon_k)\int X^\varepsilon(s,x)^{2p}dx + 1(s > V^\varepsilon_k)(\bar{X}^\varepsilon_s)^{2p'}K(k)\Big]\,ds \\ &\equiv \int_\varepsilon^{t+\varepsilon}\langle N\rangle'(s)\,ds.\end{aligned}$$

By the definition of V^ε_k we may apply Theorem 5 with $(\alpha,\beta,C) = (2p, 1/4, k)$ and conclude that

$$\begin{aligned}\langle N\rangle'(s) &\ge 1(s\le V^\varepsilon_k)K(k)\Big[\int X^\varepsilon(s,x)\,dx\Big]^{((p/2)+1)/(5/4)} \\ &\quad + 1(s > V^\varepsilon_k)(\bar{X}^\varepsilon_s)^{2p'}K(k) \\ &= K(k)(\bar{X}^\varepsilon_s)^{2p'},\end{aligned} \tag{3.36}$$

where (3.31) is used in the last line.

Define a random time change τ_t by

$$t = \int_0^{\tau_t}\frac{\langle N\rangle'(s+\varepsilon)}{K(k)(\bar{X}^\varepsilon_{s+\varepsilon})^{2p'}}\,ds \equiv A(\tau_t), \quad t < A(\bar{T}^\varepsilon_1 - \varepsilon). \tag{3.37}$$

The restriction on t ensures we are not dividing by zero in the above integrand because $\bar{T}_1^\varepsilon$ is the hitting time of 0 by $\bar{X}^\varepsilon$. Clearly (3.36) implies

$$\tau'(t) \le 1 \text{ for } t < A(\bar{T}_1^\varepsilon - \varepsilon). \tag{3.38}$$

For $t < A(\bar{T}_1^\varepsilon - \varepsilon)$, let

$$\hat{X}(t) = \bar{X}^\varepsilon(\tau_t + \varepsilon) = 2b\varepsilon + b\tau(t) + \hat{N}(t), \tag{3.39}$$

where $\hat{N}_t = N(\tau_t)$ is continuous $(\mathcal{F}_{\tau_t+\varepsilon})$-local martingale such that

$$\langle \hat{N}\rangle_t = \int_\varepsilon^{\tau_t+\varepsilon} \langle N\rangle'(s)\,ds = \int_0^{\tau_t} \langle N\rangle'_{s+\varepsilon}\,ds = K(k)\int_0^t \hat{X}(r)^{2p'}\,dr.$$

This follows by using the substitution $s = \tau_r$ and calculating the differential $d\tau(r)$ from (3.37). Note also that if $\hat{T}_x = \inf\{t \ge 0 : \hat{X}(t) = x\}$, then $A(\bar{T}_1^\varepsilon - \varepsilon) = \hat{T}_0$. Therefore by (3.39) we may assume there is a Brownian motion $\hat{B}$ so that

$$\hat{X}(t\wedge \hat{T}_0) = 2b\varepsilon + b\tau(t\wedge\hat{T}_0) + \int_0^{t\wedge\hat{T}_0} \sqrt{K(k)}\hat{X}(s)^{p'}\,d\hat{B}(s). \tag{3.40}$$

The scale function for a diffusion defined by a similar formula, but with $t\wedge\hat{T}_0$ in place of $\tau(t\wedge\hat{T}_0)$, is

$$s_k(x) = \int_0^x \exp\Big\{-\frac{2by^{1-2p'}}{K(k)(1-2p')}\Big\}\,dy.$$

That is, s_k satisfies

$$\frac{K(k)x^{2p'}}{2}s_k''(x) + bs_k'(x) = 0 \text{ on } [0,\infty),\ \ s_k(0) = 0. \tag{3.41}$$

By Itô's Lemma

$$\begin{aligned} s_k(\hat{X}(t\wedge\hat{T}_0)) = s_k(2b\varepsilon) &+ \int_0^{t\wedge\hat{T}_0} b\tau'(u)s_k'(\hat{X}(u)) + \frac{K(k)\hat{X}(u)^{2p'}}{2}s_k''(\hat{X}(u))\,du \\ &+ \int_0^{t\wedge\hat{T}_0} s_k'(\hat{X}(u))d\hat{N}(u). \end{aligned}$$

(3.38) and (3.41) show that the integrand in the drift term above is non-positive, and so $s_k(\hat{X}(t\wedge\hat{T}_0\wedge\hat{T}_1))$ is a supermartingale which therefore satisfies

$$E\big(s_k(\hat{X}(t\wedge\hat{T}_0\wedge\hat{T}_1)\big) \le s_k(2b\varepsilon).$$

This implies that

$$\begin{aligned} Q_1(\bar{X}_t^\varepsilon = 1 \text{ for some } t < \bar{T}_1^\varepsilon) &= Q_1(s_k(\hat{X}(\cdot\wedge\hat{T}_1\wedge\hat{T}_0)) \text{ hits } s_k(1) \text{ before } 0) \\ &= \lim_{t\to\infty} Q_1\Big(s_k(\hat{X}(t\wedge\hat{T}_0\wedge\hat{T}_1))/s_k(1)\Big) \\ &\le s_k(2b\varepsilon)/s_k(1). \end{aligned} \tag{3.42}$$

Under Q_0 add the equations in (3.16) to see that (we write $\bar{X}^\varepsilon$ for $\tilde{X}^\varepsilon$),

$$\bar{X}^\varepsilon_{t+\varepsilon} = 2b\varepsilon + bt + \int_\varepsilon^{t+\varepsilon} (\bar{X}^\varepsilon_s)^{p'} \sqrt{K(k)}\, dB_s,\ t+\varepsilon \le \bar{T}^\varepsilon_1 = \inf\{t : \bar{X}^\varepsilon_t = 0\}.$$

This is equation (3.40) with t in place of τ_t and so the previous calculation applies to again give us (3.42) with Q_0 in place of Q_1.

Under either Q_i, $\bar{X}^\varepsilon_t = \bar{X}^\varepsilon_t \vee \bar{Y}^\varepsilon_t$ and so we conclude

$$Q_i(\bar{X}^\varepsilon_t \vee \bar{Y}^\varepsilon_t \text{ hits } 1 \text{ for } t < \bar{T}^\varepsilon_1) \le s_k(2b\varepsilon)/s_k(1),\ i = 1, 2. \tag{3.43}$$

We next consider the escape probability for $\bar{D}^\varepsilon$ under Q_1. Let $x_0 \in (2b\varepsilon, 1)$ and

$$T_{\bar{D}}(0, x_0) = \inf\{t : \bar{D}^\varepsilon_t = 0 \text{ or } x_0\} \le \bar{T}^\varepsilon_1 \quad Q_1 - a.s.,$$

the last since $\bar{D}^\varepsilon_t = \bar{X}^\varepsilon_t - \bar{Y}^\varepsilon_t \le \bar{X}^\varepsilon_t$ for $t \le \bar{T}^\varepsilon_1$ under Q_1. It follows from (3.2), (3.4) and (3.19) that $\bar{D}^\varepsilon_t = 2b(t \wedge \varepsilon)$ for $t \le \varepsilon$, and for $t + \varepsilon \le \bar{T}^\varepsilon_1$ we have,

$$\begin{aligned}\bar{D}^\varepsilon_{t+\varepsilon} = 2b\varepsilon &+ \int_\varepsilon^{t+\varepsilon} \int (X^\varepsilon(s,x)^p - Y^\varepsilon(s,x)^p) 1(s \le V^\varepsilon_k)\, dW(s,x) \\ &+ \int_\varepsilon^{t+\varepsilon} 1(s > V^\varepsilon_k)((\bar{X}^\varepsilon_s)^{p'} - (\bar{Y}^\varepsilon_s)^{p'})\sqrt{K(k)}\, dB_s,\end{aligned} \tag{3.44}$$

which is a non-negative local martingale in t. We have

$$\begin{aligned}&Q_1(\exists t < \bar{T}^\varepsilon_1 : \bar{D}^\varepsilon_t \ge x_0) \\ &\quad \ge E\Big(\bar{D}^\varepsilon(T_{\bar{D}}(0,x_0)) x_0^{-1} 1(T_{\bar{D}}(0,x_0) < \infty, \bar{T}^\varepsilon_1 < \infty)\Big) \\ &\quad = E\Big(\bar{D}^\varepsilon(T_{\bar{D}}(0,x_0) \wedge \bar{T}^\varepsilon_1) x_0^{-1}\Big) - E\Big(\bar{D}^\varepsilon(T_{\bar{D}}(0,x_0)) x_0^{-1} 1(\bar{T}^\varepsilon_1 = \infty)\Big).\end{aligned} \tag{3.45}$$

The first term on the right-hand side is the terminal element of a bounded martingale and so

$$E\Big(\bar{D}^\varepsilon(T_{\bar{D}}(0,x_0) \wedge \bar{T}^\varepsilon_1) x_0^{-1}\Big) = 2b\varepsilon/x_0. \tag{3.46}$$

It follows from (3.35) that on $\{\bar{T}^\varepsilon_1 = \infty\}$,

$$\{\limsup_{t\to\infty} \bar{X}^\varepsilon_t < \infty\} \subset \{\lim_{t\to\infty} N_t = -\infty\}, \tag{3.47}$$

which is a Q_1-null set by the Dubins-Schwarz theorem which asserts that a continuous martingale is a time-changed Brownian motion. Therefore

$$\bar{X}^\varepsilon_{t+\varepsilon} \text{ hits 0 or 1 for some } t \le \bar{T}^\varepsilon_1 - \varepsilon, t < \infty,\ Q_1 - a.s., \tag{3.48}$$

and therefore

$$\begin{aligned}E\Big(\bar{D}^\varepsilon(T_{\bar{D}}(0,x_0)) x_0^{-1} 1(\bar{T}^\varepsilon_1 = \infty)\Big) &\le Q_1(\bar{T}^\varepsilon_1 = \infty) && (3.49)\\ &\le Q_1(\bar{X}^\varepsilon_t \text{ hits 1 for } t < \bar{T}^\varepsilon_1) \\ &\le s_k(2b\varepsilon)/s_k(1), && (3.50)\end{aligned}$$

the last by (3.42).

Since $\lim_{x\to 0+} s_k(x)/x = 1$, there is an $\varepsilon_0(k) > 0$ and $x_0 = x_0(k) \in (0,1)$, such that

$$\varepsilon \le \varepsilon_0 \text{ implies } s_k(2b\varepsilon) < 3b\varepsilon \text{ and } 2b\varepsilon < x_0 \le s_k(1)/6. \tag{3.51}$$

So for $\varepsilon \le \varepsilon_0$ and x_0 as above we may use (3.46) and (3.49) in (3.45) and conclude

$$Q_1(\exists t \in [\varepsilon, \bar{T}_1^\varepsilon) : \bar{D}_t^\varepsilon \ge x_0) \ge (2b\varepsilon/x_0) - (s_k(2b\varepsilon)/s_k(1)) > (b\varepsilon)/x_0.$$

Virtually the same proof (it is actually simpler) works for Q_0. Under Q_i, $\bar{D}_t^\varepsilon = |\bar{X}_t^\varepsilon - \bar{Y}_t^\varepsilon|$ for $t \le \bar{T}_1^\varepsilon = \inf\{t : \bar{X}_t^\varepsilon \vee \bar{Y}_t^\varepsilon = 1\}$ and so we have proved for x_0 as above,

$$Q_i(\exists t \in [\varepsilon, \bar{T}_1^\varepsilon] : |\bar{X}_t^\varepsilon - \bar{Y}_t^\varepsilon| \ge x_0) \ge \frac{b\varepsilon}{x_0} \text{ for } i = 1,2 \text{ and } 0 < \varepsilon \le \varepsilon_0, \tag{3.52}$$

and (see (3.48) for $i = 1$)

$$\bar{X}_t^\varepsilon \vee \bar{Y}_t^\varepsilon \text{ hits 0 or 1 for } t \le \bar{T}_1^\varepsilon,\ t \text{ finite } Q_i - a.s.,\ i = 1,2. \tag{3.53}$$

Let

$$\mathcal{N}_1 = \min\{j : (\bar{X}^\varepsilon \vee \bar{Y}^\varepsilon)(t + \bar{T}_j^\varepsilon) \text{ hits 1 for } t < \bar{T}_{j+1}^\varepsilon - \bar{T}_j^\varepsilon\},$$

and

$$\mathcal{N}_2 = \min\{j : |\bar{X}^\varepsilon - \bar{Y}^\varepsilon|(t + \bar{T}_j^\varepsilon) \text{ hits } x_0 \text{ for } t < \bar{T}_{j+1}^\varepsilon - \bar{T}_j^\varepsilon\}.$$

Use (3.32), (3.33) and (3.43) to see that

$$\begin{aligned}
&P(\mathcal{N}_1 > n) \\
&\quad = E\Big(1(\mathcal{N}_1 > n-1) P\big(\bar{X}^\varepsilon \vee \bar{Y}^\varepsilon((\bar{T}_{n-1}^\varepsilon + \cdot) \wedge \bar{T}_n^\varepsilon) \text{ doesn't hit } 1 \big| \mathcal{F}_{\bar{T}_{n-1}^\varepsilon}\big)\Big) \\
&\quad \ge P(\mathcal{N}_1 > n-1)\Big(1 - \frac{s_k(2b\varepsilon)}{s_k(1)}\Big).
\end{aligned}$$

Therefore, if $p_1 = \frac{s_k(2b\varepsilon)}{s_k(1)}$, then

$$P(\mathcal{N}_1 > n) \ge (1 - p_1)^{n+1}. \tag{3.54}$$

Similar reasoning using (3.52) in place of (3.43) shows that if $p_2 = \frac{b\varepsilon}{x_0}$, then for $\varepsilon \le \varepsilon_0$,

$$P(\mathcal{N}_2 > n) \le (1 - p_2)^{n+1}. \tag{3.55}$$

Note that (3.51) shows that

$$\frac{p_2}{p_1} = \frac{b\varepsilon}{s_k(2b\varepsilon)} \frac{s_k(1)}{x_0} \ge \frac{1}{3} \frac{s_k(1)}{x_0} \ge 2. \tag{3.56}$$

If $n = \lceil p_1^{-1} \rceil$ we get for $\varepsilon \le \varepsilon_0$

$$\begin{aligned} P(\mathcal{N}_2 < \mathcal{N}_1) \ge P(\mathcal{N}_1 > n) - P(\mathcal{N}_2 > n) &\ge (1-p_1)^{n+1} - (1-p_2)^{n+1} \\ &\ge (1-p_1)^{n+1} - (1-2p_1)^{n+1} \\ &\ge \frac{1}{2}(e^{-1} - e^{-2}), \end{aligned}$$

where the last inequality holds by decreasing $\varepsilon_0(k)$, if necessary. If

$$\bar{\mathbf{t}}_\varepsilon = \inf\{t : \bar{X}_t^\varepsilon \vee \bar{Y}_t^\varepsilon \ge 1\},$$

then the above bound implies that for $\varepsilon \le \varepsilon_0$,

$$P(\sup_{t \le \bar{\mathbf{t}}_\varepsilon} |\bar{X}_t^\varepsilon - \bar{Y}_t^\varepsilon| \ge x_0) \ge \frac{1}{2}(e^{-1} - e^{-2}). \tag{3.57}$$

Now let

$$\mathbf{t}_\varepsilon = \inf\{t : \langle X_t^\varepsilon, 1\rangle \vee \langle Y_t^\varepsilon, 1\rangle \ge 1\}.$$

Then (3.31) shows that

if $\mathbf{t}_\varepsilon < V_k^\varepsilon$, then $\bar{\mathbf{t}}_\varepsilon = \mathbf{t}_\varepsilon$ and $(\bar{X}_t^\varepsilon, \bar{Y}_t^\varepsilon) = (\langle X_t^\varepsilon, 1\rangle, \langle Y_t^\varepsilon, 1\rangle)$ for all $t \le \mathbf{t}_\varepsilon$,

and so by (3.57) for $\varepsilon \le \varepsilon_0$,

$$P(\sup_{t \le \mathbf{t}_\varepsilon} |\langle X_t^\varepsilon, 1\rangle - \langle Y_t^\varepsilon, 1\rangle| \ge x_0) \ge \frac{1}{2}(e^{-1} - e^{-2}) - P(V_k^\varepsilon \le \mathbf{t}_\varepsilon). \tag{3.58}$$

Now recall we have $\varepsilon_n \downarrow 0$ so that $(X^{\varepsilon_n}, Y^{\varepsilon_n}, W) \to (X, Y, W)$ weakly on $C(\mathbb{R}_+, (C_{rap}^+)^2 \times C_{tem})$, where X and Y are C_{rap}^+-valued solutions of (3.1). Arguing as in (3.47) and using Dubins-Schwarz, we see that

$$\limsup_{t \to \infty} \langle X_t, 1\rangle = \limsup_{t \to \infty} \langle Y_t, 1\rangle = \infty \quad a.s. \tag{3.59}$$

Standard weak convergence arguments now show that $\{\mathbf{t}_{\varepsilon_n}\}$ are stochastically bounded. Lemma 6 therefore shows that we may choose a fixed k sufficiently large so that

$$P(V_k^{\varepsilon_n} \le \mathbf{t}_{\varepsilon_n}) \le \frac{1}{4}(e^{-1} - e^{-2}) \text{ for all } n.$$

Using this fixed k throughout we see from (3.58) that for large enough n

$$P\Big(\sup_{t \le \mathbf{t}_{\varepsilon_n}} \big|\langle X_t^{\varepsilon_n}, 1\rangle - \langle Y_t^{\varepsilon_n}, 1\rangle\big| \ge x_0\Big) \ge \frac{1}{4}(e^{-1} - e^{-2}).$$

If $\mathbf{t}' = \inf\{t : \langle X_t, 1\rangle \vee \langle Y_t, 1\rangle \ge 2\} < \infty$ a.s., by (3.59), then the above implies

$$P\Big(\sup_{t \le \mathbf{t}'} \big|\langle X_t, 1\rangle - \langle Y_t, 1\rangle\big| \ge x_0/2\Big) \ge \frac{1}{4}(e^{-1} - e^{-2}),$$

and so $P(X \ne Y) \ge \frac{1}{4}(e^{-1} - e^{-2})$. □

Lemma 6. *For any $M \in \mathbb{N}$, $\lim_{k \to \infty} \sup_{0 < \varepsilon \le 1} P(V_k^\varepsilon \le M) = 0$.*

Proof. The proof depends on a standard argument in the spirit of Kolmogorov's continuity lemma, so we will omit some details.

Fix the time interval $[0, M]$. Define

$$\begin{aligned} V_k^\varepsilon(X) = \inf\{s \geq 0 : \exists x, x' \in \mathbb{R} \text{ such that} \\ |X^\varepsilon(s,x) - X^\varepsilon(s,x')| > k|x-x'|^{1/4}\}. \end{aligned}$$

and $V_k^\varepsilon(Y)$ likewise. It suffices to prove Lemma 6 for V_k^ε replaced by $V_k^\varepsilon(X)$ and $V_k^\varepsilon(Y)$ and so clearly we only need consider $V_k^\varepsilon(X)$. Recall that $\{X^{\varepsilon_n}\}$ is tight in $C(\mathbb{R}_+, C_{rap}^+)$. So it suffices to choose a constant $K > 0$ and prove the lemma for $X^\varepsilon(t,x) \wedge (Ke^{-|x|})^{1/p}$ in place of X^ε. Considering the integral equation for X^ε, and using the fact that $\psi \in C_c^1(\mathbb{R})$ we see that it is enough to prove Lemma 6 with X^ε replaced by the stochastic convolution

$$N^\varepsilon(t,x) = \int_0^t p_{t-s}(x-y)\varphi^\varepsilon(s,y)dW(s,y).$$

Here one can use Lemma 6.2 of [Shi94] to handle the drift terms. The term $\varphi^\varepsilon(s,y)$ is a predictable random field satisfying

$$|\varphi^\varepsilon(t,x)| \leq Ke^{-|x|}$$

for all $t \in [0, M]$, $x \in \mathbb{R}$ almost surely. Since our estimates are uniform in ε, we will omit the superscript on φ and N from now on. The constants below may depend on M and K.

Now we rely on some standard estimates which are easy to verify. We claim that there exist constants q_0, K_0 such that for $0 \leq t \leq t+\delta \leq M$ and $x \in \mathbf{R}$, and for $\delta < 1$,

$$\begin{aligned} \int_0^\delta \int_{\mathbf{R}} p_s^2(x-y)e^{-2p|y|}dyds &\leq K_0\delta^{\frac{1}{2}}e^{-q_0|x|}, \qquad (3.60) \\ \int_0^t \int_{\mathbf{R}} [p_{t-s+\delta}(x-y) - p_{t-s}(x-y)]^2 e^{-2p|y|}dyds &\leq K_0\delta^{\frac{1}{2}}e^{-q_0|x|}, \\ \int_0^t \int_{\mathbf{R}} [p_{t-s}(x-y+\delta) - p_{t-s}(x-y)]^2 e^{-2|y|}dyds &\leq K_0\delta e^{-q_0|x|}. \end{aligned}$$

From these inequalities, it follows in a standard way that for some positive constants q_1, C_0, C_1, we have

$$\begin{aligned} P\left(|N(t+\delta,x) - N(t,x)| \geq \lambda\right) &\leq C_0 \exp\left(-\frac{C_1\lambda^2}{\delta^{\frac{1}{2}}}\right)e^{-q_1|x|}, \qquad (3.61) \\ P\left(|N(t,x) - N(t,x+\delta)| \geq \lambda\right) &\leq C_0 \exp\left(-\frac{C_1\lambda^2}{\delta}\right)e^{-q_1|x|}. \end{aligned}$$

For example, if we write

$$\hat{M}_r = \int_0^r \int_{\mathbf{R}} [p_{t-s}(x-y+\delta) - p_{t-s}(x-y)]\varphi(s,y)W(dy,ds)$$

then $\hat{M}_r$ is a continuous martingale and hence a time changed Brownian motion, with time scale

$$\begin{aligned} E(r) &= \int_0^r \int_{\mathbf{R}} [p_{t-s}(x-y+\delta) - p_{t-s}(x-y)]^2 \varphi^2(s,y) dy ds \\ &\leq \int_0^r \int_{\mathbf{R}} [p_{t-s}(x-y+\delta) - p_{t-s}(x-y)]^2 K^2 e^{-2|y|} dy ds. \end{aligned}$$

Thus,

$$\begin{aligned} P\left(|N(t,x) - N(t,x+\delta)| \geq \lambda\right) &= P(|\hat{M}_t| \geq \lambda) \\ &\leq P\left(\sup_{0 \leq s \leq E(t)} |B_s| \geq \lambda \right) \end{aligned}$$

and then the reflection principle for Brownian motion and the third inequality in (3.60) (to bound $E(t)$ for $t \leq M$) gives the second inequality in (3.61).

Now we outline a standard chaining argument, and for simplicity assume that $M = 1$. Let $\mathcal{G}_n$ be the grid of points

$$\mathcal{G}_n = \left\{ \left(\frac{k}{2^{2n}}, \frac{\ell}{2^n} \right) : 0 \leq k \leq 2^{2n}, \ell \in \mathbf{Z} \right\}.$$

The Borel-Cantelli lemma along with (3.61) now implies that for large enough (random) K_1, if $n \geq K_1$ and p_1, p_2 are neighboring grid points in $\mathcal{G}_n$, then

$$|N(p_1) - N(p_2)| \leq 2^{-\frac{n}{4}}. \tag{3.62}$$

Now suppose that $q_i = (t_i, x_i)$ with $|x_1 - x_2| \leq 1$, and that each point q_i lies in some grid $\mathcal{G}_n$. From the above, there is a path from q_1 to q_2 utilizing edges in grids $\mathcal{G}_{n'}$, with $n' \leq n$, each edge in the path being a nearest neighbor edge in $\mathcal{G}_{n'}$, and with at most 8 edges from a given grid index n'. Let n_0 be the least grid index used in this path. We claim that for some constants $C > c > 0$, such a path exists with n_0 satisfying

$$\begin{aligned} c2^{-2n_0} &< |t_1 - t_2| < C2^{-2n_0}, \\ c2^{-n_0} &< |x_1 - x_2| < C2^{-n_0}. \end{aligned}$$

Using the triangle inequality to sum differences of $N(t,x)$ over edges of the path, we arrive at a geometric series, and conclude that

$$|N(q_1) - N(q_2)| \leq C_1 2^{-\frac{n_0}{4}} \text{ if } n_0 \geq K_1. \tag{3.63}$$

Although we have only proved the above for grid points, such points are dense in $[0,T] \times \mathbf{R}$, and $N(t,x)$ has a continuous version because $X(t,x)$ is continuous, and the drift contribution is smooth. Therefore it follows for all points in $[0,1] \times \mathbb{R}$. We have proved (3.63) for $\|q_1 - q_2\| \leq C2^{-K_1}$ where K_1 is stochastically bounded uniformly in ε. The required result follows. $\square$

Sketch of Proof of Theorem 4. We carry out an excursion construction of an approximate solution X^ε to (1.2) by starting the ith excursion at $(-1)^i\varepsilon\psi$, and then run each independent excursion according to a fixed law of a C^+_{rap}-valued solution to (1.2) with $X_0 = \varepsilon\psi$, if i is even, and its negative if i is odd, until the total mass hits 0. At this point a new excursion is started in the same manner. Theorem 5 is used to time change $X^\varepsilon_t(1)$ into an approximate solution $Y^\varepsilon(t) = X^\varepsilon_{\tau^\varepsilon_t}(1)$ of Girsanov's equation

$$dY_t = |Y_t|^{p'} dB_t, \tag{3.64}$$

with $p < p' < 1/2$ and $\frac{d\tau^\varepsilon(t)}{dt} \le 1$. There will be an additional term $A^\varepsilon(t)$ arising from all the excursion signed initial values up to time t but it will converge to 0 uniformly in t due to the alternating nature of the sum. We now proceed as in the excursion-based construction of non-zero solutions to Girsanov's sde (3.64) to show that one of the excursions of the approximate solutions will hit ±1 before time T with probability close to 1 as T gets large, uniformly in ε. Let N^ε be the number of excursions of Y^ε until one hits ±1 and let $N_\varepsilon(T)$ be the number of excursions of Y^ε completed by time T. N^ε is geometric with mean ε^{-1} by optional stopping. Let $U_i(\varepsilon)$ be the time to completion of the ith excursion of Y^ε. Assuming $\sqrt{T}\varepsilon^{-1} \in \mathbb{N}$, we have

$$\begin{aligned} P(\sup_{s\le T}|Y^\varepsilon_s| \ge 1) &\ge P(N_\varepsilon(T) \ge N_\varepsilon) \\ &\ge P(N_\varepsilon(T) \ge \sqrt{T}\varepsilon^{-1}) - P(N_\varepsilon > \sqrt{T}\varepsilon^{-1}) \\ &\ge P(U_{\sqrt{T}\varepsilon^{-1}}(\varepsilon) \le T) - (1-\varepsilon)^{\sqrt{T}\varepsilon^{-1}}. \end{aligned} \tag{3.65}$$

A key step now is to use diffusion theory to show that if Y satisfies (3.64) (pathwise unique until it hits zero) then

$$P(Y_t > 0 \text{ for all } t \le T | Y_0 = 1)) \sim cT^{-1/(2(1-p))} \text{ as } T \to \infty. \tag{3.66}$$

If $U_i(1)$ is the time of completion of the ith excursion of Y where the excursions now start at ±1, then scaling shows that

$$P(U_{\sqrt{T}\varepsilon^{-1}}(\varepsilon) \le T) = P(U_{\sqrt{T}\varepsilon^{-1}}(1) \le \varepsilon^{-(2-2p)}T).$$

(3.66) shows that $U_{\sqrt{T}\varepsilon^{-1}}(1)/(\sqrt{T}\varepsilon^{-1})^{2(1-p)}$ converges weakly as $\varepsilon \downarrow 0$ to a stable subordinator of index $\alpha = (2(1-p))^{-1}$ and so for any $\eta > 0$ we may choose T large enough so that for small enough ε (by (3.65)) we have

$$\begin{aligned} P(\sup_{s\le T}|Y^\varepsilon_s| \ge 1) &\ge P(U_{\sqrt{T}\varepsilon^{-1}}(\varepsilon) \le T) - (1-\varepsilon)^{\sqrt{T}\varepsilon^{-1}} \\ &\ge P\left(\frac{U_{\sqrt{T}\varepsilon^{-1}}(1)}{(\sqrt{T}\varepsilon^{-1})^{2(1-p)}} \le T^p\right) - e^{-\sqrt{T}} \ge 1-\eta. \end{aligned}$$

The fact that $(\tau^\varepsilon)'(t) \le 1$ allows us to conclude that with probability at least $1-\eta$, uniformly in ε, the total mass of our approximate solution X^ε_t will hit

± 1 for some $t \le T$. By taking a weak limit point of the X^ε we obtain the required non-zero solution to (1.2). □

4. Pathwise Non-uniqueness and Uniqueness in Law for an SDE.

The stochastic differential equation corresponding to (3.1) would be

$$X_t = X_0 + bt + \int_0^t (X_s)^p \, dB_s, \quad X_t \ge 0 \ \forall t \ge 0 \ \ a.s. \tag{4.1}$$

Here $b > 0$, $0 < p < 1/2$, B is a standard $(\mathcal{F}_t)$-Brownian motion on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ and X_0 is $\mathcal{F}_0$-measurable. A much simpler argument than that used to prove pathwise non-uniqueness in (3.1) allows one to establish pathwise non-uniqueness in (4.1). One only needs to apply the idea behind construction of $(\bar{X}_t^\varepsilon, \bar{Y}_t^\varepsilon)$ for $t \ge V_k^\varepsilon$. In any case the result is undoubtedly known, given the well-known Girsanov examples (see, e.g., Section V.26 in [RW]) and so we omit the proof.

Theorem 7. *There is a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ carrying a standard $\mathcal{F}_t$-Brownian motion and two solutions, X^1 and X^2, to (4.1) with $X_0^1 = X_0^2 = X_0 = 0$ such that $P(X^1 \ne X^2) > 0$.*

Weak existence of solutions to (4.1) for a given initial law may be constructed through approximation by Lipschitz coefficients. This is in fact how Shiga [Shi94] constructed solutions to (1.1) and hence is the method used in Theorem 1. As we were not able to verify whether or not uniqueness in law holds in (3.1) it is perhaps interesting that it does hold in (4.1). That is, the law of X is uniquely determined by the law of X_0. We have not been able to find this result in the literature and since the solutions to (4.1) turn out to have a particular sticky boundary condition at 0 which was not immediately obvious to us, we include the elementary proof here.

Theorem 8. *Any solution to (4.1) is the diffusion on $[0, \infty)$ with scale function*

$$s(x) = \int_0^x \exp\Big\{\frac{-2b|y|^{1-2p}}{1-2p}\Big\}\, dy \tag{4.2}$$

(with inverse function s^{-1} on $[0, s(\infty))$), speed measure

$$m(dx) = \frac{dx}{s'(s^{-1}(x))^2 s^{-1}(x)^{2p}} + b^{-1}\delta_0(dx) \ \text{ on } [0, s(\infty)), \tag{4.3}$$

and starting with the law of X_0. In particular if $T_0 = \inf\{t : X_t = 0\}$, then

$$T_0 < \infty \text{ implies } \int_{T_0}^{T_0+\varepsilon} 1(X_s = 0)\, ds > 0 \ \ \forall \varepsilon > 0 \ a.s., \tag{4.4}$$

and solutions to (4.1) are unique in law.

Proof. The last statement is immediate from the first assertion.

To prove X is the diffusion described above, by conditioning on X_0 we may assume $X_0 = x_0$ is constant. We will show directly that X is the appropriate scale and time change of a reflecting Brownian motion. Note that s is strictly increasing on $[0, \infty)$ (in fact on the entire real line) so that s^{-1} is well-defined. Note also that

$$s'(x) = \exp\Big\{\frac{-2b|x|^{1-2p}}{1-2p}\Big\} \text{ is of bounded variation and continuous,}$$

and

$$s''(x) = \begin{cases} -s'(x)2bx^{-2p} & \text{if } x > 0, \\ s'(x)2b|x|^{-2p} & \text{if } x < 0. \end{cases} \tag{4.5}$$

If $L_t^X(x)$ is the semimartingale local time of X, Meyer's generalized Itô formula (see Section IV.45 of [RW]) shows that

(4.6)

$$Y_t \equiv s(X_t) = Y_0 + \int_0^t s'(X_u)X_u^p\, dB_u + b\int_0^t s'(X_u)\, du + \frac{1}{2}\int L_t^X(x)ds'(x).$$

Since s' is continuous at 0,

$$\begin{aligned} \frac{1}{2}\int L_t^X(x)ds'(x) &= \frac{1}{2}\int 1(x>0)L_t^X(x)ds'(x) \\ &= \frac{1}{2}\int 1(x>0)L_t^X(x)s''(x)dx \\ &= -b\int_0^t s'(X_u)X_u^{-2p}X_u^{2p}1(X_u>0)\, du \quad \text{(by (4.5))} \\ &= -b\int_0^t s'(X_u)1(X_u>0)\, du. \end{aligned} \tag{4.7}$$

So (4.6) and $s'(0) = 1$ imply

$$Y_t = Y_0 + \int_0^t s'(X_u)X_u^p\, dB_u + b\int_0^t 1(X_u = 0)\, du. \tag{4.8}$$

Define $U = \int_0^\infty s'(X_u)^2 X_u^{2p}\, du$ and a random time change $\alpha : [0, U) \to [0, \infty)$ by

$$\int_0^{\alpha(t)} s'(X_u)^2 X_u^{2p}\, du = t. \tag{4.9}$$

Clearly α is strictly increasing and is also continuous since X cannot be 0 on any interval. If $R(t) = Y(\alpha(t))$ for $t < U$ we now show that R is a reflecting Brownian motion on $[0, \infty)$, starting at Y_0, where we extend the definition for $t \geq U$ by appending a conditionally independent reflecting Brownian motion starting at the appropriate point. In what follows we may assume $t < U$ as the values of $R(t)$ for $t \geq U$ will not be relevant. We have from (4.8)

$$R(t) = \beta_t + b \int_0^{\alpha(t)} 1(X_u = 0)\, du \equiv \beta_t + A_t,$$

where $\langle \beta \rangle_t = t$ and so β is a Brownian motion starting at Y_0. A is continuous non-decreasing and supported by $\{t : X(\alpha(t)) = 0\} = \{t : R(t) = 0\}$. By uniqueness of the Skorokhod problem (see Section V.6 in [RW]) R is a reflecting Brownian motion and $A_t = L_t^R(0)$, that is,

$$b \int_0^{\alpha(t)} 1(X_u = 0)\, du = L_t^R(0). \tag{4.10}$$

Let $\alpha^{-1} : [0, \infty) \to [0, U)$ denote the inverse function to α. Now differentiate (4.9) to see that

$$\text{if } X_u > 0, \text{ then } (\alpha^{-1})'(u) = s'(X_u)^2 X_u^{2p}. \tag{4.11}$$

We may use (4.10) and (4.11) to conclude that

$$\begin{aligned} t &= \int_0^t 1(X_u > 0)\, du + \int_0^t 1(X_u = 0)\, du \\ &= \int_0^t \frac{1(X_u > 0)}{s'(X_u)^2 X_u^{2p}}\, d(\alpha^{-1}(u)) + b^{-1} L^R_{\alpha^{-1}(t)}(0) \\ &= \int_0^{\alpha^{-1}(t)} \frac{1(R(v) > 0)}{s'(s^{-1}(R(v)))^2 s^{-1}(R(v))^{2p}}\, dv + b^{-1} L^R_{\alpha^{-1}(t)}(0), \end{aligned}$$

where we have set $u = \alpha(v)$ in the last. Therefore if m is as in (4.3), then

$$t = \int_{[0,\infty)} L^R(\alpha^{-1}(t), x) dm(x),$$

and

$$X(t) = s^{-1}(R(\alpha^{-1}(t)).$$

This identifies X as the diffusion on $[0, \infty)$ with the given scale function and speed measure. $\square$

Remarks. (1) One can also argue in the opposite direction. That is, given a diffusion X with speed measure and scale function as above and a given initial law on $[0, \infty)$, one can build a Brownian motion B, perhaps on an enlarged probability space, so that X satisfies (4.1), giving us an alternative weak existence proof.

(2) One can construct solutions as weak limits of difference equations or equivalently as standard parts of an infinitesimal difference equation. Here one cuts off the martingale part when the solution overshoots into the negative half-line and lets the positive drift with slope b bring it back to $\mathbb{R}_+$. The smaller the b the longer it takes to become positive, the more time the solution will

spend at zero and so the larger the atom of the speed measure at 0. A short calculation shows that at $p = 1/2$ the overshoot reduces to Δt (the time step in the difference equation) and so there is no time spent at 0 in the limit. (See Section V.48 of [RW] for the standard analysis.)

(3) It would appear that (4.1) is not a particular effective tool to study diffusions with drift b on the positive half-line. By just extending the equation to $[0, \infty)$ we inadvertently pick out a particular case of Feller's possible boundary behaviors at 0 among all diffusions satisfying (4.1) on $(0, \infty)$. (This is certainly not a novel observation—see the comments in Section V.48 in [RW].) Presumably things can only get worse for the stochastic pde (3.1). In the next section we scratch the surface of this issue and show that all solutions to this stochastic pde spend positive time in the (infinite-dimensional) zero state.

5. Proof of Theorem 3

Let X be a solution of (3.1) and define

$$V_k = \inf\{s : \exists x', x \text{ such that } |X(s,x) - X(s,x')| > k|x - x'|^{1/4}\}.$$

As in Lemma 6 (but as there is no ε it is a bit easier), $\lim_k V_k = \infty$ a.s. As in Section 3, we set $p' = \frac{p+2}{5}$ and $K(k) = K_5(1/4, k)$. If we define

$$R(u) = \begin{cases} \frac{\int X(u,x)^{2p}\,dx}{K(k)\langle X(u),1\rangle^{2p'}} & \text{if } \langle X(u), 1\rangle > 0 \text{ and } u \le V_k, \\ 1 & \text{otherwise,} \end{cases}$$

then by Theorem 5,

$$R(u) \ge 1 \text{ for all } u \ge 0.$$

Introduce a random time change τ given by

$$\int_0^{\tau(t)} R(u) 1(\langle X_u, 1\rangle > 0) + 1(\langle X_u, 1\rangle = 0)\,du = t. \tag{5.1}$$

Clearly τ is strictly increasing, continuous and well-defined for all $t \ge 0$. Differentiate (5.1) to see that

$$\tau'(t) R_{\tau(t)} 1(\langle X_{\tau(t)}, 1\rangle > 0) + \tau'(t) 1(\langle X_{\tau(t)}, 1\rangle = 0) = 1 \quad \text{for a.a. } t \ge 0$$

(a.a. is with respect to Lebesgue measure), and therefore

$$\tau'(t) = R^{-1}_{\tau(t)} 1(\langle X_{\tau(t)}, 1\rangle > 0) + 1(\langle X_{\tau(t)}, 1\rangle = 0) \le 1 \quad \text{for a.a. } t \ge 0. \tag{5.2}$$

Now let

$$Y(t) = \langle X(\tau(t)), 1\rangle = b\tau(t) + M(t),$$

where M is a continuous local martingale satisfying

$$\begin{aligned}\langle M\rangle_t &= \int_0^{\tau(t)}\int X(u,x)^{2p}\,dx\,du\\ &= \int_0^t\int X(\tau(r),x)^{2p}\,dx\Big[R(\tau(r))^{-1}1(Y(r)>0)+1(Y(r)=0)\Big]dr\\ &= \int_0^t K(k)Y_r^{2p'}\,dr \qquad \text{for } \tau(t)\le V_k.\end{aligned}$$

We have used (5.2) in the second line. If $T_k=\tau^{-1}(V_k)$ (a stopping time w.r.t the time-changed filtration), we may therefore assume there is a Brownian motion B so that

$$Y(t\wedge T_k)=b\tau(t\wedge T_k)+\int_0^{t\wedge T_k}\sqrt{K(k)}Y_r^{p'}\,dB_r.$$

If $b'=K(k)^{1/(2(p'-1))}b$, then $\hat Y(t)=K(k)^{1/(2(p'-1))}Y(t)$ satisfies

$$\hat Y(t\wedge T_k)=b'\tau(t\wedge T_k)+\int_0^{t\wedge T_k}\hat Y_r^{p'}\,dB_r.$$

If $s(x)=\int_0^x \exp\Big\{\frac{-2b'|y|^{1-2p'}}{1-2p'}\Big\}\,dy$, then an application of Meyer's generalized Itô's formula shows that if $Z(t)=s(\hat Y(t))$, then for $t\le T_k$,

$$Z(t)=\int_0^t s'(\hat Y_r)\hat Y_r^{p'}\,dB_r+b'\int_0^t s'(\hat Y_r)\tau'(r)\,dr+\frac12\int L_t^{\hat Y}(x)ds'(x).$$

Here, as before, $L^{\hat Y}$ is the semimartingale local time of $\hat Y$. Now argue as in (4.7) to see that for $t\le T_k$,

$$\frac12\int L_t^{\hat Y}(x)ds'(x)=-b'\int_0^t s'(\hat Y_r)1(\hat Y_r>0)\,dr.$$

Therefore if $N(t)=\int_0^t s'(\hat Y_r)\hat Y_r^{p'}\,dB_r$ and

$$A(t)=b'\int_0^t s'(\hat Y_r)(1-\tau'(r))1(\hat Y_r>0)\,dr,$$

then for $t\le T_k$,

$$\begin{aligned}Z(t)&=N(t)-A(t)+b'\int_0^t s'(0)\tau'(r)1(\hat Y(r)=0)\,dr\\ &=N(t)-A(t)+b'\int_0^t 1(Y(r)=0)\,dr, \qquad (5.3)\end{aligned}$$

where we used $s'(0)=1$ and (5.2) in the last line. A is a non-decreasing continuous process by (5.2), N is a continuous local martingale, and $N(0)=$

$A(0) = 0$. Fix k and assume $V_k > 0$, and so $T_k > 0$ because $T_k \geq V_k$. If

$$T_+ = \inf\Big\{t : \int_0^{t\wedge T_k} 1(Y(r) = 0)\, dr > 0\Big\},$$

then by (5.3), $Z(t \wedge T_+)$ is a continuous non-negative local supermartingale starting at 0 and so is identically zero. This means $Z(r) = 0$ for $r \leq T_+$ and so the same holds for $Y(r)$, which by the definition of T_+ and assumption that $T_k > 0$ implies that $T_+ = 0$ a.s. Since $V_k \uparrow \infty$ a.s. we have shown that w.p. 1

$$\int_0^t 1(\langle X(\tau(r)), 1\rangle = 0)\, dr = \int_0^t 1(Y(r) = 0)\, dr > 0 \ \forall t > 0.$$

Setting $\tau(r) = u$ and using (5.2) again (to show $\tau'(r) = 1$ on $\{\langle X(\tau(r)), 1\rangle = 0\}$ for a.a. r), we see that the above implies

$$\int_0^t 1(\langle X_u, 1\rangle = 0)\, du > 0 \quad \forall t > 0 \quad a.s.$$

The proof is complete. □

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Department of Mathematics, U. of Washington, Seattle, USA
E-mail address: burdzy@math.washington.edu

Department of Mathematics, U. of Rochester, Rochester, USA
E-mail address: http://www.math.rochester.edu/people/faculty/cmlr/

Department of Mathematics, UBC, Vancouver, Canada
E-mail address: perkins@math.ubc.ca