On Uniqueness in Law for Parabolic SPDEs and Infinite-dimensional SDEs

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Abstract

Abstract: We give a sufficient conditions for uniqueness in law for the stochastic partial differential equation

$$\frac{\partial u}{\partial t}(x,t) = \tfrac{1}{2}\frac{\partial^2 u}{\partial x^2}(x,t) + A(u(\cdot,t))\dot W_{x,t},$$

where A is an operator mapping $C[0,1]$ into itself and $\dot W$ is a space-time white noise. The approach is to first prove uniqueness for the martingale problem for the operator

$$\mathcal{L}f(x) = \sum_{i,j=1}^{\infty} a_{ij}(x)\frac{\partial^2 f}{\partial x^2}(x) - \sum_{i=1}^{\infty} \lambda_i x_i \frac{\partial f}{\partial x_i}(x),$$

where $\lambda_i = ci^2$ and the a_{ij} is a positive definite bounded operator in Toeplitz form.

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1 Introduction

Our goal is to obtain a uniqueness in law result for parabolic stochastic partial differential equations (SPDEs) of the form

$$\frac{\partial u}{\partial t} = \tfrac{1}{2}\frac{\partial^2 u}{\partial x^2}(x,t) + A(u(\cdot,t))(x)\dot{W}_{x,t}, \qquad x \in [0,1],\ t \geq 0, \tag{1.1}$$

where $\dot{W}$ is a space-time white noise on $[0,1] \times [0,\infty)$, suitable boundary conditions are imposed at 0 and 1, and A is an appropriate operator from $C[0,1]$ to $C[0,1]$ which is bounded above and away from zero. A common approach to (1.1) (see, e.g., Chapter 3 of Walsh [16]) is to convert it to a Hilbert space-valued stochastic differential equation (SDE) by setting

$$X^j(t) = \langle u_t, e_j \rangle,$$

where $\{e_j\}$ is a complete orthonormal sequence of eigenfunctions for the Laplacian (with the above boundary conditions) on $L^2[0,1]$ with eigenvalues $\{-\lambda_j\}$, $u_t(\cdot) = u(\cdot,t)$, and $\langle \cdot,\cdot \rangle$ is the usual inner product on $L^2[0,1]$. This will convert the SPDE (1.1) to the ℓ^2-valued SDE

$$dX^j(t) = -\lambda_j X^j(t)dt + \sum_k \sigma_{jk}(X_t)dW_t^j, \tag{1.2}$$

where $\{W^j\}$ are i.i.d. one-dimensional Brownian motions, $\sigma(x) = \sqrt{a(x)}$, $\mathcal{L}_+(\ell^2,\ell^2)$ is the space of positive definite bounded self-adjoint operators on ℓ^2, and $a : \ell^2 \to \mathcal{L}_+(\ell^2,\ell^2)$ is easily defined in terms of A (see (1.3) below). (1.2) has been studied extensively (see, for example, Chapters 4 and 5 of Kallianpur and Xiong [10] or Chapters I and II of Da Prato and Zabczyk [7]) but, as discussed in the introduction of Zambotti [18], we are still far away from any uniqueness theory that would allow us to characterize solutions to (1.1), except of course in the classical Lipschitz setting.

There has been some interesting work on Stroock-Varadhan type uniqueness results for equations such as (1.2). These focus on Schauder estimates, that is, smoothing properties of the resolvent, for the constant coefficient case which correspond to infinite-dimensional Ornstein-Uhlenbeck processes, and produce uniqueness under appropriate Hölder continuity conditions on a. For example Zambotti [18] and Athreya, Bass, Gordina and Perkins [1] consider the above equation and Cannarsa and Da Prato [6] considers the slightly different setting where there is no restorative drift but (necessarily) a trace class condition on the driving noise. Cannarsa and Da Prato [6] and Zambotti [18] use clever interpolation arguments to derive their Schauder estimates. However, none of the above results appear to allow one to establish uniqueness in equations arising from the SPDE (1.1). In [18] a is assumed to be a small trace class perturbation of a constant operator (see (9) and (10) of that reference) and in [6] the coefficient of the noise is essentially a Hölder continuous trace class perturbation of the identity. If we take $e_j(y) = \exp(2\pi ijy)$, $j \in \mathbb{Z}$ (periodic boundary conditions) and $\lambda_j = 2\pi^2 j^2$, then it is not hard to see that in terms of these coordinates the corresponding operator $a = (a_{jk})$ associated with the SPDE (1.1) is

$$a_{jk}(x) = \int_0^1 A(u(x))(y)^2 e^{2\pi i(j-k)y}\, dy, \qquad j,k \in \mathbb{Z}, \tag{1.3}$$

where $u = \sum_j x_j e_j$. In practice we will in fact work with cosine series and Neumann boundary conditions and avoid complex values – see (10.7) in Section 10 for a more careful derivation. Note that a is a Toeplitz matrix, that is, a_{jk} depends only on $j-k$. In particular $a_{jj}(x) = \int_0^1 A(u(x))(y)^2\, dy$ and $a(x)$ will not be a trace class perturbation of a constant operator unless A itself is constant. In [1] this restriction manifests itself in a condition (5.3) which in particular forces the α-Hölder norms $|a_{ii}|_{C^\alpha}$ to approach zero at a certain rate as $i \to \infty$; a condition which evidently fails unless A is constant.

Our main results for infinite-dimensional SDEs (Theorems 2.1 and 10.1 below) in fact will use the Toeplitz form of a (or more precisely its near Toeplitz form for our cosine series) to obtain a uniqueness result under an appropriate Hölder continuity condition on a. See the discussion prior to (3.3) in Section 3 to see how the Toeplitz condition is used. As a result these results can be used to prove a uniqueness in law result for the SPDE (1.1) under a certain Hölder continuity condition on $A(\cdot)$ (see Theorem 2.3 and Theorem 2.4).

There is a price to be paid for this advance. First, the Hölder continuity of a in the e_k direction must improve as k gets large, that is, for appropriate $\beta > 0$

$$|a_{ij}(y + he_k) - a_{ij}(y)| \le \kappa_\beta k^{-\beta} |h|^\alpha. \tag{1.4}$$

Secondly, we require $\alpha > 1/2$. Finally, to handle the off-diagonal terms of a, we assume that for appropriate $\gamma > 0$,

$$|a_{ij}(x)| \leq \frac{\kappa_\gamma}{1 + |i-j|^\gamma}. \tag{1.5}$$

To handle the SPDE, these conditions on the a_{ij} translate to assumptions on A. The operator A will have two types of smoothness. The more interesting type of smoothness is the Hölder continuity of the map $u \mapsto A(u)$. In order that (1.4) be satisfied, we require Hölder continuity of the map $u \mapsto A(u)$ of order $\alpha > 1/2$ and with respect to a weak Wasserstein norm involving sufficiently smooth test functions (see (2.10) in Theorem 2.3 and (10.19) in Theorem 2.4). The other type of smoothness is that of $A(u)(x)$ as a function of x. In order that the a_{ij} satisfy (1.5), we require that A map $C[0,1]$ into a bounded subset of C^γ for sufficiently large γ.

A consequence of the fact that A must be Hölder continuous with respect to a weak Wasserstein norm is that $A(u)(x)$ cannot be a Hölder continuous function of point values $u(x+x_i,t)$, $i = 1,\dots,n$ but can be a Hölder continuous function of $\langle u, \phi_i \rangle$, $i = 1,\dots,n$, for sufficiently smooth test functions as in Corollary 2.6. One can of course argue that all measurements are averages of u and so on physical grounds this restriction could be reasonable in a number of settings. Although dependence on point values is not a strong feature of our results, it is perhaps of interest to see what can be done in this direction. Let $\{\psi_\varepsilon : \varepsilon > 0\}$ be a C^∞ compactly supported even approximate identity so that $\psi_\varepsilon * h(x) \to h(x)$ as $\varepsilon \to 0$ for any bounded continuous h. Here $*$ is convolution on the line as usual. Let $f : \mathbb{R}^n \to [a,b]$ $(0 < a < b < \infty)$ be Hölder continuous of index $\alpha > \frac{1}{2}$ and $x_1,\dots,x_n \in [0,1]$. Then a special case of Corollary 2.7 implies uniqueness in law for (1.1) with Neumann boundary conditions if

$$A(u)(y) = \psi_\delta * (f(\psi_\varepsilon * \overline{u}(x_1 + \cdot), \dots, \psi_\varepsilon * \overline{u}(x_n + \cdot)))(y), \tag{1.6}$$

where $\overline{u}(y)$ is the even 2-periodic extension of u to $\mathbb{R}$. As $\delta, \varepsilon \downarrow 0$ the above approaches

$$\tilde{A}(u)(y) = f(\overline{u}(x_1 + y), \dots, \overline{u}(x_n + y)). \tag{1.7}$$

Proving uniqueness in (1.1) for $A = \tilde{A}$ remains unresolved for any $\alpha < 1$ unless $n = 1$ and $x_1 = 0$. In this case and for the equation (1.1) on the line, Mytnik and Perkins [13] established pathwise uniqueness, and hence uniqueness in law for $A(u)(y) = f(u(y))$ when f is Hölder continuous of index $\alpha > 3/4$, while Mueller, Mytnik and Perkins [12] showed uniqueness in law may fail in general for $\alpha < 3/4$. These latter results are infinite-dimensional extensions of the classical pathwise uniqueness results of Yamada and Watanabe [17] and a classical example of Girsanov (see e.g. Section V.26 of [14]), respectively. As in the finite-dimensional case, Mytnik and Perkins [13] does not require any non-degeneracy condition on f but is very much confined to the diagonal case in which $A(u)(y)$ depends on $u(y)$. In particular this result certainly cannot deal with A as in (1.6).

Due to the failure of standard perturbation methods to produce a uniqueness result for (1.2) which is applicable to (1.1), we follow a different and more recent approach used to prove well-posedness of martingale problems, first for jump processes in Bass[2], for uniformly elliptic finite dimensional diffusions in Bass and Perkins [5], and recently for a class of degenerate diffusions in Menozzi [11]. Instead of perturbing off a constant coefficient Ornstein-Uhlenbeck operator, the method perturbs off of a mixture of such operators. Further details are provided in Section 3.

We have not spent too much effort on trying to minimize the coefficients β and γ appearing in (1.4) and (1.5), and it would be nice to either get rid of γ altogether or produce examples showing some condition here is needed. Our current hypothesis in Theorems 2.1 and 2.3 require $\gamma \to \infty$ as $\alpha \downarrow 1/2$. Do the results here remain valid if the strengthened Hölder conditions (1.4), or (for the SPDE), (2.10) or (2.13), are replaced with standard Hölder continuity conditions? Are there examples showing that $\alpha > 1/2$ is needed (with or without these additional regularity conditions on A) for uniqueness to hold in (1.1)? Most of the motivating examples for [13] from population genetics and measure-valued diffusions had a Hölder coefficient of $\alpha = 1/2$. (The counter-examples in [12] are for $A(u)(x) = |u(x)|^{(3/4)-\epsilon}$ and so do not satisfy our non-degeneracy condition on A.)

The main existence and uniqueness results for (1.2) and (1.1) are stated in Section 2. Section 3 contains a more detailed description of our basic method using mixtures of Ornstein-Uhlenbeck densities. Section 4 collects some linear algebra results and elementary inequalities for Gaussian densities, and Section 5 presents Jaffard's theorem and some useful applications of it. The heavy lifting is done in Sections 6 and 7 which give bounds on the mixtures of Ornstein-Uhlenbeck process and their moments, and the second order derivatives of these quantities, respectively. Section 8 then proves the main estimate on smoothing properties of our mixed semigroup. The main uniqueness result for Hilbert space-valued SDEs (Theorem 2.1) is proved in Section 9. Finally Section 10 proves the slightly more general uniqueness result for SDEs, Theorem 10.1, and uses it to establish the existence and uniqueness results for the SPDE (1.1) (Theorem 2.3 and Theorem 2.4) and then some specific applications (Corollaries 2.6 and 2.7).

We often use c_1 for constants appearing in statements of results and use c_2, c_2', c_3, c_3' etc. for constants appearing in the proofs.

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2 Main results

We use $D_i f$ for the partial derivative of f in the i^{th} coordinate direction and $D_{ij} f$ for the corresponding second derivatives. We denote the inner product in $\mathbb{R}^d$ and the usual inner product in $L^2[0,1]$ by $\langle \cdot, \cdot \rangle$; no confusion should result.

Let us say $f \in \mathcal{T}_k^2$ if there exists an $f_k \in C_b^2(\mathbb{R}^k)$ such that $f(x) = f_k(x_1, \dots, x_k)$ and we let $\mathcal{T}_k^{2,C}$ be the set of such f where f_k is compactly supported. Let $\mathcal{T}^2 = \cup_k \mathcal{T}_k^2$ be the class of functions in $C_b^2(\ell^2)$ which depend only on countably many coordinates. We let $X_t(\omega) = \omega(t)$ denote the coordinate maps on $C(\mathbb{R}_+, \ell^2)$.

We are interested in the Ornstein-Uhlenbeck type operator

$$\mathcal{L}f(x) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}(x) D_{ij} f(x) - \sum_{i=1}^{\infty} \lambda_i x_i D_i f(x), \tag{2.1}$$

for $f \in \mathcal{T}^2$. Here $\{\lambda_i\}$ is a sequence of positive numbers satisfying

$$\kappa_\lambda i^2 \le \lambda_i \le \kappa_\lambda^{-1} i^2 \tag{2.2}$$

for all $i = 1, 2, \dots$, where κ_λ is a fixed positive finite constant. We assume throughout that a is a map from ℓ^2 to $\mathcal{L}_+(\ell^2, \ell^2)$ so that there exist $0 < \Lambda_0 \le$

$\Lambda_1 < \infty$ satisfying

$$\Lambda_0 |w|^2 \le \langle a(x)w, w \rangle \le \Lambda_1 |w|^2 \quad \text{for all } x, w \in \ell^2. \tag{2.3}$$

Later on we will suppose there exist $\gamma > 1$ and a constant κ_γ such that

$$|a_{ij}(x)| \le \frac{\kappa_\gamma}{1 + |i-j|^\gamma} \tag{2.4}$$

for all $x \in \ell^2$ and all i, j. We will also suppose there exist $\alpha \in (\frac{1}{2}, 1]$, $\beta > 0$ and a constant κ_β such that for all $i, j, k \ge 1$ and $y \in \ell^2$,

$$|a_{ij}(y + he_k) - a_{ij}(y)| \le \kappa_\beta |h|^\alpha (1+k)^{-\beta} \text{ for all } h \in \mathbb{R}, \tag{2.5}$$

where e_k is the unit vector in the x_k direction.

Recall that a_{ij} is of Toeplitz form if a_{ij} depends only on $i - j$.

A probability $\mathbb{P}$ on $C(\mathbb{R}_+, \ell^2)$ satisfies the martingale problem for $\mathcal{L}$ starting at $v \in \ell^2$ if $\mathbb{P}(X_0 = v) = 1$ and

$$M^f(t) = f(X_t) - f(X_0) - \int_0^t \mathcal{L} f(X_s)\, ds$$

is a martingale under $\mathbb{P}$ for each $f \in \mathcal{T}^2$.

Our main theorem on countable systems of SDEs, and the theorem whose proof takes up the bulk of this paper, is the following.

Theorem 2.1 *Suppose $\alpha \in (\frac{1}{2}, 1]$, $\beta > \frac{9}{2} - \alpha$, and $\gamma > 2\alpha/(2\alpha - 1)$. Suppose the a_{ij} satisfy (2.3), (2.4), and (2.5) and that the a_{ij} are of Toeplitz form. Let $v \in \ell^2$. Then there exists a solution to the martingale problem for $\mathcal{L}$ starting at v and the solution is unique.*

It is routine to derive the following corollary from Theorem 2.1.

Corollary 2.2 *Let $\{W^i\}$, $i = 1, 2, \ldots$ be a sequence of independent Brownian motions. Let σ_{ij} be maps from ℓ^2 into $\mathbb{R}$ such that if*

$$a_{ij}(x) = \tfrac{1}{2} \sum_{k=1}^{\infty} \sigma_{ik}(x)\sigma_{kj}(x),$$

then the a_{ij} satisfy the assumptions of Theorem 2.1. Then the ℓ^2-valued continuous solution to the system of SDEs

$$dX_t^i = \sum_{j=1}^{\infty} \sigma_{ij}(X_t)\, dW_t^j - \lambda_i X_t^i\, dt, \qquad i = 1, 2, \ldots, \tag{2.6}$$

is unique in law.

Uniqueness in law has the usual meaning here. If there exists another process $\overline{X}$ with the same initial condition and satisfying

$$d\overline{X}_t^i = \sum_{j=1}^{\infty} \sigma_{ij}(\overline{X}_t)\, d\overline{W}_t^j - \lambda_i \overline{X}_t^i\, dt,$$

where $\{\overline{W}\}$ is a sequence of independent Brownian motions, then the joint laws of (X, W) and $(\overline{X}, \overline{W})$ are the same.

We now turn to the stochastic partial differential equation (SPDE) that we are considering:

$$\frac{\partial u}{\partial t}(x,t) = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(x,t) + A(u_t)(x)\, \dot{W}_{x,t}, \qquad x \in [0,1], \tag{2.7}$$

where $u_t(x) = u(x,t)$ and $\dot{W}_{x,t}$ is an adapted space-time Brownian motion on $[0,1] \times \mathbb{R}_+$ defined on some filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Here A maps continuous functions on $[0,1]$ to continuous functions on $[0,1]$. We impose Neumann boundary conditions at the endpoints. Following Chapter 3 of [16], this means that a continuous $C[0,1]$-valued adapted process $t \to u(t, \cdot)$ is a solution to (2.7) if and only if

$$\langle u_t, \varphi \rangle = \langle u_0, \varphi \rangle + \int_0^t \langle u_s, \varphi''/2 \rangle\, ds + \int_0^t \int \varphi(x) A(u_s)(x)\, dW_{x,s} \tag{2.8}$$

for all $t \geq 0$. whenever $\varphi \in C^2[0,1]$ satisfies $\varphi'(0) = \varphi'(1) = 0$. Solutions to (2.7) are unique in law if and only if for a given $u_0 \in C[0,1]$ the laws of any two solutions to (2.7) on $C(\mathbb{R}_+, C[0,1])$ coincide.

We specialize our earlier notation and let $e_k(x) = \sqrt{2}\cos(k\pi x)$ if $k \geq 1$, and $e_0(x) \equiv 1$, so that $\{e_k\}$ is a complete orthonormal system for $L^2[0,1]$. Here is our theorem for SPDEs. It is proved in Section 10 along with the remaining results in this section.

Theorem 2.3 *Assume*

$$u_n \to u \text{ in } C[0,1] \text{ implies } \|A(u_n) - A(u)\|_2 \to 0. \tag{2.9}$$

Suppose there exist

$$\alpha \in \Big(\frac{1}{2}, 1\Big], \qquad \gamma > \frac{2\alpha}{2\alpha - 1}, \qquad \beta > \Big(\Big(\frac{9}{2}\Big) - \alpha\Big) \vee \Big(\frac{\gamma}{2-\gamma}\Big),$$

and also positive constants κ_1, κ_2 and κ_3 such that for all $u \in C[0,1]$,

$$\|A(u + he_k) - A(u)\|_2 \leq \kappa_1 |h|^\alpha (k+1)^{-\beta} \qquad \text{for all } k \geq 0,\ h \in \mathbb{R}, \tag{2.10}$$

$$0 < \kappa_2 \leq A(u)(x) \leq \kappa_2^{-1}, \qquad \text{for all } x \in [0,1], \tag{2.11}$$

and

$$|\langle A(u)^2, e_k\rangle| \leq \frac{\kappa_3}{1 + (k+1)^\gamma} \qquad \text{for all } k \geq 0. \tag{2.12}$$

Then for any $u_0 \in C([0,1])$ there is a solution of (2.7) and the solution is unique in law.

Here the existence of the solution is in the sense of weak existence.

To give a better idea of what the above conditions (2.10) and (2.12) entail we formulate some regularity conditions on $A(u)$ which will imply them.

For $\delta \in [0,1)$ and $k \in \mathbb{Z}_+$, $\|u\|_{C^{k+\delta}}$ has the usual definition:

$$\|u\|_{C^{k+\delta}} = \sum_{i=0}^{k} \|u^{(i)}\|_\infty + 1_{(\delta>0)} \sup_{x \neq y; x,y \in [0,1]} \frac{|u^{(k)}(y) - u^{(k)}(x)|}{|y-x|^\delta},$$

where $u^{(i)}$ is the i^{th} derivative of u and we consider the 0^{th} derivative of u to just be u itself. C^k is the usual space of k times continuously differentiable functions equipped with $\|\cdot\|_{C^k}$ and $C^{k+\delta} = \{u \in C^k : \|u\|_{C^{k+\delta}} < \infty\}$ with the norm $\|u\|_{C^{k+\delta}}$.

If $f \in C([0,1])$ let $\overline{f}$ be the extension of f to $\mathbb{R}$ obtained by first reflecting to define an even function on $[-1,1]$, and then extending to $\mathbb{R}$ as a 2-periodic continuous function. That is, $\overline{f}(-x) = f(x)$ for $0 < x \leq 1$ and $\overline{f}(x+2) = \overline{f}(x)$ for all x. In order to be able to work with real valued processes and functions, we introduce the space

$$C^\zeta_{per} = \{f \in C^\zeta([0,1]) : \overline{f} \in C^\zeta(\mathbb{R})\},$$

that is, the set of f whose even extension to the circle of circumference 2 is in C^ζ. A bit of calculus shows that $f \in C^\zeta_{per}$ if and only if $f \in C^\zeta([0,1])$ and $f^{(k)}(0) = f^{(k)}(1) = 0$ for all odd $k \le \zeta$. Such f will be even functions, and consequently their Fourier coefficients (considered on the interval $[-1,1]$) will be real.

The following theorem is a corollary to Theorem 2.3.

Theorem 2.4 *Suppose there exist*

$$\alpha \in \Big(\frac{1}{2}, 1\Big], \qquad \gamma > \frac{2\alpha}{(2\alpha-1)}, \qquad \overline{\beta} > \Big(\Big(\frac{9}{2\alpha}\Big) - 1\Big) \vee \Big(\frac{\gamma}{\alpha(2-\gamma)}\Big),$$

and also positive constants κ_1, κ_2 and κ_3 such that for all u, v continuous on $[0,1]$,

$$\|A(u) - A(v)\|_2 \le \kappa_1 \sup_{\varphi \in C^{\overline{\beta}}_{per}, \|\varphi\|_{C^{\overline{\beta}}} \le 1} |\langle u - v, \varphi\rangle|^\alpha, \tag{2.13}$$

$$0 < \kappa_2 \le A(u)(x) \le \kappa_2^{-1}, \qquad x \in [0,1], \tag{2.14}$$

and

$$A(u) \in C^\gamma_{per} \text{ and } \|A(u)\|_{C^\gamma} \le \kappa_3. \tag{2.15}$$

Then for any $u_0 \in C([0,1])$ there is a solution of (2.7) *and the solution is unique in law.*

Note that (2.13) is imposing Hölder continuity in a certain Wasserstein metric.

Remark 2.5 The above conditions on α, β and γ hold if $\gamma > \frac{2\alpha}{2\alpha-1} \vee \frac{14}{5}$, and $\overline{\beta} > \frac{9}{2\alpha} - 1$.

As a consequence of Theorem 2.4, we give a class of examples. Let $\alpha \in (\frac{1}{2}, 1]$. Suppose $n \ge 1$ and $\varphi_1, \dots, \varphi_n$ are functions in $C^{\overline{\beta}}_{per}$ for $\overline{\beta} > \frac{9}{2\alpha} - 1$. Suppose $f : [0,1] \times \mathbb{R}^n \to [0,\infty)$ is bounded above and below by positive constants, and f as a function of the first variable is in C^γ_{per} for $\gamma > \frac{2\alpha}{2\alpha-1} \vee \frac{14}{5}$ and satisfies $\sup_{y_1,\dots,y_n} \|f(\cdot, y_1, \dots, y_n)\|_\gamma \le \kappa$. Assume also that f is Hölder continuous of order α with respect to its second through $(n+1)^{st}$ variables:

$$\begin{aligned} |f(x, y_1, \dots, y_{i-1}, y_i + h, y_{i+1}, \dots, y_n) - f(x, y_1, \dots, y_{i-1}, y_i, y_{i+1}, \dots, y_n)| \\ \le c|h|^\alpha, \qquad \text{for } 1 \le i \le n, \end{aligned}$$

where c does not depend on $x, y_1, \dots, y_n$.

Corollary 2.6 *With f and $\varphi_1, \dots, \varphi_n$ as above, let*

$$A(u)(x) = f(x, \langle u, \varphi_1 \rangle, \dots, \langle u, \varphi_n \rangle).$$

Then a solution to (2.7) *exists and is unique in law.*

A second class of examples can be built from convolution operators. If f, g are real-valued functions on the line, $f * g$ is the usual convolution of f and g.

Corollary 2.7 *Assume $\psi : \mathbb{R} \to \mathbb{R}_+$ and $\phi_1, \phi_2, \dots \phi_n : \mathbb{R} \to \mathbb{R}$ are even C^∞ functions with compact support and ψ is not identically* 0. *Suppose also that for some $0 < a \le b < \infty$ and some $\alpha \in (1/2, 1]$, $f : \mathbb{R}^n \to [a, b]$ satisfies*

$$|f(x) - f(x')| \le c_f \|x - x'\|_\infty^\alpha \qquad \text{for all } x, x' \in \mathbb{R}^n. \tag{2.16}$$

If

$$A(u)(x) = \psi * (f(\phi_1 * \overline{u}(\cdot), \dots, \phi_n * \overline{u}(\cdot)))(x), \tag{2.17}$$

then there is a solution to (2.7) *and the solution is unique in law.*

3 Overview of proof

In this section we give an overview of our argument. For most of this overview, we focus on the stochastic differential equation (1.2) where a is of Toeplitz form, that is, a_{ij} depends only on $i - j$. This is where the difficulties lie and puts us in the context of Theorem 2.1.

Assume we have a $K \times K$ matrix a that is of Toeplitz form, and we will require all of our estimates to be independent of K. Define

$$\mathcal{M}^z f(x) = \sum_{i,j=1}^K a_{ij}(z) D_{ij} f(x) - \sum_{i=1}^K \lambda_i x_i D_i f(x),$$

where λ_i satisfies (2.2). Let $p^z(t, x, y)$ be the corresponding transition probability densities and let $r_\theta^z(x, y)$ be the resolvent densities. Thus $\mathcal{L} f(x) = \mathcal{M}^x f(x)$.

We were unable to get the standard perturbation method to work and instead we used the method described in [5]. The idea is to suppose there are two solutions $\mathbb{P}_1$ and $\mathbb{P}_2$ to the martingale problem and to let $S_i f = \mathbb{E}_i \int_0^\infty e^{-\theta t} f(X_t)\, dt$. Some routine calculations show that $S_i(\theta - \mathcal{L})f = f$, and so $S_\Delta(\theta - \mathcal{L})f = 0$, where S_Δ is the linear functional $S_1 - S_2$. If

$$f(x) = \int r_\theta^y(x,y) g(y)\, dy$$

were in the domain of $\mathcal{L}$ when g is C^∞ with compact support, we would have

$$\begin{aligned}(\theta - \mathcal{L})f(x) &= \int (\theta - \mathcal{M}^y) r_\theta^y(x,y) g(y)\, dy + \int (\mathcal{M}^y - \mathcal{M}^x) r_\theta^y(x,y) g(y)\, dy \\ &= g(y) + \int (\mathcal{M}^y - \mathcal{M}^x) r_\theta^y(x,y) g(y)\, dy.\end{aligned}$$

Such f need not be in the domain of $\mathcal{L}$, but we can do an approximation to get around that problem.

If we can show that

$$\Big| \int (\mathcal{M}^y - \mathcal{M}^x) r_\theta^y(x,y) g(y)\, dy \Big| \le \tfrac{1}{2} \|g\|_\infty, \tag{3.1}$$

for θ large enough, we would then get

$$|S_\Delta g| \le \tfrac{1}{2} \|S_\Delta\|\, \|g\|_\infty,$$

which implies that the norm of the linear functional S_Δ is zero. It is then standard to obtain the uniqueness of the martingale problem from this.

We derive (3.1) from a suitable bound on

$$\int \Big| (\mathcal{M}^y - \mathcal{M}^x) p^y(t,x,y) \Big|\, dy. \tag{3.2}$$

Our bound needs to be independent of K, and it turns out the difficulties are all when t is small.

When calculating $D_{ij} p^y(t,x,y)$, where the derivatives are with respect to the x variable, we obtain a factor $e^{-(\lambda_i + \lambda_j)t}$ (see (7.1)), and thus by (2.2), when summing over i and j, we need only sum from 1 to $J \approx t^{-1/2}$ instead of from 1 to K. When we estimate (3.2), we get a factor t^{-1} from $D_{ij} p^y(t,x,y)$

and we get a factor $|y-x|^\alpha \approx t^{\alpha/2}$ from the terms $a_{ij}(y) - a_{ij}(x)$. If we consider only the main diagonal, we have J terms, but they behave somewhat like sums of independent mean zero random variables, so we get a factor $\sqrt{J} \approx t^{-1/4}$ from summing over the main diagonal where $i = j$ ranges from 1 to J. Therefore when $\alpha > 1/2$, we get a total contribution of order $t^{-1+\eta}$ for some $\eta > 0$, which is integrable near 0. The Toeplitz form of a allows us to factor out $a_{ii}(y) - a_{ii}(x)$ from the sum since it is independent of i and so we are indeed left with the integral in y of

$$\left| \sum_{i=1}^{J} D_{ii} p^y(t,x,y) \right|. \tag{3.3}$$

Let us point out a number of difficulties. All of our estimates need to be independent of K, and it is not at all clear that

$$\int_{\mathbb{R}^K} p^y(t,x,y)\, dy$$

can be bounded independently of K. That it can is Theorem 6.3. We replace the $a_{ij}(y)$ by a matrix that does not depend on y_K. This introduces an error, but not too bad a one. We can then integrate over y_K and reduce the situation from the case where a is a $K \times K$ matrix to where it is $(K-1) \times (K-1)$ and we are now in the $(K-1) \times (K-1)$ situation. We do an induction and keep track of the errors.

From (3.3) we need to handle

$$\int \Big| \sum_{i=1}^{J} D_{ii} p^y(t,x,y) \Big|\, dy,$$

and here we use Cauchy-Schwarz, and get an estimate on

$$\int \sum_{i,j=1}^{J} D_{ii} p^y(t,x,y) D_{jj} p^y(t,x,y)\, dy.$$

This is done in a manner similar to bounding $\int p^y(t,x,y)\, dy$, although the calculations are of course more complicated.

We are assuming that $a_{ij}(x)$ decays at a rate at least $(1+|i-j|)^\gamma$ as $|i-j|$ gets large. Thus the other diagonals besides the main one can be handled in a similarly manner and $\gamma > 1$ allows us to then sum over the diagonals.

A major complication that arises is that $D_{ij}p^y(t,x,y)$ involves a^{-1} and we need a good off-diagonal decay on a^{-1} as well as on a. An elegant linear algebra theorem of Jaffard gives us the necessary decay, independently of the dimension.

To apply the above, or more precisely its cousin Theorem 10.1, to the SPDE (1.1) with Neumann boundary conditions, we write a solution $u(\cdot,t)$ in terms of a Fourier cosine series with random coefficients. Let $e_n(x) = \sqrt{2}\cos(\pi n x)$ if $n \geq 1$, and $e_0(x) \equiv 1$, $\lambda_n = n^2\pi^2/2$ and define $X^n(t) = \langle u(\cdot,t), e_n\rangle$. Then it is easy to see that $X = (X^n)$ satisfies (1.2) with

$$a_{jk}(x) = \int_0^1 A(u(x))^2(y)e_j(y)e_k(y)\,dy, \qquad x \in \ell^2(\mathbb{Z}_+),$$

where $u(x) = \sum_0^\infty x_n e_n$. We are suppressing some issues in this overview, such as extending the domain of A to L^2. Although (a_{jk}) is not of Toeplitz form it is easy to see it is a small perturbation of a Toeplitz matrix and satisfies the hypotheses of Theorem 10.1. This result then gives the uniqueness in law of X and hence of u.

4 Some linear algebra

Define $g_r = rI$, where I is the identity matrix and let $E(s)$ be the diagonal matrix whose (i,i) entry is $e^{-\lambda_i s}$ for a given sequence of positive reals $\lambda_1 \leq \cdots \leq \lambda_m$. Given an $m \times m$ matrix a, let

$$a(t) = \int_0^t E(s)\,a\,E(s)\,ds \tag{4.1}$$

be the matrix whose (i,j) entry is

$$a_{ij}(t) = a_{ij}\frac{1-e^{-(\lambda_i+\lambda_j)t}}{\lambda_i+\lambda_j}.$$

Note $\lim_{t\to 0} a_{ij}(t)/t = a_{ij}$, and we may view a as $a'(0)$.

Given a nonsingular matrix a, we use A for a^{-1}. When we write $A(t)$, this will refer to the inverse of $a(t)$. Given a matrix b or g_r, we define $B, G_r, b(t), g_r(t), B(t)$, and $G_r(t)$ analogously. If $r = 1$ we will write G for G_1 and g for g_1.

Let $\|a\|$ be the usual operator norm, that is, $\|a\| = \sup\{\|aw\| : \|w\| \leq 1\}$. If C is a $m \times m$ matrix, recall that the determinant of C is the product of the eigenvalues and the spectral radius is bounded by $\|C\|$. Hence

$$|\det C| \leq \|C\|^m. \tag{4.2}$$

If a and b are non-negative definite matrices, we write $a \geq b$ if $a - b$ is also non-negative definite. Recall that if $a \geq b$, then $\det a \geq \det b$ and $B \geq A$. This can be found, for example, in [8, Corollary 7.7.4].

Lemma 4.1 *Suppose a is a matrix with $a \geq g_r$. Then $a(t) \geq g_r(t)$.*

Proof. Using (4.1),

$$a(t) = \int_0^t E(s)\, a\, E(s)\, ds \geq \int_0^t E(s)\, g_r\, E(s)\, ds = g_r(t).$$

□

Lemma 4.2 *Suppose $a \geq g_r$. Then $\det a(t) \geq \det g_r(t)$.*

Proof. By Lemma 4.1, $a(t) \geq g_r(t)$, and the result follows. □

For arbitrary square matrices a we let

$$\|a\|_s = \max\{\sup_i \sum_j |a_{ij}|, \sup_j \sum_i |a_{ij}|\}.$$

Schur's Lemma (see e.g., Lemma 1 of [9]) states that

$$\|a\| \leq \|a\|_s. \tag{4.3}$$

As an immediate consequence we have:

Lemma 4.3 *If a is a $m \times m$ matrix, then*

$$|\langle x, ay\rangle| \leq \|x\|\, \|ay\| \leq \|x\|\, \|y\| \|a\|_s.$$

Lemma 4.4 *For all λ_i, λ_j,*

$$\Big(\frac{2\lambda_i}{1-e^{-2\lambda_i t}}\Big)^{1/2}\Big(\frac{1-e^{-(\lambda_i+\lambda_j)t}}{\lambda_i+\lambda_j}\Big)\Big(\frac{2\lambda_j}{1-e^{-2\lambda_j t}}\Big)^{1/2} \le 1. \tag{4.4}$$

Proof. This is equivalent to

$$\int_0^t e^{-(\lambda_i+\lambda_j)s}\,ds \le \Big(\int_0^t e^{-2\lambda_i s}\,ds\Big)^{1/2}\Big(\int_0^t e^{-2\lambda_j s}\,ds\Big)^{1/2}$$

and so is immediate from Cauchy-Schwarz. □

Define

$$\widetilde{a}(t) = G(t)^{1/2}a(t)G(t)^{1/2},$$

so that

$$\widetilde{a}_{ij}(t) = G_{ii}(t)^{1/2}a_{ij}(t)G_{jj}(t)^{1/2}. \tag{4.5}$$

Let $\widetilde{A}(t)$ be the inverse of $\widetilde{a}(t)$, that is,

$$\widetilde{A}(t) = g(t)^{1/2}A(t)g(t)^{1/2}. \tag{4.6}$$

A calculus exercise will show that for all positive λ, t,

$$\frac{1+\lambda t}{2t} \le \frac{2\lambda}{1-e^{-2\lambda t}} \le \frac{2(1+\lambda t)}{t}. \tag{4.7}$$

Lemma 4.5 *If a be a positive definite matrix with $g_{\Lambda_1} \ge a \ge g_{\Lambda_0}$, and*

$$\overline{\Lambda}_0(t) = \Lambda_0\Big(\frac{1-e^{-2\lambda_m t}}{2\lambda_m}\Big),\ \overline{\Lambda}_1(t) = \Lambda_1\Big(\frac{1-e^{-2\lambda_1 t}}{2\lambda_1}\Big),$$

then for all $t>0$,

$$g_{\Lambda_1} \ge \widetilde{a}(t) \ge g_{\Lambda_0} \quad g_{\overline{\Lambda}_1(t)} \ge a(t) \ge g_{\overline{\Lambda}_0(t)}.$$

Proof. Our definitions imply

$$\begin{aligned}
\langle \widetilde{a}(t)x, x\rangle &= \langle G(t)^{1/2}\int_0^t E(s)aE(s)\,ds\,G(t)^{1/2}x, x\rangle \\
&= \int_0^t \langle aE(s)G(t)^{1/2}x, E(s)G(t)^{1/2}x\rangle\,ds \\
&\ge \Lambda_0\int_0^t \langle E(s)G(t)^{1/2}x, E(s)G(t)^{1/2}x\rangle\,ds,
\end{aligned}$$

by the hypotheses on a. The right side is

$$\Lambda_0 \int_0^t \sum_i e^{-2\lambda_i s} \frac{2\lambda_i}{1 - e^{-2\lambda_i t}} |x_i|^2 \, ds = \Lambda_0 \|x\|^2.$$

The upper bound is similar. The bounds on $a(t)$ are a reformulation of Lemma 4.1 and the analogous upper bound. □

Lemma 4.6 *Let a and b be positive definite matrices with $g_{\Lambda_1} \geq a, b \geq g_{\Lambda_0}$. Then*

$$\|\widetilde{a}(t) - \widetilde{b}(t)\| \leq \|\widetilde{a}(t) - \widetilde{b}(t)\|_s \leq \|a - b\|_s, \tag{4.8}$$

$$\|\widetilde{A}(t) - \widetilde{B}(t)\| \leq \Lambda_0^{-2} \|a - b\|_s, \tag{4.9}$$

and for all w, w',

$$|\langle w, (\widetilde{A}(t) - \widetilde{B}(t)) w' \rangle| \leq \Lambda_0^{-2} \|w\| \|w'\| \|a - b\|_s. \tag{4.10}$$

Proof. The first inequality in (4.8) follows from (4.3). The second inequality holds since

$$\begin{aligned}
&\|\widetilde{a}(t) - \widetilde{b}(t)\|_s \\
&\quad = \|G(t)^{1/2} (a(t) - b(t)) G(t)^{1/2}\|_s \\
&\quad = \sup_i \sum_j G_{ii}(t)^{1/2} \Big(\frac{1 - e^{(\lambda_i + \lambda_j)t}}{\lambda_i + \lambda_j} \Big) G_{jj}(t)^{1/2} |a_{ij} - b_{ij}| \\
&\quad \leq \sup_i \sum_j |a_{ij} - b_{ij}| = \|a - b\|_s,
\end{aligned}$$

where Lemma 4.4 is used in the last line and symmetry is used in the next to last line.

Turning to (4.9), we have

$$\begin{aligned}
\|\widetilde{A}(t) - \widetilde{B}(t)\| &= \|\widetilde{A}(t)(\widetilde{b}(t) - \widetilde{a}(t))\widetilde{B}(t)\| \\
&\leq \|\widetilde{A}(t)\| \|\widetilde{B}(t)\| \|\widetilde{b}(t) - \widetilde{a}(t)\|.
\end{aligned} \tag{4.11}$$

The lower bound on $\widetilde{a}(t)$ (and hence $\widetilde{b}(t)$) in Lemma 4.5 implies that

$$\|\widetilde{A}(t)\| \|\widetilde{B}(t)\| \leq \Lambda_0^{-2}.$$

Use this and (4.8) in (4.11) to derive (4.9). (4.10) is then immediate. □

Lemma 4.7 *Let a and b be positive definite matrices with $g_{\Lambda_1} \geq a, b \geq g_{\Lambda_0}$, and set $\theta = \Lambda_0^{-1} m \|a-b\|_s$. Then*

$$\Big|\frac{\det \widetilde{b}(t)}{\det \widetilde{a}(t)} - 1\Big| \leq \theta e^{\theta}.$$

Proof. We write

$$\begin{aligned} \frac{\det \widetilde{b}(t)}{\det \widetilde{a}(t)} &= \det(\widetilde{b}(t)\widetilde{A}(t)) = \det(I + (\widetilde{b}(t)\widetilde{A}(t) - I)) \\ &= \det(I + (\widetilde{b}(t) - \widetilde{a}(t))\widetilde{A}(t)). \end{aligned} \tag{4.12}$$

Clearly

$$\|I + (\widetilde{b}(t) - \widetilde{a}(t))\widetilde{A}(t)\| \leq \|I\| + \|\widetilde{b}(t) - \widetilde{a}(t)\| \, \|\widetilde{A}(t)\|. \tag{4.13}$$

Use the lower bound on $\widetilde{a}(t)$ in Lemma 4.5 to see that $\|\widetilde{A}(t)\| \leq \Lambda_0^{-1}$, and then use (4.8) in the above to conclude that

$$\|I + (\widetilde{b}(t) - \widetilde{a}(t))\widetilde{A}(t)\| \leq 1 + \Lambda_0^{-1}\|a - b\|_s.$$

Hence from (4.12) and (4.2) we have the bound

$$\begin{aligned} \Big|\frac{\det \widetilde{b}(t)}{\det \widetilde{a}(t)}\Big| &\leq \|I + (\widetilde{b}(t) - \widetilde{a}(t))\widetilde{A}(t)\|^m \\ &\leq \Big(1 + \Lambda_0^{-1}\|a-b\|_s\Big)^m \\ &\leq e^{\Lambda_0^{-1} m \|a-b\|_s}. \end{aligned}$$

Observe that $\widetilde{a}(t)$ and $\widetilde{b}(t)$ are positive definite, so $\det \widetilde{a}(t)$ and $\det \widetilde{b}(t)$ are positive real numbers. We now use the inequality $e^x \leq 1 + xe^x$ for $x > 0$ to obtain

$$\frac{\det \widetilde{b}(t)}{\det \widetilde{a}(t)} \leq 1 + \theta e^{\theta}.$$

Reversing the roles of a and b,

$$\frac{\det \widetilde{a}(t)}{\det \widetilde{b}(t)} \leq 1 + \theta e^{\theta},$$

and so,

$$\frac{\det \widetilde{b}(t)}{\det \widetilde{a}(t)} \geq \frac{1}{1+\theta e^{\theta}} \geq 1-\theta e^{\theta}.$$

□

Let us introduce the notation

$$Q_m(w,C) = (2\pi)^{-m/2}(\det C)^{1/2}e^{-\langle w,Cw/2\rangle}, \tag{4.14}$$

where C is a positive definite $m \times m$ matrix, and $w \in \mathbb{R}^m$.

Proposition 4.8 *Assume a, b are as in Lemma 4.7. Set*

$$\theta = \Lambda_0^{-1} m\|a-b\|_s \text{ and } \phi = \Lambda_0^{-2}\|w\|^2\|a-b\|_s.$$

For any $M > 0$ there is a constant $c_1 = c_1(M)$ so that if $\theta, \phi < M$, then

$$\left|\frac{Q_m(w,\widetilde{A}(t))}{Q_m(w,\widetilde{B}(t))} - 1\right| \leq c_1(\phi+\theta).$$

Proof. Using the inequality

$$|e^x - 1| \leq |x|e^{(x^+)}, \tag{4.15}$$

we have from Lemma 4.6,

$$\left|e^{-\langle w,(\widetilde{A}(t)-\widetilde{B}(t))w\rangle/2} - 1\right| \leq \phi e^{\phi}.$$

Using the inequalities

$$|1-\sqrt{x}| \leq |1-x|, \qquad x \geq 0,$$

and

$$|xy-1| \leq |x|\,|y-1| + |x-1|, \qquad x,y \geq 0,$$

the proposition now follows by Lemmas 4.6 and 4.7 with
$c_1 = e^M(1+Me^M)^{1/2}$.

□

We note that if a, b are $m \times m$ matrices satisfying $\sup_{i,j}|a_{ij} - b_{ij}| \leq \delta$, we have the trivial bound

$$\|a-b\|_s \leq m\delta. \tag{4.16}$$

Lemma 4.9 *Suppose a is a $(m+1)\times(m+1)$ positive definite matrix, A is the inverse of a, B is the $m\times m$ matrix defined by*

$$B_{ij} = A_{ij} - \frac{A_{i,m+1}A_{j,m+1}}{A_{m+1,m+1}}, \qquad i,j\le m. \tag{4.17}$$

Let b be the $m\times m$ matrix defined by $b_{ij}=a_{ij}$, $i,j\le m$. Then $b=B^{-1}$.

Proof. Let δ_{ij} be 1 if $i=j$ and 0 otherwise. If $i,j\le m$, then

$$\begin{aligned}
\sum_{k=1}^m b_{ik}B_{kj} &= \sum_{k=1}^m a_{ik}A_{kj} - \sum_{k=1}^m a_{ik}\frac{A_{k,m+1}A_{j,m+1}}{A_{m+1,m+1}}\\
&= \sum_{k=1}^{m+1} a_{ik}A_{kj} - a_{i,m+1}A_{m+1,j} - \sum_{k=1}^{m+1} a_{ik}\frac{A_{k,m+1}A_{j,m+1}}{A_{m+1,m+1}}\\
&\qquad + a_{i,m+1}\frac{A_{m+1,m+1}A_{j,m+1}}{A_{m+1.m+1}}\\
&= \delta_{ij} - \frac{\delta_{i,m+1}A_{j,m+1}}{A_{m+1,m+1}} = \delta_{ij}.
\end{aligned}$$

The last equality holds because $i\le m$. □

5 Jaffard's theorem

We will use the following result of Jaffard (Proposition 3 in [9]). Throughout this section $\gamma>1$ is fixed.

Proposition 5.1 *Assume b is an invertible $K\times K$ matrix satisfying $\|b\|\le\Lambda_1$, $\|B\|\le\Lambda_0^{-1}$, and*

$$|b_{ij}|\le\frac{c_1}{1+|i-j|^\gamma} \qquad \text{for all } i,j,$$

where $B=b^{-1}$. There is a constant c_2, depending only on c_1, γ, Λ_0 and Λ_1, but not K, such that

$$|B_{ij}|\le\frac{c_2}{1+|i-j|^\gamma} \qquad \text{for all } i,j.$$

The dependence of c_2 on the given parameters is implicit in the proof in [9].

We now suppose that a is a positive definite $K \times K$ matrix such that for some positive Λ_0, Λ_1,

$$g_{\Lambda_1} \geq a \geq g_{\Lambda_0}. \tag{5.1}$$

We suppose also that (2.4) holds. Our estimates and constants in this section may depend on Λ_i and κ_γ, but will be independent of K, as is the case in Proposition 5.1.

Recall $a(t)$ and $\widetilde{a}(t)$ are defined in (4.1) and (4.5), respectively, and $A(t)$ and $\widetilde{A}(t)$, respectively, are their inverses.

Lemma 5.2 *For all $t > 0$,*

$$|\widetilde{a}_{ij}(t)| \leq \frac{\kappa_\gamma}{1 + |i-j|^\gamma} \qquad \textit{for all } i, j.$$

Proof. Since

$$G_{ii}(t) = \frac{2\lambda_i}{1 - e^{-2\lambda_i t}},$$

then

$$\widetilde{a}_{ij}(t) = \Big(\frac{2\lambda_i}{1 - e^{-2\lambda_i t}}\Big)^{1/2} a_{ij} \Big(\frac{1 - e^{-(\lambda_i + \lambda_j)t}}{\lambda_i + \lambda_j}\Big) \Big(\frac{2\lambda_j}{1 - e^{-2\lambda_j t}}\Big)^{1/2}.$$

Using (2.4) and Lemma 4.4, we have our result. □

Lemma 5.3 *There exists a constant c_1, depending only on κ_γ, Λ_0 and Λ_1, so that*

$$|\widetilde{A}_{ij}(t)| \leq \frac{c_1}{1 + |i-j|^\gamma}.$$

Proof. This follows immediately from Lemma 4.5, Lemma 5.2, and Jaffard's theorem (Proposition 5.1). □

We set

$$L(i, j, t) = \Big(\frac{1 + \lambda_i t}{t}\Big)^{1/2} \Big(\frac{1 + \lambda_j t}{t}\Big)^{1/2}.$$

The proposition we will use in the later parts of the paper is the following.

Proposition 5.4 *There exists a constant c_1, depending only on κ_γ, Λ_0 and Λ_1, such that*

$$(2\Lambda_1)^{-1} L(i,i,t) 1_{(i=j)} \le |A_{ij}(t)| \le L(i,j,t)\Big(\frac{c_1}{1+|i-j|^\gamma}\Big).$$

Proof. Since $\widetilde{a}(t) = G(t)^{1/2} a(t) G(t)^{1/2}$, then

$$a(t) = g(t)^{1/2} \widetilde{a}(t) g(t)^{1/2},$$

and hence

$$A(t) = G(t)^{1/2} \widetilde{A}(t) G(t)^{1/2}.$$

Therefore

$$A_{ij}(t) = \Big(\frac{2\lambda_i}{1-e^{-2\lambda_i t}}\Big)^{1/2} \widetilde{A}_{ij}(t) \Big(\frac{2\lambda_j}{1-e^{-2\lambda_j t}}\Big)^{1/2}. \tag{5.2}$$

The upper bound now follows from Lemma 5.3 and (4.7).

For the left hand inequality, by (5.2) and the lower bound in (4.7) it suffices to show

$$\widetilde{A}_{ii}(t) \ge \Lambda_1^{-1}, \tag{5.3}$$

and this is immediate from the uniform upper bound on $\widetilde{a}(t)$ in Lemma 4.5.

□

6 A Gaussian-like measure

Let us suppose K is a fixed positive integer, $0 < \Lambda_0 \le \Lambda_1 < \infty$, and that we have a $K \times K$ symmetric matrix-valued function $a : \mathbb{R}^K \to \mathbb{R}^{K\times K}$ with

$$\Lambda_0 \sum_{i=1}^K |y_i|^2 \le \sum_{i,j=1}^K a_{ij}(x) y_i y_j \le \Lambda_1 \sum_{i=1}^K |y_i|^2, \quad x \in \mathbb{R}^K, y = (y_1, \ldots, y_K) \in \mathbb{R}^K.$$

It will be important that all our bounds and estimates in this section will not depend on K. We will assume $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_K$ satisfy (2.2). As usual, $A(x)$ denotes the inverse to $a(x)$, and we define

$$a_{ij}(x,t) = a_{ij}(x) \int_0^t e^{-(\lambda_i+\lambda_j)s}\, ds,$$

and then $A(x,t)$ to be the inverse of $a(x,t)$. Let $\widetilde{a}(x,t)$ and $\widetilde{A}(x,t)$ be defined as in (4.5) and (4.6), respectively. When $x = (x_1, \ldots, x_K)$, define $x' = (x'_1, \ldots, x'_K)$ by

$$x'_i = e^{-\lambda_i t} x_i,$$

and set $w = y - x'$. For $j \leq K$, define $\pi_{j,x} : \mathbb{R}^K \to \mathbb{R}^K$ by

$$\pi_{j,x}(y) = (y_1, y_2, \ldots, y_j, x'_{j+1}, \ldots, x'_K),$$

and write π_j for $\pi_{j,x}$ if there is no ambiguity. Recall that

$$Q_K(w, A(y,t)) = (2\pi)^{-K/2} (\det A(y,t))^{1/2} \exp\Big(- \langle w, A(y,t)w\rangle/2 \Big). \tag{6.1}$$

The dependence of A on y but not x is not a misprint; $y \to Q_K(y - x', A(y,t))$ will not be a probability density. It is however readily seen to be integrable; we show more below.

The choice of K in the next result is designed to implement a key induction argument later in this section.

Lemma 6.1 *Assume $K = m+1$ and $a(y) = a(\pi_m(y))$, that is, $a(y)$ does not depend on y_{m+1}. Let $b(y)$ be the $m \times m$ matrix with $b_{ij}(y) = a_{ij}(y)$ for $i, j \leq m$, and let $B(y)$ be the inverse of $b(y)$. Then for all x,*
(a) we have

$$\int Q_{m+1}(w, A(y))\, dy_{m+1} = Q_m(w, B(y)).$$

(b) If $y_1, \ldots, y_m$ are held fixed, $Q_{m+1}(w, A(y))/Q_m(w, B(y))$ equals the density of a normal random variable with mean

$$\mu(y_1, \ldots, y_m) = - \frac{\sum_{i=1}^m w_i A_{i,m+1}(y)}{A_{m+1,m+1}(y)}$$

and variance $\sigma^2(y_1, \ldots, y_m) = (A_{m+1,m+1}(y))^{-1}$.

Proof. Lemma 4.9 and some algebra show that

$$\begin{aligned} \sum_{i,j=1}^{m+1} (y_i - x'_i)(y_j - x'_j) A_{ij}(y) = \sum_{i,j=1}^{m} (y_i - x'_i)(y_j - x'_j) B_{ij}(y) \\ + A_{m+1,m+1}(y) |y_{m+1} - x'_{m+1} - \mu|^2. \end{aligned} \tag{6.2}$$

Let $C(y)$ be the $(m+1)\times(m+1)$ matrix such that

$$\begin{cases} C_{ij}(y) = B_{ij}(y), & i,j \le m; \\ C_{i,m+1}(y) = 0, & i \le m; \\ C_{m+1,j}(y) = A_{m+1,j}(y), & j \le m+1. \end{cases}$$

If $\mathrm{row}_i(D)$ denotes the i^{th} row of a matrix D, note that

$$\mathrm{row}_i(C(y)) = \mathrm{row}_i(A(y)) - 1_{(i\le m)} \frac{A_{i,m+1}}{A_{m+1,m+1}} \mathrm{row}_{m+1}(A(y)).$$

Therefore $\det C(y) = \det A(y) > 0$, and it follows that

$$\det A(y) = \det C(y) = A_{m+1,m+1}(y) \det B(y). \tag{6.3}$$

Part (a) now follows from (6.2), (6.3), and evaluating the standard Gaussian integral. Part (b) is then immediate from (6.2) and (6.3). $\square$

Let $B_0 = 8\log(\Lambda_1/\Lambda_0) + 4\log 2$ and for $B > 0$ let

$$S_{B,K} = S_B = \{z \in \mathbb{R}^K : \|z\|^2 < B\Lambda_1 K\}. \tag{6.4}$$

Recalling that $w = y - x'$, we will often use the further change of variables

$$w' = G(t)^{1/2} w = G(t)^{1/2}(y - x'). \tag{6.5}$$

Note that when integrating $Q_K(w', A(y,t))$ with respect to w', y is an implicit function of w'.

Lemma 6.2 *(a) For any $p \ge 0$ there is a $c_p = c_p(\Lambda_1)$ such that if $B \ge B_0$ and F is a $K \times K$ symmetric matrix-valued function of w with $G_{\Lambda_0} \ge F \ge G_{\Lambda_1}$, then*

$$\int_{S_B^c} \|w\|^{2p} Q_K(w, F)\, dw \le c_p K^p e^{-BK/16}.$$

(b) For all x,

$$\int_{S_B^c} \|w'\|^{2p} Q_K(w', \widetilde{A}(y,t))\, dw' \le c_p K^p e^{-BK/16}.$$

Proof. (a) We have $G_{\Lambda_0} = (\Lambda_1/\Lambda_0)G_{\Lambda_1}$, and so

$$\begin{aligned} Q_K(w,F) &\le (2\pi)^{-K/2}(\det G_{\Lambda_0})^{1/2} e^{-\langle w, G_{\Lambda_1} w\rangle/2} \\ &= \Big(\frac{\Lambda_1}{\Lambda_0}\Big)^{K/2} Q_K(w, G_{\Lambda_1}). \end{aligned} \tag{6.6}$$

Let Z_i be i.i.d. mean zero normal random variables with variance 1 and let

$$Y_i = \sqrt{\Lambda_1} Z_i.$$

From (6.6) we have

$$\int_{S_B^c} \|w\|^{2p} Q_K(w,F)\,dw \le \Big(\frac{\Lambda_1}{\Lambda_0}\Big)^{K/2} \int_{S_B^c} \|w\|^{2p} Q_K(w, G_{\Lambda_1})\,dw.$$

The right hand side is the same as

$$\begin{aligned} &\Big(\frac{\Lambda_1}{\Lambda_0}\Big)^{K/2} \mathbb{E}\,\Big[\Big(\sum_{i=1}^K \Lambda_1 |Z_i|^2\Big)^p ; \sum_{i=1}^K \Lambda_1 |Z_i|^2 \ge B\Lambda_1 K\Big] \\ &\qquad \le \Big(\frac{\Lambda_1}{\Lambda_0}\Big)^{K/2} (\Lambda_1)^p \mathbb{E}\,\Big[\Big(\sum_{i=1}^K |Z_i|^2\Big)^p ; \sum_{i=1}^K |Z_i|^2 \ge BK\Big] \\ &\qquad \le \Big(\frac{\Lambda_1}{\Lambda_0}\Big)^{K/2} (\Lambda_1)^p \Big[\mathbb{E}\,\Big(\Big(\sum_{i=1}^K |Z_i|^2\Big)^{2p}\Big]^{1/2} \\ &\qquad\qquad \times \Big[\mathbb{E}\,\exp\Big(\sum_{i=1}^K |Z_i|^2/4\Big)\Big]^{1/2} e^{-BK/8} \\ &\qquad \le c_p K^p \Big[\Big(\frac{\Lambda_1}{\Lambda_0}\Big)^{1/2} \mathbb{E}\,(\exp(|Z_1|^2/4))^{1/2} e^{-B/8}\Big]^K. \end{aligned}$$

Since $\mathbb{E}\, e^{|Z_1|^2/4} = \sqrt{2}$, our choice of B shows that the above is at most

$$c_p K^p \exp(-BK/16).$$

(b) By Lemma 4.5, $g_{\Lambda_0} \le \widetilde{a}(y,t) \le g_{\Lambda_1}$, so $G_{\Lambda_0} \ge \widetilde{A}(y,t) \ge G_{\Lambda_1}$. Hence (b) follows from (a). □

For $m \le K$ we let $a^m(y,t)$, respectively $\widetilde{a}^m(y,t)$, be the $m \times m$ matrices whose (i,j) entry is $a_{ij}(\pi_{m,x'}(y),t)$, respectively $\widetilde{a}_{ij}(\pi_{m,x'}(y),t)$. We use $A^m(y,t)$ and $\widetilde{A}^m(y,t)$ to denote their respective inverses.

The main theorem of this section is the following.

Theorem 6.3 *Suppose* (2.5) *holds with* $\beta > 3-\alpha$. *Let* $w' = G(t)^{1/2}(y-x')$. *Then there exists a constant* c_1 *depending on* α, β, κ_β, p, Λ_0, *and* Λ_1 *but not* K, *such that for all* $t > 0$ *and* $x \in \mathbb{R}$:

(a) For all $1 \le j \le K$,

$$\int_{\mathbb{R}^K} |w'_j|^{2p} Q_K(w', \widetilde{A}(y,t))\, dw' \\ \le c_1 \Big[\int_{\mathbb{R}^j} |w'_j|^{2p} Q_j(w', \widetilde{A}^j(y,t))\, dw' + 1 \Big].$$

(b)

$$\int_{\mathbb{R}^K} Q_K(w', \widetilde{A}(y,t))\, dw' \le c_1,$$

and

$$\int_{\mathbb{R}^K} Q_K(y-x', A(y,t))\, dy \le c_1.$$

Remark 6.4 This is one of the more important theorems of the paper. In the proof of (a) we will define a geometrically decreasing sequence $K_0, ..., K_N$ with $K_0 = K$ and $K_N = j$ and let C_m be the expression on the right-hand side of (a) but with K_m in place of K and $\widetilde{A}^{K_m}$ in place of $\widetilde{A}$. We will bound C_m inductively in terms of C_{m+1} by using Lemma 6.2 and Proposition 4.8. This will give (a) and reduce (b) to the boundedness in the $K = 1$ case, which is easy to check.

Proof of Theorem 6.3. All constants in this argument may depend on $\alpha, \beta, \kappa_\beta, \Lambda_0, \Lambda_1$, and p. Let $K_0, K_1, \ldots, K_N$ be a decreasing sequence of positive integers such that $K_0 = K$, $K_N = j$, and $\frac{5}{4} \le K_m/K_{m+1} \le 4$ for each $0 \le m < N$.

Let

$$C_m = \int |w'_j|^{2p} Q_{K_m}(w', \widetilde{A}^{K_m}(y,t))\, dw'. \tag{6.7}$$

Our plan is to bound C_m inductively over m. Write

$$\begin{aligned} C_m &= \int_{S^c_{B_0,K_m}} |w'_j|^{2p} Q_{K_m}(w', \widetilde{A}^{K_m}(y,t))\, dw' \\ &\qquad + \int_{S_{B_0,K_m}} |w'_j|^{2p} Q_{K_m}(w', \widetilde{A}^{K_m}(y,t))\, dw' \\ &= I_1 + I_2. \end{aligned} \tag{6.8}$$

Assume $m < N$. We can bound I_1 using Lemma 6.2 and conclude

$$I_1 \le c_p K_m^p e^{-B_0 K_m/16} \le c'_p e^{-B_0 K_m/17}. \tag{6.9}$$

Turning to I_2, we see that by our hypothesis on a, we have

$$\begin{aligned} |a_{ij}^{K_m}(y,t) - &a_{ij}^{K_m}(\pi_{K_{m+1}}(y),t)| \\ &\le \kappa_\beta \sum_{k=K_{m+1}+1}^{K_m} |w_k|^\alpha k^{-\beta} \\ &= \kappa_\beta \sum_{k=K_{m+1}+1}^{K_m} |w'_k|^\alpha g_{kk}(t)^{\alpha/2} k^{-\beta} \\ &\le c_1 (t^{\alpha/2} \wedge K_m^{-\alpha}) \|w'\|^\alpha \Big[\sum_{k=K_{m+1}+1}^{K_m} k^{-2\beta/(2-\alpha)} \Big]^{(2-\alpha)/2}. \end{aligned}$$

In the last line we use Hölder's inequality and also the bound

$$g_{kk}(t) = \int_0^t e^{-2\lambda_k s}\, ds \le t \wedge (2\lambda_k)^{-1} \le c_2 (t \wedge k^{-2}), \tag{6.10}$$

by (2.2). We also used the geometric decay of the $\{K_m\}$.

If $w' \in S_{B_0,K_m}$ so that $\|w'\|^\alpha \le (B_0 \Lambda_1 K_m)^{\alpha/2}$, some elementary arithmetic shows there is a constant c_3 so that

$$\begin{aligned} |a_{ij}^{K_m}(y,t) - a_{ij}^{K_m}(\pi_{K_{m+1}}(y),t)| &\le c_3 (t^{\alpha/2} \wedge K_m^{-\alpha}) K_m^{\alpha/2} [K_m^{1-(2\beta/(2-\alpha))}]^{(2-\alpha)/2} \\ &\le c_3 (t^{\alpha/2} \wedge K_m^{-\alpha}) K_m^{1-\beta}. \end{aligned} \tag{6.11}$$

Set $\delta = c_3 K_m^{1-\beta-\alpha}$. We now apply Proposition 4.8 for $w' \in S_{B_0,K_m}$ with $a = a^{K_m}(y,t)$ and $b = a^{K_m}(\pi_{K_{m+1}}(y),t)$. In view of (4.16) and (6.11), we may take

$$\theta = \Lambda_0^{-1} K_m^2 \delta \qquad \text{and} \qquad \phi = \Lambda_0^{-2} \Lambda_1 B_0 K_m^2 \delta,$$

so that

$$\theta \vee \phi \le c_3 K_m^{3-\beta-\alpha} \le c_3.$$

Proposition 4.8 shows that for $w' \in S_{B_0,K_m}$,

$$\Big| \frac{Q_{K_m}(w', \widetilde{A}^{K_m}(y,t))}{Q_{K_m}(w', \widetilde{A}^{K_m}(\pi_{K_{m+1}}(y),t))} - 1 \Big| \le c_4 K_m^{3-\beta-\alpha}. \tag{6.12}$$

Therefore we have

$$I_2 \le (1 + c_4 K_m^{3-\beta-\alpha}) \int |w_j'|^{2p} Q_{K_m}(w', \widetilde{A}^{K_m}(\pi_{K_{m+1}}(y),t))\, dw'.$$

Recall $m+1 \le N$ so that $j \le K_{m+1}$. Integrate over w'_{K_m} using Lemma 6.1, then over w'_{K_m+1} using Lemma 6.1 again, and continue until we have integrated over $w'_{K_{m+1}+1}$ to see that

$$\int |w_j'|^{2p} Q_{K_m}(w', \widetilde{A}^{K_m}(\pi_{K_{m+1}}(y),t))\, dw' = C_{m+1}, \tag{6.13}$$

and hence

$$I_2 \le (1 + c_4 K_m^{3-\beta-\alpha}) C_{m+1}. \tag{6.14}$$

This and (6.9) together show that (6.8) implies that for $0 \le m < N$,

$$C_m \le c_p' e^{-B_0 K_m/17} + (1 + c_4 K_m^{3-\beta-\alpha}) C_{m+1}. \tag{6.15}$$

This and a simple induction imply

$$\begin{aligned} C_0 &\le \exp\Big(c_4 \sum_{m=0}^{N-1} K_m^{3-\beta-\alpha}\Big) C_N \\ &\qquad + c_p' \sum_{m=0}^{N-1} e^{-B_0 K_m/17} \exp\Big(\sum_{\ell=1}^{m-1} c_4 K_\ell^{3-\beta-\alpha}\Big) \\ &\le c_5(p)[C_N + 1], \end{aligned} \tag{6.16}$$

since $\beta > 3 - \alpha$. Part (a) follows.

For (b), we may apply (a) with $p = 0$ and $j = 1$ to get

$$\begin{aligned} \int Q_K(w', &\widetilde{A}(y,t))\, dw' \\ &\le c_6 \Big[\int_{-\infty}^{\infty} Q_1(w', \widetilde{A}^1(y,t))\, dw + 1 \Big]. \end{aligned} \tag{6.17}$$

Recall from Lemma 4.5 that the scalar $\widetilde{A}^1(y,t)$ satisfies $(\Lambda_1)^{-1} \le |\widetilde{A}^1(y,t)| \le (\Lambda_0)^{-1}$ and so the above integral is at most

$$\Big(\frac{\Lambda_1}{\Lambda_0}\Big)^{1/2} \int Q_1(w, \Lambda_1 t)\, dw = (\Lambda_1/\Lambda_0)^{1/2}.$$

The first bound in (b) follows from this and (6.17). Using the change of variables $w' = G(t)^{1/2} w$, we see that the second integral in (b) equals the first.

$\square$

Proposition 6.5 *Under the hypotheses of Theorem 6.3,*

$$\int Q_K(y - x', A^K(y,t))\, dy \to 1$$

as $t \to 0$, *uniformly in* K *and* x.

Proof. We will use the notation of the proof of Theorem 6.3 with $j = 1$, $p = 0$, and $t < 1$. Using the change of variables $w' = G(t)^{1/2}(y - x')$, it suffices to prove

$$\int Q_K(w', \widetilde{A}^K(y,t))\, dw'$$

converges to 1 uniformly as $t \to 0$.

We define a decreasing sequence $K_0, \ldots, K_N$ as in the proof of Theorem 6.3 with $K_0 = K$ and $K_N = 1$, we let

$$C_m(t) = \int Q_{K_m}(w', \widetilde{A}^{K_m}(y,t))\, dw',$$

we let $R > 0$ be a real number to be chosen later, and we write

$$|C_0(t) - 1| \le |C_N(t) - 1| + \sum_{m=0}^{N-1} |C_m(t) - C_{m+1}(t)|. \tag{6.18}$$

We will bound each term on the right hand side of (6.18) appropriately, and that will complete the proof.

Using (6.13) and with $S_{B,K}$ defined by (6.4), we write

$$\begin{aligned}
&|C_m(t) - C_{m+1}(t)| \\
&\quad \le \int_{S^c_{R,K_m}} [Q_{K_m}(w', \widetilde{A}^{K_m}(y,t)) + Q_{K_m}(w', \widetilde{A}^{K_m}(\pi_{K_{m+1}}(y),t))]\, dw' \\
&\qquad + \int_{S_{R,K_m}} \Big| \frac{Q_{K_m}(w', \widetilde{A}^{K_m}(y,t))}{Q_{K_m}(w', \widetilde{A}^{K_m}(\pi_{K_{m+1}}(y),t))} - 1 \Big| \\
&\qquad\qquad \times Q_{K_{m+1}}(w', \widetilde{A}^{K_m}(\pi_{K_{m+1}}(y),t))\, dw' \\
&\quad = J_1(t) + J_2(t).
\end{aligned}$$

By Lemma 6.2(a),

$$J_1(t) \le c_1 e^{-c_2 R K_m}.$$

Choose $0 < \eta < \beta-(3-\alpha)$ and note that (6.11) implies there exists $c_2 = c_2(R)$ such that

$$|a^{K_m}_{ij}(y,t) - a^{K_m}_{ij}(\pi_{K_{m+1}}(y),t)| \le c_2 t^\eta K_m^{1-\beta-\alpha+\eta} \equiv \delta.$$

Follow the argument in the proof of Theorem 6.3 with this value of δ to see that

$$\begin{aligned}
J_2(t) &\le c_2 t^{\eta/2} K_m^{3-\beta-\alpha+\eta} \int Q_{K_{m+1}}(w', \widetilde{A}^{K_{m+1}}(y,t))\, dw' \\
&= c_2 t^{\eta/2} K_m^{3-\beta-\alpha+\eta} C_{m+1}(t) \\
&\le c_3 t^{\eta/2} K_m^{3-\beta-\alpha+\eta}.
\end{aligned}$$

We used the uniform boundedness of C_{m+1} from Theorem 6.3 for the last inequality.

A very similar argument shows that

$$\Big| C_N(t) - \int Q_1(w', \widetilde{A}^1(x',t))\, dw \Big| \le c_4 e^{-c_4 R} + c_5 t^{\alpha/2},$$

where c_5 depends on R. For example, in bounding the analog of $J_2(t)$, we may now take $\delta = c_6 R^\alpha t^{\alpha/2}$ by adjusting the argument leading up to (6.11). Now use that $Q_1(w', \widetilde{A}(x',t))$ is the density of a normal random variable, so that $\int Q_1(w', \widetilde{A}(x',t))\, dw' = 1$. Substituting in (6.18), we obtain

$$\begin{aligned}
|C_N(t) - 1| &\le c_4 e^{-c_4 R} + c_5 t^{\alpha/2} + \sum_{m=0}^{N-1} [c_1 e^{-c_2 R K_m} + c_3 t^{\eta/2} K_m^{3-\beta-\alpha+\eta}] \\
&\le c_7 e^{-c_7 R} + c_8 (t^{\alpha/2} + t^{\eta/2});
\end{aligned}$$

c_8 depends on R but c_7 does not. For the second inequality recall that $3-\beta-\alpha+\eta<0$ and the K_m were chosen in the proof of Theorem 6.3 so that $\frac{5}{4}\le K_m/K_{m+1}\le 4$. Given $\varepsilon>0$, choose R large so that $c_7e^{-c_7R}<\varepsilon$ and then take t small enough so that $c_8(t^{\alpha/2}+t^{\eta/2})<\varepsilon$. $\square$

Corollary 6.6 *Assume the hypotheses of Theorem 6.3. For any $p\ge 0$ there exists $c_1=c_1(p)>0$ such that*

$$\int \|w'\|^{2p}Q_K(w',\widetilde{A}(y,t))\,dw\le c_1K^p$$

for all $t>0$.

Proof. Bound the above integral by

$$\int_{S^c_{B_0}}\|w'\|^{2p}Q_K(w',\widetilde{A}(y,t))\,dw'+(B_0\Lambda_1K)^p\int_{S_{B_0}}Q_K(w',\widetilde{A}(y,t))\,dw'.$$

The first term is at most $c_pK^pe^{-B_0K/16}$ by Lemma 6.2. The integral in the second term is at most c_1 by Theorem 6.3 (b). The result follows. $\square$

Lemma 6.7 *If $r>0,\gamma>1$, then there exists $c_1=c_1(r,\gamma)$ such that for all N,*

$$\sum_{m=1}^{N}\frac{m^r}{1+|m-k|^\gamma}\le c_1\Big[N^{(1+r-\gamma)^+}+1_{(\gamma=1+r)}\log N+k^r\Big].$$

Proof. The above sum is bounded by

$$c_2\Big[\sum_{m=1}^{N}\frac{(m-k)^r}{1+|m-k|^\gamma}+k^r\sum_{m=1}^{N}\frac{1}{1+|m-k|^\gamma}\Big].$$

The first term is at most $c_3\sum_{n=1}^{N}n^{r-\gamma}$ and the second term is at most c_4k^r. The result follows. $\square$

For the remainder of this subsection, except for Theorem 6.12, we take $p\ge 1/2$, $\alpha>1/2$, $\gamma>3/2$, $\beta>(2-\alpha/2+p)\vee(3-\alpha)$, and assume (2.4) holds. With a bit of additional work the condition on γ may be weakened to $\gamma > 1$ but in Section 8 we will need stronger conditions on γ so we made no attempt to optimize here.

For $p \geq 1/2$ define

$$\|f(w)\|_{2p} = \Big[\int |f(w')|^{2p} Q_j(w', \widetilde{A}(y,t))\, dw' \Big]^{1/2p},$$

the L^{2p} norm of f.

We start with a rather crude bound. We write $\widetilde{A}w'$ for $\widetilde{A}(y,t)w'$.

Lemma 6.8 *There exists c_1 such that for all $1 \leq k \leq j \leq K$,*

$$\|(\widetilde{A}w')_k\|_{2p} \leq c_1 j^{1/2}.$$

Proof. By (2.4) and Lemma 5.3 we have

$$\begin{aligned}\|(\widetilde{A}w')_k\|_{2p} &\leq c_2 \Big\| \sum_{m=1}^{j} \frac{|w'_m|}{1+|m-k|^\gamma} \Big\|_{2p} \\ &\leq c_3 \sum_{m=1}^{j} \Big(\frac{1}{1+|m-k|^\gamma} \Big) \|w'_m\|_{2p}.\end{aligned}$$

We can use Corollary 6.6 with $K = j$ to bound $\|w'_m\|_{2p}$ by

$$\|(\|w'\|)\|_{2p} \leq c_4 j^{1/2}.$$

The bound follows. $\square$

Lemma 6.9 *Assume there exists $c_1 > 0$ such that*

$$\int |(\widetilde{A}(y,t)w')_k|^{2p} Q_j(w', \widetilde{A}(y,t))\, dw' \leq c_1 \tag{6.19}$$

for all $j \geq k \geq ((j/2) \vee 2)$ and $t > 0$. Then there is a constant c_2, so that for all $1 \leq j \leq K$ and all $t > 0$,

$$\int |w'_j|^{2p} Q_K(w', \widetilde{A}(y,t))\, dw' \leq c_2. \tag{6.20}$$

Proof. If $z = \widetilde{A}(y,t)w'$, then by Lemma 5.2

$$\begin{aligned} \|w'_j\|_{2p} &= \Big\| \sum_{k=1}^{j} \widetilde{a}_{jk} z_k \Big\|_{2p} \\ &\le \sum_{k=1}^{j} \frac{\kappa_\gamma}{1+|k-j|^\gamma} \|z_k\|_{2p}. \end{aligned} \tag{6.21}$$

Use Lemma 6.8 to bound $\|z_k\|_{2p}$ for $k \le (j/2) \vee 1$ and (6.19) to bound it for $k > (j/2) \vee 1$. This leads to

$$\|w'_j\|_{2p} \le c_3 \Big[\sum_{k=1}^{\lfloor j/2 \rfloor \vee 1} j^{-\gamma} j^{1/2} + \sum_{k=\lceil j/2 \rceil \vee 2}^{j} \Big(\frac{1}{1+|k-j|^\gamma} \Big) \Big] \le c_4,$$

where $\gamma > 3/2$ is used in the last line. This gives (6.20). □

To establish (6.19) we argue in a way similar to that of Theorem 6.3. For $j \ge k$, as in (6.19), define $\overline{\pi} : \mathbb{R}^j \to \mathbb{R}^j$ by

$$\overline{\pi}(y_1, \ldots, y_j) = (y_1, \ldots, y_{k-1}, x'_k, y_{k+1}, \ldots, y_j)$$

and

$$b(y,t) = a(\overline{\pi}(y), t), \qquad B(y,t) = A(\overline{\pi}(y), t).$$

As usual

$$\widetilde{b}(y,t) = G(t)^{1/2} b(y,t) G(t)^{1/2}$$

with inverse

$$\widetilde{B}(y,t) = g(t)^{1/2} B(y,t) g(t)^{1/2}.$$

Lemma 6.10 *There exists c_1 such that for all $K \ge j \ge k \ge j/2 > 0$,*

$$\int |(\widetilde{B}w')_k|^{2p} [Q_j(w', \widetilde{A}(y,t)) - Q_j(w', \widetilde{B}(y,t))]\, dw \le c_1.$$

Proof. As usual, $w' = G(t)^{1/2}(y - x')$. If j, k are as above, then by (2.5) and (6.10)

$$\begin{aligned} |a_{mn}(y,t) - b_{mn}(y,t)| &\le \kappa_\beta |w_k|^\alpha k^{-\beta} \\ &\le c_2 \|w'\|^\alpha k^{-\alpha-\beta} \end{aligned} \tag{6.22}$$

by (6.10) and $k \geq 2$. So for $w' \in S_{B_0,j}$ we can use $k \geq j/2$ to conclude

$$|a_{mn}(y,t) - b_{mn}(y,t)| \leq c_3 k^{-\alpha/2-\beta},$$

and therefore using $k \geq j/2$ again,

$$\|a(y,t) - b(y,t)\|_s \leq 2c_3 k^{1-\alpha/2-\beta}.$$

For $w' \in S_{B_0,j}$ we may therefore apply Proposition 4.8 with

$$\theta + \phi \leq c_4 k^{2-\alpha/2-\beta} \leq c_4. \tag{6.23}$$

It follows from Proposition 4.8 and the first inequality in (6.23) that

$$\Big|\frac{Q_j(w', \widetilde{A}(y,t))}{Q_j(w', \widetilde{B}(y,t))} - 1\Big| \leq c_5 k^{2-\alpha/2-\beta} \quad \text{ for } w' \in S_{B_0,j}. \tag{6.24}$$

By our off-diagonal bound (2.4) and Lemma 5.3 we have

$$|\widetilde{B}_{km}| \leq c_6 (1 + |k-m|^\gamma)^{-1}, \tag{6.25}$$

and so (the constants below may depend on p)

$$\begin{aligned} |(\widetilde{B}w')_k|^{2p} &\leq \Big|\sum_{m=1}^{j} \widetilde{B}_{km}^2\Big|^p \|w'\|^{2p} \\ &\leq c_7 \|w'\|^{2p}. \end{aligned} \tag{6.26}$$

Use (6.24) and (6.26) to bound the required integral by

$$\begin{aligned} &\int_{S_{B_0}^c} |(\widetilde{B}w')_k|^{2p} Q_j(w', \widetilde{A}(y,t))\, dw \\ &\qquad + \int_{S_{B_0}} |(\widetilde{B}w')_k|^{2p} Q_j(w', \widetilde{B}(y,t))\, dw'\, c_5 k^{2-\alpha/2-\beta} \\ &\quad \leq c_7 \int_{S_{B_0}^c} \|w'\|^{2p} Q_j(w', \widetilde{A}(y,t))\, dw' \\ &\qquad + c_8 k^{2-\alpha/2+p-\beta} \int_{S_{B_0}} Q_j(w', \widetilde{B}(y,t))\, dw. \end{aligned}$$

The first term is at most $c_p j^p e^{-B_0 j/16}$ by Lemma 6.2, and the last term is bounded by $c_9 k^{2-\alpha/2+p-\beta}$, thanks to Theorem 6.3. Adding the above bounds gives the required result because $\beta \geq 2 - \alpha/2 + p$. $\square$

Lemma 6.11 *There exists a constant c_1 such that for all $j \geq k \geq (j/2) \vee 2$,*

$$\int |((\widetilde{A} - \widetilde{B})w')_k|^{2p} Q_j(w', \widetilde{A}(y,t))\, dw' \leq c_1.$$

Proof. We use

$$\|\widetilde{A} - \widetilde{B}\| = \|\widetilde{A}(\widetilde{b} - \widetilde{a})\widetilde{B}\| \leq \|\widetilde{A}\|\, \|\widetilde{b} - \widetilde{a}\|\, \|\widetilde{B}\|.$$

Lemma 4.5 implies

$$\|\widetilde{B}\|\, \|\widetilde{A}\| \leq \Lambda_0^{-2},$$

and Lemma 4.6 and (6.22) show that

$$\|\widetilde{b} - \widetilde{a}\| \leq \|b - a\|_s \leq c_2 \|w'\|^\alpha k^{1-\beta-\alpha}.$$

These bounds give

$$\begin{aligned} \int & |((\widetilde{A}(y,t) - \widetilde{B}(y,t))w')_k|^{2p} Q_j(w', \widetilde{A}(y,t))\, dw' \\ &\leq \int \|\widetilde{A}(y,t) - \widetilde{B}(y,t)\|^{2p} \|w'\|^{2p} Q_j(w', \widetilde{A}(y,t))\, dw' \\ &\leq c_3 k^{2p(1-\beta-\alpha)} \int \|w'\|^{2p(1+\alpha)} Q_j(w', \widetilde{A}(y,t))\, dw'. \end{aligned}$$

By Corollary 6.6 this is at most $c_4 k^{p(3-2\beta-\alpha)}$, which gives the required bound since $\beta > 3 - \alpha \geq (3-\alpha)/2$. □

Theorem 6.12 *Assume* (2.4) *for some $\gamma > 3/2$ and* (2.5) *for some $\alpha > 1/2$ and*

$$\beta > (2 - \alpha/2 + p) \vee (\tfrac{7}{2} - \alpha/2) \vee (3 - \alpha).$$

Let $p \geq 0$. Let $w' = G(t)^{1/2}(y - x')$. Then there is a $c_1 = c_1(p)$ such that for all $i \leq j \leq K$, $t > 0$, and $x \in \mathbb{R}^K$,

$$\int |w'_j|^{2p} Q_K(w', \widetilde{A}(y,t))\, dw' \leq c_1, \tag{6.27}$$

and

$$\int |w_j|^{2p} Q_K(w, A(y,t))\, dw \leq c_1 \frac{t^p}{(1+\lambda_j t)^p} \leq c_1 t^p. \tag{6.28}$$

Proof. Consider (6.27). First assume $p \geq 1/2$. As $\beta > 3-\alpha$, Theorem 6.3(a) allows us to assume $K = j$. Lemma 6.9 reduces the proof to establishing (6.19) in Lemma 6.9 for j and k as in that result, so assume $j \geq k \geq (j/2)\vee 2$. Lemmas 6.10 and 6.11 imply that

$$\begin{aligned}
&\int |(\widetilde{A}(y,t)w')_k|^{2p} Q_j(w', \widetilde{A}(y,t))\, dw' \qquad (6.29)\\
&\leq c_2\Big[\int |\widetilde{A}(y,t) - \widetilde{B}(y,t))w')_k|^{2p} Q_j(w', \widetilde{A}(y,t))\, dw' \\
&\qquad + \int |(\widetilde{B}(y,t)w')_k|^{2p} Q_j(w', \widetilde{A}(y,t))\, dw'\Big] \\
&\leq c_3\Big[1 + \int |(\widetilde{B}(y,t)w')_k|^{2p} Q_j(w', \widetilde{B}(y,t))\, dw'\Big] \\
&\equiv c_4[1 + I].
\end{aligned}$$

To evaluate the integral I, note that

$$\begin{aligned}
I = \int |\widetilde{B}_{kk}(y,t)|^{2p} \cdot \Big| w'_k + \sum_{m\neq k} \frac{\widetilde{B}_{km}(y,t)w'_m}{\widetilde{B}_{kk}(y,t)}\Big|^{2p} \\
\times Q_j(w', \widetilde{B}(y,t))\, dw'.
\end{aligned}$$

Changing the indices in Lemma 6.1 with $\widetilde{a}$ and $\widetilde{b}$ playing the roles of a and b, respectively, we see that provided we hold the coordinates $\widehat{w} = (w'_j)_{j\neq k}$ fixed, if $\widehat{y} = (y_j)_{j\neq k}$ and $\widehat{B}(\widehat{y},t)$ is the inverse of $(\widetilde{b}_{mn}(y,t))_{m\neq k, n\neq k}$, then $Q_j(w', \widetilde{B}(y,t))/Q_{j-1}(\widehat{w}, \widehat{B}(\widehat{y},t))\, dw'$ as a function of w'_k is the density of a normal random variable with mean

$$\mu = -\sum_{m\neq k} \frac{\widetilde{B}_{km}(y,t)}{\widetilde{B}_{kk}(y,t)}$$

and variance $\sigma^2 = \widetilde{B}_{kk}(y,t)^{-1}$. So if we integrate over w'_k, Lemma 6.1 implies

$$\begin{aligned}
I &= \int |\widetilde{B}_{kk}(y,t)|^p c_p Q_{j-1}(\widehat{w}, \widehat{B}(\widehat{y},t))\, d\widehat{w} \\
&\leq c_p \int Q_{j-1}(\widehat{w}, \widehat{B}(\widehat{y},t))\, d\widehat{w},
\end{aligned}$$

where we used Lemma 5.3 in the last line. Finally we use Theorem 6.3(b) to bound the above integral by c'_p. Put this bound into (6.29) to complete the proof of (6.27) when $p \geq 1/2$.

For $p < 1/2$, we write

$$\int |w'_j|^{2p} Q_K(w', \widetilde{A}(y,t))\, dw' \leq \int (1 + |w'_j|) Q_K(w', \widetilde{A}(y,t))\, dw'$$

and apply the above and Theorem 6.3(b).

The change of variables $w' = G(t)^{1/2} w$ shows that

$$\int |w_j|^{2p} Q_j(w, A(y,t))\, dw = g_{jj}(t)^p \int |w'_j|^{2p} Q_j(w', \widetilde{A}(y,t))\, dw'.$$

Now use (4.7) for $\lambda_j > 0$ and $g_{jj}(t) = t$ if $\lambda_j = 0$, to see that

$$g_{jj}(t) \leq \frac{2t}{1 + \lambda_j t}.$$

This and (6.27) now give (6.28).

$\square$

7 A second derivative estimate

We assume $0 \leq \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_K$ satisfies (2.2) for all $i \leq K$. Our goal in this section is to bound the second derivatives

$$D_{jk} Q_K(y - x', A(y,t)) = \frac{\partial^2}{\partial x_j \partial x_k} Q_K(y - x', A(y,t))$$

uniformly in K. Here $a(y,t)$ and $A(y,t) = a(y,t)^{-1}$ are as in Section 6, and we assume (2.5) for appropriate β and (2.4) for $\gamma > 3/2$ throughout. The precise conditions on β will be specified in each of the results below. The notations A^m, $\widetilde{A}^m$, $\widetilde{A}$ from Section 6 are also used.

A routine calculation shows that for $j, k \leq K$,

$$\begin{aligned} D_{jk} Q_K(y - x', A(y,t)) = e^{-(\lambda_j + \lambda_k)t} & S_{jk}(w, A(y,t)) \\ & \times Q_K(w, A(y,t)), \end{aligned} \tag{7.1}$$

where $w = y - x'$ and for a $K \times K$ matrix A,

$$S_{jk} = S_{jk}(w, A) = \Big(\sum_{n=1}^{K} A_{jn} w_n\Big)\Big(\sum_{n=1}^{K} A_{kn} w_n\Big) - A_{jk}$$
$$= (Aw)_j (Aw)_k - A_{jk}.$$

We use the same notation if A is an $m \times m$ matrix for $m \le K$, but then our sums are up to m instead of K.

We will need a bound on the L^2 norm of a sum of second derivatives. The usual change of variables $w' = G(t)^{1/2}(y - x')$ will reduce this to bounds on

$$I^K_{jk\ell} = \int_{\mathbb{R}^K} S_{j,j+\ell} S_{k,k+\ell}(w', \widetilde{A}(y,t))\, Q_K(w', \widetilde{A}(y,t))\, dw'.$$

These bounds will be derived by induction as in Theorem 6.3 and so we introduce for $m \le K$,

$$I^m_{jk\ell} = \int_{\mathbb{R}^m} S_{j,j+\ell} S_{k,k+\ell}(w', \widetilde{A}^m(y,t)) Q_m(w', \widetilde{A}^m(y,t))\, dw'.$$

As the argument is more involved than the one in the proof of Theorem 6.3, to simplify things we will do our induction from m to $m-1$ rather than using geometric blocks of variables. This leads to a slightly stronger condition on β in Proposition 7.6 below than would otherwise be needed.

If A is an $m \times m$ matrix, we set $A_{ij} = 0$ if i or j is greater than m. This means, for example, that $S_{jk}(w, A) = 0$ if $j \vee k > m$. In what follows x is always fixed, all bounds are uniform in x, and when integrating over w'_j, we will be integrating over $y_j = y_j(w'_j)$ as well.

Since $w' = G(t)^{1/2} w$ we have from (6.11)

$$|w_n| = g_{nn}(t)^{1/2} |w'_n| \le \begin{cases} c_1(\sqrt{t} \wedge (n^{-1}))|w'_n| & \text{if } n \ge 2 \\ c_1\sqrt{t}|w'_n| & \text{if } n = 1. \end{cases} \tag{7.2}$$

Lemma 7.1 *Assume $\beta > \frac{5}{2}$. There exists c_1 such that for all $m, j, k > 0$ and $\ell \ge 0$ satisfying $(j \vee k) + \ell \le m \le K$ and $m \ge 2$,*

$$\int |S_{j,j+\ell} S_{k,k+\ell}(w', \widetilde{A}^m(y,t)) - S_{j,j+\ell} S_{k,k+\ell}(w', \widetilde{A}^m(\pi_{m-1}(y), t))|$$
$$\times\, Q_m(w, \widetilde{A}^m(y,t))\, dw'$$
$$\le c_1 m^{5/2-\beta-\alpha}.$$

Proof. Let j, k, ℓ and m be as above. The pointwise bound on $\widetilde{A}^m$ in Lemma 5.3 implies

$$\begin{aligned} &|S_{k,k+\ell}(w', \widetilde{A}^m(y,t))| \\ &\quad \le c_2\Big[\sum_{n=1}^{m}\sum_{\nu=1}^{m}(1+|n-k|^{\gamma})^{-1}(1+|\nu-k-\ell|^{\gamma})^{-1}|w'_n|\,|w'_\nu| \\ &\qquad + (1+\ell^{\gamma})^{-1}\Big], \end{aligned} \tag{7.3}$$

and so

$$|S_{k,k+\ell}(w', \widetilde{A}^m(y,t))| \le c_3(\|w'\|^2+1). \tag{7.4}$$

The triangle inequality gives

$$\begin{aligned} &|S_{j,j+\ell}(w', \widetilde{A}^m(y,t)) - S_{j,j+\ell}(w', \widetilde{A}^m(\pi_{m-1}(y),t))| \\ &\quad \le |((\widetilde{A}^m(y,t) - \widetilde{A}^m(\pi_{m-1}(y),t))w')_j|\,|(\widetilde{A}^m(y,t)w')_{j+\ell}| \\ &\qquad + |(\widetilde{A}^m(\pi_{m-1}(y),t)w')_j|\,|((\widetilde{A}^m(y,t) - \widetilde{A}^m(\pi_{m-1}(y),t))w')_{j+\ell}| \\ &\qquad + |\widetilde{A}^m_{j,j+\ell}(y,t) - \widetilde{A}^m_{j,j+\ell}(\pi_{m-1}(y),t)|. \end{aligned} \tag{7.5}$$

By (4.9) in Lemma 4.6, for $i \le m$,

$$\begin{aligned} |(\widetilde{A}^m(y,t) - &\widetilde{A}^m(\pi_{m-1}(y),t)w')_i| \\ &\le \|\widetilde{A}^m(y,t) - \widetilde{A}^m(\pi_{m-1}(y),t)\|\,\|w'\| \\ &\le \Lambda_0^{-2}\|\widetilde{a}^m(y,t) - \widetilde{a}^m(\pi_{m-1}(y),t)\|_s\,\|w'\| \\ &\le \Lambda_0^{-2}\sum_{j=1}^{m}\kappa_\beta|w_m|^{\alpha}m^{-\beta}\|w'\| \\ &\le c_4\|w'\|\,|w'_m|^{\alpha}m^{1-\beta-\alpha}, \end{aligned} \tag{7.6}$$

where (7.2) and $m \ge 2$ are used in the last line.

Lemma 5.3 implies that for $i \le m$,

$$|(\widetilde{A}^m(y,t)w')_i| \le c_5\sum_{\nu=1}^{m}(1+|\nu-i|^{\gamma})^{-1}|w'_\nu|, \tag{7.7}$$

and (4.10) together with the calculation in (7.6) implies

$$\begin{aligned} |\widetilde{A}^m(y,t)_{j,j+\ell} - \widetilde{A}^m(\pi_{m-1}(y),t)_{j,+\ell}| &\le \Lambda_0^{-2}\|a^m(y,t) - a^m(\pi_{m-1}(y),t)\|_s \\ &\le c_6|w'_m|^{\alpha}m^{1-\beta-\alpha}, \end{aligned} \tag{7.8}$$

as in (7.6) above.

Now use (7.6), (7.7), and (7.8) in (7.5) and then appeal to (7.4) to conclude that

$$\begin{aligned}
&\int |S_{j,j+\ell}(w', \widetilde{A}^m(y,t)) - S_{j,j+\ell}(w', \widetilde{A}^m(\pi_{m-1}(y),t))|\, |S_{k,k+\ell}(w', \widetilde{A}^m(y,t))| \\
&\qquad \times Q_m(w', \widetilde{A}^m(y,t))\, dw' \qquad\qquad (7.9)\\
&\quad \le c_7 m^{1-\beta-\alpha}\Big\{ \int (\|w'\|^2+1)\, |w'_m|^\alpha Q_m(w', \widetilde{A}^m(y,t))\, dw' \\
&\qquad\qquad + \sum_{\nu=1}^m \Big((1+|\nu-j|^\gamma)^{-1} + (1+|\nu-j-\ell|^\gamma)^{-1} \Big) \\
&\qquad\qquad \times \int |w'_\nu||w'_m|^\alpha [\|w'\|^3 + \|w'\|] \Big\} Q_m(w', \widetilde{A}^m(y,t))\, dw'
\end{aligned}$$

There are several integrals to bound but the one giving the largest contribution and requiring the strongest condition on β will be

$$I = \int |w'_\nu|\, |w'_m|^\alpha\, \|w'\|^3 Q_m(w', \widetilde{A}^m(y,t))\, dw'.$$

Apply Hölder's inequality for triples with $p = \frac{1+\alpha}{1-\varepsilon}$, $q = \frac{1+\alpha}{\alpha(1-\varepsilon)}$ and $r = \varepsilon^{-1}$ to conclude

$$\begin{aligned}
I &\le \Big[\int |w'_\nu|^p Q_m(w', \widetilde{A}^m(y,t))\, dw' \Big]^{1/p} \Big[\int |w'_m|^{\alpha q} Q_m(w', \widetilde{A}^m(y,t))\, dw' \Big]^{1/q} \\
&\qquad \times \Big[\int \|w'\|^{3r} Q_m(w', \widetilde{A}^m(y,t))\, dw' \Big]^{1/r} \\
&\le c_8 m^{3/2}.
\end{aligned}$$

Here we used Corollary 6.6, Theorem 6.12 and the fact that $\beta > 5/2$ means the hypotheses of this last result are satisfied for ε small enough. The other integrals on the right-hand side of (7.9) lead to smaller bounds and so the left-hand side of (7.9) is at most $c_9 m^{5/2-\beta-\alpha}$. A similar bound applies with the roles of j and k reversed, and so the required result is proved. □

Lemma 7.2 *Assume $\beta > 2-(\alpha/2)$. There exists c_1 such that for all j,k,ℓ,m as in Lemma 7.1 and satisfying $2 \le m$,*

$$\begin{aligned}\int |S_{j,j+\ell}S_{k,k+\ell}(w',\widetilde{A}^m(\pi_{m-1}(y),t))|&\\ \times |Q_m(w',\widetilde{A}^m(y,t)) - Q_m(w',\widetilde{A}^m(\pi_{m-1}(y),t))|\,dw'&\\ \le c_1 m^{2-(\alpha/2)-\beta}.&\end{aligned}$$

Proof. Recall that B_0 is as in Lemma 6.2. Use (7.4) on $S^c_{B_0,M}$ and (7.3) on $S_{B_0,m}$ to bound the above integrand by

$$\begin{aligned}&c_2\Big[\int_{S^c_{B_0,m}}(\|w'\|^4+1)[Q_m(w',\widetilde{A}^m(y,t)) + Q_m(w',\widetilde{A}^m(\pi_{m-1}(y),t))]\,dw'\\ &\quad+\int_{S_{B_0,m}}\Big[\Big(\sum_{n=1}^m\sum_{\nu=1}^m(1+|n-k|^\gamma)^{-1}(1+|\nu-k-\ell|^\gamma)^{-1}|w'_n|\,|w'_\nu|\Big)+1\Big]\\ &\qquad\times\Big|\frac{Q_m(w',\widetilde{A}^m(y,t))}{Q_m(w',\widetilde{A}^m(\pi_{m-1}(y),t))}-1\Big|\,Q_m(w',\widetilde{A}^m(\pi_{m-1}(y),t))\Big]\,dw'\Big]\\ &\quad= c_2(I_1(t)+I_2(t)).\end{aligned}$$

By Lemma 6.2,

$$I_1(t) \le c_3 m^2 e^{-B_0 m/16} \le c_4 e^{-B_0 m/17}.$$

We bound $I_2(t)$ as in the proof of Theorem 6.3 but with m in place of K_m. This requires some minor changes. Now for $w' \in S_{B_0,m}$ the δ coming from (6.11) is less than or equal to

$$c_5|w'_m|^\alpha m^{-\beta-\alpha} \le c_6 m^{-\alpha/2-\beta},$$

and

$$\phi\vee\theta \le c_7 m^{2-\alpha/2-\beta} \le c_7.$$

So for $w' \in S_{B_0.m}$, applying Proposition 4.8 as before, we get

$$\Big|\frac{Q_m(w',\widetilde{A}^m(y,t))}{Q_m(w',\widetilde{A}^m(\pi_{m-1}(y),t))}-1\Big| \le c_8 m^{2-\alpha/2-\beta},$$

and therefore

$$\begin{aligned} I_2(t) \le c_8 m^{2-\alpha/2-\beta} \int \Big[\Big(\sum_{n=1}^m \sum_{\nu=1}^m (1+|n-k|^\gamma)^{-1}(1+|\nu-k-\ell|^\gamma)^{-1} \\ \times |w'_n|\,|w'_\nu|\Big) + 1\Big] Q_m(w', \widetilde{A}^m(\pi_{m-1}(y),t))\,dw' \\ \le c_9 m^{2-\alpha/2-\beta}, \end{aligned}$$

where Theorem 6.12 and Cauchy-Schwarz are used in the last line. The lower bound on β shows the hypotheses of Theorem 6.12 are satisfied. Combining the bounds on $I_1(t)$ and $I_2(t)$ completes the proof. □

Note that if Z is a standard normal random variable, then $\mathbb{E}\,[(Z^2-1)^2] = 2$.

Lemma 7.3 *If j,k,ℓ,m are as in Lemma 7.1 and for $w' \in \mathbb{R}^m$,*

$$r_{m-1}w' = (w'_1, \ldots, w'_{m-1}),$$

then

$$\begin{aligned} &\int S_{j,j+\ell}S_{k,k+\ell}(w', \widetilde{A}^m(\pi_{m-1}(y),t)) \frac{Q_m(w', \widetilde{A}^m(\pi_{m-1}(y),t))}{Q_{m-1}(r_{m-1}w', \widetilde{A}^{m-1}(y,t))}\,dw'_m \\ &= \Big\{S_{j,j+\ell}S_{k,k+\ell}(w', \widetilde{A}^{m-1}(y,t))1_{((j\vee k)+\ell\le m-1)}\Big\} \qquad (7.10) \\ &\quad + \Big\{[\widetilde{A}^m_{jm}(\pi_{m-1}(y),t)(\widetilde{A}^{m-1}(y,t)r_{m-1}w')_{j+\ell} \\ &\qquad\qquad + \widetilde{A}^m_{j+\ell,m}(\pi_{m-1}(y),t)(\widetilde{A}^{m-1}(y,t)r_{m-1}w')_j] \\ &\qquad \times [\widetilde{A}^m_{km}(\pi_{m-1}(y),t)(\widetilde{A}^{m-1}(y,t)r_{m-1}w')_{k+\ell} \\ &\qquad\qquad + \widetilde{A}^m_{k+\ell,m}(\pi_{m-1}(y),t)(\widetilde{A}^{m-1}(y,t)r_{m-1}w')_k] \\ &\qquad \times \widetilde{A}^m_{mm}(\pi_{m-1}(y),t)^{-1}\Big\} \\ &\quad + \Big\{2(\widetilde{A}^m_{jm}\widetilde{A}^m_{j+\ell,m}\widetilde{A}^m_{km}\widetilde{A}^m_{k+\ell,m})(\pi_{m-1}(y),t)\widetilde{A}^m_{mm}(\pi_{m-1}(y),t)^{-2}\Big\} \\ &= V^1(j,k,\ell,m) + V^2(j,k,\ell,m) + V^3(j,k,\ell,m). \end{aligned}$$

Proof. We apply Lemma 6.1 with m in place of $m+1$ and $\widetilde{a}^m(\pi_{m-1}(y),t)$ playing the role of $a(y)$. Then under

$$G_m(y,t) = \frac{Q_m(w', \widetilde{A}^m(\pi_{m-1}(y),t))}{Q_{m-1}(r_{m-1}w', \widetilde{A}^{m-1}(y,t))}, \qquad (7.11)$$

w'_m has a normal distribution with mean

$$\mu = -\sum_{i=1}^{m-1} \frac{\widetilde{A}^m_{mi}(\pi_{m-1}(y),t)w'_i}{\widetilde{A}^m_{mm}(\pi_{m-1}(y),t)}$$

and variance $\sigma^2 = \widetilde{A}^m_{mm}(\pi_{m-1}(y),w')^{-1}$. Set $\widehat{w}'_m = w_m - \mu$,

$$R^m_j = \sum_{i=1}^{m-1} \widetilde{A}^m_{ji}(\pi_{m-1}(y),t)w'_i, \qquad j \le m,$$

$$R^{m-1}_j = \sum_{i=1}^{m-1} \widetilde{A}^{m-1}_{ji}(y,t)w'_i, \quad \text{for } j \le m-1, \qquad R^{m-1}_m = 0,$$

$$\text{and} \qquad C_j = \widetilde{A}^m_{mj}(\pi_{m-1}(y),t), \qquad j \le m.$$

Lemma 4.9 with $a = \widetilde{a}^m(\pi_{m-1}(y),t)$ and m in place of $m-1$ gives

$$\widetilde{A}^m_{ji}(\pi_{m-1}(y),t) = \widetilde{A}^{m-1}_{ji}(y,t) + C_jC_i\sigma^2, \qquad j,i \le m,$$

where we recall that by convention $\widetilde{A}^{m-1}_{ji}(y,t) = 0$ if i or j is greater than $m-1$. Therefore

$$R^m_j = R^{m-1}_j - C_j\mu, \qquad j \le m,$$

and so for j,k,ℓ,m as in the lemma,

$$\begin{aligned}
&S_{j,j+\ell}S_{k,k+\ell}(w',\widetilde{A}^m(\pi_{m-1}(y),t)) \\
&\quad = [(R^m_j + C_jw'_m)(R^m_{j+\ell} + C_{j+\ell}w'_m) - \widetilde{A}^m_{j,j+\ell}(\pi_{m-1}(y),t))] \\
&\qquad \times [(R^m_k + C_kw'_m)(R^m_{k+\ell} + C_{k+\ell}w'_m) - \widetilde{A}^m_{k,k+\ell}(\pi_{m-1}(y),t))] \\
&\quad = [(R^{m-1}_j + C_j\widehat{w}'_m)(R^{m-1}_{j+\ell} + C_{j+\ell}\widehat{w}'_m) - \widetilde{A}^{m-1}_{j,j+\ell}(y,t) - C_jC_{j+\ell}\sigma^2] \\
&\qquad \times [(R^{m-1}_k + C_k\widehat{w}'_m)(R^{m-1}_{k+\ell} + C_{k+\ell}\widehat{w}'_m) - \widetilde{A}^{m-1}_{k,k+\ell}(y,t) - C_kC_{k+\ell}\sigma^2].
\end{aligned}$$

Rearranging terms, we see that the above equals

$$\begin{aligned}
&[R^{m-1}_jR^{m-1}_{j+\ell} - \widetilde{A}^{m-1}_{j,j+\ell}(y,t) \\
&\qquad + \widehat{w}'_m(C_jR^{m-1}_{j+\ell} + C_{j+\ell}R^{m-1}_j) + (|\widehat{w}'_m|^2 - \sigma^2)C_jC_{j+\ell}] \qquad (7.12) \\
&\quad \times [R^{m-1}_kR^{m-1}_{k+\ell} - \widetilde{A}^{m-1}_{k,k+\ell}(y,t) + \widehat{w}'_m(C_kR^{m-1}_{k+\ell} + C_{k+\ell}R^{m-1}_k) \\
&\qquad + (|\widehat{w}'_m|^2 - \sigma^2)C_kC_{k+\ell}] \\
&= (R^{m-1}_jR^{m-1}_{j+\ell} - \widetilde{A}^{m-1}_{j,j+\ell}(y,t))(R^{m-1}_kR^{m-1}_{k+\ell} - \widetilde{A}^{m-1}_{k,k+\ell}(y,t)) \\
&\qquad + |\widehat{w}'_m|^2(C_jR^{m-1}_{j+\ell} + C_{j+\ell}R^{m-1}_j)(C_kR^{m-1}_{k+\ell} + C_{k+\ell}R^{m-1}_k) \\
&\qquad + (|\widehat{w}'_m|^2 - \sigma^2)^2C_jC_{j+\ell}C_kC_{k+\ell} + \text{ off-diagonal terms.}
\end{aligned}$$

When we multiply each off-diagonal term by $G_m(y,t)$ and integrate over w'_m, we get zero. This is because the conditional normal distribution of w'_m under $G_m(y,t)$ implies that each of

$$\begin{aligned}
&\int \widehat{w}'_m G_m(y,t)\,dw'_m,\\
&\int (|\widehat{w}'_m|^2-\sigma^2)G_m(y,t)\,dw'_m, \text{ and}\\
&\int (\widehat{w}'_m)(|\widehat{w}'_m|^2-\sigma^2)G_m(y,t)\,dw'_m
\end{aligned}$$

equals zero.

Now integrate the remaining terms on the right hand side of (7.12) with respect to $G_m(y,t)\,dw'_m$, noting that R_i^{m-1}, C_i, and $\widetilde{A}_{ij}^{m-1}$ do not depend on w'_m. Use the fact that

$$\int |\widehat{w}'_m|^2 G_m(y,t)\,dw'_m = \sigma^2 = \widetilde{A}^m(\pi_{m-1}(y),t)^{-1}$$

and

$$\int (|\widehat{w}'_m|^2-\sigma^2)^2 G_m(y,m)\,dw'_m = 2\sigma^4 = 2\widetilde{A}^m(\pi_{m-1}(y),t)^{-2}$$

to obtain the desired expression. In particular note that

$$\begin{aligned}
&(R_j^{m-1}R_{j+\ell}^{m-1}-\widetilde{A}_{j,j+\ell}^{m-1}(y,t))(R_k^{m-1}R_{k+\ell}^{m-1}-\widetilde{A}_{k,k+\ell}^{m-1}(y,t))\\
&\qquad = S_{j,j+\ell}S_{k,k+\ell}(r_{m-1}w',\widetilde{A}^{m-1}(y,t))1_{((j\vee k)+\ell\le m-1)}.
\end{aligned}$$

□

We treat V^2 and V^3 in (7.10) as error terms and so introduce

$$\begin{aligned}
E^1(j,k,\ell,m) &= \int_{\mathbb{R}^{m-1}} |V^2(j,k,\ell,m)|\,dw',\\
E^2(j,k,\ell,m) &= \int_{\mathbb{R}^{m-1}} |V^3(j,k,\ell,m)|\,dw',
\end{aligned}$$

and

$$E(j,k,l,m) = E^1(j,k,\ell,m) + E^2(j,k,\ell,m).$$

We are ready for our inductive bounds on the integral $I_{jk\ell}^m$, defined at the beginning of this section.

Proposition 7.4 *Assume $\beta > \frac{7}{2} - \alpha$. There exists c_1 such that for all integers j, k, ℓ such that $1 \le j \le k \le k + \ell \le K$,*

$$I^K_{jk\ell} \le c_1(k+\ell)^{(7/2)-\alpha-\beta} + \sum_{m=(k+\ell)\vee 2}^{K} E(j,k,\ell,m).$$

Proof. If $K \ge m \ge 2 \vee (k+\ell)$, we can combine Lemmas 7.1, 7.2 and 7.3 to see that

$$I^m_{jk\ell} \le I^{m-1}_{jk\ell} 1_{(k+\ell \le m-1)} + c_2 m^{5/2-\beta-\alpha} + c_3 m^{2-\alpha/2-\beta} + E(j,k,\ell,m).$$

Therefore by induction

$$\begin{aligned} I^K_{jk\ell} \le I^{1\vee(k+\ell-1)}_{jk\ell} 1_{(k+\ell \le 1 \vee (k+\ell-1))} + c_4 \sum_{m=2\vee(k+\ell)}^{K} m^{(5/2)-\beta-\alpha} \\ + \sum_{m=2\vee(k+\ell)}^{K} E(i,j,k,\ell). \end{aligned} \tag{7.13}$$

The first term in the above is $I^1_{110} 1_{(k+\ell=1)}$. For $m = 1$, $\widetilde{A}^1(y,t)$ is a scalar and an argument similar to that in (b) of Theorem 6.3 shows that

$$\begin{aligned} I^1_{110} =& S_{11}(w', \widetilde{A}^1(y,t))^2\, Q_1(w', \widetilde{A}^1(y,t))dw' \\ \le& c_5 \int (1 + \|w'\|^4) Q_1(w', \widetilde{A}^1(y,t))dw' \qquad \text{(by (7.4))} \\ \le& c_6. \end{aligned} \tag{7.14}$$

Use (7.14) to bound the first term in (7.13) and then bound the second terms in the obvious manner to complete the proof. □

To use the above bound we of course will have to control the $E(j,k,\ell,m)$'s. If $\zeta > 0$, set

$$J = J_\zeta(t) = \lceil (\zeta \log(t^{-1}+1))/t)^{1/2} \rceil. \tag{7.15}$$

Lemma 7.5 *Assume $\beta > 3 - (\alpha/2)$. There exists a c_1 such that for all $0 \le \ell \le K$,*

$$\sum_{1 \le j \le k \le J_\zeta(t)} \sum_{m=2\vee(k+\ell)}^{K} E(j,k,\ell,m) \le c_1 J_\zeta(t).$$

Proof. We consider $E^1(j,k,\ell,m)$. There is a product giving rise to four terms, all of which are handled in a similar way. We consider only

$$
\begin{aligned}
&E_1^1(j,k,\ell,m) \\
&= \int_{\mathbb{R}^{m-1}} |\widetilde{A}_{j+\ell,m}(\pi_{m-1}(y),t)(\widetilde{A}^{m-1}(y,t)w')_j \widetilde{A}_{k+\ell,m}(\pi_{m-1}(y),t) \\
&\qquad \times (\widetilde{A}^{m-1}(y,t)w')_k|\widetilde{A}^m_{mm}(\pi_{m-1}(y),t)^{-1}Q_{m-1}(w',\widetilde{A}^{m-1}(y,t))\,dw',
\end{aligned}
$$

as this is the worst term. Use the upper bound on $\widetilde{A}^m_{ij}$ and the lower bound on $\widetilde{A}^m_{ii}$ from Lemma 5.3 to see that

$$
\begin{aligned}
E_1^1(j,k,\ell,m) \le c_2(1+|m-j-\ell|^\gamma)^{-1}&(1+|m-k-\ell|^\gamma)^{-1} \\
\times \sum_{n=1}^{m-1}\sum_{\nu=1}^{m-1}&(1+|n-j|^\gamma)^{-1}(1+|\nu-k|^\gamma)^{-1} \\
\times \int &|w_\nu|\,|w'_\nu|Q(w',\widetilde{A}^{m-1}(y,t))\,dw'.
\end{aligned}
$$

An application of Cauchy-Schwarz and Theorem 6.12 shows that for our value of β the last integral is bounded by c_3. This leads to

$$
E_1^1(j,k,\ell,m) \le c_4(1+|m-j-\ell|^\gamma)^{-1}(1+|m-k-\ell|^\gamma)^{-1}.
$$

Now sum over j,m, and k in that order to see that

$$
\begin{aligned}
&\sum_{1\le j\le k\le J}\ \sum_{m=2\vee(k+\ell)}^{K} E_1^1(j,k,\ell,m) \\
&\qquad \le \sum_{k=1}^{J}\sum_{m=k+\ell}^{K}\sum_{j=1}^{k}(1+|m-j-\ell|^\gamma)^{-1}(1+|m-k-\ell|^\gamma)^{-1}c_4 \\
&\qquad \le \sum_{k=1}^{J}\sum_{m=k+\ell}^{K}(1+|m-k-\ell|^\gamma)^{-1}c_5 \\
&\qquad \le c_6 J.
\end{aligned}
$$

The other terms making up $E^1(j,k,\ell,m)$ are bounded in a similar manner.

Consider now $E^2(j,k,\ell,m)$. Again the upper and lower bounds in Lemma 5.3 and Theorem 6.3(b) imply that for $j \le k \le k+\ell \le m$,

$$\begin{aligned}E^2(j,k,\ell,m) &\le c_7(1+|m-j|^\gamma)^{-1}(1+|m-k|^\gamma)^{-1}(1+|m-j-\ell|^\gamma)^{-1}\\&\quad\times(1+|m-k-\ell|^\gamma)^{-1}\\&\le c_7(1+|m-j-\ell|^\gamma)^{-1}(1+|m-k-\ell|^\gamma)^{-1}.\end{aligned}$$

Again sum over j then m and then k to see

$$\sum_{1\le j\le k\le J}\ \sum_{m=2\vee(k+\ell)}^{K} E^2(j,k,\ell,m) \le c_8 J.$$

Combining the above bounds gives the required result. $\square$

Proposition 7.6 *Assume $\beta > \frac{9}{2}-\alpha$. There exists c_1 so that for any $0 \le \ell \le J$,*

$$\begin{aligned}\int \Big(\sum_{j=1}^{J} e^{-\lambda_j t-\lambda_{j+\ell}t} S_{j,j+\ell}(y-x',A(y,t))\Big)^2 Q_K(y-x',A(y,t))\,dy\\ \le c_1 J t^{-2}.\end{aligned}$$

Proof. As usual we set $w = g(t)^{1/2}w'$, which leads to

$$\begin{aligned}S_{j,j+\ell}(w,A(y,t)) &= S_{j,j+\ell}(g(t)^{1/2}w',A(y,t))\\&= G_{jj}(t)^{1/2}S_{j,j+\ell}(w',\widetilde{A}(y,t))G_{j+\ell,j+\ell}(t)^{1/2}.\end{aligned}$$

Let $H_i(t) = e^{-\lambda_i t}G_{ii}(t)^{1/2}$, so that

$$0 \le H_i(t) = \Big(\int_0^t e^{2\lambda_i(t-s)}\,ds\Big)^{-1/2} \le t^{-1/2}. \tag{7.16}$$

The integral we have to bound now becomes

$$\begin{aligned}\int \Big(\sum_{j=1}^{J} H_j(t)S_{j,j+\ell}(w',\widetilde{A}(y,t))H_{j+\ell}(t)\Big)^2\\ \times Q_K(w',\widetilde{A}(y,t))\,dw'\\ = \sum_{j,k=1}^{J} H_j(t)H_k(t)H_{j+\ell}(t)H_{k+\ell}(t)I^K_{jk\ell}.\end{aligned}$$

Now use the upper bound on H_i, Lemma 7.5, Proposition 7.4 for $j \le k$, and symmetry in (j,k) to bound the above by

$$c_2 t^{-2}\Big\{\sum_{j\le k\le J}\Big[(k+\ell)^{(7/2)-\beta-\alpha}+\sum_{m=2\vee(k+\ell)}^{K} E(j,k,\ell,m)\Big]\Big\}$$
$$\le c_3 t^{-2} J[\ell^{(9/2)-\beta-\alpha}+1]$$

where Lemma 7.5 and the condition on β are used in the last line. □

We need a separate (and much simpler) bound to handle the absolute values of $D_{jk}Q_K(y-x', A(y,t))$ for $j\vee k \ge J_\zeta(t)$.

Lemma 7.7 *Assume $\beta > 3-\frac{\alpha}{2}$. There exists c_1 such that for all $i,j,k \le K$ and $p \ge 0$,*

$$\int |w_i'|^{2p}|S_{jk}(w', \widetilde{A}(y,t))|Q_K(w', \widetilde{A}(y,t))\,dw' \le c_1.$$

Proof. By (7.3) the above integral is at most

$$c_2\int\Big(\sum_{n=1}^{m}\sum_{\nu=1}^{m}(1+|n-j|^\gamma)^{-1}(1+|\nu-k|^\gamma)^{-1}|w_n'|\,|w_\nu'|+1\Big)$$
$$\times |w_i'|^{2p}Q_K(w', \widetilde{A}(y,t))\,dw'.$$

Now apply Theorem 6.12 and Cauchy-Schwarz to obtain the required bound.
□

The proof of the following is left to the reader.

Lemma 7.8 *There exists a constant c_1 so that for all $\theta > 0$, $r \ge 1$,*

$$\sum_{|j|+|k|\ge r} e^{-\theta j^2}e^{-\theta k^2} \le \frac{c_1}{\theta}e^{-\theta r^2/4}.$$

Proposition 7.9 *Assume $\beta > 3-\frac{\alpha}{2}$. There exists c_1 such that for all i,j,k and p*

$$\int_{\mathbb{R}^K}|w_i|^{2p}|D_{jk}Q_K(y-x', A(y,t))|\,dy \le c_1 t^{-1+p}.$$

Proof. As in the proof of Proposition 7.6, if $H_i(t) = e^{-\lambda_i t} G_{ii}(t)^{1/2}$, then the substitution $w = g(t)^{1/2} w'$ leads to

$$\begin{aligned}
\int |w_i|^{2p} |D_{jk} Q_K(y - x', A(y,t))| \, dy \\
&= \int |w_i|^{2p} e^{-(\lambda_j + \lambda_k)t} |S_{jk}(w, A(y,t))| Q_K(w, A(y,t)) \, dw \\
&\le t^p H_j(t) H_k(t) \int |w_i'|^{2p} |S_{jk}(w', \widetilde{A}(y,t))| Q_K(w', \widetilde{A}(y,t)) \, dw' \\
&\le c_2 t^p H_j(t) H_k(t),
\end{aligned}$$

the last by Lemma 7.7.

A bit of calculus shows that

$$H_j(t) = \Big(\int_0^t e^{2\lambda_j(t-s)} \, ds \Big)^{-1/2} \le e^{-\lambda_j t/2} t^{-1/2}.$$

$\square$

Proposition 7.10 *Assume $\beta > 3 - \frac{\alpha}{2}$. There are constants ζ_0 and c_1 such that if $\zeta \ge \zeta_0$ and $J = J_\zeta(t)$, then*

$$\sum_{j=1}^K \sum_{k=1}^K 1_{(j \vee k > J)} \int_{\mathbb{R}^K} |D_{jk} Q_K(y - x', A(y,t))| \, dy \le c_1 (t+1)^{-2}.$$

Proof. Using Proposition 7.9, the sum is at most

$$\begin{aligned}
c_2 \sum_{j=1}^K \sum_{k=1}^K 1_{(j \vee k > J)} e^{-(\lambda_j + \lambda_k)t/2} t^{-1} &\le c_2 \sum_{j=1}^K \sum_{k=1}^K 1_{(j \vee k > J)} e^{-c_3(j^2 + k^2)t} t^{-1} \\
&\le c_4 e^{-c_4 J^2 t} t^{-2}.
\end{aligned}$$

Lemma 7.8 is used in the last line, and (2.2) and $j \vee k > J \ge 1$ are used in the next to the last line. The above bound is at most

$$c_5 (t^{-1} + 1)^{-c_4 \zeta} t^{-2}.$$

Now take $\zeta_0 = 2/c_4$ to complete the proof. $\square$

8 Main estimate

We assume now that a satisfying (2.3) is also of Toeplitz form. For a point v in ℓ^2 define $v'_k = e^{-\lambda_k t}v_k$ and, abusing our earlier notation slightly, define $\pi_k = \pi_k : \ell^2 \to \ell^2$ by

$$\pi_k(x) = (x_i, \dots, x_k, 0, 0, \dots).$$

For $1 \le i, j \le K$ we let

$$a^K_{|i-j|}(x) \equiv a^K_{ij}(x) = a_{ij}(\pi_K(x)), \quad a^K_{ij}(x,t) = a^K_{ij}(x)\int_0^t e^{-(\lambda_i+\lambda_j)s}ds,\ x \in \ell^2,$$

and let $A^K(x,t)$ be the inverse of $a^K(x,t)$. We will apply the results of Sections 6 and 7 to these $K \times K$ matrices. We will sometimes write $\overline{x}_K$ for $(x_1, \dots, x_K)$, and when convenient will identify $\pi_K(x)$ with $\overline{x}_K$. It will be convenient now to work with the notation

$$N_K(t,x,y) = Q_K(\pi_K(y-x'), A^K(y,t)), \tag{8.1}$$

so that

$$N_K(t,x,y) = N_K(t, \pi_K(x), \pi_K(y)),\ x, y \in \ell^2. \tag{8.2}$$

As before $D_{ij}N_K(t,x,y)$ denotes second order partial derivatives in the x variable.

Our goal in this section is to prove the following:

Theorem 8.1 *Assume $(a_{ij}(y))$ satisfies (2.5) and (2.4) for all $i, j, k \in \mathbb{N}$, for some $\alpha \in (\frac{1}{2}, 1]$, $\beta > \frac{9}{2} - \alpha$, and $\gamma > \frac{2\alpha}{2\alpha-1}$. Then there is a $c_1 > 0$ and $\eta_1 = \eta_1(\alpha,\gamma) > 0$ so that for all $x \in \ell^2$, $K \in \mathbb{N}$, and $t > 0$,*

$$\begin{aligned}\int_{\mathbb{R}^K} \Big| \sum_{i,j=1}^{\infty} [a^K_{ij}(x) - a^K_{ij}(y)] D_{ij}N_K(t,x,y)\Big|\, d\overline{y}_K& \\ \le c_1 t^{-1+\eta_1}(1 + \|x\|^\alpha_\infty).&\end{aligned} \tag{8.3}$$

Proof. Note first that by (8.2) $D_{ij}N_K = 0$ if $i \vee j > K$ and so by the symmetry of $a(x)$ and the Toeplitz form of a, the integral we need to bound is

$$
\begin{aligned}
I \equiv & \int_{\mathbb{R}^K} \Big| \sum_{i=1}^{K} \sum_{j=1}^{K} (a^K_{ij}(x) - a^K_{ij}(y)) D_{ij} N_K(t,x,y) \Big| \, d\overline{y}_K \\
\leq & 2 \int_{\mathbb{R}^K} \Big| \sum_{\ell=1}^{K-1} \sum_{j=1}^{K} (a^K_{\ell}(x) - a^K_{\ell}(y)) D_{j+\ell,j} N_K(t,x,y) \Big| \, d\overline{y}_K \\
& + \int_{\mathbb{R}^K} |a^K_0(x) - a^K_0(y)| \Big| \sum_{j=1}^{K} D_{jj} N_K(t,x,y) \Big| \, d\overline{y}_K.
\end{aligned}
$$

Now let $J = J_\zeta(t)$ where ζ is as in Proposition 7.10. If $j > J$ or $\ell \geq J$ then clearly $i = j + \ell > J$, so that

$$
\begin{aligned}
I \leq & \, 2 \int \sum_{\ell=0}^{J-1} |a^K_{\ell}(x) - a^K_{\ell}(y)| \Big| \sum_{j=1}^{J} D_{j+\ell,j} N_K(t,x,y) \Big| \, d\overline{y}_K \qquad (8.4) \\
& + \sum_{i=1}^{K} \sum_{j=1}^{K} 1_{(i \vee j \geq J)} \int |(a^K_{ij}(x) - a^K_{ij}(y)) D_{ij} N_K(t,x,y)| \, d\overline{y}_K \\
= & \, 2I_1 + I_2.
\end{aligned}
$$

Proposition 7.10 implies that

$$
I_2 \leq 2\Lambda_1 c_2 (t+1)^{-2}. \qquad (8.5)
$$

Recalling that $x'_k = e^{-\lambda_k t} x_k$, we can write

$$
\begin{aligned}
I_1 \leq & \sum_{\ell=0}^{J-1} \int |a^K_{\ell}(x') - a^K_{\ell}(y)| \Big| \sum_{j=1}^{J} D_{j,j+\ell} N_K(t,x,y) \Big| \, d\overline{y}_K \\
& + \sum_{\ell=0}^{J-1} |a^K_{\ell}(x) - a^K_{\ell}(x')| \int \Big| \sum_{j=1}^{J} D_{j,j+\ell} N_K(t,x,y) \Big| \, d\overline{y}_K \qquad (8.6) \\
\equiv & I_{1,1} + I_{1,2}. \qquad (8.7)
\end{aligned}
$$

Let

$$
d_{\alpha,\beta}(x,y) = \sum_{n=1}^{K} |x_n - y_n|^{\alpha} n^{-\beta}. \qquad (8.8)
$$

By (2.5) and (2.4),

$$|a^K_\ell(x') - a^K_\ell(y)| \le c_3 \min((1+\ell^\gamma)^{-1}, d_{\alpha,\beta}(x',y)). \tag{8.9}$$

Therefore by (7.1)

$$\begin{aligned}
I_{1,1} =& \sum_{\ell=0}^{J-1} \int |a^K_\ell(x') - a^K_\ell(y)| \Big| \sum_{j=1}^{J} \exp(-(\lambda_j + \lambda_{j+l})t) \\
&\qquad \times S_{j,j+\ell}(\pi_K(y-x'), A^K(y,t)) \Big| N_K(t,x,y)\, d\overline{y}_K \qquad (8.10)\\
\le& c_4 \sum_{\ell=0}^{J-1} \Big[\int \Big((1+\ell^\gamma)^{-2} \wedge d_{\alpha,\beta}(x',y)^2\Big) N_K(t,x,y)\, d\overline{y}_K\Big]^{1/2} \\
&\qquad \times \Big[\int \Big(\sum_{j=1}^{J} \exp(-(\lambda_j+\lambda_{j+\ell})t) S_{j,j+\ell}(\pi_K(y-x'), A^K(y,t))\Big)^2 \\
&\qquad \times N_K(t,x,y)\, d\overline{y}_K\Big]^{1/2} \\
\le& c_5 \Big(\sum_{\ell=0}^{J-1} \Big((1+\ell^\gamma)^{-1} \wedge \Big[\int \Big(\sum_{n=1}^{K} |x'_n - y_n|^{2\alpha} n^{-\beta}\Big) N_K(t,x,y)\, d\overline{y}_K\Big]^{1/2}\Big)\Big) \\
&\qquad \times \sqrt{J} t^{-1}.
\end{aligned}$$

In the last line we used Proposition 7.6 on the second factor and the Cauchy-Schwarz inequality on the sum in the first factor and then Theorem 6.3(b) to bound the total mass in this factor. Next use Theorem 6.12 with $p = \alpha$ to conclude that

$$\int |x'_n - y_n|^{2\alpha} N_K(t,x,y)\, d\overline{y}_K \le c_6 t^\alpha.$$

It now follows from (8.10) and the choice of J that $I_{1,1}$ is at most

$$\begin{aligned}
& c_7 \Big\{\sum_{\ell=0}^{J-1} ((1+\ell^\gamma)^{-1} \wedge (t^{\alpha/2}))\Big\} \Big(\log\Big(\frac{1}{t}+1\Big)\Big)^{1/4} t^{-5/4} \\
&\qquad \le c_8 \Big\{\sum_{\ell=1}^{J} (\ell^{-\gamma} \wedge t^{\alpha/2})\Big\} \Big(\log\Big(\frac{1}{t}+1\Big)\Big)^{1/4} t^{-5/4}. \qquad (8.11)
\end{aligned}$$

By splitting the above sum up at $\ell = \lfloor t^{-\alpha/2\gamma} \rfloor$ we see that

$$\sum_{\ell=1}^{J} (\ell^{-\gamma} \wedge t^{\alpha/2}) \le c_9 \Big(t^{\alpha/2} \Big)^{(\gamma-1)/\gamma}. \tag{8.12}$$

Using this in (8.11), we may bound $I_{1,1}$ by

$$c_{10} \Big(\log\Big(\frac{1}{t} + 1 \Big) \Big)^{1/4} t^{(\alpha(\gamma-1)/2\gamma)-5/4} \le c_{11} t^{-1+\eta}, \tag{8.13}$$

for some $\eta = \eta(\alpha, \gamma) > 0$ because $\gamma > \frac{2\alpha}{2\alpha-1}$.

Turning to $I_{1,2}$, note that

$$\begin{aligned} d_{\alpha,\beta}(x', x) &= \sum_{n=1}^{K} |x_n|^\alpha |1 - e^{-\lambda_n t}|^\alpha n^{-\beta} \\ &\le \|x\|_\infty^\alpha t^\alpha \sum_{n=1}^{\infty} \lambda_n^\alpha n^{-\beta} \\ &\le c_{12} \|x\|_\infty^\alpha t^\alpha, \end{aligned} \tag{8.14}$$

where (2.2) and $\beta - 2\alpha > 1$ are used in the last line. Therefore (8.9) now gives

$$|a_\ell^K(x') - a_\ell^K(x)| \le c_{13} \min((1+\ell^\gamma)^{-1}, \|x\|_\infty^\alpha t^\alpha). \tag{8.15}$$

As in (8.10) we now get (again using Proposition 7.6)

$$\begin{aligned} I_{1,2} \le & \sum_{\ell=0}^{J-1} c_{13} \min((1+\ell^\gamma)^{-1}, \|x\|_\infty^\alpha t^\alpha) \\ & \times \Big[\int \Big(\sum_{j=1}^{J} e^{-(\lambda_j + \lambda_{j+\ell})t} S_{j,j+\ell}(\pi_K(y - x'), A^K(y,t)) \Big)^2 N_K(t,x,y)\, d\overline{y}_K \Big]^{1/2} \\ & \le c_{14} \sqrt{J} t^{-1} \sum_{\ell=0}^{J-1} \min((1+\ell^\gamma)^{-1}, \|x\|_\infty^\alpha t^\alpha). \end{aligned} \tag{8.16}$$

Now use (8.12) with $\|x\|_\infty^\alpha t^\alpha$ in place of $t^{\alpha/2}$ to conclude that

$$I_{1,2} \le c_{15} \Big(\log\Big(\frac{1}{t} + 1 \Big) \Big)^{1/4} t^{-5/4} (\|x\|_\infty^\alpha t^\alpha)^{(\gamma-1)\gamma} \le c_{16} (\|x\|_\infty^\alpha + 1) t^{-1+\eta} \tag{8.17}$$

for some $\eta = \eta(\alpha, \gamma) > 0$ because $\gamma > \frac{2\alpha}{2\alpha-1} > \frac{4\alpha}{4\alpha-1}$.

Finally use the above bound on $I_{1,2}$ and the bound on $I_{1,1}$ in (8.13) to bound I_1 by the right-hand side of (8.3). Combining this with the bound on I_2 in (8.5) completes the proof. □

For $R > 0$ let $p_R : \mathbb{R} \to \mathbb{R}$ be given by $p_R(x) = (x \wedge R) \vee (-R)$ and define a truncation operator $\tau_R : \ell^2 \to \ell^2$ by $(\tau_R x)_n = p_R(x_n)$. Define a^R by

$$a^R(x) = a(\tau_R x). \tag{8.18}$$

Clearly $a^R(x) = a(x)$ whenever $\|x\|_\infty \equiv \sup_n |x_n| \le R$. We write $a^{K,R}$ for the $K \times K$ matrix $(a^R)^K$.

Lemma 8.2 *For any $\lambda \ge 0$ and $t, R > 0$, $\sup_{x\in\mathbb{R}} |p_R(x) - p_R(xe^{-\lambda t})| \le R\lambda t$.*

Proof. Assume without loss of generality that $x > 0$ and set $x' = e^{-\lambda t}x$. If $x' \ge R$, $p_R(x) = p_R(x') = R$, and if $x \le R$, then

$$|p_R(x) - p_R(x')| = |x - x'| = (1 - e^{-\lambda t})x \le \lambda t R.$$

Finally if $x' < R < x$, then

$$|p_R(x) - p_R(x')| = R - x' = R - e^{-\lambda t}x \le R(1 - e^{-\lambda t}) \le \lambda t R.$$

□

Lemma 8.3 *If a satisfies (2.3), (2.5), and (2.4) and is of Toeplitz form, then for any $R > 0$, a^R satisfies the same conditions with the same constants.*

Proof. This is elementary and so we only consider (2.5). For this note that

$$\begin{aligned}|a^R_{ij}(y + he_k) - a^R_{ij}(y)| &\le \kappa_\beta |p_R(x_k + h) - p_R(x_k)|^\alpha k^{-\beta}\\ &\le \kappa_\beta |h|^\alpha k^{-\beta},\end{aligned}$$

as required. □

Corollary 8.4 *Assume the hypotheses of Theorem 8.1. Then for all $x \in \ell^2$, $K \in \mathbb{N}$ and $R, t > 0$,*

$$\int_{\mathbb{R}^K} \Big| \sum_{i,j=1}^{\infty} [a_{ij}^{K,R}(x) - a_{ij}^{K,R}(y)] D_{ij} N_K(t,x,y) \Big| \, d\overline{y}_K \qquad (8.19)$$
$$\leq c_1 t^{-1+\eta_1}(1 + R^\alpha).$$

Proof. We use the notation in the proof of Theorem 8.1. By Lemma 8.3 and the proof of Theorem 8.1 it suffices to show that we have

$$I_{1,2} \leq c_2(R^\alpha + 1)t^{-1+\eta} \qquad (8.20)$$

instead of (8.17). We have by Lemma 8.2

$$\begin{aligned} d_{\alpha,\beta}(\tau_R x', \tau_R x) &= \sum_{n=1}^{K} |p_R(x_n) - p_R(e^{-\lambda_n t} x_n)|^\alpha n^{-\beta} \\ &\leq \sum_{n=1}^{K} (R\lambda_n t)^\alpha n^{-\beta} \\ &\leq (Rt)^\alpha c_3 \sum_{n=1}^{K} n^{2\alpha - \beta} \\ &\leq c_4 R^\alpha t^\alpha. \qquad (8.21) \end{aligned}$$

The fact that $\beta - 2\alpha > 1$ is used in the last line. Now use (8.21) in place of (8.14) and argue exactly as in the proof of (8.17) to derive (8.20) and so complete the proof. □

9 Uniqueness

In this section we prove Theorem 2.1. Recall the definitions of $\mathcal{T}_k^2$ and $\mathcal{T}_k^{2,C}$ and the definition of the martingale problem for the operator $\mathcal{L}$ from Section 2. Throughout this section we assume the hypotheses of Theorem 2.1 are in force.

Lemma 9.1 *There exists c_1 so that for all $x, y \in \ell^2$,*

$$\|a(x) - a(y)\| \leq \|a(x) - a(y)\|_s \leq c_1 |x - y|^{\alpha/2}.$$

Proof. We need only consider the second inequality by (4.3). Our hypotheses (2.5) and (2.4) imply

$$\begin{aligned} |a_{ij}(x) - a_{ij}(y)| &\leq \min\Big(\frac{2\kappa_\gamma}{1 + |i-j|^\gamma}, \kappa_\beta \sum_k |x_k - y_k|^\alpha k^{-\beta}\Big) \\ &\leq c_2 (1 + |i-j|^{-\gamma/2}) \Big(\sum_k |x_k - y_k|^\alpha k^{-\beta}\Big)^{1/2}. \end{aligned}$$

We have $\gamma > 2$ and $2\beta > 2 - \alpha$ by (2.5), and so

$$\begin{aligned} \sum_j |a_{ij}(x) - a_{ij}(y)| &\leq c_3 \Big(\sum_k |x_k - y_k|^\alpha k^{-\beta}\Big)^{1/2} \\ &\leq c_3 \Big(\sum_k |x_k - y_k|^2\Big)^{\alpha/4} \Big(\sum_k k^{-2\beta/(2-\alpha)}\Big)^{(2-\alpha)/4} \\ &\leq c_4 \|x - y\|^{\alpha/2}. \end{aligned}$$

□

Proposition 9.2 *For each $v \in \ell^2$ there is a solution to the martingale problem for $\mathcal{L}$ starting at v.*

Proof. This is well known and follows, for example from the continuity of a given by Lemma 9.1 and Theorem 4.2 of [1]. □

We turn to uniqueness. Let $\mathcal{L}^R(x)$ be defined in terms of a^R analogously to how $\mathcal{L}$ is defined in terms of a.

Lemma 9.3 *For any $R > 0$ and $v \in \ell^2$ there is a unique solution to the martingale problem for $\mathcal{L}_R$ starting at v.*

Proof. By Lemma 8.3 and Proposition 9.2 we only need show uniqueness.

We fix $R > 0$ and for $K \in \mathbb{N}$ define

$$\mathcal{M}_K^x f(z) = \sum_{i,j \le K} a_{ij}^R(x) D_{ij} f(z) - \sum_{j \le K} \lambda_j z_j D_j f(z).$$

Note that if $f \in \mathcal{T}_k^2$ and $K \ge k$, then

$$\mathcal{L}_R f(x) = \mathcal{M}_K^x f(x). \tag{9.1}$$

Let

$$\gamma_K(dy) = m(dy_1) \cdots m(dy_K) \delta_0(dy_{K+1}) \delta_0(dy_{K+2}) \cdots,$$

where m is Lebesgue measure on $\mathbb{R}$ and δ_z is point mass at z. Define $\|f\|_{C_0} = \sup_z |f(z)|$.

Suppose $\mathbb{P}_1, \mathbb{P}_2$ are two solutions to the martingale problem for $\mathcal{L}_R$ started at some fixed point v. For $\theta > 0$ and f bounded and measurable on ℓ^2, let

$$S_\theta^i f = \mathbb{E}_i \int_0^\infty e^{-\theta t} f(X_t)\, dt, \qquad i = 1, 2,$$

and $S_\Delta f = S_\theta^1 f - S_\theta^2 f$. Set

$$\Gamma = \sup_{\|f\|_{C_0} \le 1} |S_\Delta f|.$$

Note

$$\Gamma < \infty \tag{9.2}$$

by the definition of $S_\theta^i f$.

If $f \in \mathcal{T}^2$, we have

$$f(X_t) - f(X_0) = M^f(t) + \int_0^t \mathcal{L}_R f(X_s)\, ds$$

where M^f is a martingale under each $\mathbb{P}_i$. Taking expectations, multiplying both sides by $\theta e^{-\theta t}$, and integrating over t from 0 to ∞, we see that

$$f(v) = S_\theta^i(\theta f - \mathcal{L}_R f).$$

Now take differences in the above to get

$$S_\Delta(\theta f - \mathcal{L}_R f) = 0. \tag{9.3}$$

Next let $g \in \mathcal{T}_k^{2,C}$ and for $K \geq k$ set

$$f_{\varepsilon K}(x) = \int e^{\theta\varepsilon} \int_\varepsilon^\infty e^{-\theta t} N_K(t,x,y) g(y)\, dt\, \gamma_K(dy).$$

Since $N_K(t,x,y)$ is smooth in x, bounded uniformly for $t \geq \varepsilon$ and $N_K(t,x,y)$ depends on x only through $\pi_K(x)$, we see that $f_{\varepsilon K} \in \mathcal{T}_K^2$.

If we write

$$W_{\varepsilon K}(x,y) = e^{\theta\varepsilon} \int_\varepsilon^\infty e^{-\theta t} N_K(t,x,y)\, dt, \tag{9.4}$$

then

$$f_{\varepsilon K}(x) = \int W_{\varepsilon K}(x,y) g(y)\, \gamma_K(dy).$$

Holding y fixed and viewing $N_K(t,x,y)$ and $W_{\varepsilon K}(x,y)$ as functions of x, we see by Kolmogorov's backward equation for the Ornstein-Uhlenbeck process with diffusion matrix $(a_{ij}(y))_{i,j\leq K}$ that

$$\mathcal{M}_K^{\pi_K(y)} N_K(t,x,y) = \frac{\partial}{\partial t} N_K(t,x,y).$$

Alternatively, one can explicitly calculate the derivatives. Differentiating under the integral in (9.4) gives

$$(\theta - \mathcal{M}_K^{\pi_K(y)}) W_{\varepsilon K}(x,y) = N_K(\varepsilon,x,y). \tag{9.5}$$

By (9.1) for all x and $K \geq k$

$$\begin{aligned}
(\theta - \mathcal{L}_R) f_{\varepsilon K}(x) &= (\theta - \mathcal{M}_K^x) f_{\varepsilon K}(x) \qquad (9.6)\\
&= \int (\theta - \mathcal{M}_K^{\pi_K(y)}) W_{\varepsilon K}(x,y) g(y)\, \gamma_K(dy)\\
&\quad - \int (\mathcal{M}_K^{\pi_K(x)} - \mathcal{M}_K^{\pi_K(y)}) W_{\varepsilon K}(x,y) g(y)\, \gamma_K(dy)\\
&\quad - \int (\mathcal{M}_K^x - \mathcal{M}_K^{\pi_K(x)}) W_{\varepsilon K}(x,y) g(y)\, \gamma_K(dy)\\
&= g(x) + \Big[\int N_K(\varepsilon,x,y) g(y)\, \gamma_K(dy) - g(x) \Big]\\
&\quad - \int (\mathcal{M}_K^{\pi_K(x)} - \mathcal{M}_K^{\pi_K(y)}) W_{\varepsilon K}(x,y) g(y)\, \gamma_K(dy)\\
&\quad - \int (\mathcal{M}_K^x - \mathcal{M}_K^{\pi_K(x)}) W_{\varepsilon K}(x,y) g(y)\, \gamma_K(dy)\\
&= g(x) + I_1(\varepsilon,K,x) + I_2(\varepsilon,K,x) + I_3(\varepsilon,K,x).
\end{aligned}$$

We used (9.5) in the third equality.

For $x \in \ell^2$ fixed we first claim that

$$I_1(\varepsilon, K, x) \to 0 \tag{9.7}$$

boundedly and uniformly in $K \geq k$ as $\varepsilon \to 0$. By virtue of Proposition 6.5, it suffices to show

$$\int N_K(\varepsilon, x, y)[g(y) - g(x)]\, \gamma_K(dy) \to 0$$

boundedly and pointwise as $\varepsilon \to 0$, uniformly in $K \geq k$. The boundedness is immediate from Theorem 6.3. Since $g \in \mathcal{T}_k^2$, given η there exists δ such that $|g(y) - g(x)| \leq \eta$ if $|\pi_k(y - x)| \leq \delta$, and using Theorem 6.3, it suffices to show

$$\int_{\{y: \sum_{i=1}^k |y_i - x_i|^2 \geq \delta^2\}} N_K(\varepsilon, x, y)\, \gamma_K(dy) \to 0$$

pointwise as $\varepsilon \to 0$, uniformly in K. Since $e^{-\lambda_i \varepsilon} x_i \to x_i$ for $i \leq k$ as $\varepsilon \to 0$, it suffices to show (recall $\overline{y}_K = (y_1, \ldots, y_K)$)

$$\int_{\{y: \sum_{i=1}^k |y_i - x_i'|^2 \geq \delta^2/2\}} N_K(\varepsilon, x, y)\, d\overline{y}_K \to 0 \tag{9.8}$$

as $\varepsilon \to 0$ uniformly in $K \geq k$. By Theorem 6.12 the above integral is at most

$$\int \sum_{i=1}^k \frac{|y - x_i'|^2}{\delta^2/2} N_K(\varepsilon, x, y)\, d\overline{y}_K \leq \frac{c_1 k \varepsilon}{\delta^2/2}$$

and (9.7) is established.

Next we claim that for each $\varepsilon > 0$

$$\lim_{K \to \infty} \sup_x |I_3(\varepsilon, K, x)| = 0. \tag{9.9}$$

Since $t \geq \varepsilon$ in the integral defining $W_\varepsilon(x, y)$ we can differentiate through the integral and conclude that

$$\begin{aligned} &|I_3(\varepsilon, K, x)| \qquad (9.10)\\ &\quad \leq \int_\varepsilon^\infty e^{-\theta(t-\varepsilon)} \sum_{i,j \leq K} |a_{ij}^R(x) - a_{ij}^R(\pi_K(x))|\, \|g\|_{C_0} \\ &\qquad\qquad \times \int_{\mathbb{R}^K} e^{-(\lambda_i + \lambda_j)t} |S_{ij}(w, A^K(y, t))| Q(w, A^K(y, t)) dw\, dt. \end{aligned}$$

As in the proof of Proposition 7.6, the substitution $w' = G(t)^{1/2}w$ shows that the integral over $\mathbb{R}^K$ in (9.10) equals

$$\int_{\mathbb{R}^K} H_i(t)H_j(t)\,|S_{ij}(w', \widetilde{A}^K(y,t)|Q_K(w'\widetilde{A}^K(y,t))\,dw' \le c_2 t^{-1}, \tag{9.11}$$

where (7.16) and Lemma 7.7 are used in the above. By (2.5) we have

$$\begin{aligned}\sum_{i,j\le K}|a_{ij}^R(x) - a_{ij}^R(\pi_K(x))| &\le \sum_{i,j\le K}\sum_{\ell>K}\kappa_\beta|p_R(x_\ell)|^\alpha \ell^{-\beta} \\ &\le \kappa_\beta K^2 R^\alpha \sum_{\ell>K}\ell^{-\beta} \\ &\le c_3 R^\alpha K^{3-\beta}.\end{aligned} \tag{9.12}$$

Use (9.11) and (9.12) in (9.10) to get

$$\begin{aligned}|I_3(\varepsilon, K, x)| &\le \int_\varepsilon^\infty e^{-\theta(t-\varepsilon)} c_4 t^{-1} R^\alpha K^{3-\beta}\,dt \\ &\le c_4\theta^{-1}\varepsilon^{-1}R^\alpha K^{3-\beta},\end{aligned}$$

which proves (9.9) by our hypothesis on β.

Finally for I_2, we use Corollary 8.4 and multiply both sides of (8.19) by $e^{-\theta(t-\varepsilon)}$, and then integrate over t from ε to ∞ to obtain by Fubini

$$\begin{aligned}&|I_2(\varepsilon, K, x)| \\ &= \Big|\int_{y\in\mathbb{R}^K}(\mathcal{M}_K^{\pi_K(x)} - \mathcal{M}_K^{\pi_K(y)})\Big[e^{\theta\varepsilon}\int_\varepsilon^\infty e^{-\theta t}N_K(t,\cdot,y)g(y)\,dt\Big](x)\gamma_K(dy)\Big| \\ &\le \tfrac{1}{4}\|g\|_{C_0},\end{aligned} \tag{9.13}$$

for all $\varepsilon \in (0,1)$ and $K \ge k$, provided we choose $\theta > \theta_0 \ge 1$, where θ_0 depends on R and the c_1 and η_1 of Theorem 8.1. This implies that for $\theta > \theta_0$,

$$\sup_{\varepsilon\in(0,1),K\ge k}|S_\Delta(I_2(\varepsilon, K, \cdot))| \le \tfrac{1}{2}\Gamma\|g\|_{C_0}. \tag{9.14}$$

Using (9.3) and (9.6) for $K \ge k$, we have

$$|S_\Delta g| \le |S_\Delta(I_1(\varepsilon, K, \cdot))| + |S_\Delta(I_2(\varepsilon, K, \cdot))| + |S_\Delta(I_3(\varepsilon, K, \cdot))|.$$

Now let $K \to \infty$ and use (9.9) and (9.14) to conclude that

$$\begin{aligned}|S_\Delta g| &\le \limsup_{K\to\infty} |S_\Delta(I_1(\varepsilon, K, \cdot))| + \limsup_{K\to\infty} |S_\Delta(I_2(\varepsilon, k, \cdot))| \\ &\le \limsup_{K\to\infty} |S_\Delta(I_1(\varepsilon, K, \cdot))| + \tfrac{1}{2}\Gamma \|g\|_{C_0}.\end{aligned}$$

Then letting $\varepsilon \to 0$ and using (9.7), we obtain

$$|S_\Delta g| \le \tfrac{1}{2}\Gamma \|g\|_{C_0},$$

provided $g \in \mathcal{T}_k^{2,C}$. By a monotone class argument and the fact that S_Δ is the difference of two finite measures, we have the above inequality for $g \in \mathcal{T}$. The σ-field we are using is generated by the cylindrical sets, so another application of the monotone class theorem leads to

$$|S_\Delta g| \le \tfrac{1}{2}\Gamma \|g\|_{C_0}$$

for all bounded g which are measurable with respect to $\sigma(\cup_j \mathcal{T}_j)$. Taking the supremum over all such g bounded by 1, we obtain

$$\Gamma \le \tfrac{1}{2}\Gamma.$$

Since $\Gamma < \infty$ by (9.2), then $\Gamma = 0$ for every $\theta > \theta_0$.

This proves that $S^1_\theta f = S^2_\theta f$ for every bounded and continuous f. By the uniqueness of the Laplace transform, this shows that the one-dimensional distributions of X_t are the same under $\mathbb{P}_1$ and $\mathbb{P}_2$. We now proceed as in [15, Chapter 6] or [3, Chapter 5] to obtain uniqueness of the martingale problem for $\mathcal{L}_R$. $\square$

We now complete the proof of the main result for infinite-dimensional stochastic differential equations from the introduction.

Proof of Theorem 2.1. We have existence holding by Proposition 9.2. Uniqueness follows from Lemma 9.3 by a standard localization argument; see [4, Section 6]. $\square$

To derive Corollary 2.2 from Theorem 2.1 is completely standard and is left to the reader.

10 SPDEs

Before proving our uniqueness result for our SPDE, we first need need a variant of Theorem 2.1 for our application to SPDEs. Let $\lambda_0 = 0$ and now let

$$\mathcal{L}' f(x) = \sum_{i,j=0}^{\infty} a_{ij}(x) D_{ij} f(x) - \sum_{i=0}^{\infty} \lambda_i x_i D_i f(x).$$

In this case $\ell^2 = \ell^2(\mathbb{Z}_+)$.

Theorem 10.1 *Suppose α, β, γ, and the λ_i are as in Theorem 2.1 and in addition $\beta > \gamma/(\gamma - 2)$. Suppose a satisfies (2.3) and a can be written as $a_{ij} = a_{ij}^{(1)} + a_{ij}^{(2)}$, where the $a_{ij}^{(1)}$ satisfy (2.4) and (2.5) and is of Toeplitz form, and $a_{ij}^{(2)}$ satisfies (2.5) and there exists a constant κ'_γ such that*

$$|a_{ij}^{(2)}(x)| \le \frac{\kappa'_\gamma}{1 + (i+j)^\gamma} \tag{10.1}$$

for all $x \in \ell^2$ and $i, j \ge 0$. Then if $v \in \ell^2$, there exists a solution to the martingale problem for $\mathcal{L}'$ starting at v and the solution is unique in law.

Proof. First, all the arguments of the previous sections are still valid when we let our indices run over $\{0, 1, 2, \ldots\}$ instead of $\{1, 2, \ldots\}$ provided

(1) we replace expressions like $2\lambda_i/(1 - e^{-2\lambda_i t})$ by $1/t$ when $\lambda_i = 0$, which happens only when $i = 0$, and

(2) we replace expressions like $n^{-\beta}$ by $(1+n)^{-\beta}$.

Existence follows from Theorem 4.2 of [1] as in the proof of Theorem 2.1.

Define N_K in terms of a and its inverse A as in (8.1). We prove the following analog of (8.19) exactly as in the proof of Corollary 8.4 and Theorem 8.1:

$$\begin{aligned} \int_{\mathbb{R}^K} \Big| \sum_{i,j=1}^{\infty} [(a^{(1)})_{ij}^{K,R}(x) - (a^{(1)})_{ij}^{K,R}(y)] D_{ij} N_K(t,x,y) \Big| \, dy \qquad & (10.2) \\ \le c_1 t^{-1+\eta_1}(1 + R^\alpha). \qquad & \end{aligned}$$

Here note that the proof uses the bounds on N_K and $D_{ij}N_K$ from Sections 6 and 7 and the regularity properties of $a^{(1)}$ (which are the same as those of a in the proof of Theorem 2.1) separately. If we prove the analog of (10.2) with $a^{(1)}$ replaced by $a^{(2)}$, we can then proceed exactly as in Section 9 to obtain our theorem. That is, it suffices to fix K and R and to show that for some $c_1, \eta_1 > 0$,

$$\int_{\mathbb{R}^K} \Big| \sum_{i,j=1}^{J} [(a^{(2)})_{ij}^{K,R}(x) - (a^{(2)})_{ij}^{K,R}(y)] D_{ij} N_K(t,x,y) \Big| \, dy \le c_1 t^{-1+\eta_1}. \quad (10.3)$$

Very similarly to the derivation of (8.15) (see also that of (8.21)), we have

$$|(a^{(2)})_{ij}^{K,R}(x) - (a^{(2)})_{ij}^{K,R}(x')| \le c_1 \min((1+i+j)^{-\gamma}, R^\alpha t^\alpha).$$

Since $\alpha \in (1/2, 1]$ and $\gamma > 2\alpha/(2\alpha - 1)$, then $\gamma > 2$. We can choose $\eta_2 \in (0,1)$ such that $\gamma(1-\eta_2) > 2$, and then

$$|(a^{(2)})_{ij}^{K,R}(x) - (a^{(2)})_{ij}^{K,R}(x')| \le c_1 (1+i+j)^{-\gamma(1-\eta_2)} R^{\alpha\eta_2} t^{\alpha\eta_2}.$$

Using this and Proposition 7.9 with $p = 0$ and observing that $(a^{(2)})^{K,R}$ satisfies all the hypotheses in Section 7, we conclude that

$$\sum_{i,j=0}^{J} \int |(a^{(2)})_{ij}^{K,R}(x) - (a^{(2)})_{ij}^{K,R}(x')| \, |D_{ij} N_K(t,x,y)| \, dy \quad (10.4)$$
$$\le c_2 \sum_{i,j=0}^{J} (1+i+j)^{-\gamma(1-\eta_2)} t^{\alpha\eta_2 - 1}.$$

The condition $\beta > \gamma/(\gamma-2)$, allows us to find η_3 such that $\gamma(1-\eta_3) > 2$ and $\beta\eta_3 > 1$. Fix i and j for the moment and let $d_{\alpha,\beta}(x,y)$ be defined as in (8.8). We write

$$\begin{aligned}
\int d_{\alpha,\beta}(x', y)^{\eta_3} |D_{ij} N_K(t,x,y)| \, dy \\
&\le \int \sum_{n=0}^{\infty} |x'_n - y_n|^{\alpha\eta_3} (n+1)^{-\beta\eta_3} |D_{ij} N_K(t,x,y)| \, dy \\
&\le \sum_{n=0}^{\infty} (n+1)^{-\beta\eta_3} t^{\alpha\eta_3/2 - 1} \\
&\le c_3 t^{\alpha\eta_3/2-1},
\end{aligned}$$

using Proposition 7.9. Since

$$|(a^{(2)})_{ij}^{K,R}(x') - (a^{(2)})_{ij}^{K,R}(y)| \le c_4 \min((1+i+j)^{-\gamma}, d_{\alpha,\beta}(x',y)),$$

then

$$|(a^{(2)})_{ij}^{K,R}(x') - (a^{(2)})_{ij}^{K,R}(y)| \le c_4 (1+i+j)^{-\gamma(1-\eta_3)} d_{\alpha,\beta}(x',y)^{\eta_3}.$$

Consequently

$$\begin{aligned} \sum_{i,j=0}^{J} \int |(a^{(2)})_{ij}^{K,R}(x') - (a^{(2)})_{ij}^{K,R}(y)|\, |D_{ij} N_K(t,x,y)|\, dy \qquad\qquad (10.5) \\ \le c_5 \sum_{i,j=0}^{J} (1+i+j)^{-\gamma(1-\eta_3)} \sup_{i,j} \int d_{\alpha,\beta}(x',y)^{\eta_3} |D_{ij} N_K(t,x,y)|\, dy \\ \le c_5 t^{\alpha\eta_3/2-1}. \end{aligned}$$

Combining with (10.4) gives (10.3), as required. □

Before proving Theorem 2.3, we need the following lemma. Recall that $e_n = \sqrt{2}\cos n\pi x$ for $n \ge 1$ and $e_0 \equiv 1$.

Lemma 10.2 *Suppose $f \in C^\zeta_{per}$ and $\|f\|_{C^\zeta} \le 1$. There exists a constant c_1 depending only on ζ such that*

$$|\langle f, e_n\rangle| \le \frac{c_1}{1+n^\zeta} \qquad \textit{for all } n \in \mathbb{Z}_+.$$

Proof. Let T be the circle of circumference 2 obtained by identifying ± 1 in $[-1,1]$. Since we can extend the domain of f to T so that f is C^ζ on T and $\cos y = \frac{1}{2}(e^{iy} + e^{-iy})$, it suffices to show that the Fourier coefficients of a C^ζ function on T decay at the rate $|n|^{-\zeta}$. If $\zeta = k + \delta$ for $k \in \mathbb{Z}_+$ and $\delta \in [0,1)$, [19, II.2.5] says that the n^{th} Fourier coefficients of $f^{(k)}$ is $c_2|n|^k$ times the n^{th} Fourier coefficient of f. Writing $\widehat{g}$ for the Fourier coefficients of g, we then have $|\widehat{f}(n)| \le c_3 |n|^{-k} |\widehat{f^{(k)}}(n)|$. By [19, II.4.1],

$$|\widehat{f^{(k)}}(n)| \le c_4 |n|^{-\delta}.$$

Combining proves the lemma. □

We now prove Theorem 2.3.

Proof of Theorem 2.3. Our first job will be to use the given $A : C[0,1] \to C[0,1]$ to build a corresponding mapping $a : \ell^2 \to \mathcal{L}_+(\ell^2, \ell^2)$, where $\mathcal{L}_+(\ell^2, \ell^2)$ is the space of self-adjoint bounded positive definite mappings on ℓ^2, so that a satisfies the hypotheses of Theorem 10.1.

We first argue that A has a unique continuous extension to a map $\overline{A} : L^2[0,1] \to L^2[0,1]$. Let $\mathcal{S}$ be the space of finite linear combinations of the $\{e_k\}$. If $u = \sum_{i=0}^N x_i e_i$, $v = \sum_{i=0}^N y_i e_i \in \mathcal{S}$, then by (2.10) and Hölder's inequality we have

$$\begin{aligned}
\|A(u) - A(v)\|_2 &\le \kappa_1 \sum_{i=0}^N |x_i - y_i|^\alpha (i+1)^{-\beta} \\
&\le \kappa_1 \|u - v\|_2^\alpha \left(\sum_{i=0}^N (i+1)^{-2\beta/(2-\alpha)} \right)^{(2-\alpha)/2} \\
&\le c_1 \|u - v\|_2^\alpha,
\end{aligned}$$

because $\beta > \frac{9}{2} - \alpha > (2-\alpha)/2$. Using (2.9), we have

$$\|A(u) - A(v)\|_2 \le c_1 \|u - v\|_2^\alpha \tag{10.6}$$

for $u, v \in C[0,1]$. Therefore A, whose domain is $C[0,1]$, is a bounded operator with respect to the L^2 norm. Thus there is a unique extension of A to all of L^2. By continuity, it is clear that the extension satisfies (2.10), (2.11) (for almost every x with respect to Lebesgue measure), and (2.12).

If $x = \{x_j\} \in \ell^2$, let $u(x) = \sum_{j=0}^\infty x_j e_j \in L^2$ and define a symmetric operator on ℓ^2 by

$$a_{jk}(x) = \int_0^1 A(u(x))(y)^2 e_j(y) e_k(y)\, dy.$$

If $z \in \ell^2$, then

$$\begin{aligned}
\sum_{i,j} z_i a_{ij}(x) z_j &= \sum_{i,j} \int_0^1 z_i e_i(y) A(u)(y)^2 z_j e_j(y)\, dy \\
&= \int_0^1 \Big(\sum_i z_i e_i(y) \Big)^2 A(u)(y)^2\, dy
\end{aligned}$$

$$
\begin{aligned}
&\geq \kappa_2^2 \int_0^1 \Big(\sum_i z_i e_i(y)\Big)^2 \, dy \\
&= \kappa_2^2 \sum_{i=0}^{\infty} z_i^2,
\end{aligned}
$$

using the lower bound in (2.11) and the fact that the e_i are an orthonormal basis. The upper bound is done in the very same fashion, and thus (2.3) holds.

Using the identity

$$
\cos A \cos B = \tfrac{1}{2}[\cos(A-B) + \cos(A+B)],
$$

we see that if $i, j \geq 1$,

$$
\begin{aligned}
a_{ij}(x) &= \int_0^1 A(u)(y)^2 e_i(y) e_j(y) \, dy = 2 \int_0^1 A(u)(y)^2 \cos(i\pi y) \cos(j\pi y) \, dy \\
&= \int_0^1 A(u)(y)^2 \cos((i-j)\pi y) \, dy + \int_0^1 A(u)(y) \cos((i+j)\pi y) \, dy \\
&= a_{ij}^{(1)}(x) + a_{ij}^{(2)}(x).
\end{aligned}
$$

If i or j is 0, there is a trivial adjustment of a multiplicative constant. Note both $a^{(1)}$ and $a^{(2)}$ are symmetric because cosine is an even function, and that $a^{(1)}$ is of Toeplitz form. Also (2.12) now shows that $a^{(1)}$ satisfies (2.4) and $a^{(2)}$ satisfies (10.1).

Finally we check (2.5). We have

$$
\begin{aligned}
|a_{ij}^{(1)}(x + he_k) - a_{ij}^{(1)}(x)|^2 &\leq |\langle A(u + he_k)^2 - A(u)^2, e_{i-j} \rangle|^2 \\
&\leq \|A(u + he_k)^2 - A(u)^2\|_2^2 \\
&\leq 4\kappa_2^{-2} \|A(u + he_k) - A(u)\|_2^2 \\
&\leq 4\kappa_2^{-2} \kappa_1^2 |h|^{2\alpha} (k+1)^{-2\beta}
\end{aligned}
$$

by (2.11) and (2.10). This establishes (2.5) for $a^{(1)}$ and virtually the same argument gives it for $a^{(2)}$. Hence a satisfies the hypotheses of Theorem 10.1.

Turning next to uniqueness in law, let u satisfy (2.7) with $u_0 \in C[0,1]$ and define $X_n(t) = \langle u(\cdot, t), e_n \rangle$. The continuity of $t \to u(t, \cdot)$ in $C[0,1]$ shows that $t \to X_t \equiv \{X_n(t)\}$ is a continuous ℓ^2-valued process. Applying (2.8) with $\varphi = e_k$, we see that

$$X_k(t) = X_k(0) - \int_0^t \frac{k^2\pi^2}{2} X_k(s)\, ds + M_k(t),$$

where $M_k(t)$ is a martingale such that

$$\langle M_j, M_k \rangle_t = \int_0^t \langle A(u_s)e_j, A(u_s)e_k \rangle\, ds = \int_0^t a_{jk}(X(s))\, ds. \tag{10.7}$$

Thus we see that $\{X_k\}$ satisfies (2.6) with $\lambda_i = i^2\pi^2/2$.

Since u_t is the L^2 limit of the sums $\sum_{k=0}^n X_k(t)e_k(x)$ and u_t is continuous in x, then u_t is easily seen to be a Borel measurable function of $X(t)$. Thus to prove uniqueness in law of u, it suffices to prove uniqueness in law of X. It is routine to show the equivalence of uniqueness in law of (2.6) to uniqueness of the martingale problem for $\mathcal{L}$. Since the a_{ij} satisfy the hypotheses of Theorem 10.1, we have uniqueness of the martingale problem for $\mathcal{L}$.

Finally, the proof of Theorem 2.3 will be complete once we establish the existence of solutions to (2.7). We sketch a proof, which follows along standard lines. Let $X_t^n = \langle u_t, e_n \rangle$. By Theorem 10.1 there is a unique continuous ℓ^2-valued solution X to (2.6) with $\lambda_n = n^2\pi^2/2$, where a is constructed from A as above. If

$$u(s,x) = \sum_{n=0}^{\infty} X^n(s)e_n(x), \tag{10.8}$$

then the continuity of $X(t)$ in ℓ^2 shows that the above series converges in $L^2[0,1]$ for all $s \geq 0$ a.s. and $s \to u(s,\cdot)$ is a continuous L^2-valued stochastic process. It follows from (2.6) that

$$X^n(t) = \langle u_0, e_n \rangle + M_n(t) - \lambda_n \int_0^t X^n(s)\, ds, \tag{10.9}$$

where each M_n is a continuous square integrable martingale such that

$$\langle M_m, M_n \rangle_t = \int_0^t a_{mn}(X_s)\, ds = \int_0^t \int_0^1 A(u_s)(y)^2 e_m(y)e_n(y)\, dy\, ds. \tag{10.10}$$

We next verify that u satisfies (2.8). Let $\phi \in C^2[0,1]$ satisfy $\phi'(0) = \phi'(1) = 0$. Note that

$$u^N(s,x) \equiv \sum_{n=0}^{N} X^n(s)e_n(x) \to u(s,x) \text{ in } L^2[0,1] \tag{10.11}$$

as $N \to \infty$ for all $s \geq 0$ a.s. By (10.9) we have

$$\langle u_t^N, \phi\rangle = \sum_{n=0}^{N} \langle u_0, e_n\rangle\langle \phi, e_n\rangle + \sum_{n=0}^{N} M_n(t)\langle \phi, e_n\rangle - \int_0^t \sum_{n=1}^{N} \lambda_n X^n(s)\langle e_n, \phi\rangle\, ds \tag{10.12}$$

$$= I_1^N(\phi) + M_t^N(\phi) + V_t^N(\phi).$$

Parseval's equality shows that

$$\lim_{N\to\infty} I_1^N(\phi) = \langle u_0, \phi\rangle. \tag{10.13}$$

Integrating by parts twice in $\langle \phi, e_n\rangle$, and using the boundary conditions of ϕ, we find that

$$V_t^N(\phi) = \int_0^t \langle u_s^N, \phi''/2\rangle\, ds.$$

Now $\sup_{s\leq t} \|u_s^N\|_2 \leq \sup_{s\leq t} \|u_s\|_2 < \infty$ for all $t > 0$ and so by dominated convergence we see from the above and (10.11) that

$$\lim_{N\to\infty} V_t^N(\phi) = \int_0^t \langle u_s, \phi''/2\rangle\, ds \quad \text{for all } t \geq 0 \quad a.s. \tag{10.14}$$

If $N_2 > N_1$, then by (10.10) and (2.11) we have

$$\begin{aligned}
\langle (M^{N_2} - M^{N_1})(\phi)\rangle_t &= \int_0^t \int_0^1 A(u_s)(y)^2 \Big(\sum_{n=N_1+1}^{N_2} \langle e_n, \phi\rangle e_n(y)\Big)^2 dy\, ds \\
&\leq \kappa_2^{-2} \int_0^t \int_0^1 \Big(\sum_{n=N_1+1}^{N_2} \langle e_n, \phi\rangle e_n(y)\Big)^2 dy\, ds \\
&= \kappa_2^{-2} t \sum_{n=N_1+1}^{N_2} \langle e_n, \phi\rangle^2 \to 0 \text{ as } N_1, N_2 \to \infty.
\end{aligned}$$

It follows that there is a continuous L^2 martingale $M_t(\phi)$ such that for any $T > 0$,

$$\sup_{t\leq T} |M_t^N(\phi) - M_t(\phi)| \to 0 \text{ in } L^2,$$

and

$$\begin{aligned}\langle M(\phi)\rangle_t &= L^1 - \lim_{N\to\infty} \langle M^N(\phi)\rangle_t \\ &= L^1 - \lim_{N\to\infty} \int_0^t \int_0^1 A(u_s)(y)^2 \Big(\sum_0^N \langle e_n, \phi\rangle e_n(y)\Big)^2 \, dy \, ds \\ &= \int_0^t \int_0^1 A(u_s)(y)^2 \phi(y)^2 \, dy \, ds.\end{aligned}$$

Since A is bounded, M is an orthogonal martingale measure in the sense of Chapter 2 of [16] and so is a continuous orthogonal martingale measure in the sense of Chapter 2 of [16]. This (see especially Theorem 2.5 and Proposition 2.10 of [16]) and the fact that A is bounded below means one can define a white noise $\dot{W}$ on $[0,1]\times[0,\infty)$ on the same probability space in the obvious manner, so that

$$M_t(\phi) = \int_0^t \int_0^1 A(u_s)(y)\phi(y) \, dW_{s,y} \quad \text{for all } t \ge 0 \ a.s. \quad \text{for all } \phi \in L^2[0,1].$$

Therefore we may take limits in (10.12) and use the above, together with (10.13) and (10.14), to conclude that u satisfies (2.8).

It remains to show that there is a jointly continuous version of $u(t,x)$. Note first that

$$X^n(t) = e^{-\lambda_n t}\langle u_0, e_n\rangle + \int_0^t e^{-\lambda_n(t-s)} \, dM_n(s), \tag{10.15}$$

and so

$$\begin{aligned}u^N(t,x) &= \sum_{n=0}^N e^{-\lambda_n t}\langle u_0, e_n\rangle e_n(x) + \sum_{n=0}^N \int_0^t e^{-\lambda_n(t-s)} \, dM_n(s) \, e_n(x) \\ &\equiv \hat{u}^N(t,x) + \tilde{u}^N(t,x).\end{aligned} \tag{10.16}$$

Let $p(t,x,y)$ denote the fundamental solution of $\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2} p(t,x,y)$ with Neumann boundary conditions, and let P_t be the corresponding semigroup. By Mercer's theorem,

$$p(t,x,y) = \sum_{n=0}^\infty e^{-\lambda_n t}\phi_n(x)\phi_n(y),$$

where the series converges uniformly on $t \geq \varepsilon$, $x, y \in [0,1]$ for every $\varepsilon > 0$. It follows that

$$\hat{u}_t^N(x,y) \to P_t u_0(x) \text{ for all } t > 0,\ x \in [0,1].$$

An easy $L^2(\mathbb{P})$ convergence argument using square functions shows there is a jointly measurable random field $\{\tilde{u}(t,x) : t \geq 0, x \in [0,1]\}$ so that $\tilde{u}(0,\cdot) \equiv 0$ and

$$\tilde{u}^N(t,x) \to \tilde{u}(t,x) \text{ in } L^2(\mathbb{P}) \text{ uniformly in } (t,x),$$

and so for some subsequence

$$\tilde{u}^{N_k}(t,x) \to \tilde{u}(t,x) \ a.s. \text{ for each } (t,x).$$

So let $N = N_k \to \infty$ in (10.16) to conclude

$$\lim_k u^{N_k}(t,x) = P_t u_0(x) + \tilde{u}(t,x) \quad a.s. \quad \text{for all } t > 0,\ x \in [0,1].$$

It now follows easily from (10.11) that

$$u(t,x) = P_t u_0(x) + \tilde{u}(t,x) \qquad \text{a.a. } x,\ \mathbb{P} - a.s. \text{ for all } t \geq 0, \tag{10.17}$$

where the equality holds trivially for all x if $t = 0$.

Clearly $P_t u_0(x)$ is jointly continuous by the continuity of u_0, and so we next show there is a continuous version of $\tilde{u}(t,x)$. This is done in a standard way using Burkholder's inequality to obtain an appropriate bound on $\mathbb{E}\left(|\tilde{u}^N(t,x) - \tilde{u}^N(s,y)|^p\right)$, uniformly in N, and then using Fatou's lemma to get a corresponding bound for $\tilde{u}$ and Kolmogorov's lemma to obtain a continuous version of $\tilde{u}$ satisfying $\tilde{u}(0,\cdot) \equiv 0$. After a few initial calculations using the definition of A, the details become similar to the proof of Corollary 3.4 in [16]. Alternatively, one can derive the mild form of (2.7) from our L^2-valued solution u and compare to (10.17), to conclude that

$$\tilde{u}(t,x) = \int_0^t \int_0^1 p(t-s,x,y) A(u_s)(y) dW_{s,y},\ a.s. \text{ for all } (t,x),$$

and then the argument becomes virtually the same as in Corollary 3.4 of [16].

We have shown that there is a jointly continuous process $v(t,x)$ such that

$$u(t,x) = v(t,x) \text{ a.a. } x \text{ for all } t \geq 0, \text{ and } v(0,\cdot) = u_0(\cdot),\ \mathbb{P} - a.s.$$

Here the continuity in t in L^2 of both sides allows us to combine the temporal null sets. As A has been continuously extended to a map from L^2 to L^2, we have $A(u_s) = A(v_s)$ in $L^2[0,1]$ for all $s \geq 0$ a.s. and so the white noise integral in (2.8) remains unchanged if u is replaced by v. It now follows easily that (2.8) remains valid with v in place of u. Therefore v is the required continuous $C[0,1]$-valued solution of (2.7). $\square$

Proposition 10.3 *Let $\alpha, \beta, \gamma > 0$.*
(a) If

$$A : C[0,1] \to C^\gamma_{per} \text{ and } \sup_{u \in C[0,1]} \|A(u)\|_{C^\gamma} \leq \kappa_3', \tag{10.18}$$

then (2.12) *holds for some κ_3 depending on κ_3' and γ.*

(b) If

$$\|A(u) - A(v)\|_2 \leq \kappa_1' \sup_{\varphi \in C^{\beta/\alpha}_{per}, \|\varphi\|_{C^{\beta/\alpha}} \leq 1} |\langle u - v, \varphi \rangle|^\alpha \tag{10.19}$$

for all u, v continuous on $[0,1]$*, then* (2.9) *holds and* (2.10) *holds for some κ_1, depending on κ_1', α and β.*

Proof. (a) It follows easily from Leibniz's formula that

$$\|A^2(u)\|_{C^\gamma} \leq c_\gamma \|A(u)\|^2_{C^\gamma}.$$

It is also clear that $A(u) \in C^\gamma_{per}$ implies that the same is true of $A(u)^2$. The result now follows from Lemma 10.2.

(b) Cauchy-Schwarz shows the left-hand side of (10.19) is bounded above by $\kappa_1' \|u - v\|_2^\alpha$ and so (2.9) follows. By (10.19) and Lemma 10.2 we have

$$\begin{aligned}
\|A(u + he_k) - A(u)\|_2 &\leq \kappa_1' \sup_{\varphi \in C^{\beta/\alpha}_{per}, \|\varphi\|_{C^{\beta/\alpha}} \leq 1} |h|^\alpha |\langle e_k, \varphi \rangle|^\alpha \\
&\leq \kappa_1' |h|^\alpha \left(\frac{c_1(\beta/\alpha)}{1 + k^{\beta/\alpha}} \right)^\alpha \\
&\leq \kappa_1' c_2(\alpha, \beta) |h|^\alpha (1 + k)^{-\beta}.
\end{aligned}$$

$\square$

Proof of Theorem 2.4. This is an immediate consequence of Theorem 2.3 and Proposition 10.3. $\square$

Proof of Corollary 2.6. By our assumptions on f, $A(u)(x)$ is bounded above and below by positive constants, is in C^{γ}_{per}, and is bounded in C^{γ} norm uniformly in u. By our assumptions on f,

$$\begin{aligned} |A(u)(x) - A(v)(x)| &\leq c_1 \sum_{j=1}^{n} |\langle u - v, \varphi_j \rangle|^{\alpha} \\ &\leq c_2 \sup_{\varphi \in C^{\overline{\beta}}_{per}, \|\varphi\|_{C^{\overline{\beta}}} \leq 1} |\langle u - v, \varphi \rangle|^{\alpha}. \end{aligned}$$

Squaring and integrating over $[0,1]$ shows that A satisfies (2.13) and we can then apply Theorem 2.4. $\square$

Proof of Corollary 2.7. We verify the hypotheses of Theorem 2.3. Use (2.17) to define $A(u)(x)$ for all x in the line, not just $[0,1]$. It is clear that for any $u \in C[0,1]$, $A(u)$ is then an even C^{∞} function on $\mathbb{R}$ with period two, and so in particular

$$A : C[0,1] \to C^{\infty}_{per} \equiv \cap_k C^k_{per}.$$

Moreover the kth derivative of $A(u)(x)$ is bounded uniformly in x and u. If we choose γ and β large enough so that the conditions of Theorem 2.3 are satisfied, we see from the above and Proposition 10.3(a) that (2.12) holds.

Turning to the boundedness condition (2.11), we have

$$A(u)(x) \geq a \int \psi(x - y)\, dy = a\|\psi\|_1 > 0,$$

and the corresponding upper bound is similar.

For (2.10), note that by the Hölder continuity of f,

$$
\begin{aligned}
&\sup_{x\in[0,1]} |A(u+he_k)(x) - A(u)(x)| \\
&\qquad \le \sup_{x\in[0,1]} \Big| \int \psi(x-y)[f(\phi_1 * (\overline{u+he_k})(y), \ldots, \phi_n * (\overline{u+he_k})(y)) \\
&\qquad\qquad\qquad - f(\phi_1 * \overline{u}(y), \ldots, \phi_n * \overline{u}(y))]\, dy \Big| \\
&\qquad \le \|\psi\|_1 c_f \sup_{y\in\mathbb{R}, j\le n} |h|^\alpha |\phi_j * e_k(y)|^\alpha. \qquad (10.20)
\end{aligned}
$$

In the last inequality we use the linearity of $u \to \overline{u}$ and $\overline{e}_k = e_k$. Since ϕ_j is smooth with compact support, its Fourier transform decays faster than any power, and so

$$
\Big| \int \phi_j(w) e^{-iw2\pi x}\, dw \Big| \le c_{\beta/\alpha,j}(1+|2\pi x|)^{-\beta/\alpha} \qquad \text{for all } x. \qquad (10.21)
$$

Now for $k \ge 0$,

$$
\begin{aligned}
|\phi_j * e_k(y)| &\le \sqrt{2} \Big| \int \phi_j(y-z)\cos(2\pi k z)\, dz \Big| \\
&\le \sqrt{2} \Big| \int \phi_j(y-z) e^{i2\pi k z}\, dz \Big| \\
&= \sqrt{2} \Big| \int \phi_j(w) e^{-i2\pi k w}\, dw\, e^{i2\pi k y} \Big| \\
&\le \sqrt{2} c_{\beta/\alpha,j}(1+k)^{-\beta/\alpha},
\end{aligned}
$$

by (10.21). Use this in (10.20) to obtain (2.10). Finally, the proof of (2.9) is easy and should be clear from (10.20). The result now follows from Theorem 2.3. □

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