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A CRITERION FOR CONVERGENCE TO SUPER-BROWNIAN MOTION ON PATH SPACE

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We give a sufficient condition for tightness for convergence of rescaled critical spatial structures to the canonical measure of super-Brownian motion. This condition is formulated in terms of the r-point functions for $r = 2, \dots, 5$. The r-point functions describe the expected number of particles at given times and spatial locations, and have been investigated in the literature for many high-dimensional statistical physics models, such as oriented percolation and the contact process above 4 dimensions and lattice trees above 8 dimensions. In these settings, convergence of the finite-dimensional distributions is known through an analysis of the r-point functions, but the lack of tightness has been an obstruction to proving convergence on path space.

We apply our tightness condition first to critical branching random walk to illustrate the method as tightness here is well-known. We then use it to prove tightness for sufficiently spread-out lattice trees above 8 dimensions, thus proving that the measure-valued process describing the distribution of mass as a function of time converges in distribution to the canonical measure of super-Brownian motion. We conjecture that the criterion will also apply to other statistical physics models.

CONTENTS

1. Introduction and main results. In the past twenty years, many critical high-dimensional spatial branching models have been conjectured or proved to converge to super-Brownian motion (SBM). Perhaps the earliest were Rick Durrett's conjecture on long-range contact processes in 1 dimension converging to super-Brownian motion with logistic growth (see Section 1 in [46]) and David Aldous's conjecture on lattice trees above 8 dimensions converging to Integrated Super-Brownian Excursion (see Section 4.2 in [1]). Attempts to understand the large-scale behaviour of lattice trees in high dimensions date back even further [44, 40].

Significant progress has been made in a number of settings, including (i) lattice trees (LT) above 8 dimensions [18, 12, 13, 34], (ii) oriented percolation (OP) above 4 spatial dimensions

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[33] (see also [3, 47, 48]), (iii) the contact process (CP) above 4 spatial dimensions in [29, 30] (see also [2]), (iv) the contact process in lower dimensions when the range of the process increases with the rescaling [7], (v) the voter model in 2 or more dimensions [8, 5, 9], and (vi) the Lotka-Volterra model [10]. These results suggest that SBM is a universal scaling limit of critical interacting particle systems above a critical dimension. See [11, 49] for detailed surveys of super-processes and convergence towards them, and [14, 15, 43] for introductions to super-processes and continuous-state branching processes.

In each of the cases (i)-(iii) above, what has been proved is the rescaled convergence for large dimensions of the (Fourier-transforms of the) so-called r-point functions, which describe the expected number of particles at given times and spatial locations. Together with the results in [35] and the identification of the survival probability in high-dimensions in [25] (see also [23, 24] for sharper results in the context of oriented percolation), these results prove the convergence of the finite-dimensional distributions to those of the canonical measure of super-Brownian motion (CSBM), thus showing that CSBM is the only possible limiting càdlàg *process* in these models. Proving tightness and hence a full statement of weak convergence on path space in these settings has remained a major open problem. For example, in the setting (i) of lattice trees, see the discussion in Section 1 of [34], and in the setting (ii) of oriented percolation see the discussion following Corollary 1.3 in [33]. Verifying tightness in these limit theorems for random mass distributions at criticality has remained open due to the lack of any general method to bound higher moments of increments of the rescaled discrete processes. In (iv) the weaker (i.e. long range) setting makes the issue easier to handle, and for the approximate voter model settings (v, vi), the coalescing random walk dual allows one to resolve the problem. In the case of self-avoiding walk, tightness was established using sub-additivity (see Lemma 6.6.3 and (6.6.42) in [45]). The convergence of the finite-dimensional distributions in settings (i) to (iii) relies on subtle cancellations. It was not even clear that such cancellations are valid uniformly in time and hence at least one of us believed tightness would fail in some settings.

In this paper we resolve this issue. We give conditions for tightness of the rescaled empirical measures of a general class of discrete time particle systems on the space of càdlàg measure-valued paths, and hence establish convergence to CSBM on path space if convergence of finite-dimensional distributions is known as in the cases above. This involves formulating an expansion (with bounds) for moment measures which leads to control of the moments of the *increments* of the Fourier transforms of the rescaled empirical measures of our discrete mass distributions (Conditions 3.1 and 3.2 in Section 3.) The key condition is formulated in terms of the r-point functions for $r \leq 5$.

As a test case we first verify the conditions for critical branching random walk, reproving the fact that branching random walk converges on path space to CSBM (see e.g., [49]). We then go on to show that lattice trees above 8 spatial dimensions also satisfy our conditions, hence giving the first example of a high-dimensional "self-interacting" model in (i)-(iii) above for which convergence to CSBM on the space of càdlàg paths is proved. We expect that the general method developed herein can also be used to prove convergence on path space for (ii) and that an appropriate continuous-time modification of it should enable a corresponding proof in the case of (iii).

1.1. *General setting.* We work with general discrete-space particle systems/models in statistical mechanics. Each particle α in the system has an associated spatial location $\phi(\alpha) \in \mathbb{Z}^d$ and temporal component $|\alpha| \in \mathbb{Z}_+$. (For the processes we will study, the labels α will take values in a finite rooted tree.) Let $N_n(x)$ be the number of particles alive at time n located at position x, and N_n denote the total population at time n, i.e.,

$$N_n(x) = \#\{\alpha\colon \phi(\alpha) = x, |\alpha| = n\}, \quad \text{and} \quad N_n = \sum_{x\in\mathbb{Z}^d} N_n(x). \tag{1.1}$$

Let $\mathbb{P}$ and $\mathbb{E}$ denote the probability measure describing the law of the model and its expectation, respectively. We start from a single initial particle located at the origin o, and assume that once the total population reaches 0 it remains at 0 thereafter, i.e.

$$N_0 = N_0(o) = 1, \qquad \text{and} \qquad N_n = 0 \text{ for all } n \geq S \equiv \inf\{m > 0 : N_m = 0\}, \quad \mathbb{P}\text{-almost surely.} \tag{1.2}$$

For critical models of branching random walk, lattice trees, oriented percolation and the contact process (for sufficiently spread-out kernels above the respective critical dimensions), it is known (see e.g. [49, 25]) that there exist model dependent positive constants A and V so that the survival probabilities satisfy

$$n\mathbb{P}(N_n > 0) \to \frac{2}{AV}, \text{ as } n \to \infty. \tag{1.3}$$

We will in fact assume such a convergence in general for our tightness result (see (2.3) below). The r-point functions for $r \geq 2$, $\vec{x} \in \mathbb{Z}^{d(r-1)}$ and $\vec{n} \in \mathbb{Z}_+^{r-1}$ are defined by

$$t_{\vec{n}}^{(r)}(\vec{x}) = \mathbb{E}\Big[\prod_{i=1}^{r-1} N_{n_i}(x_i)\Big]. \tag{1.4}$$

Models such as lattice trees and oriented percolation have single occupancy, i.e. $N_n(x) \in \{0,1\}$ for every $x \in \mathbb{Z}^d$ and $n \in \mathbb{Z}_+$, $\mathbb{P}$-almost surely. In such cases, letting $\{o \xrightarrow{n} x\} = \{N_n(x) > 0\}$ be the event that there exists a connection to vertex x at time n, we recover the following expression that typically appears in the lace-expansion literature

$$t_{\vec{n}}^{(r)}(\vec{x}) = \mathbb{P}(o \xrightarrow{n_i} x_i \forall i = 1, \dots, r-1). \tag{1.5}$$

This includes some degeneracy, such as $t^{(3)}_{(n,n)}(x,x) = t^{(2)}_n(x)$.

Let $\mathcal{M}_F(E)$ be the space of finite measures on a Polish space E, equipped with the topology of weak convergence. Define a sequence of $\mathcal{M}_F(\mathbb{R}^d)$-valued processes $\{X_t^{(n)}\}_{t\geq 0}$, for $n \in \mathbb{N}$, by

$$X_t^{(n)}(\cdot) = \frac{1}{A^2Vn} \sum_{x\in\mathbb{Z}^d} N_{nt}(x)\delta_{x/\sqrt{vn}}(\cdot), \tag{1.6}$$

where $v > 0$ is another model-dependent constant, and $N_{nt} \equiv N_{\lfloor nt\rfloor}$. The scaling of time and space by n and $n^{-1/2}$ respectively is the scaling under which random walk converges to Brownian motion. The scaling of the measure by n^{-1} occurs since typically (see [25]) if the population is alive at time n, it is of size n. However, since the population survives until time n with probability of order $1/n$ by (1.3), we would see nothing at time $t = 1$ in the limit as $n \to \infty$. Therefore to get a non-trivial scaling limit we define a sequence of finite (but non-probability) measures μ_n on the Skorokhod space $\mathcal{D}(\mathcal{M}_F(\mathbb{R}^d))$ of càdlàg $\mathcal{M}_F(\mathbb{R}^d)$-valued paths $\{X_t\}_{t\geq 0}$ by

$$\mu_n(\cdot) = nAV\mathbb{P}(\{X_t^{(n)}\}_{t\geq 0} \in \cdot). \tag{1.7}$$

Clearly this sequence of measures cannot converge to a finite measure. For $X = \{X_t\}_{t\geq 0} \in \mathcal{D}(\mathcal{M}_F(\mathbb{R}^d))$, let $S = S(X) = \inf\{t > 0 \colon X_t = 0_M\}$, denote the extinction time of the process, where 0_M is the zero-measure. We let $\mathbb{N}_0$ be the canonical measure of super-Brownian motion as defined in Section II.7 of [49] with the branching parameter $\gamma = 1$ and the underlying spatial motion there being standard (variance parameter is 1) d-dimensional Brownian motion. Recall that $\mathbb{N}_0$ is a σ-finite measure on $\mathcal{D}(\mathcal{M}_F(\mathbb{R}^d))$ supported by the set of continuous paths X starting at 0_M which stick at 0_M after the extinction time $S > 0$ and such that $\mathbb{N}_0(S > \varepsilon) = 2/\varepsilon$ for all $\varepsilon > 0$. Then, for the critical branching random walk model starting from a single particle at the origin, there are A, V, v (specified in Section 1.2.1 below) so that

$$\mu_n(\cdot, S > \varepsilon) \xrightarrow{w} \mathbb{N}_0(\cdot, S > \varepsilon) \quad \text{on } \mathcal{D}(\mathcal{M}_F(\mathbb{R}^d)) \text{ for every } \varepsilon > 0, \tag{1.8}$$

where the convergence is weak convergence of finite measures. See, for example [49, Theorem II.7.3(a)] (the argument given there for branching Brownian motion is easy to adapt to branching random walk) or Theorem 1.1 below. In general we abbreviate (1.8) as $\mu_n \xrightarrow{w} \mathbb{N}_0$.

1.2. *Specific models.* In each case below the model is defined in terms of a symmetric random walk kernel $D\colon\mathbb{Z}^d \to [0,1]$ satisfying $\sum_{x\in\mathbb{Z}^d} D(x) = 1$. Although the results hold slightly more generally, let us assume that D is uniform on a box of radius L in $\mathbb{Z}^d$, i.e.,

$$D(x) = \mathbb{1}_{\{0<\|x\|_\infty\leq L\}}/((2L+1)^d - 1). \tag{1.9}$$

Throughout the paper we use $|x|$ to denote the Euclidean norm of a vector x. Let

$$\sigma^2 = \sum_{x\in\mathbb{Z}^d} |x|^2 D(x) < \infty, \tag{1.10}$$

so that the covariance matrix of D is $(\sigma^2/d)I_d$. In both of our models the random branching objects can be defined in terms of pairs $(\mathbf{T}, \phi)$, where $\mathbf{T}$ is a finite abstract tree (see Section 1.2.1 below) and $\phi : \mathbf{T} \to \mathbb{Z}^d$ is an embedding of the vertices of that tree. For a vertex $\alpha \in \mathbf{T}$ we let $|\alpha|$ denote the distance of α from the root in $\mathbf{T}$. Edges in $\mathbf{T}$ will be denoted by $\alpha\alpha'$ where α' is a child of α in the tree.

The two models differ markedly in the probabilities assigned to different $(\mathbf{T}, \phi)$.

1.2.1. *Branching random walk.* We follow the construction in [4] and [20]. Let $W = \bigcup_{n=0}^\infty \{0\} \times \mathbb{N}^n$ be the set of finite words starting with a 0. If $\alpha = 0\alpha_1\cdots\alpha_n \in W \setminus \{0\}$, the parent of α is $\pi(\alpha) = 0\cdots\alpha_{n-1} \in W$. A (finite) rooted tree $\mathbf{T}$ is a finite subset of W such that

(i) $0 \in \mathbf{T}$ (call 0 the root of $\mathbf{T}$) and,
(ii) If $\alpha = 0\alpha_1 \cdots \alpha_n \in \mathbf{T}$, then $\pi(\alpha) \in \mathbf{T}$ and $\{0\cdots\alpha_{n-1}j : j < \alpha_n\} \subset \mathbf{T}$.

If $\alpha = 0\cdots\alpha_n \in \mathbf{T}$, $\mathbf{T} \cap \pi^{-1}(\alpha)$ are the children of α and $|\alpha| = n$ is the generation of α and, as noted above, is also the graph distance in $\mathbf{T}$ between α and the root. We say $\beta \in \mathbf{T}$ is a descendant of $\alpha \in \mathbf{T}$, or α is an ancestor of β, if $|\beta| \geq |\alpha| = n$ and $0\beta_1 \cdots \beta_n = \alpha$. Let $\alpha \wedge \beta$ be the most recent common ancestor of α and β (i.e., $|\alpha \wedge \beta|$ is maximal). We write $\overline{\mathbf{T}}$ to denote the set all (finite) rooted trees.

Next we introduce a probability law on $\overline{\mathbf{T}}$ to construct the critical Galton-Watson tree with a critical offspring distribution $(p_m)_{m\geq 0}$ satisfying

$$\sum_{m\geq 0} m p_m = 1, \qquad V = \sum_{m \geq 0} m(m-1)p_m < \infty. \tag{1.11}$$

To avoid the trivial offspring law δ_1 we assume that $p_0 > 0$. We will also assume that

$$\sum_{m\geq 0} m^{\ell} p_m < \infty \text{ for all natural numbers } \ell, \tag{1.12}$$

although for the tightness part of our arguments $\ell = 4$ suffices in the above. We begin with a single individual whose offspring distribution is given by $(p_m)_{m\geq 0}$. Each of the offspring then independently has offspring of its own, with the same critical distribution $(p_m)_{m\geq 0}$. Mathematically this leads to a probability on $\overline{\mathbf{T}}$ given by

$$P(\mathbf{T}) = \prod_{\alpha \in \mathbf{T}} p_{\xi_\alpha}, \text{ for } \mathbf{T} \in \overline{\mathbf{T}}, \tag{1.13}$$

where $\xi_\alpha = \xi_\alpha(\mathbf{T}) \in \mathbb{Z}_+$ is the number of children of α in $\mathbf{T}$. In this case $N_n = \#\{\alpha \in \mathbf{T} : |\alpha| = n\}$ is the size of the nth generation, and Kolmogorov [41] showed that (1.3) holds with $A = 1$ and V as in (1.11) (see, e.g. [49, Theorem II.1.1]).

We define a random embedding ϕ of $\mathbf{T}$ into $\mathbb{Z}^d$ to be a mapping from the vertices of $\mathbf{T}$ into $\mathbb{Z}^d$ such that the root is mapped to the origin and, given that α is mapped to $x \in \mathbb{Z}^d$, each child α' of α is mapped to $y \in \mathbb{Z}^d$ with probability $D(y-x)$, independently of the other children. Branching random walk is then defined to be the random configuration $(\mathbf{T}, \phi)$ with probability law

$$\mathbb{P}(\mathbf{T}, \phi) = P(\mathbf{T}) \prod_{\alpha\alpha' \in \mathbf{T}} D(\phi(\alpha') - \phi(\alpha)). \tag{1.14}$$

Recall that $\alpha\alpha' \in \mathbf{T}$ means that α' is a child of α in the tree $\mathbf{T}$. In particular, the path in $\mathbb{Z}^d$ from the origin to $\phi(\alpha)$, where $\alpha = 0\alpha_1\alpha_2 \ldots \alpha_m \in \mathbf{T}$ is a random walk path of length $|\alpha| = m$ with transition probabilities given by D. The counting process $N_n(x)$ associated with $(\mathbf{T}, \phi)$ is now given by (1.1). For this model we take $v = \sigma^2 d^{-1}$ in (1.6), and re-prove the following well-known result:

THEOREM 1.1. *For critical branching random walk,* $\mu_n \xrightarrow{w} \mathbb{N}_0$.

This result has a long history, for which we refer the reader to [49] and the references therein. We include the proof in Section 4 to illustrate our convergence criterion in a less technical setting. The additional moment condition (1.12) that we have imposed on the offspring distribution is not needed for the convergence result above (see [49]), but as our general approach is based on convergence of mean moment measures this seems unavoidable without additional truncation arguments.

1.2.2. *Lattice trees.* A lattice tree $\mathcal{T}$ in $\mathbb{Z}^d$ is a finite connected set of lattice edges xx' (and their associated end vertices x, x') containing no cycles. We write $xx' \in \mathcal{T}$ (resp. $x \in \mathcal{T}$) if xx' is a edge in $\mathcal{T}$ (resp. x is a vertex in $\mathcal{T}$). Here x and x' denote vertices in $\mathbb{Z}^d$. We will choose a random lattice tree according to a weight function on the embedded objects, defined by

$$W_z(\mathcal{T}) = \prod_{xx' \in \mathcal{T}} zD(x'-x), \tag{1.15}$$

and we define the two-point function by $\rho_z(x) = \sum_{\mathcal{T} \ni o,x} W_z(\mathcal{T})$. A subadditivity argument shows that the radius of convergence z_c of $\sum_{x\in\mathbb{Z}^d} \rho_z(x)$ is finite [40]. Let $W(\cdot) = W_{z_c}(\cdot)$ and $\rho = \rho_{z_c}(o)$. We define a probability measure on the (countable) set of lattice trees containing the origin by $\mathbb{P}(\mathcal{T}) = \rho^{-1} W(\mathcal{T})$.

Let us now describe the model in terms of embeddings of abstract rooted trees, for comparison with the branching random walk situation. If $\mathbf{T}$ is a finite rooted tree, an injection $\phi : \mathbf{T} \to \mathbb{Z}^d$ is an embedding of a lattice tree $\mathcal{T}$ iff it maps the root of $\mathbf{T}$ to the origin, has range $\mathcal{T}$, and $\alpha\alpha' \in \mathbf{T}$ iff $\phi(\alpha)\phi(\alpha') \in \mathcal{T}$. We call $(\mathbf{T}, \phi)$ a tree embedding. Two tree embeddings, $(\mathbf{T}, \phi)$ and $(\mathbf{T}', \phi')$, of the same lattice tree $\mathcal{T}$ are equivalent in the space of all tree embeddings. Clearly we may identify the lattice tree $\mathcal{T}$ with the equivalence class $[\mathbf{T}, \phi]$ of all its tree embeddings. The weight (1.15) of a lattice tree $\mathcal{T}$ can then be expressed in terms of any tree embedding $(\mathbf{T}, \phi)$ in its equivalence class as $W_z(\mathcal{T}) = \prod_{\alpha\alpha' \in \mathbf{T}} zD(\phi(\alpha') - \phi(\alpha))$, and thus $\mathbb{P}([\mathbf{T}, \phi]) = \rho^{-1} z_c^{|\mathbf{T}|} \prod_{\alpha\alpha' \in \mathbf{T}} D(\phi(\alpha') - \phi(\alpha))$. Note also that $N_n(x) = \#\{\alpha \in \mathbf{T} : \phi(\alpha) = x, |\alpha| = n\} \in \{0, 1\}$ is invariant under equivalence of $(\mathbf{T}, \phi)$.

Our main result for lattice trees is the following:

THEOREM 1.2. *For the lattice trees model with $d > 8$ and step distribution (1.9), for sufficiently large L, depending on d, there are positive constants A, V, v such that $\mu_n \xrightarrow{w} \mathbb{N}_0$.*

1.3. *Some applications and open problems.* Let $\{X_t\}_{t\geq 0}$ denote a canonical path in $\mathcal{D}(\mathcal{M}_F(\mathbb{R}^d))$. Weak convergence on path space implies weak convergence of such real-valued functionals as $\sup_{t\in I} \psi(X_t)$ and $\int_I \psi(X_s)\, ds$ where I is a bounded interval of non-negative reals and $\psi : \mathcal{M}_F(\mathbb{R}^d) \to \mathbb{R}$ is bounded and continuous. We give a pair of typical applications, the first of which uses a continuity property of the limiting super-Brownian motion.

COROLLARY 1.3. *Assume $\mu_n \xrightarrow{w} \mathbb{N}_0$. Let I be a bounded interval of non-negative reals and $E \subset \mathbb{R}^d$ have Lebesgue null boundary. Then*

$$\mu_n\Big(\sup_{s\in I} X_s^{(n)}(E) \in \cdot,\ S > \varepsilon\Big) \xrightarrow{w} \mathbb{N}_0\Big(\sup_{s\in I} X_s(E) \in \cdot,\ S > \varepsilon\Big), \quad \textit{for all } \varepsilon > 0.$$

PROOF. It is sufficient to show that X is a continuity point of $X \mapsto \sup_{t\in I} X_t(E)$ for $\mathbb{N}_0$-a.a. X. Let E° and $\bar{E}$ be the interior and closure of E, respectively. It follows easily from [49, Theorem III.5.1] that

$$X_t(\partial E)=0\ \forall t\ge 0,\ \text{and } t\mapsto X_t(\bar E) \text{ is continuous } \mathbb{N}_0-a.a.X. \tag{1.16}$$

Hence it suffices to choose a continuous $X\in\mathcal{D}(\mathcal{M}_F(\mathbb{R}^d))$ satisfying the properties in (1.16) and show that if $X^{(n)}\to X$ in $\mathcal{D}(\mathcal{M}_F(\mathbb{R}^d))$, then

$$\lim_{n\to\infty}\sup_{t\in I}X_t^{(n)}(E)=\sup_{t\in I}X_t(E). \tag{1.17}$$

We first show that

$$\limsup_{n\to\infty}\sup_{t\in I}X_t^{(n)}(\bar E)\le\sup_{t\in I}X_t(\bar E). \tag{1.18}$$

Choose $t_n\in I$ so that $X_{t_n}^{(n)}(\bar E)\ge\sup_{t\in I}X_t^{(n)}(\bar E)-\frac{1}{n}$. We can find a subsequence $\{n_k\}_{k\in\mathbb{N}}$ for which the left-hand side of (1.18) is the limit through the subsequence, and so that $t_{n_k}\to t\in\bar I$. Since X is continuous, we have $X_{t_{n_k}}^{(n_k)}\to X_t$ in $\mathcal{M}_F(\mathbb{R}^d)$. It follows that

$$\begin{aligned}\limsup_{n\to\infty}\sup_{t\in I}X_t^{(n)}(\bar E)&=\lim_{k\to\infty}\sup_{t\in I}X_t^{(n_k)}(\bar E)\le\limsup X_{t_{n_k}}^{(n_k)}(\bar E)\\&\le X_t(\bar E)\le\sup_{t\in\bar I}X_t(\bar E)=\sup_{t\in I}X_t(\bar E),\end{aligned}$$

where the continuity of $t\mapsto X_t(\bar E)$ is used in the last equality. An even simpler argument, left for the reader, shows that

$$\liminf_{n\to\infty}\sup_{t\in I}X_t^{(n)}(E^\circ)\ge\sup_{t\in I}X_t(E^\circ). \tag{1.19}$$

(If $\varepsilon>0$, start by choosing $t_0\in I$ so that $\sup_{t\in I}X_t(E^\circ)\le X_{t_0}(E^\circ)+\varepsilon$). Now use (1.18), (1.19) and the fact that $X_t(E^\circ)=X_t(\bar E)$ for all t, to derive (1.17) and so complete the proof. ∎

The second application can often be obtained without tightness, but becomes relatively straightforward in the presence of tightness.

Corollary 1.4. *Assume* $\mu_n\xrightarrow{w}\mathbb{N}_0$, *and*

$$\sup_{n\in\mathbb{N},1\ge s>0}E_{\mu_n}[X_s^{(n)}(1)]\le C. \tag{1.20}$$

Then

$$\mu_n\Big(\int_0^\infty X_s^{(n)}(1)ds\ge t\Big)\to\mathbb{N}_0\Big(\int_0^\infty X_s(1)ds\ge t\Big)=\sqrt{2/(\pi t)},\quad \textit{for all } t>0.$$

Remark 1.5. *The conclusion of Corollary 1.4 is equivalent to* $\sqrt{n}\mathbb{P}(\sum_{k\ge0}N_k\ge n)\to\sqrt{2/(V\pi)}$. *The condition (1.20) will hold in all of the models we have in mind. It is a consequence of Condition 3.1(a) below and (1.2). See the discussion after Condition 3.1 for the references showing that it holds for a range of models including lattice trees (it is trivial for branching random walk).*

Proof. The equality for the limiting measure is a consequence of the well-known connection between the canonical measure of super-Brownian motion and Itô's excursion measure (see Chapter III of [43]). It also can be derived from Theorem 1.1 and the exact asymptotics for the total progeny of a Galton-Watson process (see Theorem I.13.1 of [16]).

Next, we show that $\mu_n\Big(\int_0^\infty X_s^{(n)}(1)ds \geq t, S \leq \varepsilon\Big)$ vanishes uniformly in n when $\varepsilon \downarrow 0$. For this, we note that, by the Markov inequality and (1.20) for $\varepsilon \leq 1$,

$$\begin{aligned}\mu_n\Big(\int_0^\infty X_s^{(n)}(1)ds \geq t, S \leq \varepsilon\Big) &\leq \mu_n\Big(\int_0^\varepsilon X_s^{(n)}(1)ds \geq t\Big)\\ &\leq t^{-1}E_{\mu_n}\Big[\int_0^\varepsilon X_s^{(n)}(1)ds\Big]\\ &\leq Ct^{-1}\varepsilon.\end{aligned} \tag{1.21}$$

Further by (1.8)

$$\limsup_n \mu_n\Big(\int_0^\infty X_s^{(n)}(1)ds \geq t, S > 1/\varepsilon\Big) \leq \limsup_n \mu_n(S > 1/\varepsilon) = \mathbb{N}_0(S > 1/\varepsilon) = O(\varepsilon), \tag{1.22}$$

the last, e.g. by Remark II.7.4 of [49]. Since $\int_0^{1/\varepsilon} X_s(1)ds$ is a continuous functional of the path we have (by differencing (1.8)),

$$\mu_n\Big(\int_0^{1/\varepsilon} X_s^{(n)}(1)ds \in \cdot, S \in (\varepsilon, 1/\varepsilon]\Big) \xrightarrow{w} \mathbb{N}_0\Big(\int_0^{1/\varepsilon} X_s(1)ds \in \cdot, S \in (\varepsilon, 1/\varepsilon]\Big). \tag{1.23}$$

As the limiting distribution has no atoms (this follows easily from the fact that $\int_0^\infty X_s(1)ds$ has no atoms under $\mathbb{N}_0$ by the equality of the limit already noted) the above implies that

$$\mu_n\Big(\int_0^\infty X_s^{(n)}(1)ds \geq t, S \in (\varepsilon, 1/\varepsilon]\Big) \to \mathbb{N}_0\Big(\int_0^\infty X_s(1)ds \geq t, S \in (\varepsilon, 1/\varepsilon]\Big) \text{ for all } t > 0. \tag{1.24}$$

Putting the pieces together, and using that (see the above argument)

$$\mathbb{N}_0\Big(\int_0^\infty X_s(1)ds \geq t, S \in [\varepsilon, 1/\varepsilon]\Big) = \mathbb{N}_0\Big(\int_0^\infty X_s(1)ds \geq t\Big) + O(\varepsilon), \tag{1.25}$$

we may complete the proof. ∎

For some applications the weak topology on the space of finite measures on $\mathbb{R}^d$ is too weak. For example weak convergence of measures does not imply convergence of their supports as the support functional is only lower semi-continuous. To illustrate this issue consider the "one-arm probabilities" which have attracted considerable attention recently, particularly in the context of percolation. For lattice trees, let $\|\mathcal{T}\| = \max\{|x| : x \in \mathcal{T}\}$, and let B_r be the open Euclidean ball of radius r, centred at the origin in $\mathbb{R}^d$. For a measure-valued process $X = \{X_t\}_{t\geq 0}$, let $\|X\| = \sup\{r : \cup_{t\geq 0}\{X_t(B_r^c) > 0\} \neq \varnothing\}$. We make the following conjecture about the asymptotics of the extrinsic one-arm probability $\mathbb{P}(\|\mathcal{T}\| > r)$ for lattice trees.

Conjecture 1.6 (Extrinsic one-arm probability). *For $d > 8$ and sufficiently large L,*

$$\lim_{r\to\infty} r^2 \mathbb{P}(\|\mathcal{T}\| > r) = \frac{v}{AV}\mathbb{N}_0(\|X\| > 1). \tag{1.26}$$

See [42] for a proof that the extrinsic one-arm probability for percolation above 6 dimensions decays like $1/r^2$ (although the precise constant in the asymptotics is not identified there).

While (1.26) does not follow from the weak convergence result $\mu_n \xrightarrow{w} \mathbb{N}_0$ established in this paper, the lower bound in the following corollary (proved in Section 5.2) is almost immediate:

COROLLARY 1.7 (Extrinsic one-arm lower bound). *Let $d > 8$. Then for sufficiently large L,*

$$\liminf_{r\to\infty} r^2 \mathbb{P}(\|\mathcal{T}\| > r) \ge \frac{v}{AV}\mathbb{N}_0(\|X\| > 1). \tag{1.27}$$

In Section 2 we discuss the relationship between convergence of the r-point functions and weak convergence of the finite-dimensional distributions. We then give convenient general abstract conditions, notably bounds on the fourth moments of the increments of their Fourier transforms, for weak convergence to CSBM on path space (Theorem 2.2). This result is proved in Section 5 along with Corollary 1.7. In Section 3 we state the two conditions on the r-point functions (Conditions 3.1 and 3.2) and formulate our general tightness result, Theorem 3.3, for general discrete time models. Condition 3.1 consists of bounds on the 2-point function that are known to hold for both lattice trees and oriented percolation in high dimensions–see the discussion in Section 3, and Lemma 3.5. Theorem 3.3 is then proved in Section 6 by verifying the conditions of Theorem 2.2. In Section 6 we prove Theorem 1.2 by verifying Condition 3.2 for lattice trees (with $d > 8$ and L sufficiently large) through lace expansion techniques. The proof here relies on certain diagrammatic bounds for coefficients arising in the lace expansion, Proposition 7.3, which in turn is established in Appendix A by modifying the arguments in [34].

Application to other models:. For oriented percolation and the contact process above 4 dimensions, convergence of the finite-dimensional distributions has been proved, due to the survival probability and r-point function asymptotics provided in [25, 23, 24] and [33, 28, 30] respectively. Tightness remains an open problem in each case.

CONJECTURE 1.8 (Convergence for oriented percolation). *Condition 3.2, and hence $\mu_n \xrightarrow{w} \mathbb{N}_0$, is valid for oriented percolation when $d > 4$ and L is sufficiently large.*

By the above discussion this reduces to the verification of Condition 3.2, which would be through a lace expansion analysis. We believe that a similar approach can be used to prove weak convergence to super-Brownian motion for the contact process as well, but one first needs versions of the tightness criterion given in Section 3 that are appropriate for continuous-time processes.

The picture is much less complete in the case of ordinary (non-oriented) percolation and $d > 6$, where even convergence of the r-point functions (hence finite-dimensional distributions) and validity of Condition 3.1 below is not yet known.

2. Weak convergence of measure-valued processes. We work in the general framework of Section 1.1. Let $\mathcal{D} = \mathcal{D}(\mathcal{M}_F(\mathbb{R}^d))$ be the (Polish) space of càdlàg $\mathcal{M}_F(\mathbb{R}^d)$-valued processes $\{X_t\}_{t\geq 0}$ equipped with the Skorokhod topology. The space $\mathcal{M}_F(\mathcal{D})$ is also Polish. For $X_t \in \mathcal{M}_F(\mathbb{R}^d)$, $f: \mathbb{R}^d \to \mathbb{C}$ and $k \in \mathbb{R}^d$, define

$$X_t(f) = \int_{\mathbb{R}^d} f(x) X_t(dx), \quad \text{and} \quad \hat{X}_t(k) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}k\cdot x} X_t(dx). \tag{2.1}$$

Recall the definition (1.8) of weak convergence to the canonical measure $\mathbb{N}_0$. Much is known about $\mathbb{N}_0$, for example, by [49, Theorem II.7.2(iii)], for every $\varepsilon > 0$,

$$\mathbb{N}_0\left(X_\varepsilon(1) \in G \setminus \{0_M\}\right) = \left(\frac{2}{\varepsilon}\right)^2 \int_G \mathrm{e}^{-\frac{2}{\varepsilon}x} dx, \quad \text{and therefore} \quad \mathbb{N}_0(S > \varepsilon) = \mathbb{N}_0(X_\varepsilon(1) > 0) = \frac{2}{\varepsilon}. \tag{2.2}$$

For each fixed ε, (1.8) is a statement about convergence of a sequence of finite measures on a Polish space. By considering the test function 1, (1.8) implies that for each $\varepsilon > 0$,

$$\mu_n(S > \varepsilon) \to \mathbb{N}_0(S > \varepsilon). \tag{2.3}$$

In other words, convergence of the survival probabilities is necessary for weak convergence. It is easy to check from the definitions (notably (1.6) and (1.7)) and (2.2) that (2.3) for any fixed $\varepsilon > 0$ is equivalent to (1.3) which in turn is equivalent to (2.3) for all $\varepsilon > 0$. Convergence of the survival probabilities for branching random walk reduces to a statement about Galton-Watson branching processes and is a well-known result due to Kolmogorov [41] (see [16, Section I.10] or [49, Theorem II.1.1]). The corresponding property (2.3) for lattice trees, oriented percolation, and the contact process is a very recent result [23, 24, 25]. Due to this fact, the weak convergence in (1.8) can easily be translated into a statement about convergence of the corresponding (conditional) probability measures on the Polish space $\mathcal{D}$,

$$P_n^{\varepsilon} \xrightarrow{w} P_{\mathbb{N}_0}^{\varepsilon} \qquad \text{for all } \varepsilon > 0, \tag{2.4}$$

where $P_n^{\varepsilon}(\cdot) = \mu_n(\cdot | S > \varepsilon) \equiv \mu_n(\cdot, S > \varepsilon)/\mu_n(S > \varepsilon)$, and $P_{\mathbb{N}_0}^{\varepsilon}(\cdot) = \mathbb{N}_0(\cdot | S > \varepsilon)$.

Let $l \geq 1$, and $\vec{t} = (t_1, \dots, t_l) \in [0, \infty)^l$. We use $\pi_{\vec{t}} : \mathcal{D} \to \mathcal{M}_F(\mathbb{R}^d)^l$ to denote the projection map satisfying $\pi_{\vec{t}}(X) = (X_{t_1}, \dots X_{t_l})$. The *finite-dimensional distributions* (f.d.d) of $\nu \in \mathcal{M}_F(\mathcal{D})$ are the measures $\nu_{\vec{t}} \in \mathcal{M}_F(\mathcal{M}_F(\mathbb{R}^d)^l)$ given by

$$\nu_{\vec{t}}(H) \equiv \nu\left(\{X : \pi_{\vec{t}}(X) \in H\}\right), \qquad H \in \mathcal{B}(\mathcal{M}_F(\mathbb{R}^d)^l),$$

where $\mathcal{B}(E)$ denotes the Borel sets of E.

The probability measure $P_{\mathbb{N}_0}^{\varepsilon}$ on $\mathcal{D}$ is uniquely determined by its finite-dimensional distributions. By [35, Proposition 2.4], convergence of the finite-dimensional distributions follows from convergence of the survival probabilities and convergence of the mean moment measures

$$E_{\mu_n}\left[\prod_{i=1}^{r-1} \hat{X}_{t_i}^{(n)}(k_i)\right] \to E_{\mathbb{N}_0}\left[\prod_{i=1}^{r-1} \hat{X}_{t_i}(k_i)\right], \qquad \text{for } r \geq 2, \vec{t} \in [0,\infty)^{r-1}, \vec{k} \in \mathbb{R}^{d(r-1)}. \tag{2.5}$$

Again note that under (2.3), the convergence of the f.d.d. of the normalized or unnormalized measures (written $P^{\varepsilon}_{n} \xrightarrow{fdd} P^{\varepsilon}_{\mathbb{N}_0}$ for each $\varepsilon > 0$, and $\mu_n \xrightarrow{fdd} \mathbb{N}_0$, respectively) are equivalent. Using the lace expansion, the limits (2.5) have been established for LT, OP, and CP in terms of asymptotic formulae for the Fourier transforms of the r-point functions (1.4). See Section 3 for more details and Section 1 for references. Thus weak convergence in (2.4) would follow from relative (sequential) compactness of the sequence of probability measures P^{ε}_{n}.

For a Polish space E, let $\mathcal{D}(E)$ be the space of càdlàg functions from $\mathbb{R}_+$ to E with the Skorokhod topology, and recall that $\mathcal{D} = \mathcal{D}(\mathcal{M}_F(\mathbb{R}^d))$.

Definition 2.1 ($\mathcal{C}$-relatively compact). *A set of probabilities $\{P_\alpha\}_{\alpha\in I}$ on $\mathcal{D}(E)$ is $\mathcal{C}$-relatively compact on $\mathcal{D}(E)$ if it is relatively compact (every sequence has a convergent subsequence) and every weak limit point P_∞ satisfies $P_\infty(\mathcal{C}^c) = 0$ where $\mathcal{C}$ is the set of continuous functions in $\mathcal{D}(E)$.*

Our first result reduces the $\mathcal{C}$-relative compactness of the probabilities P^{ε}_{n} to a fourth moment condition for increments of the processes. The proof is given in Section 5.1. Recall that we are in the general setting of Section 1.1.

Theorem 2.2. *Suppose that (2.3) holds for some $\varepsilon > 0$, and that the following hold:*

(i) $\sup_{n\in\mathbb{N}} n^{-1}\sum_{x\in\mathbb{Z}^d}|x|^2\mathbb{E}[N_n(x)] < \infty$,

(ii) there exist $\zeta > 0$, $\eta > 0$ such that for each $T > 0$ and $k \in \mathbb{R}^d$ there exists $C_{k,T} > 0$ with $\sup_{|k|<\eta} C_{k,T} < \infty$ for each T, such that for all $n \in \mathbb{N}$,

$$n\mathbb{E}\Big[\big|\hat{X}^{(n)}_{l/n}(k) - \hat{X}^{(n)}_{j/n}(k)\big|^4\Big] \le C_{k,T}\left(\frac{|l-j|}{n}\right)^{1+\zeta}, \quad \text{for all } j,l \in \mathbb{Z}_+ \text{ satisfying } j,l \le nT. \tag{2.6}$$

Then the probability measures $\{P^{\varepsilon}_{n}\}_{n\in\mathbb{N}}$ are $\mathcal{C}$-relatively compact on $\mathcal{D}$ for all $\varepsilon > 0$.

If, in addition, (2.5) holds, then $\mu_n \xrightarrow{w} \mathbb{N}_0$.

3. The r-point functions and a general criterion for tightness. In this section, we continue to work in the general setting of Section 1.1 and replace the tightness condition (2.6) by a criterion in terms of the Fourier transforms of the r-point functions (1.4). Firstly note that

$$\hat{X}^{(n)}_t(k) = \frac{1}{A^2Vn}\int e^{\mathrm{i}k\cdot y}\sum_{x\in\mathbb{Z}^d} N_{nt}(x)\delta_{x/\sqrt{vn}}(dy) = \frac{1}{A^2Vn}\sum_{x\in\mathbb{Z}^d} e^{\mathrm{i}\frac{k}{\sqrt{nv}}\cdot x}N_{nt}(x). \tag{3.1}$$

Therefore

$$\begin{aligned} E_{\mu_n}\left[\prod_{i=1}^{r-1}\hat{X}^{(n)}_{t_i}(k_i)\right] &= \frac{nAV}{(A^2Vn)^{r-1}}\sum_{\vec{x}\in\mathbb{Z}^{d(r-1)}} e^{\mathrm{i}\frac{\vec{k}}{\sqrt{nv}}\cdot\vec{x}}\mathbb{E}\Big[\prod_{j=1}^{r-1}N_{nt_j}(x_j)\Big] \\ &= \frac{1}{A(A^2Vn)^{r-2}}\sum_{\vec{x}\in\mathbb{Z}^{d(r-1)}} e^{\mathrm{i}\frac{\vec{k}}{\sqrt{nv}}\cdot\vec{x}}t^{(r)}_{n\vec{t}}(\vec{x}). \end{aligned} \tag{3.2}$$

Defining

$$\hat{t}^{(r)}_{\vec{n}}(\vec{k}) = \sum_{\vec{x}} \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}} t^{(r)}_{\vec{n}}(\vec{x}), \tag{3.3}$$

we see that

$$E_{\mu_n}\left[\prod_{i=1}^{r-1} \hat{X}^{(n)}_{t_i}(k_i)\right] = \frac{1}{A(A^2Vn)^{r-2}} \hat{t}^{(r)}_{n\vec{t}}\Big(\frac{\vec{k}}{\sqrt{vn}}\Big). \tag{3.4}$$

When $r \geq 3$, $\hat{t}^{(r)}_{n\vec{t}}$ includes a small contribution from degenerate cases arising from some of the x_j being equal. We assume the following bounds on the increments $\hat{t}^{(2)}_{j+1}(k) - \hat{t}^{(2)}_{j}(k)$ and on the second spatial moments $\sum_{x\in\mathbb{Z}^d} |x|^2 t^{(2)}_n(x)$:

CONDITION 3.1 (Two-point function assumptions).

(a) There is a constant $a > 1$ so that for any $K' > 0$ there is a K so that for all integers $j \geq 0$

$$\text{(3.5)} \qquad \text{for all } |k|^2 \leq K' j^{-1}, \quad |\hat{t}^{(2)}_{j+1}(k) - \hat{t}^{(2)}_{j}(k)| \leq K((j+1)^{-a} + |k|^2).$$

(b) Theorem 2.2(i) holds, that is, there is a constant $K > 0$ so that

$$\text{(3.6)} \qquad \sum_{x \in \mathbb{Z}^d} |x|^2 t^{(2)}_n(x) \leq Kn, \qquad \text{for all } n \geq 1.$$

The above bounds are important players in the *inductive approach* to the lace expansion, as studied in various guises in [21, 26, 28, 31]. Lemma 3.5 below verifies that Condition 3.1 holds for both lattice trees and oriented percolation in high dimensions.

Our next condition involves various lace expansions for the r-point function for $r = 3, 4, 5$, and forms the heart of our analysis. It is a technical condition which will likely be difficult to digest upon first reading. Its verification for branching random walk in Section 4 below should help, and its relation with other expansions in the literature, especially in the context of lattice trees, is discussed in Section 7.1 below. In its statement, for a vector $\vec{n} = (n_1, \ldots, n_{r-1})$, $\bar{n}$ denotes the largest coordinate of $\vec{n}$, and $\underline{n}$ the smallest coordinate of $\vec{n}$. If $I \subset J_r = \{1, \ldots, r-1\}$, let $\vec{n}_I = (n_i)_{i \in I}$, and for $m \in \mathbb{Z}_+$ let $\vec{n} - m = (n_1 - m \ldots, n_{r-1} - m)$. Finally, define

$$\text{(3.7)} \qquad m_1 \vartriangle m_2 \equiv (m_1 \vee m_2) - (m_1 \wedge m_2) + 1 = |m_1 - m_2| + 1.$$

CONDITION 3.2 (Lace expansion and its bounds). *There exist $\hat{\chi}^{(1)}_{m_1,m_2}(\vec{k}_{I_0}, \vec{k}_{I_1}, \vec{k}_{I_2})$, $\hat{\chi}^{(2)}_{m_1,m_2;n_1}(\vec{k}_{I_0}, \vec{k}_{I_1}, \vec{k}_{I_2})$, $\hat{\chi}^{(3)}_{m_1,m_2;n_1,n_2}(\vec{k}_{I_0}, \vec{k}_{I_1}, \vec{k}_{I_2})$, $\hat{\kappa}^{(3)}_{\vec{n}}(\vec{k})$, $\hat{\kappa}^{(4)}_{\vec{n}}(\vec{k})$, $\hat{\kappa}^{(5)}_{\vec{n}}(\vec{k})$, and constants $a > 1$ and $p \in [0,1)$ such that, for $r \in \{3,4,5\}$ and all $\vec{n} \in \mathbb{N}^{r-1}$, $\vec{k} \in \mathbb{R}^{(r-1)d}$,*

$$\text{(3.8)} \qquad \hat{t}^{(r)}_{\vec{n}}(\vec{k}) = \sum_{I_0, I_1, I_2} \sum_{1 \leq m_1 \leq \underline{n_{I_1}}, 1 \leq m_2 \leq \underline{n_{I_2}}} \hat{\chi}^{(|I_0|+1)}_{m_1,m_2;\vec{n}_{I_0}}(\vec{k}_{I_0}, \vec{k}_{I_1}, \vec{k}_{I_2}) \hat{t}^{(|I_1|+1)}_{\vec{n}_{I_1}-m_1}(\vec{k}_{I_1}) \hat{t}^{(|I_2|+1)}_{\vec{n}_{I_2}-m_2}(\vec{k}_{I_2}) + \hat{\kappa}^{(r)}_{\vec{n}}(\vec{k}),$$

where the sum over I_0, I_1, I_2 is over all partitions of $\{1, \ldots, r-1\}$ such that $1 \in I_1$, $I_2 \neq \varnothing$ and $i_2 = \min\{i \in \{1, \ldots, r-1\} \setminus I_1\} \in I_2$. The lace expansion coefficients satisfy the bounds

$$\text{(3.9)} \qquad |\hat{\kappa}^{(3)}_{\vec{n}}(\vec{k})| \leq C, \qquad |\hat{\kappa}^{(4)}_{\vec{n}}(\vec{k})| \leq C\bar{n}, \qquad |\hat{\kappa}^{(5)}_{\vec{n}}(\vec{k})| \leq C\Big(|\bar{n} - \underline{n}|^2 + \bar{n}^{3-a}\Big),$$

and, uniformly in I_0, I_1, I_2 (with $(m \vee n)_* \equiv m_1 \vee m_2 \vee n_*$ in (3.11) and $(m \vee n)_{**} \equiv m_1 \vee m_2 \vee n_* \vee n_{**}$ in (3.12)),

$$\text{(3.10)} \qquad |\hat{\chi}^{(1)}_{m_1,m_2}(\vec{k})| \leq C(m_1 \vartriangle m_2)^{-a},$$

$$\text{(3.11)} \qquad |\hat{\chi}^{(2)}_{m_1,m_2;n_*}(\vec{k})| \leq C(m_1 \vartriangle m_2)^{-a}\left((m_1 \vartriangle m_2) + (m \vee n)_*^p\right),$$

$$\text{(3.12)} \qquad |\hat{\chi}^{(3)}_{m_1,m_2;n_*,n_{**}}(\vec{k})| \leq C(m \vee n)_{**}(m_1 \vartriangle m_2)^{-a}\left((m_1 \vartriangle m_2) + (m \vee n)_{**}^p\right).$$

We may, and shall, assume $a \in (1,2)$. As an example, consider the expansion (3.8) for $r = 3$. In this case we must have $I_0 = \varnothing$, $I_1 = \{1\}$, $I_2 = \{2\}$ and so (3.8) becomes

$$\hat{t}^{(3)}_{\vec{n}}(\vec{k}) = \sum_{1 \le m_1 \le n_1, 1 \le m_2 \le n_2} \hat{\chi}^{(1)}_{m_1,m_2}(\vec{k}) \hat{t}^{(2)}_{n_1-m_1}(k_1) \hat{t}^{(2)}_{n_2-m_2}(k_2) + \hat{\kappa}^{(3)}_{\vec{n}}(\vec{k}). \tag{3.13}$$

Clearly Theorem 2.2(ii) holds (with $1+\zeta = a \wedge (2-p)$) if for every $T > 0$, there exists $C_T > 0$ such that

$$n\mathbb{E}\Big[\Big|\hat{X}^{(n)}_{l/n}(k) - \hat{X}^{(n)}_{j/n}(k)\Big|^4\Big] \le C_T n^{-(a\wedge(2-p))} |l-j|^{a \wedge (2-p)} \tag{3.14}$$
$$\text{for all } j, l \in \mathbb{Z}_+, \text{ and } k \in \mathbb{R}^d \text{ satisfying } j \le l \le nT \text{ and } |k| \le T.$$

Thus, the following result implies that Theorem 2.2(ii) follows from Conditions 3.1 and 3.2.

Theorem 3.3 (Tightness criterion). *Assume that Conditions 3.1 and 3.2 hold. Then (3.14) holds. As a result, conditions (i) and (ii) of Theorem 2.2 hold, and so if (2.3) also holds for some $\varepsilon > 0$, then $\{P^\varepsilon_n\}$ are $\mathcal{C}$-relatively compact on $\mathcal{D}$ for all $\varepsilon > 0$.*

The final result is immediate from Theorems 2.2 and 3.3.

Corollary 3.4. *Assume Conditions 3.1 and 3.2 as well as (2.3) and (2.5). Then $\mu_n \xrightarrow{w} \mathbb{N}_0$.*

We give a short overview of the proof of Theorem 3.3. To abbreviate notation, for a function $f^{(r)}_{\vec{j}} : \mathbb{N}^{r-1} \to \mathbb{R}$, and for $l, j \in \mathbb{N}$ such that $l \ge j$, we write

$$\Delta f^{(r)}_{j,l} = (-1)^{r-1} \sum_{\vec{\sigma} \in \{0,1\}^{r-1}} (-1)^{\sigma_1 + \cdots + \sigma_{r-1}} f^{(r)}_{\vec{\sigma}_{j,l}}, \tag{3.15}$$

where $(\vec{\sigma}_{j,l})_i = j + \sigma_i(l-j)$. In particular, $\Delta f^{(2)}_{j,l} = f_l - f_j$, so Condition 3.1(a) is a bound on $\Delta \hat{t}^{(2)}_{j,j+1}(k)$, where the variable $k \in \mathbb{R}^d$ is fixed in this notation. Note also that for $r = 1$, $\Delta f^{(r)}_{j,l} = f^{(r)}$, and for $r \ge 2$, $\Delta f^{(r)}_{j,j} = 0$.

We claim that for $j \le l$ in $\mathbb{N}$,

$$n\mathbb{E}\Big[\Big|\hat{X}^{(n)}_{l/n}(k) - \hat{X}^{(n)}_{j/n}(k)\Big|^4\Big] = (A^2V)^{-4} n^{-3} \Delta \hat{t}^{(5)}_{j,l}\big((k,k,-k,-k)/\sqrt{vn}\big), \tag{3.16}$$

where $k \in \mathbb{R}^d$ is again fixed in the above notation. To see this, let $X_j = \hat{X}^{(n)}_{j/n}(k)$ and let $\bar{X}_j$ denote the complex conjugate of X_j. Then the left-hand side of (3.16) is

$$\begin{aligned} & n\mathbb{E}\left[(X_l - X_j)^2(\bar{X}_l - \bar{X}_j)^2\right] \\ &= (-1)^4 n\mathbb{E}\left[((-1)X_l + X_j)^2((-1)\bar{X}_l + \bar{X}_j)^2\right] \\ &= (AV)^{-1}(-1)^4 \sum_{\vec{\sigma} \in \{0,1\}^4} (-1)^{\sigma_1 + \cdots + \sigma_4} E_{\mu_n}\left[\prod_{i=1}^{2} \hat{X}^{(n)}_{(\sigma_{j,l})_i}(k) \prod_{i'=3}^{4} \hat{X}^{(n)}_{(\sigma_{j,l})_{i'}}(-k)\right] \\ &= \frac{1}{(A^2V)^4 n^3} \sum_{\vec{\sigma} \in \{0,1\}^4} (-1)^{\sigma_1 + \cdots + \sigma_4} \hat{t}^{(5)}_{\vec{\sigma}_{j,l}}((k,k,-k,-k)/\sqrt{vn}), \end{aligned} \tag{3.17}$$

where (3.4) is used in the last line. This proves (3.16). Therefore, to prove Theorem 3.3, it suffices to show that for each $T > 0$ there exists a constant $C_T > 0$ such that,

$$(3.18)\quad |\Delta\hat{t}^{(5)}_{j,l}(\vec{k})| \le C_T l^3 \Big|\frac{l-j}{l}\Big|^{a\wedge(2-p)} \quad \text{for all } j \le l \in \mathbb{Z}_+, \text{ and } \vec{k} \in \mathbb{R}^{4d} \text{ satisfying } l|\vec{k}|^2 \le T^3.$$

By applying the linear operator Δ to (3.8), we obtain that $\Delta\hat{t}^{(5)}_{j,l}(\vec{k})$ satisfies

(3.19)
$$\Delta\hat{t}^{(5)}_{j,l}(\vec{k}) = \sum_{\vec{I}}\sum_{m_1,m_2\le j} \Delta\hat{\chi}^{(|I_0|+1)}_{m_1,m_2;j,l}(\vec{k})\Delta\hat{t}^{(|I_1|+1)}_{j-m_1,l-m_1}(\vec{k}_{I_1})\Delta\hat{t}^{(|I_2|+1)}_{j-m_2,l-m_2}(\vec{k}_{I_2}) + \hat{\gamma}^{(5)}_{j,l}(\vec{k}) + \Delta\hat{\kappa}^{(5)}_{j,l}(\vec{k}),$$

where $\hat{\gamma}^{(5)}_{j,l}(\vec{k})$ denotes the contribution due to terms where $m_1 \in [j+1,l]$ or $m_2 \in [j+1,l]$. To check this, one should note, for example, that if $f^{(5)}_{n_1,n_2,n_3,n_4} = g^{(2)}_{n_1}h^{(2)}_{n_2}k^{(3)}_{n_3,n_4}$, then $\Delta f^{(5)}_{j,l} = \Delta g^{(2)}_{j,l}\Delta h^{(2)}_{j,l}\Delta k^{(3)}_{j,l}$.

In Section 6, we bound the contributions to (3.19) one by one. In order to bound $\Delta\hat{t}^{(5)}_{j,l}(\vec{k})$, we need to rely on bounds on $\Delta\hat{t}^{(2)}_{j,l}(\vec{k})$, $\Delta\hat{t}^{(3)}_{j,l}(\vec{k})$ and $\Delta\hat{t}^{(4)}_{j,l}(\vec{k})$, respectively. These bounds are similar in spirit to the bound on $\Delta\hat{t}^{(5)}_{j,l}(\vec{k})$ required for Theorem 3.3, and we start by proving these simpler bounds using Condition 3.2.

Lemma 3.5 (Verification of Condition 3.1 for LT and OP). *For all L sufficiently large, Condition 3.1 holds for lattice trees in dimensions $d > 8$ and oriented percolation in dimensions $d > 4$.*

Proof. For any model satisfying the assumptions of [26] (which is a generalization of the inductive method derived in [21, 31]) for a given set of parameters (including $\theta > 2$ and $\beta = L^{d/p^*}$ for some $p^* \ge 1$), (3.6) in Condition 3.1(b) is immediate from the third bound in [26, (8)].

We next verify Condition 3.1(a). Fixing $K' > 0$ and assuming that $|k|^2 \le K'/j$, we note that the claim (3.5) holds trivially for all small j by taking K large enough. To establish the claim for large j, let $a(k) = 1 - \hat{D}(k)$, where we recall that $D(\cdot)$ is the single step kernel. Then [26, (5),(6)] implies that either $a(k) \le (\gamma\log(j+1))/(j+1)$ or $a(k) > \eta$ (where γ and η are positive constants). If $a(k) \le (\gamma\log(j+1))/(j+1)$, then [26, (H3)] gives the claim (3.5) with $a = \theta - 1$ (provided that $d/p^* \ge 2$), by using [26, (5)] when $\|k\|_\infty \le L^{-1}$ and [26, (6)] when $\|k\|_\infty > L^{-1}$, and using the uniform bound on $|\hat{t}^{(2)}_j(k)|$ provided by the second bound in [26, (8)]. Otherwise $a(k) > (\gamma\log(j+1))/(j+1)$ and $a(k) > \eta$, so by [26, (H4)], the claim (3.5) holds with $a = \theta$ (so also with $a = \theta - 1$).

The assumptions of [26] with $\theta = d/2 > 2$ and $p^* = 1$ are proved in [33, Section 2.1.2] for oriented percolation (see also [28] for the contact process) when $d > 4$. For lattice trees, the assumptions of [26] are verified in [34, Section 3.3] with $\theta = (d-4)/2$ and $p^* = 2$, when $d > 8$. ■

4. Proof of Theorem 1.1. Recall that the survival probability of BRW satisfies $n\mathbb{P}(N_n > 0) \to 2/V$ (Kolmogorov [41]), and the r-point functions scale to those of SBM when the branching law has all moments (see e.g., [20, Theorem 3.2]). By [35, Proposition 2.4], $\mu_n \xrightarrow{fdd} \mathbb{N}_0$, so by Theorems 3.3 and 2.2 it remains to verify Conditions 3.1 and 3.2. For this we will only require (1.12) for $\ell = 4$.

Let $(\lambda_j)_{j\geq 0}$ denote the factorial moments of the distribution $(p_m)_{m\geq 0}$, i.e. $\lambda_1 = 1$, $\lambda_2 = V$ and more generally,

$$\lambda_j = \sum_{m=j}^{\infty} \frac{m!}{(m-j)!} p_m < \infty \text{ for } j \leq 4 \text{ by (1.12) with } \ell = 4. \tag{4.1}$$

Also, we write $\mathcal{P}_j$ for the set of partitions of $J_r = \{1,\ldots,r-1\}$ into j non-empty sets $\vec{I}^* = (I_1^*,\ldots,I_j^*)$ where we order the elements of $\vec{I}^* \in \mathcal{P}_j$ by ordering the smallest elements so that I_1^* contains the element 1, I_2^* contains the smallest element of $J_r \setminus I_1^*$ etc. Then, [20, Proposition 2.3] states that for every $\vec{x} \in \mathbb{Z}^{d(r-1)}$ and every $\vec{n} = (n_1,\ldots,n_{r-1})$ with $n_i \geq 1$ for all $i = 1,\ldots,r-1$,

$$t_{\vec{n}}^{(r)}(\vec{x}) = \sum_{j=1}^{r-1} \lambda_j \sum_{\vec{I}^* \in \mathcal{P}_j} \prod_{s=1}^{j} (D * t_{\vec{n}_{I_s^*}-1}^{(|I_s^*|+1)})(\vec{x}_{I_s^*}), \tag{4.2}$$

where $(D * t_{\vec{n}_{I_s^*}-1}^{(|I_s^*|+1)})(\vec{x}_{I_s^*}) = \sum_{z\in\mathbb{Z}^d} D(z) t_{\vec{n}_{I_s^*}-1}^{(|I_s^*|+1)}(\vec{x}_{I_s^*} - (z,\ldots,z))$. Let $k(I) = \sum_{j\in I} k_j$. From (4.2) and an induction it is possible to derive (3.8) (see below), where if $\underline{J} = J_r \setminus \{j\}$ when $\underline{n} = n_j$ with j minimal, then for $r = 3,4,5$,

$$\hat{\kappa}_{\vec{n}}^{(r)}(\vec{k}) = \hat{D}(k(J_r))^{\underline{n}} \hat{t}_{\vec{n}_{\underline{J}}-\underline{n}}^{(r-1)}(\vec{k}_{\underline{J}}), \tag{4.3}$$

and

$$\hat{\chi}_{m_1,m_2}^{(1)}(\vec{k}_{I_1},\vec{k}_{I_2}) = \lambda_2 \delta_{m_1,m_2} \hat{D}(k(J_r))^{m_1-1} \hat{D}(k(I_1)) \hat{D}(k(I_2)), \tag{4.4}$$

$$\begin{aligned} \hat{\chi}_{m_1,m_2;n_*}^{(2)}(\vec{k}_{I_0},\vec{k}_{I_1},\vec{k}_{I_2}) &= \lambda_3 \delta_{m_1,m_2} \mathbb{1}_{\{m_1\leq n_*\}} \hat{D}(k(J_r))^{m_1-1} \\ &\quad \times \hat{D}(k(I_1)) \hat{D}(k(I_2)) \hat{D}(k(I_0))^{n_*-m_1+1}, \end{aligned} \tag{4.5}$$

$$\begin{aligned} \hat{\chi}_{m_1,m_2;n_*,n_{**}}^{(3)}(\vec{k}) &= \lambda_4 \delta_{m_1,m_2} \mathbb{1}_{\{m_1\leq n_*\wedge n_{**}\}} \hat{D}(k(J_r))^{m_1-1} \\ &\quad \times \hat{D}(k_1)\hat{D}(k_2)\hat{D}(k_3)^{n_*-m_1+1}\hat{D}(k_4)^{n_{**}-m_1+1} \\ &\quad + \lambda_3 \delta_{m_1,m_2} \mathbb{1}_{\{m_1\leq n_*\wedge n_{**}\}} \hat{D}(k(J_r))^{m_1-1} \\ &\quad \times \hat{D}(k_1)\hat{D}(k_2)\hat{D}(k_3+k_4)\hat{t}_{n_*-m_1,n_{**}-m_1}^{(3)}(k_3,k_4). \end{aligned} \tag{4.6}$$

(In most cases, $k_j \in \mathbb{R}^d$ is the jth coordinate of $\vec{k}$; the few cases where $k_j \in \mathbb{R}$ is the jth coordinate of $k \in \mathbb{R}^d$ will be clear from the context.)

In particular, note that $m_1 = m_2$ in the summation in (3.8). The form of the final formula is different but whenever it is used we shall see below that it will be the case that $I_0 = \{3,4\}$, $I_1 = \{1\}$, and $I_2 = \{2\}$ and so it may be recast as the previous formulae. We now give a direct derivation of (3.8) without using (4.2) to give some intuition for (3.8) in the simplest possible case.

Consider an individual $\alpha = 0\alpha_1\ldots\alpha_{n_1}$ in the Galton-Watson tree $\mathbf{T}$. Then by (1.14),

$$\mathbb{P}(\phi(\alpha) = x_1 | \alpha \in \mathbf{T}) = D^{*n_1}(x_1),$$

i.e., the path in $\mathbf{T}$ from 0 to α is embedded as a simple random walk path with step distribution D. Therefore,

$$
\begin{aligned}
t^{(2)}_{n_1}(x_1) = \mathbb{E}(N_{n_1}(x_1)) &= \mathbb{E}\Big[\sum_{|\alpha|=n_1} \mathbb{1}_{\{\phi(\alpha)=x_1,\alpha\in\mathbf{T}\}}\Big] \\
&= \sum_{|\alpha|=n_1} D^{*n_1}(x_1)P(\alpha\in\mathbf{T}) \\
&= D^{*n_1}(x_1)\mathbb{E}[N_{n_1}] = D^{*n_1}(x_1),
\end{aligned}
\tag{4.7}
$$

which is also immediate from (4.2) and induction.

We have for $r \in \{2,3,4,5\}$,

$$
\begin{aligned}
t^{(r)}_{\vec{n}}(\vec{x}) &= \mathbb{E}\Big[\prod_{i=1}^{r-1} N_{n_i}(x_i)\Big] \\
&= \sum_{|\alpha_1|=n_1}\cdots\sum_{|\alpha_{r-1}|=n_{r-1}} \mathbb{P}(\phi(\alpha_i)=x_i, \alpha_i\in\mathbf{T},\ \forall i<r) \\
&= \sum_{|\alpha_1|=n_1}\cdots\sum_{|\alpha_{r-1}|=n_{r-1}} \mathbb{P}(\phi(\alpha_i)=x_i,\ \forall i<r|\alpha_i\in\mathbf{T},\ \forall i<r)P(\alpha_i\in\mathbf{T},\ \forall i<r).
\end{aligned}
$$

Taking Fourier transforms, we get

$$
\begin{aligned}
\hat{t}^{(r)}_{\vec{n}}(\vec{k}) = &\sum_{|\alpha_1|=n_1}\cdots\sum_{|\alpha_{r-1}|=n_{r-1}} P(\alpha_i\in\mathbf{T},\ \forall i<r) \\
&\qquad\times\sum_{\vec{x}} \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}}\mathbb{P}(\phi(\alpha_i)=x_i,\ \forall i<r|\alpha_i\in\mathbf{T},\ \forall i<r) \\
\equiv &\sum_{|\alpha_1|=n_1}\cdots\sum_{|\alpha_{r-1}|=n_{r-1}} P(\alpha_i\in\mathbf{T},\ \forall i<r)\hat{m}_{\alpha_1,\dots,\alpha_{r-1}}(\vec{k}).
\end{aligned}
\tag{4.8}
$$

Here $\hat{m}_{\alpha_1,\dots,\alpha_{r-1}}(\vec{k})$ is the Fourier transform of the random vector in $\mathbb{Z}^{d(r-1)}$ obtained by running independent random walks (with kernel D) along the branches of the tree generated by $\alpha_1,\dots,\alpha_{r-1}$ and considering the final positions of the $r-1$ leaves in $\mathbb{Z}^d$. (Although the reader should note from the above that we use the term "leaves" loosely as we include the cases where α_i is an ancestor of α_j.) This follows from the independence in (1.14).

We let $\alpha\vee\beta = 0\alpha_1\dots\alpha_{|\alpha|}\beta_1\dots\beta_{|\beta|}$ denote the concatenation of two words in the set of finite words W and $\alpha|m = 0\alpha_1\dots\alpha_m$ denote the ancestor of α in generation m. Fix $\alpha_1,\dots,\alpha_{r-1}\in W$ with $|\alpha_i| = n_i$ and let $i_2\in\{2,\dots,r-1\}$ be such that $m_*\equiv|\alpha_1\wedge\alpha_{i_2}|$ is minimal among all α_i, $i\ge 2$ (take i_2 minimal if there is more than one choice). If α_1 is a descendant of α_{i_2}, or conversely, then $\underline{n} = |\alpha_1|\wedge|\alpha_{i_2}|\equiv n_{\underline{i}}$ (take $\underline{i}=1$ if $\alpha_1=\alpha_{i_2}$). To find the contribution to (4.8) from these terms, set $\underline{J}=\{1,\dots,r-1\}\smallsetminus\{\underline{i}\}$ and sum over $\beta=\alpha_1\wedge\alpha_{i_2}$ and use the independence properties of the Galton-Watson tree from (1.14) to see that this contribution equals

$$\sum_{|\beta|=\underline{n}} \sum_{|\alpha_i'|=n_i-\underline{n}, i\neq \underline{i}} P(\beta \vee \alpha_i' \in \mathbf{T},\, i \in \underline{J})$$

$$\times \sum_{y} \sum_{x_i', i \in \underline{J}} \mathrm{e}^{\mathrm{i}k_{\underline{i}}\cdot y} \mathrm{e}^{\mathrm{i}(\vec{k}_{\underline{J}}\cdot (y+x_i')_{i\in \underline{J}})} \mathbb{P}(\phi(\beta) = y, \phi(\beta\vee\alpha_i') - \phi(\beta) = x_i' | \beta \vee \alpha_i' \in \mathbf{T},\, i \in \underline{J})$$

$$= \Big[\sum_{|\beta|=\underline{n}} P(\beta \in \mathbf{T}) \sum_{y} \mathrm{e}^{\mathrm{i}(\sum_i k_i \cdot y)} \mathbb{P}(\phi(\beta)=y) \Big] \cdot \sum_{|\alpha_i'|=n_J-\underline{n}, i\neq \underline{i}} P(\alpha_i' \in \mathbf{T}, i \in \underline{J}) \tag{4.9}$$

$$\times \sum_{x_i', i\in \underline{J}} \mathrm{e}^{\mathrm{i}(\vec{k}_{\underline{J}}\cdot x_{\underline{J}}')} \mathbb{P}(\phi(\alpha_i') = x_i', i \in \underline{J} | \alpha_i' \in \mathbf{T}, i \in \underline{J})$$

$$= \hat{D}(k(J_r))^{\underline{n}} \hat{t}^{(r-1)}_{\vec{n}_{\underline{J}}-\underline{n}}(\vec{k}_{\underline{J}}),$$

which by (4.3) is $\hat{\kappa}^{(r)}_{\vec{n}}(\vec{k})$. In the last line we have used (4.7) and (4.8). Note that some of the coordinates of $\vec{n}_{\underline{J}} - \underline{n}$ may be zero. Therefore $\hat{\kappa}^{(r)}_{\vec{n}}(\vec{k})$ represents the contribution from the terms when α_1 is a descendant of α_{i_2} or conversely.

Assume now that α_1 is not a descendant of α_{i_2} and α_{i_2} is not a descendant of α_1. This means that there is a proper branch point between α_1 and $\{\alpha_2, \dots, \alpha_{r-1}\}$ at time $m_* \in \{0, \dots, \underline{n}-1\}$. Let $\beta = \alpha_1 | m_*$ be the parent of the branching children. Let $j \in \{2, \dots, r-1\}$ denote the number of children of β which are ancestors of some α_i, $i \le r-1$. Let $m_1 = m_* + 1$ and $w_1 = \alpha_1 | m_1$ be the child of β which is the ancestor of α_1, and let $w_2 \neq w_1$ be the child of β which is the ancestor of α_{i_2}.

When $j \ge 3$ we let $w_3, \dots, w_j$ denote the other $j-2$ children of β which are ancestors of $\{\alpha_i : i \neq 1, i_2\}$; use the tree order in W to order them. For $s = 1, \dots, j$, let $I_s' \subset \{1, \dots, r-1\}$ denote the set of i such that w_s is an ancestor of α_i. Therefore $1 \in I_1'$ and $i_2 = \min\{i \in \{1, \dots, r-1\} \setminus I_1'\} \in I_2'$.

We now derive (3.8) for $r = 3$. In this case $j = 2$, $I_1' = \{1\}$, $I_2' = \{2\} = \{i_2\}$, and so

$$\underline{n} = \underline{n_{I_1'}} \wedge \underline{n_{I_2'}}. \tag{4.10}$$

If m is the total number of children of β then the number of ways we may choose w_1 and w_2 is $\frac{m!}{(m-2)!}$ and so the remaining contribution to (4.8) is

$$\sum_{m_1=1}^{\underline{n}} \sum_{m=2}^{\infty} p_m \frac{m!}{(m-2)!} \sum_{|\beta|=m_1-1} P(\beta \in \mathbf{T}) \hat{m}_\beta(k_1+k_2) \hat{D}(k_1)\hat{D}(k_2)$$

$$\times \sum_{|\alpha_i'|=n_i-m_1, i=1,2} P(\alpha_1' \in \mathbf{T}) P(\alpha_2' \in \mathbf{T}) \hat{m}_{\alpha_1'}(k_1) \hat{m}_{\alpha_2'}(k_2)$$

$$= \lambda_2 \sum_{m_1=1}^{\underline{n}} \Big[\sum_{|\beta|=m_1-1} P(\beta \in \mathbf{T}) \hat{m}_\beta(k_1+k_2) \Big] \hat{D}(k_1)\hat{D}(k_2) \hat{t}^{(2)}_{n_1-m_1}(k_1) \hat{t}^{(2)}_{n_2-m_1}(k_2).$$

In the first line the product $\hat{D}(k_1)\hat{D}(k_2)$ arises from the steps taken by w_1 and w_2 from their parent. The term in square brackets equals $\hat{D}(k_1+k_2)^{m_1-1}$ by (4.7) and (4.8). Letting $(I_0, I_1, I_2) = (\varnothing, I_1', I_2')$ and using (4.10), we see this establishes (3.8) for $r = 3$ with $\hat{\chi}^{(1)}$ as in (4.4). (It is also easy to derive the above by iterating (4.2).)

We omit the derivation of (3.8) for $r=4$ and move straight to the more complex $r=5$ case. Then $j\le 4$, $I_0=\cup_{s=3}^{j} I'_s$ (I_0 as in (3.8)), $1\in I_1$, and $i_2\in I_2\neq\varnothing$. The number of ways to choose $w_1,\dots,w_j$ when β has m children is $\frac{m!}{(m-j)!}$. The contribution from $I_0=\varnothing$ (which implies $j=2$ and is the main term), and from the case where there is a true branch point, may be calculated as in the $r=3$ case and is

$$\begin{aligned}
&\sum_{I_1,I_2}\ \sum_{m_1\le \underline{n}=\underline{n_{I_1}}\wedge \underline{n_{I_2}}}\ \sum_{m=2}^{\infty} p_m\frac{m!}{(m-2)!}\sum_{|\beta|=m_1-1}\ \sum_{|\alpha'_i|=n_i-m_1,i\in I_1}\ \sum_{|\alpha''_i|=n_i-m_1,i\in I_2} P(\beta\in\mathbf{T})\hat m_\beta(k(J_r))\\
&\qquad\times P(\alpha'_i\in\mathbf{T},i\in I_1)P(\alpha''_i\in\mathbf{T},i\in I_2))\hat D(k(I_1))\hat D(k(I_2))\hat m_{\alpha'_{I_1}}(\vec k_{I_1})\hat m_{\alpha''_{I_2}}(\vec k_{I_2})\\
&=\sum_{I_1,I_2}\ \sum_{m_1\le \underline{n_{I_1}}\wedge\underline{n_{I_2}}}\lambda_2\hat D(k(J_r))^{m_1-1}\hat D(k(I_1))\hat D(k(I_2))\hat t^{(|I_1|+1)}_{\vec n_{I_1}-m_1}(\vec k_{I_1})\hat t^{(|I_2|+1)}_{\vec n_{I_2}-m_1}(\vec k_{I_2}).
\end{aligned}\tag{4.11}$$

Here the first sum is over I_1,I_2 a partition of $\{1,\dots,4\}$ into non-empty sets such that $1\in I_1$ and $i_2\in I_2$ and in the last line we have again used (4.8). This gives the $I_0=\varnothing$ contribution to (3.8) with $\hat\chi^{(1)}$ as in (4.4).

Consider next the contribution from $|I_0|=1$, in which case $j=3$, $|I_j|=1$ or 2 for $j=1,2$ and $i_2\in\{2,3\}$. For the sake of definiteness assume that $i_2=2$. Let $n_0=\vec n_{I_0}$. Then the contribution to (4.8) is the sum over I_0,I_1,I_2 a partition of $\{1,\dots,4\}$ into non-empty sets such that $|I_0|=1$, $1\in I_1$ and $i_2=2\in I_2$ of

$$\begin{aligned}
&\sum_{m_1\le\underline{n}}\ \sum_{m=3}^{\infty}p_m\frac{m!}{(m-3)!}\sum_{|\beta|=m_1-1}\ \sum_{|\alpha'_i|=n_0-m_1,i\in I_0}\ \sum_{|\alpha''_i|=n_i-m_1,i\in I_1}\ \sum_{|\alpha'''_i|=n_i-m_1,i\in I_2}\\
&\quad P(\beta\in\mathbf{T})\hat m_\beta(k(J_r))P(\alpha'_i\in\mathbf{T},i\in I_0)P(\alpha''_i\in\mathbf{T},i\in I_1)P(\alpha'''_i\in\mathbf{T},i\in I_2)\\
&\quad\times\hat D(k(I_0))\hat D(k(I_1))\hat D(k(I_2))\hat m_{\alpha'}(k(I_0))\hat m_{\alpha''}(k(I_1))\hat m_{\alpha'''}(k(I_2))\\
&=\lambda_3\sum_{m_1\le\underline{n_{I_1}}\wedge\underline{n_{I_2}}}\mathbb{1}_{\{m_1\le n_0\}}\hat D(k(J_r))^{m_1-1}\hat D(k(I_0))\hat D(k(I_1))\hat D(k(I_2))\hat D(k(I_0))^{n_0-m_1}\\
&\qquad\times\hat t^{(|I_1|+1)}_{\vec n_{I_1}-m_1}(\vec k_{I_1})\hat t^{(|I_2|+1)}_{\vec n_{I_2}-m_1}(\vec k_{I_2}).
\end{aligned}$$

This gives the $|I_0|=1$ contribution to (3.8) with $\hat\chi^{(2)}$ as in (4.5) (and $n_*=n_0$).

Finally consider the $|I_0|=2$ contribution. In this case $j=3$ or 4, $I_1=\{1\}$, $I_2=\{2\}$ and $I_0=\{3,4\}$. The contribution from $j=3$ means there is a later split in the I_0 part of the tree, which results in a 3-point function in the above calculations. So after a short calculation this contribution becomes

$$\begin{aligned}
&\lambda_3\sum_{m_1\le\underline{n_{I_1}}\wedge\underline{n_{I_2}}}\mathbb{1}_{\{m_1\le n_3\wedge n_4\}}\hat D(k(J_r))^{m_1-1}\hat D(k_1)\hat D(k_2)\hat D(k_3+k_4)\\
&\qquad\times\hat t^{(3)}_{n_3-m_1,n_4-m_1}(k_3,k_4)\hat t^{(2)}_{\vec n_{I_1}-m_1}(\vec k_{I_1})\hat t^{(2)}_{\vec n_{I_2}-m_1}(\vec k_{I_2}).
\end{aligned}\tag{4.12}$$

After some relabelling (note that $\vec n_{I_0}=(n_3,n_4)$) this gives rise to the second part of $\hat\chi^{(3)}$ in (4.6) (with $(n_*,n_{**})=(n_3,n_4)$). The $j=4$ case corresponds to four disjoint ancestral branches from β to $\alpha_1, \dots, \alpha_4$ and this gives rise to a contribution of

$$\lambda_4 \sum_{m_1 \le \underline{n_{I_1}} \wedge \underline{n_{I_2}}} \mathbb{1}_{\{m_1 \le n_3 \wedge n_4\}} \hat{D}(k(J_r))^{m_1-1} \left(\prod_{j=1}^{2} \hat{D}(k_j) \right) \left(\prod_{j=3}^{4} \hat{D}(k_j)^{n_j - m_1 + 1} \right) \times \hat{t}^{(2)}_{\vec{n}_{I_1}-m_1}(\vec{k}_{I_1}) \hat{t}^{(2)}_{\vec{n}_{I_2}-m_1}(\vec{k}_{I_2}). \tag{4.13}$$

After some relabelling ($\vec{n}_{I_0} = (n_3, n_4)$) this corresponds to the first part of $\hat{\chi}^{(3)}$ in (4.6). This completes the derivation of (3.8).

To verify Condition 3.1, note that from (4.7)

$$\hat{t}^{(2)}_{j+1}(k) - \hat{t}^{(2)}_j(k) = \left(\prod_{i=1}^{j} \hat{D}(k) \right) [\hat{D}(k) - 1] = \left(\prod_{i=1}^{j} \hat{D}(k) \right) \sum_x (\cos(k \cdot x) - 1) D(x).$$

Therefore by (1.10) and the fact that $|\cos(y) - 1| \le y^2/2$, (3.5) holds with $K = \sigma^2/2$ for all $K' > 0$. The condition (3.6) with $K = \sigma^2$ can be obtained from (4.7) since the left-hand side is the mean square displacement of an n-step walk in $\mathbb{Z}^d$ with kernel D and so equals $n\sigma^2$.

We next verify that $\hat{\kappa}^{(r)}$ and $\hat{\chi}^{(i)}_{m_1,m_2}$ satisfy the bounds in Condition 3.2. Since $|\hat{D}(k)| \le 1$ for all k, we see from (4.7) that $|\hat{\kappa}^{(3)}_{\vec{n}}(\vec{k})| \le 1$. For $r = 3, 4$, we use the fact that critical BRW satisfies

$$|\hat{t}^{(r)}_{\vec{n}}(\vec{k})| \le C\bar{n}^{r-2}. \tag{4.14}$$

This is true because the left-hand side is

$$\Big| \sum_{\vec{x}} \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}} \mathbb{E}\Big[\prod_{i=1}^{r-1} N_{n_i}(x_i) \Big] \Big| \le \mathbb{E}\Big[\sum_{\vec{x}} \prod_{i=1}^{r-1} N_{n_i}(x_i) \Big] = \mathbb{E}\left[\prod_{i=1}^{r-1} N_{n_i} \right] \le C\bar{n}^{r-2},$$

where the last holds by a standard moment generating function calculation as in [16, Section I.5]. Therefore, $|\hat{\kappa}^{(4)}_{\vec{n}}(\vec{k})| \le C(\bar{n} - \underline{n})$, and $|\hat{\kappa}^{(5)}_{\vec{n}}(\vec{k})| \le C(\bar{n} - \underline{n})^2$, as required. The bounds on $\hat{\chi}^{(1)}_{m_1,m_2}(\vec{k}), \hat{\chi}^{(2)}_{m_1,m_2;n_*}(\vec{k})$, and $\hat{\chi}^{(3)}_{m_1,m_2;n_*,n_{**}}(\vec{k})$ trivially hold, the latter by another application of (4.14). This completes the verification of Condition 3.2, and hence the proof of Theorem 1.1. ∎

5. Weak convergence proofs.

5.1. *Proof of Theorem 2.2.* In this section we prove Theorem 2.2 which gives a criterion for tightness of a sequence of measure-valued processes. The section concludes with the simple proof of Corollary 1.7.

The first step towards our tightness criterion is to develop a user-friendly version of what is sometimes called Jakubowski's Theorem (Theorem 5.2 below), which involves tightness of the real-valued or complex-valued processes obtained by integrating a separating class of test functions with respect to the underlying measure-valued processes. The second step will be to reduce the tightness of these $\mathbb{R}$- or $\mathbb{C}$-valued processes to a fourth moment bound.

Definition 5.1 (Separating class). *A class $D_0 \subset C_b(\mathbb{R}^d, \mathbb{C})$ is a* separating class *in $\mathcal{M}_F(\mathbb{R}^d)$ if*

$$[\mu(\phi) = \nu(\phi) \text{ for every } \phi \in D_0] \Rightarrow [\mu = \nu]\,. \tag{5.1}$$

Since the distribution of a random vector is determined by its Fourier transform, we have that $D_0 = \{x \mapsto e^{ik\cdot x} : k \in \mathbb{R}^d\}$ is a separating class.

The following theorem is standard, see e.g., [49, Theorem II.4.1].

Theorem 5.2. *Let D_0 be a separating class containing 1. A sequence of probabilities $\{P_n\}_{n\in\mathbb{N}}$ on $\mathcal{D}$ is $\mathcal{C}$-relatively compact if and only if the following two conditions hold:*

(CC) *For every $\eta > 0$ and $T > 0$ there is a compact $K_{\eta,T} \subset \mathbb{R}^d$ such that*

$$\sup_n P_n\left(\sup_{t\le T} X_t(K^c_{\eta,T}) > \eta\right) < \eta. \tag{5.2}$$

($\mathbb{C}$-vP) *For every $\phi \in D_0$, the sequence of probabilities $\{P_n^\phi\}_{n\in\mathbb{N}}$ on $\mathcal{D}(\mathbb{C})$ defined by $P_n^\phi(\bullet) = P_n(\{X_t(\phi)\}_{t\ge 0} \in \bullet)$ is $\mathcal{C}$-relatively compact.*

For $x \in \mathbb{Z}^d$ and $k \in \mathbb{R}^d$, let

$$\xi_k(x) = \cos(k\cdot x) \text{ and } \phi_k(x) = e^{ik\cdot x}. \tag{5.3}$$

Recall from (2.1) that $\hat{X}_t(k) = X_t(\phi_k)$, and in particular, $\hat{X}_t(0) = X_t(1)$. To prepare for a user-friendly version of Theorem 5.2 for $D_0 = \{\phi_k : k \in \mathbb{R}^d\}$ we start with a well-known tail estimate. We let e_j be the jth unit basis vector in $\mathbb{R}^d$ and for a finite measure, μ, on $\mathbb{R}^d$ and $M > 0$, define

$$Z(\mu, M) = M\sum_{j=1}^d \int_0^{1/M} [\mu(1) - \mu(\xi_{k_j e_j})]dk_j.$$

Lemma 5.3. *There is a $\bar{C} = \bar{C}(d)$ so that for any $\mu \in \mathcal{M}_F(\mathbb{R}^d)$,*

$$\mu(|x| > \sqrt{d}M) \le \bar{C}Z(\mu, M).$$

Proof. If μ_j is the jth marginal of μ, then the left-hand side is at most $\sum_{j=1}^d \mu_j(|x_j| > M)$. Now apply Proposition 8.29 in Breiman [6] to each marginal. ■

We will apply the following result to the conditional probabilities P_n^ε.

Theorem 5.4. *Let $\{X^{(n)}, n \in \mathbb{N}\}$ be a sequence of processes (under probability measures P_n) with sample paths in $\mathcal{D}$. Assume that*

(i) for each $k \in \mathbb{R}^d$, $\{\hat{X}^{(n)}(k), n \in \mathbb{N}\}$ is $\mathcal{C}$-tight in $\mathcal{D}(\mathbb{C})$, i.e., $\{\hat{X}_0^{(n)}(k), n \in \mathbb{N}\}$ is tight in $\mathcal{C}$, and for any $T, \varepsilon > 0$, $k \in \mathbb{R}$, there exist $\delta = \delta(\varepsilon, T, k) > 0$ and $n_0 = n_0(\varepsilon, T, k) \in \mathbb{N}$ such that

$$\sup_{n\ge n_0} P_n\Big(\sup_{s,t\le T, |s-t|\le\delta} |\hat{X}_t^{(n)}(k) - \hat{X}_s^{(n)}(k)| > \varepsilon\Big) < \varepsilon, \tag{5.4}$$

(ii) for any $T,\varepsilon>0$ there exist $\eta_0=\eta_0(\varepsilon,T)>0$, $\delta(\varepsilon,T)>0$ and $n_0(\varepsilon,T)\in\mathbb{N}$ such that

$$\sup_{|k|\le\eta_0,n\ge n_0} P_n\Big(\sup_{s,t\le T,|s-t|\le\delta}|X_t^{(n)}(\xi_k)-X_s^{(n)}(\xi_k)|>\varepsilon\Big)<\varepsilon, \tag{5.5}$$

and,

(iii) for any $T>0$ and $j=1,\dots,d$,

$$\lim_{(n,k_j)\to(\infty,0)}\sup_{t\le T}E_n\big[X_t^{(n)}(1)-X_t^{(n)}(\xi_{k_je_j})\big]=0. \tag{5.6}$$

Then $\{X^{(n)}\}$ is $\mathcal{C}$-tight in $\mathcal{D}$.

Proof. We wish to apply Theorem 5.2 and so will verify its hypotheses with $D_0=\{\phi_k\colon k\in\mathbb{R}^d\}$ as above. Note that ($\mathbb{C}$-vP) is immediate by (i).

Step 1. $\mathcal{C}$-tightness of $Z(X_t^{(n)},M)$. Let $Z_t^{M,n}=Z(X_t^{(n)},M)$. Then clearly

$$|Z_t^{M,n}-Z_s^{M,n}|\le M\sum_{j=1}^d\int_0^{1/M}|X_t^{(n)}(1)-X_s^{(n)}(1)|+|X_t^{(n)}(\xi_{k_je_j})-X_s^{(n)}(\xi_{k_je_j})|\,dk_j. \tag{5.7}$$

We first claim that for $\varepsilon,T>0$ there are $n_1\in\mathbb{N}$ and $\eta_1,\delta_1>0$ so that

$$\sup_{M\ge1/\eta_1,n\ge n_1}P_n\Big(\sup_{s,t\le T,|s-t|\le\delta_1}|Z_t^{M,n}-Z_s^{M,n}|>\varepsilon\Big)<\varepsilon. \tag{5.8}$$

Define

$$\Gamma_n(R,T)=\{\sup_{s\le T}X_s^{(n)}(1)>R\},$$

and choose $R=R_{T,\varepsilon}\ge1$ so that

$$\sup_n P_n(\Gamma_n(R,T))<\varepsilon/5.$$

This is possible due to the $\mathcal{C}$-tightness of $\{X^{(n)}(1),n\in\mathbb{N}\}$, given by *(i)*. Let n_0, η_0, and δ be as in (5.5), and set $n_1=n_0(\varepsilon^2/(5Rd),T)$, $\eta_1=\eta_0(\varepsilon^2/(5Rd))$ and $\delta_1=\delta(\varepsilon^2/(5Rd),T)$. If $n\ge n_1$ and $M\ge1/\eta_1$, then by (5.7)

$$\begin{aligned}
&P_n\Big(\sup_{s,t\le T,|s-t|\le\delta_1}|Z_t^{M,n}-Z_s^{M,n}|>\varepsilon\Big)\\
&\quad\le P_n(\Gamma_n(R,T))+\frac{M}{\varepsilon}\sum_{j=1}^d\int_0^{1/M}E_n\Big[\sup_{s,t\le T,|s-t|\le\delta_1}|X_t^{(n)}(1)-X_s^{(n)}(1)|\mathbb{1}_{\Gamma_n(R,T)^c}\Big]\\
&\qquad\qquad+E_n\Big[\sup_{s,t\le T,|s-t|\le\delta_1}|X_t^{(n)}(\xi_{k_je_j})-X_s^{(n)}(\xi_{k_je_j})|\mathbb{1}_{\Gamma_n(R,T)^c}\Big]dk_j.
\end{aligned} \tag{5.9}$$

Let $W_n=\sup_{s,t\le T,|s-t|\le\delta_1}|X_t^{(n)}(1)-X_s^{(n)}(1)|$. Write $1=\mathbb{1}_{\{W_n>\varepsilon^2/5d\}}+\mathbb{1}_{\{W_n\le\varepsilon^2/5d\}}$ and use the fact that $|W_n|\le R$ on the event $\Gamma_n(R,T)^c$, and similar reasoning for the second term, to see that the right side of (5.9) is strictly less than

$$\frac{\varepsilon}{5}+\frac{M}{\varepsilon}\sum_{j=1}^{d}\int_0^{1/M}\Big[RP_n\Big(\sup_{s,t\le T,|s-t|\le\delta_1}|X_t^{(n)}(1)-X_s^{(n)}(1)|>\varepsilon^2/(5d)\Big)+\varepsilon^2/(5d)$$
$$+RP_n\Big(\sup_{s,t\le T,|s-t|\le\delta_1}|X_t^{(n)}(\xi_{k_je_j})-X_s^{(n)}(\xi_{k_je_j})|>\varepsilon^2/(5d)\Big)+\varepsilon^2/(5d)\Big]\,dk_j.$$

Applying (ii), this is bounded by

$$\frac{\varepsilon}{5}+\frac{Md}{\varepsilon}\int_0^{1/M}\frac{2R\varepsilon^2}{5dR}+\frac{2\varepsilon^2}{5d}\,dk=\varepsilon,$$

which establishes the claim (5.8).

Step 2. We next claim that for $\varepsilon,T>0$ there are $M_3(\varepsilon,T)>0$ and $n_3(\varepsilon,T)\in\mathbb{N}$ so that

$$\sup_{M\ge M_3,n\ge n_3}P_n\Big(\sup_{t\le T}Z_t^{M,n}>\varepsilon\Big)\le\varepsilon. \tag{5.10}$$

First let $\delta_1=\delta_1(\varepsilon/2,T)$, $\eta_1=\eta_1(\varepsilon/2,T)$, and $n_1=n_1(\varepsilon/2,T)$ be as in (5.8), so that

$$\sup_{M\ge 1/\eta_1,n\ge n_1}P_n\Big(\sup_{s,t\le T,|s-t|\le\delta_1}|Z_t^{M,n}-Z_s^{M,n}|>\varepsilon/2\Big)<\varepsilon/2. \tag{5.11}$$

If $T,\varepsilon>0$, then by (5.6) for $n\ge n_2(\varepsilon,T)$ and $M\ge M_2(\varepsilon,T)$,

$$\sup_{t\le T}E_n[Z_t^{M,n}]\le\sum_{j=1}^{d}\sup_{t\le T,|k_j|\le 1/M}E_n[X_t^{(n)}(1)-X_t^{(n)}(\xi_{k_je_j})]<\varepsilon. \tag{5.12}$$

Let $n_2=n_2(\varepsilon^2\delta_1/(5T),T)$ and $M_2=M_2(\varepsilon^2\delta_1/(5T),T)$ be as in (5.12) so that

$$\sup_{n\ge n_2,M\ge M_2,i\delta_1\le T}E_n[Z_{i\delta_1}^{M,n}]<\frac{\varepsilon^2\delta_1}{5T}. \tag{5.13}$$

Then (5.11) and (5.13) together imply that for $M\ge M_3\equiv M_2\vee(1/\eta_1)$ and $n\ge n_3\equiv n_1\vee n_2$,

$$\begin{aligned}P_n(\sup_{t\le T}Z_t^{M,n}>\varepsilon)&\le P_n\Big(\max_{0\le i\delta_1\le T}Z_{i\delta_1}^{M,n}>\varepsilon/2\Big)+P_n\Big(\max_{0\le i\delta_1\le T}\sup_{t\in[i\delta_1,(i+1)\delta_1\wedge T]}|Z_t^{M,n}-Z_{i\delta_1}^{M,n}|>\varepsilon/2\Big)\\&\le\frac{T}{\delta_1(\varepsilon/2)}\max_{i\delta_1\le T}E_n[Z_{i\delta_1}^{M,n}]+\frac{\varepsilon}{2}\\&\le\frac{2T}{\delta_1\varepsilon}\cdot\frac{\varepsilon^2\delta_1}{5T}+\frac{\varepsilon}{2}<\varepsilon.\end{aligned}$$

This proves (5.10).

To conclude the proof of Theorem 5.4, we combine (5.10) with Lemma 5.3 to conclude that for $n\ge n_3(\varepsilon/\bar{C},T)=n_3$ and $M=\sqrt{d}M_3(\varepsilon/\bar{C},T)$ (and $\bar{C}\ge 1$ as in Lemma 5.3),

$$P_n\Big(\sup_{t\le T}X_t^{(n)}(|x|>M)>\varepsilon\Big)\le P_n\Big(\sup_{t\le T}Z_t^{M_3,n}>\varepsilon/\bar{C}\Big)<\frac{\varepsilon}{\bar{C}}\le\varepsilon.$$

By increasing M_3 we can handle $n < n_3$ and so $\{X^{(n)}\}$ satisfies **(CC)** in Theorem 5.2. ∎

We next verify that the tightness condition in Theorem 5.4 (iii) follows from (3.6) and (2.3).

LEMMA 5.5. *Assume that* $\sum_x |x|^2 t_n^{(2)}(x) \le Kn$, *i.e.* (3.6) *holds, and for a particular* $\varepsilon > 0$, $\mu_n(S > \varepsilon) \to \mathbb{N}_0(S > \varepsilon)$, *i.e.,* (2.3) *holds. Then Theorem 5.4(iii) holds for the measures* $\{P_n^\varepsilon : n \in \mathbb{N}\}$.

PROOF. By (2.3), it is sufficient to show that

$$\lim_{n\to\infty, k\to 0} \sup_{t\le T} E_{\mu_n}[(X_t^{(n)}(1) - X_t^{(n)}(\xi_k))\mathbb{1}_{\{S>\varepsilon\}}] \to 0.$$

Since the term in brackets in the expectation is non-negative, the expectation satisfies

$$\begin{aligned} 0 \le E_{\mu_n}[(X_t^{(n)}(1) - X_t^{(n)}(\xi_k))\mathbb{1}_{\{S>\varepsilon\}}] &\le E_{\mu_n}[X_t^{(n)}(1) - X_t^{(n)}(\xi_k)] \\ &\le C\sum_x \left| 1 - \cos\left(\frac{k}{\sqrt{vn}} \cdot x\right)\right| t_{nt}^{(2)}(x) \\ &\le \frac{C|k|^2}{n}\sum_x |x|^2 t_{nt}^{(2)}(x) \le \frac{C|k|^2}{n} Knt, \end{aligned} \tag{5.14}$$

by (3.6). This converges to zero as $k \to 0$, uniformly in $t \le T$. ∎

We finally reduce the tightness conditions in Theorem 5.4 *(i)* and *(ii)* to the fourth moment condition in Theorem 2.2.

LEMMA 5.6. *Suppose that* $\varepsilon > 0$, (2.3) *holds and* $\{X_0^{(n)}(1), n \in \mathbb{N}\}$ *is tight in* $\mathbb{R}$. *Suppose also that there exist* $\zeta, \eta_0 > 0$ *and constants* $C_{k,T} > 0$ *for* $k \in \mathbb{R}^d$, $T > 0$ *with* $\sup_{|k|\le\eta_0} C_{k,T} \le C_T < \infty$ *such that:*
For all $n \in \mathbb{N}$, $T \in \mathbb{N}$, $k \in \mathbb{R}^d$ *and all* $s, t \in H_{n,T} \equiv \{m/n : m \in \mathbb{N}, 0 \le m \le n(T+1)\}$,

$$E_n^\varepsilon\left[|\hat{X}_t^{(n)}(k) - \hat{X}_s^{(n)}(k)|^4\right] \le C_{k,T}|t-s|^{1+\zeta}. \tag{5.15}$$

Then Theorem 5.4(i) and (ii) hold for the measures $\{P_n^\varepsilon : n \in \mathbb{N}\}$.

PROOF. The tightness of $\{X_0^{(n)}(1), n \in \mathbb{N}\}$ in $\mathbb{R}$ implies that the same is true of $\{\hat{X}_0^{(n)}(k), n \in \mathbb{N}\}$ in $\mathbb{C}$ for any $k \in \mathbb{R}^d$. Note first that we may define a new probability space with probability measure P on which all of the processes $X^{(n)}$ are defined each with their respective laws P_n^ε. Since also $|X_t(\xi_k) - X_s(\xi_k)| \le |\hat{X}_t(k) - \hat{X}_s(k)|$, it is then sufficient to show that for some $\delta = \delta(\varepsilon) > 0$,

$$\sup_{n\ge n_0(k,\varepsilon,T)} P\Big(\sup_{s,t\le T, |s-t|\le\delta} |\hat{X}_t^{(n)}(k) - \hat{X}_s^{(n)}(k)| > \varepsilon\Big) < \varepsilon \text{ for each } k \in \mathbb{R}^d, \quad \text{and} \tag{5.16}$$

$$\sup_{|k|\le\eta_0, n\ge n_0'(\varepsilon,T)} P\Big(\sup_{s,t\le T, |s-t|\le\delta} |\hat{X}_t^{(n)}(k) - \hat{X}_s^{(n)}(k)| > \varepsilon\Big) < \varepsilon. \tag{5.17}$$

To derive the above from our fourth moment conditions we can use a familiar dyadic expansion argument of Lévy (see, e.g., [36, Thm. I.4.3]). We omit the details which are standard.

Proof of Theorem 2.2. Recall that (2.3) for a single $\varepsilon > 0$ implies it holds for all $\varepsilon > 0$ as both are equivalent to (1.3). Fix any $\varepsilon > 0$. By assumption, $\sup_n n^{-1} \sum_x |x|^2 \mathbb{E}[N_n(x)] < \infty$ so (3.6) holds. The tightness of $\{X_0^{(n)}(1), n \in \mathbb{N}\}$ is trivial because $X_0^{(n)}(1) = 1/n$. By Lemma 5.5, Theorem 5.4(iii) holds for the probabilities $\{P_n^\varepsilon\}$. By assumption, (2.6) holds, so (5.15) holds. Therefore by Lemma 5.6, (i) and (ii) of Theorem 5.4 hold for $\{P_n^\varepsilon\}$. Theorem 5.4 now applies to the probability measures $\{P_n^\varepsilon : n \in \mathbb{N}\}$, and shows that $\{X^{(n)}\}$ are $\mathcal{C}$-tight in $\mathcal{D}$ under these measures.

If also (2.5) holds, then by [35, Proposition 2.4], for all $\varepsilon > 0$, $P_n^\varepsilon \xrightarrow{fdd} P_{\mathbb{N}_0}^\varepsilon$, and so $P_n^\varepsilon \xrightarrow{w} P_{\mathbb{N}_0}^\varepsilon$ by the above. As $\varepsilon > 0$ is arbitrary, it follows that $\mu_n \xrightarrow{w} \mathbb{N}_0$. ∎

5.2. *Proof of Corollary 1.7.* We prove the following result, which includes Corollary 1.7.

COROLLARY 5.7 (Extrinsic one-arm lower bound). *Let $d > 8$. Then for sufficiently large L and any $s > 0$,*

$$\liminf_{n\to\infty} nVA\mathbb{P}(\|\mathcal{T}\| > R\sqrt{vn}, N_{ns} > 0) \geq \mathbb{N}_0(\|X\| > R, S > s), \quad \text{and} \tag{5.18}$$

$$\liminf_{n\to\infty} \mathbb{P}\big(\|\mathcal{T}\| > R\sqrt{vn} \big| N_{ns} > 0\big) \geq \mathbb{N}_0\big(\|X\| > R \big| S > s\big). \tag{5.19}$$

Moreover (1.27) holds.

PROOF. The first claim follows from Theorem 1.2 and the facts that

$$nVA\mathbb{P}(\|\mathcal{T}\| > R\sqrt{vn}, N_{ns} > 0) = \mu_n(\|X^{(n)}\| > R, S > s),$$

and $\phi(X) = \mathbb{1}_{\{\|X\|>R, S>s\}}$ is a lower semi-continuous function on $\mathcal{D}(\mathcal{M}_F(\mathbb{R}^d))$ (as is easy to check). The second claim then follows since $\mu_n(S > s) \to \mathbb{N}_0(S > s)$.

To establish the third claim (and hence prove Corollary 1.7), let $\delta, R > 0$. By the modulus of continuity for the supports of super-Brownian motion ([49, Corollary III.1.5]) and the fact that if Ξ is a Poisson point process with intensity $\mathbb{N}_0$, then $X_t = \int \nu_t \Xi(d\nu)$ is a super-Brownian motion starting at δ_0, we conclude that

$$\lim_{s\downarrow 0}\Big[1 - \exp\Big(-\mathbb{N}_0\Big(\int_0^s X_t(\bar{B}_R^c)\,dt > 0\Big)\Big)\Big] = \lim_{s\downarrow 0} \mathbb{N}_0\Big(\int_0^s X_t(\bar{B}_R^c)\,dt > 0\Big) = 0.$$

Therefore there exists $s > 0$ such that

$$\delta > \mathbb{N}_0\left(\int_0^s X_t(\bar{B}_R^c)dt > 0\right) \geq \mathbb{N}_0(\cup_{t\geq 0}\{X_t(\bar{B}_R^c) > 0\}, X_s(1) = 0) = \mathbb{N}_0(\|X\| > R, S \leq s).$$

It follows that for $R > 0$

$$\begin{aligned}\liminf_{n\to\infty} nAV\mathbb{P}(\|\mathcal{T}\| > R\sqrt{vn}) &\geq \liminf_{n\to\infty} \mu_n(\|X^{(n)}\| > R, S > s) \\ &\geq \mathbb{N}_0(\|X\| > R, S > s) \geq \mathbb{N}_0(\|X\| > R) - \delta.\end{aligned} \tag{5.20}$$

Thus,

$$\liminf_{n\to\infty} nAV\mathbb{P}(\|\mathcal{T}\| > R\sqrt{vn}) \geq \mathbb{N}_0(\|X\| > R),$$

so that if $r_n = R\sqrt{vn}$ and $R = 1$, we have

$$\liminf_{n\to\infty} r_n^2\mathbb{P}(\|\mathcal{T}\| > r_n) \geq \frac{v}{AV}\mathbb{N}_0(\|X\| > 1),$$

and a standard interpolation argument gives the third claim. ∎

6. Tightness: Proof of Theorem 3.3. In this section we prove Theorem 3.3. The proof is then organized as follows. We must show (3.14) which, as we saw in Section 3, would follow from (3.18). In Section 6.1, we begin by bounding sums that frequently enter our analysis. Our starting point of the analysis for $\Delta\hat{t}^{(5)}_{j,l}(\vec{k})$ is (3.19), for which we need to prove bounds on $\Delta\hat{t}^{(3)}_{j,l}(\vec{k})$ and $\Delta\hat{t}^{(4)}_{j,l}(\vec{k})$. These are established in Section 6.2 and Section 6.3, respectively. Finally, in Section 6.4, we bound $\Delta\hat{t}^{(5)}_{j,l}(\vec{k})$ and complete the proof of Theorem 3.3.

6.1. *Preparations.*

LEMMA 6.1 (Bounds on lower point functions). *Assume that Condition 3.1 holds. Then*

(a) there exists a $C > 0$ such that uniformly in k,

$$|\hat{t}^{(2)}_n(k)| \leq C, \tag{6.1}$$

and,

(b) if K and K' are as in Condition 3.1, then for all $l \geq j$ and uniformly in $k \in \mathbb{R}^d$ such that $|k|^2 l \leq K'$,

$$|\Delta\hat{t}^{(2)}_{j,l}(k)| \leq K|l-j|((j+1)^{-a} + |k|^2). \tag{6.2}$$

Assume that Conditions 3.1 and 3.2 both hold. Then

(c) there is a $C > 0$ so that for $\bar{n} > 0$, uniformly in $\vec{k}$,

$$|\hat{t}^{(3)}_{\bar{n}}(\vec{k})| \leq C\bar{n}, \tag{6.3}$$

and

$$|\hat{t}^{(4)}_{\bar{n}}(\vec{k})| \leq C\bar{n}^2. \tag{6.4}$$

REMARK 6.2. *In specific models one often derives (6.3) and (6.4) from (6.1) using an inductive argument together with model-specific combinatorial bounds. Here we will use our general Conditions 3.1 and 3.2 to derive them directly.*

PROOF. For (6.1), we use $|\hat{t}^{(2)}_n(k)| \leq \hat{t}^{(2)}_n(0)$, $\hat{t}^{(2)}_0(0) = 1$ (by (1.2)) and sum the bound in Condition 3.1 over $j \in [0, n-1]$. For (6.2), we sum the bound in Condition 3.1 over $[j, l-1]$

and note that the bound in (3.5) is decreasing in j. For (6.3), we use Condition 3.2 for $r = 3$, with the fact that

$$\sum_{m_1,m_2\le\bar n}(m_1 \vartriangle m_2)^{-a} \le 2\sum_{\substack{m,m':\\ m\le m'\le\bar n}}(m'-m+1)^{-a} \le 2\sum_{m\le\bar n}\sum_{m'-m\ge 0}(m'-m+1)^{-a}\le C\bar n, \tag{6.5}$$

and use (6.1) for the arising two-point functions.

For (6.4), consider Condition 3.2 for $r = 4$, where $\hat\kappa^{(4)}_{\vec n} \le C\bar n$. For $|I_0| = 0$, $|I_1| = 1$ and $|I_2| = 2$, use (6.3) to bound $|\hat t^{(3)}_{\bar n_{I_2}-m_2}(\vec k_{I_2})|$, (6.1) to bound $|\hat t^{(2)}_{n_1-m_1}(k_1)|$, and (3.10) to bound $|\hat\chi^{(1)}_{m_1,m_2}|$. This allows us to bound the contribution from these terms by

$$C\sum_{m_1,m_2\le\bar n}(m_1 \vartriangle m_2)^{-a}\bar n \le C\bar n^2.$$

The same bound holds if cardinalities of I_1 and I_2 are reversed. If $|I_j| = 1$ for $j = 1,2,3$, then use (6.1) to bound the $\hat t$ terms and (3.11) to obtain the same bound, recalling that $p < 1$. As these are the only possible $|I_j|$ values we are done. ■

In our analysis we frequently rely on summation bounds involving powers of $m_1 \vartriangle m_2$. These bounds are stated in the following two lemmas.

Lemma 6.3 (Summation bounds). *For $a \in (1,2)$ there is a C so that for $j \ge 1$,*

$$\sum_{m_1,m_2\le j}(m_1 \vartriangle m_2)^{-a} \le Cj \tag{6.6}$$

$$\sum_{m_1,m_2\le j}(m_1 \vartriangle m_2)^{-a}(j-m_1+1)^{-a} \le C, \tag{6.7}$$

$$\sum_{m_1,m_2\le j}(m_1 \vartriangle m_2)^{-(a-1)}(j-m_1+1)^{-a} \le Cj^{2-a}, \tag{6.8}$$

$$\sum_{m_1,m_2\le j}(m_1 \vartriangle m_2)^{-(a-1)}(j-m_1+1)^{-a}(j-m_2+1)^{-a} \le C. \tag{6.9}$$

Proof. The proof of (6.6) is given above in (6.5). By considering the two cases $0 \le m_1 - m_2 \le j$ and $0 \le m_2 - m_1 \le j$ we bound the left hand side of (6.7) by

$$\sum_{m_1\le j}(j-m_1+1)^{-a}\,2\sum_{m=0}^{j}(m+1)^{-a}\le C.$$

Similarly for (6.8) the bound is

$$\sum_{m_1\le j}(j-m_1+1)^{-a}\,2\sum_{m=0}^{j}(m+1)^{1-a}\le Cj^{2-a}.$$

For (6.9), note that $(m_1 \vartriangle m_2)^{-(a-1)} \le 1$ since $a > 1$, and that $\sum_{s=0}^{j}(j-s+1)^{-a} \le C$. ■

We note that Δ applied to the factors $\hat{\chi}$ leaves their bounds unchanged. Therefore from the bounds in Condition 3.2, for $m_1, m_2 \le l$ and $s = 1, 2, 3$, we see that

$$|\Delta\hat{\chi}^{(s)}_{m_1,m_2;j,l}(\vec{k})| \le C(m_1 \wedge m_2)^{-a}\big[(m_1 \wedge m_2) + l^p\big]^{\mathbb{1}_{\{s\ge 2\}}} l^{\mathbb{1}_{\{s=3\}}} \tag{6.10}$$

$$\le C(m_1 \wedge m_2)^{-a} l^{(s-1)}. \tag{6.11}$$

Note here that if $s = 1$ then $\Delta\hat{\chi}^{(s)}_{m_1,m_2;j,l}(\vec{k}) = \hat{\chi}^{(s)}_{m_1,m_2}(\vec{k})$ by definition.

6.2. *Bound on* $\Delta\hat{t}^{(3)}_{j,l}(\vec{k})$. To bound $\Delta\hat{t}^{(3)}_{j,l}(\vec{k})$ we will rely on the lace expansion from Condition 3.2.

PROPOSITION 6.4. *Assume that Conditions 3.1 and 3.2 hold. For each* $K' > 0$ *there is a* C *so that, for all* $j \le l$ *and uniformly in* $\vec{k}$ *such that* $|\vec{k}|^2 l \le K'$,

$$|\Delta\hat{t}^{(3)}_{j,l}(\vec{k})| \le C|l - j|. \tag{6.12}$$

PROOF. We may assume $j < l$ as otherwise the left-hand side is zero. From (3.8) we may write

$$\Delta\hat{t}^{(3)}_{j,l}(\vec{k}) = \sum_{m_1,m_2\le j} \hat{\chi}^{(1)}_{m_1,m_2}(\vec{k})\Delta\hat{t}^{(2)}_{j-m_1,l-m_1}(k_1)\Delta\hat{t}^{(2)}_{j-m_2,l-m_2}(k_2) + \hat{\gamma}^{(3)}_{j,l}(\vec{k}) + \Delta\hat{\kappa}^{(3)}_{j,l}(\vec{k}), \tag{6.13}$$

where $\hat{\gamma}^{(3)}_{j,l}(\vec{k})$ denotes the contribution due to terms where $m_1 \in [j+1, l]$ or $m_2 \in [j+1, l]$. By (3.9), $|\Delta\hat{\kappa}^{(3)}_{j,l}(\vec{k})| \le 4C$, which satisfies the required bound since $j < l$. Apply (6.1) in Lemma 6.1 to obtain

$$|\Delta\hat{t}^{(2)}_{j-m_2,l-m_2}(k_1)| \le 2C, \tag{6.14}$$

and (6.2) in Lemma 6.1 to $|\Delta\hat{t}^{(2)}_{j-m_1,l-m_1}(k_2)|$ as well as (3.10) to arrive at

$$\begin{aligned}\sum_{m_1,m_2\le j} &|\hat{\chi}^{(1)}_{m_1,m_2}(\vec{k})||\Delta\hat{t}^{(2)}_{j-m_1,l-m_1}(k_1)||\Delta\hat{t}^{(2)}_{j-m_2,l-m_2}(k_2)| \\ &\le C|l - j| \sum_{m_1,m_2\le j} (m_1 \wedge m_2)^{-a}\big(|\vec{k}|^2 + (j - m_1 + 1)^{-a}\big).\end{aligned} \tag{6.15}$$

This gives rise to 2 terms, which we bound one by one. We may use (6.6) to bound the term containing $|\vec{k}|^2$ by

$$C|k|^2|l - j| \sum_{m_1,m_2\le j} (m_1 \wedge m_2)^{-a} \le C|k|^2|l - j|j \le CK'|l - j|,$$

where we use that $|\vec{k}|^2 j \le K'$. The term containing no factor of $|\vec{k}|^2$ is (by (6.7)) bounded by

$$C|l - j| \sum_{m_1,m_2\le j} (m_1 \wedge m_2)^{-a}(j - m_1 + 1)^{-a} \le C|l - j|.$$

Both terms satisfy the required bound.

To bound $\hat{\gamma}^{(3)}_{j,l}(\vec{k})$, we note that when applying the Δ operator to $\hat{t}^{(3)}_{\vec{n}}(\vec{k})$ in (3.8), the variable $\vec{n}$ enters both in the summands and in the domain of summation for m_1, m_2. We only get a contribution to $\Delta\hat{t}^{(3)}_{j,l}(\vec{k})$ from $m_2 \in [j+1, l]$ if $n_2 = l$ and so when applying the Δ operator we only sum over $n_1 \in \{j, l\}$ and not over n_2. Similar constraints apply to $m_1 \in [j+1, l]$ and as a result,

$$
\begin{aligned}
\hat{\gamma}^{(3)}_{j,l}(\vec{k}) = &\sum_{m_1 \le j} \sum_{m_2=j+1}^{l} \hat{\chi}^{(1)}_{m_1,m_2}(\vec{k}) \Delta\hat{t}^{(2)}_{j-m_1,l-m_1}(k_1)\hat{t}^{(2)}_{l-m_2}(k_2) \\
&+ \sum_{m_2 \le j} \sum_{m_1=j+1}^{l} \hat{\chi}^{(1)}_{m_1,m_2}(\vec{k}) \Delta\hat{t}^{(2)}_{j-m_2,l-m_2}(k_2)\hat{t}^{(2)}_{l-m_1}(k_1) \\
&+ \sum_{j<m_1,m_2 \le l} \hat{\chi}^{(1)}_{m_1,m_2}(\vec{k}) \hat{t}^{(2)}_{l-m_1}(k_1)\hat{t}^{(2)}_{l-m_2}(k_2) \\
\equiv &\Sigma_1 + \Sigma_2 + \Sigma_3.
\end{aligned}
$$

By (3.10), (6.1) and (6.14), $|\Sigma_1| + |\Sigma_2|$ is at most

$$
C \sum_{m_2=j+1}^{l} \sum_{m_1 \le j} (m_2 - m_1 + 1)^{-a} \le C|l-j|.
$$

The same reasoning shows that $|\Sigma_3| \le C|l-j|$. All terms satisfy the required bound and we are done. ∎

6.3. *Bound on* $\Delta\hat{t}^{(4)}_{j,l}(\vec{k})$. We next investigate $\Delta\hat{t}^{(4)}_{j,l}(\vec{k})$:

PROPOSITION 6.5. *Suppose that Conditions 3.1 and 3.2 hold. For each $K' > 0$ there is a C so that, for all $j \le l$ and uniformly in $\vec{k}$ such that $|\vec{k}|^2 l \le K'$,*

$$
|\Delta\hat{t}^{(4)}_{j,l}(\vec{k})| \le C|l-j|l. \tag{6.16}
$$

PROOF. We may assume $j < l$ as the expression being bounded is zero if $j = l$. The analogue of (3.19) and (6.13) for $r = 4$ is

$$
\Delta\hat{t}^{(4)}_{j,l}(\vec{k}) = \sum_{I_0,I_1,I_2} \sum_{m_1,m_2 \le j} \Delta\hat{\chi}^{(|I_0|+1)}_{m_1,m_2;j,l}(\vec{k}) \Delta\hat{t}^{(|I_1|+1)}_{j-m_1,l-m_1}(\vec{k}_{I_1}) \Delta\hat{t}^{(|I_2|+1)}_{j-m_2,l-m_2}(\vec{k}_{I_2}) + \hat{\gamma}^{(4)}_{j,l}(\vec{k}) + \Delta\hat{\kappa}^{(4)}_{j,l}(\vec{k}), \tag{6.17}
$$

where $\hat{\gamma}^{(4)}_{j,l}(\vec{k})$ denotes the contribution from terms (m_1, m_2) such that $m_i \in [j+1, l]$ for $i = 1$ or 2.

It follows from (3.9) that

$$
|\Delta\hat{\kappa}^{(4)}_{j,l}(\vec{k})| \le Cl \le Cl|l-j|,
$$

(since $l > j$) and hence satisfies the required bound.

Consider next the summation term in (6.17). Note that $s = |I_0| + 1$ is 1 or 2. For $s = 2$ we have $|I_1| = |I_2| = 1$ and can use (6.14) to bound $|\Delta\hat{t}^{(2)}_{j-m_2,l-m_2}(\vec{k}_{I_2})|$, and (6.2) to bound

$|\Delta\hat{t}^{(2)}_{j-m_1,l-m_1}(\vec{k}_{I_1})|$, and (6.11) to see that for $|\vec{k}|^2 l \le K'$, the $s=2$ contribution to the sum in (6.17) is at most

$$\begin{aligned}
&C\sum_{m_1,m_2\le j}(m_1 \vartriangle m_2)^{-a} l|l-j|(|k_1|^2+(j-m_1+1)^{-a})\\
&\le C\sum_{m_1,m_2\le j}(m_1 \vartriangle m_2)^{-a} l|l-j|[K'l^{-1}+(j-m_1+1)^{-a}]\\
&\le C|l-j|l[K'jl^{-1}+1]\le C|l-j|l,
\end{aligned}$$

as required, where (6.6) and (6.7) are used in the next to last inequality.

For $s=1$ we have $|I_1|=1$, $|I_2|=2$, or conversely. Use (6.14), Proposition 6.4, (6.11), and then (6.6) to see that for $|\vec{k}|^2 l\le K'$, the $s=1$ contribution to the sum in (6.17) is at most

$$C\sum_{m_1,m_2\le j}(m_1 \vartriangle m_2)^{-a}|l-j|\le Cj|l-j|\le Cl|l-j|,$$

as required.

Turning to $\hat{\gamma}^{(4)}_{j,l}(\vec{k})$, consider first the contribution from $(m_1,m_2)\in[1,j]\times[j+1,l]$. To get a contribution to $\Delta\hat{t}^{(4)}_{j,l}(\vec{k})$ from $m_2\in[j+1,l]$ we must have $\underline{n_{I_2}}=l$ (here $\vec{n}$ is a triple of j's and l's) and so $\vec{n}|_{I_2}\equiv l$. Therefore when applying the Δ operator, the I_2 variables are held fixed at l while the others are allowed to vary through $\{j,l\}$. As a result the contribution from these terms equals the sum over I_0,I_1,I_2 of

$$\sum_{m_1=1}^{j}\sum_{m_2=j+1}^{l}\Delta\hat{\chi}^{(|I_0|+1)}_{m_1,m_2;j,l}(\vec{k}_{I_0},\vec{k}_{I_1},\vec{k}_{I_2})\Delta\hat{t}^{(|I_1|+1)}_{j-m_1,l-m_1}(\vec{k}_{I_1})\hat{t}^{(|I_2|+1)}_{l-m_2}(\vec{k}_{I_2}). \tag{6.18}$$

Here, to simplify notation, we write $t^{(r)}_n(\vec{k})$ for the r-point function where all time variables are equal to n.

The contribution to (6.18) when $|I_0|=1$ (whence $|I_1|=|I_2|=1$) is handled using (3.11) (with the fact that $n^*,m_i\le l$) together with (6.1) and (6.14), giving a bound of

$$C\sum_{m_2=j+1}^{l}\sum_{m_1=1}^{j}(m_1 \vartriangle m_2)^{-a} l\le C|l-j|l,$$

as required.

For the contribution to (6.18) from $|I_0|=0$, so $|I_1|=1,|I_2|=2$ (or $|I_1|=2,|I_2|=1$), we use (3.10), (6.14), and (6.3) (or (3.10), (6.1), and Proposition 6.4) to see that for $|\vec{k}|^2 l\le K'$, the contribution is at most

$$C\sum_{m_2=j+1}^{l}\sum_{m_1=1}^{j}(m_1 \vartriangle m_2)^{-a} l\le Cl|l-j|,$$

as required. This establishes the required bound on the contribution from terms $(m_1,m_2)\in[1,j]\times[j+1,l]$ and the terms $(m_1,m_2)\in[j+1,l]\times[1,j]$ may be handled in an identical way.

Finally consider the contribution to $\hat{\gamma}^{(4)}_{j,l}(\vec{k})$ from $(m_1, m_2) \in [j+1, l]^2$. In this case we only get a contribution to $\hat{\gamma}^{(4)}_{j,l}(\vec{k})$ if $\underline{n_{I_1}} = \underline{n_{I_2}} = l$, and so when applying the Δ operator the variables in $I_1 \cup I_2$ are set to l and only the I_0 variables vary through $\{j, l\}$. As a result the contribution here equals the sum over I_0, I_1, I_2 of

$$\sum_{m_1,m_2=j+1}^{l} \Delta \hat{\chi}^{(|I_0|+1)}_{m_1,m_2;j,l}(\vec{k}_{I_0}, \vec{k}_{I_1}, \vec{k}_{I_2}) \hat{t}^{(|I_1|+1)}_{l-m_1}(\vec{k}_{I_1}) \hat{t}^{(|I_2|+1)}_{l-m_2}(\vec{k}_{I_2}). \tag{6.19}$$

If $|I_0| = 1$, then $|I_1| = |I_2| = 1$, and so using $|\hat{t}^{(2)}_{l-m_1}(\vec{k}_{I_1})| \le C$ (by (6.1)) we see from (3.11) (with $n_*, m_i \le l$ and $p < 1$) that the contribution from $|I_0| = 1$ is at most

$$C \sum_{m_1,m_2=j+1}^{l} (m_1 \vartriangle m_2)^{-a} l \le C l |l-j|, \tag{6.20}$$

as required. If $|I_0| = 0$, then $|I_1| = 1, |I_2| = 2$, or conversely, and so using (3.10), (6.1), and (6.3) we may also bound this contribution by (6.20). This completes the proof that $|\hat{\gamma}^{(4)}_{j,l}(\vec{k})| \le Cl|l-j|$ for $|\vec{k}|^2 l \le K'$, and hence establishes the required bound on $|\Delta \hat{t}^{(4)}_{j,l}(\vec{k})|$. ∎

6.4. *Bound on* $\Delta \hat{t}^{(5)}_{j,l}(\vec{k})$*.* The following proposition verifies (3.18) (with $K' = T^3$), and hence completes the proof of Theorem 3.3.

PROPOSITION 6.6. *Suppose that Conditions 3.1 and 3.2 hold. For each $K' > 0$ there is a C so that, for all $j \le l$ and uniformly in $\vec{k}$ such that $|\vec{k}|^2 l \le K'$,*

$$|\Delta \hat{t}^{(5)}_{j,l}(\vec{k})| \le C l^3 \left| \frac{l-j}{l} \right|^{a \wedge (2-p)}. \tag{6.21}$$

PROOF. Without loss of generality, we assume throughout this proof that $l > j$. Our point of departure is (3.19). We now bound all the contributions in that decomposition of $\Delta \hat{t}^{(5)}_{j,l}(\vec{k})$ one by one.

The bound on $\Delta \hat{\kappa}^{(5)}_{j,l}(\vec{k})$*..* By (3.9) for $r = 5$, we obtain that

$$|\Delta \hat{\kappa}^{(5)}_{j,l}(\vec{k})| \le C(|l-j|^2 + l^{3-a}) \le C l^3 (|l-j|/l)^a, \tag{6.22}$$

since $a \in (1,2)$ and $1 \le l - j \le l$. The above is no more than the required bound in (6.21).

Bounds on terms with $m_1, m_2 \le j$*..* We bound the summands in (3.19) according to the value of $s \in \{1,2,3\}$ arising in $\Delta \hat{\chi}^{(s)}$. For $s = 3$ we have $|I_0| = 2$, $I_1 = \{1\}$ and $|I_2| = 1$. Use (6.14) to bound $|\Delta \hat{t}^{(2)}_{j-m_2,l-m_2}(\vec{k}_{I_2})|$ and (6.2) to bound $|\Delta \hat{t}^{(2)}_{j-m_1,l-m_1}(k_1)|$ and see that by (3.12) for $|\vec{k}|^2 l \le K'$, the $s = 3$ contribution is at most

$$\begin{aligned} &C \sum_{m_1,m_2 \le j} (m_1 \vartriangle m_2)^{-a} \big[(m_1 \vartriangle m_2) + l^p \big] l |l-j| \big(|k_1|^2 + (j - m_1 + 1)^{-a} \big) \\ &\le C|l-j| \sum_{m_1,m_2 \le j} \Big[(m_1 \vartriangle m_2)^{1-a} + (m_1 \vartriangle m_2)^{-a} l^p \Big] \big[K' + l(j-m_1+1)^{-a} \big] \\ &\le C|l-j| \Big[K' \sum_{m_1 \le j} (j - m_1 + 1)^{2-a} + l j^{2-a} + K' j l^p + l^{1+p} \Big], \end{aligned}$$

where in the last line we have used (6.8) and (6.7). Clearly the above is at most

$$C|l-j|[j^{3-a}+lj^{2-a}+jl^{p}+l^{1+p}] \le C|l-j|^{a\wedge(2-p)}[l^{3-a}+l^{1+p}] \le Cl^{3}\Big|\frac{l-j}{l}\Big|^{a\wedge(2-p)}, \tag{6.23}$$

which is the required upper bound.

We continue with the sum involving $\Delta\hat{\chi}^{(2)}$. In this case $|I_1|=1$ and $|I_2|=2$, or conversely, $|I_0|=1$, and we may use (6.10), Proposition 6.4 and (6.14) to bound the $s=2$ contribution for $|\vec{k}|^2 l\le K'$, by

$$\begin{aligned}
&C|l-j|\sum_{m_1,m_2\le j}(m_1\vartriangle m_2)^{-a}[(m_1\vartriangle m_2)+l^{p}]\\
&\le C|l-j|\sum_{m_1,m_2\le j}\Big[(m_1\vartriangle m_2)^{1-a}+l^{p}(m_1\vartriangle m_2)^{-a}\Big]\\
&\le C|l-j|[j^{3-a}+jl^{p}].
\end{aligned}$$

The last expression is less than the left-hand side of (6.23) and hence by that inequality satisfies the required bound.

Finally consider the sum involving $\hat{\chi}^{(1)}_{m_1,m_2}(\vec{k})$. In this case $|I_1|=3$ and $|I_2|=1$ (or conversely), or $|I_1|=|I_2|=2$. In the former case we may use (6.10), Proposition 6.5 and (6.2) to bound this contribution (for $|\vec{k}|^2 l\le K'$) by

$$\begin{aligned}
C|l-j|^2 l\sum_{m_1,m_2\le j}(m_1\vartriangle m_2)^{-a}[|\vec{k}|^2+(j-m_2+1)^{-a}] &\le C|l-j|^2 l(j|\vec{k}|^2+1)\\
&\le C|l-j|^2 l\le Cl^3\Big|(l-j)/l\Big|^2, \qquad (6.24)
\end{aligned}$$

which is smaller than required. In the case $|I_1|=|I_2|=2$ we may use Proposition 6.4 (twice) and (6.10) to bound this contribution (for $|\vec{k}|^2 l\le K'$) by

$$C|l-j|^2\sum_{m_1,m_2\le j}(m_1\vartriangle m_2)^{-a}\le C|l-j|^2 j\le Cl^3\Big|(l-j)/l\Big|^2,$$

which is again smaller than required.

We have shown that the summation over $m_1,m_2\le j$ in (3.19) satisfies the required bound.

Bounds on terms with $m_2\in[j+1,l]$ and $m_1\le j$ (or vice-versa).. To get a contribution to $\Delta\hat{t}^{(5)}_{j,l}(\vec{k})$ from $m_2\in[j+1,l]$ we must have $\underline{n_{I_2}}=l$ (where $\vec{n}$ is a quadruple of j's and l's) and so $\vec{n}$ is held constant at l over I_2. So when applying the Δ operator, the I_2 variables are fixed at l, while the others range over j and l as usual. As a result the contribution to this term is again the sum over I_0,I_1,I_2 of (6.18).

Using (3.12), (6.1) and (6.14), we see that the contribution due to $|I_0|=2$ is bounded by

$$\begin{aligned}
&Cl\sum_{m_1=1}^{j}\sum_{m_2=j+1}^{l}(m_1\vartriangle m_2)^{-a}[(m_1\vartriangle m_2)+l^{p}] \qquad (6.25)\\
&\le C|l-j|l\sum_{m_1=1}^{j}[(j-m_1+1)^{-(a-1)}+l^{p}(j-m_1+1)^{-a}]\\
&\le C|l-j|[lj^{2-a}+l^{1+p}],
\end{aligned}$$

where we have used the fact that here $m_1 \vartriangle m_2 = m_2 - m_1 + 1 \geq j - m_1 + 1$ for all $m_2 \geq j$ to get the first inequality. This quantity is bounded by the left-hand side of (6.23) and so by that inequality satisfies the required bound.

When $|I_0| = 1$, we have $|I_1| = 1, |I_2| = 2$ (or $|I_1| = 2, |I_2| = 1$) and we use (3.11), (6.14) or (6.1), and (6.3) or Proposition 6.4 to get a contribution of at most (6.25) (for $|\vec{k}|^2 l \leq K'$).

Finally consider $|I_0| = 0$. For $|\vec{k}|^2 l \leq K'$, the absolute value of the contribution from $\Delta\hat{t}^{(4)}_{l-m_1,j-m_1}\hat{t}^{(2)}_{l-m_2}$ is bounded by (use Proposition 6.5 and (6.1))

$$C \sum_{m_2=j+1}^{l} \sum_{m_1=1}^{j} (m_1 \vartriangle m_2)^{-a} |l-j| l \leq C|l-j|^2 l = Cl^3(|l-j|/l)^2,$$

which is less than the required bound. The contribution from $\Delta\hat{t}^{(3)}_{l-m_1,j-m_1}\hat{t}^{(3)}_{l-m_2}$ obeys an identical bound (use Proposition 6.4 and (6.3)). The contribution from $\Delta\hat{t}^{(2)}_{l-m_1,j-m_1}\hat{t}^{(4)}_{l-m_2}$ is bounded in absolute value by (use (6.4) and (6.14))

$$C \sum_{m_1=1}^{j} \sum_{m_2=j+1}^{l} (m_1 \vartriangle m_2)^{-a} (l - m_2 + 1)^2 \leq C|l-j|^3 \leq Cl^3(|l-j|/l)^3,$$

which is again less than the required bound. The contribution from $m_1 \in [j+1, l]$ and $m_2 \leq j$ is handled similarly.

Bounds on terms with $m_1, m_2 \in [j+1, l]$.. In this case we get a contribution only if $\underline{n_{I_1}} = \underline{n_{I_2}} = l$. So when applying the Δ operation the variables in I_1 and I_2 are set equal to l and only the I_0 variables vary between j and l. As a result the contribution from these terms equals the sum over I_0, I_1, I_2 of (6.19). All these contributions are of two kinds. The contributions involving $\hat{\chi}^{(1)}_{m_1,m_2}(\vec{k})$ are bounded by (use (6.10), (6.1), (6.3) and (6.4))

$$C \sum_{m_1,m_2=j+1}^{l} (m_1 \vartriangle m_2)^{-a} (l - (m_1 \wedge m_2) + 1)^2 \leq C|l-j|^3 = Cl^3(|l-j|/l)^3,$$

which satisfies the required bound. The contributions involving the terms $\Delta\hat{\chi}^{(3)}_{m_1,m_2;l,j}(\vec{k})$ and $\Delta\hat{\chi}^{(2)}_{m_1,m_2;l,j}(\vec{k})$ are each bounded by (use (6.10), (6.1) and (6.3) in separate calculations)

$$Cl \sum_{m_1,m_2=j+1}^{l} (m_1 \vartriangle m_2)^{-a} \big[(m_1 \vartriangle m_2) + l^p\big] \leq Cl\big[|l-j| l^{2-a} + l^p |l-j|\big] = C|l-j|\big[l^{3-a} + l^{1+p}\big],$$

and so again satisfies the required bound, as in (6.23).

Conclusion of the proof of Proposition 6.6.. Summing up the bounds for $\Delta\hat{\kappa}^{(5)}$ and the four cases $m_1, m_2 \leq j$, $m_1 \leq j, j+1 \leq m_2 \leq l$, $m_2 \leq j, j+1 \leq m_1 \leq l$, and $j+1 \leq m_1, m_2 \leq l$ gives the claimed bound (6.21). ■

7. Proof of Condition 3.2 for lattice trees. In this section, we use the ideas in [34] to prove Condition 3.2 for sufficiently spread-out lattice trees in dimension $d > 8$. The expansion in [34] is based on an adaption of the lace expansion on a tree for a network of self-avoiding walks, derived by the first author and Slade in [32]. We follow the same strategy as described for BRW in Section 4, and explain how the lace expansion in [34] can be used to yield the required estimates. We first derive the expansion in Section 7.2. In Section 7.3 we prove the bounds on $\hat{\chi}^{(r)}$, and in Section 7.4 we bound $\hat{\kappa}^{(r)}$. Both of these proofs are carried out assuming Proposition 7.3, which is in turn established in Appendix A.

Before beginning the proof, in Section 7.1 we relate Condition 3.2 to existing expansions in the literature and give some intuition underlying our conjecture that it should also be verifiable in other settings, notably OP. The results in Section 7.1 will not be used directly in subsequent proofs and those interested in the details of the verification for LT's may prefer to move directly to Section 7.2.

7.1. *Discussion of Condition 3.2.* In this section, we discuss the form of Condition 3.2. Condition 3.2 should be thought of as an expansion for the 3-point function (when $r = 3$, (3.8) is indeed precisely the lace expansion for the 3-point function), but one where there can be extra connections that need to be handled appropriately. We start by ignoring these extra connections, and discuss the lace expansion for the 3-point function, which has all the important features, but is notationally less cumbersome. We will indicate how some of the bounds in (3.10)–(3.12), as well as (3.9), can be derived from bounds on the lace expansion for the 3-point function that have been proved elsewhere in the literature.

To explain the idea behind the expansions and bounds in (3.8) in Condition 3.2, consider the 3-point function for lattice trees containing the origin o at time 0 and points x_1 and x_{i_2} at times (tree distances from the origin) n_1 and n_{i_2}, with $i_2 = 2$. Such a lattice tree can be considered as the minimal subtree (the *backbone*) containing these three points, plus some ribs (each rooted at a backbone vertex) that are themselves lattice trees. These ribs interact in the sense that they must avoid each other. In this context the lace expansion gives an exact expression for the 3-point function, via an inclusion-exclusion type approach to collecting and counting the interactions between the nodes along the path(s) from $(o, 0)$ to (x_1, n_1) and (x_{i_2}, n_{i_2}), working outwards from the branch point. It pretends as if certain ribs do not interact, and in doing so overcounts some of the configurations where ribs intersect each other. It then corrects this overcounting by subtracting off configurations where particular ribs intersect each other. It proceeds by assuming further ribs do not interact, and correcting, with each correction term including another pair of intersecting ribs. The variable $m_1 - 1 \in [0, n_1]$ (resp. $m_2 - 1 \in [0, n_{i_2}]$) indicates the time at which a particular contribution to the expansion from the branch point in the direction of x_1 (resp. x_{i_2}) stops, while $m_0 + 1 \in [0, (m_1 \wedge m_2) - 1]$ indicates the corresponding quantity from the origin to the branch point, and the time of the branch point is denoted $m_* \in [m_0 + 1, (m_1 \wedge m_2) - 1]$. See for example Figure 1. So roughly speaking, the m_i are locations along the backbone at which prior and subsequent ribs are independent (i.e. have no interaction). The various quantities arising from this expansion can be bounded in absolute value by certain convolutions of 2-point functions that can be expressed in terms of Feynman diagrams which indicate which ribs intersect each other.

To be a bit more precise, as a special case of what we establish in Section 7.2, the lace expansion for the 3-point function (where $i_2 = 2$) for lattice trees can be expressed in the

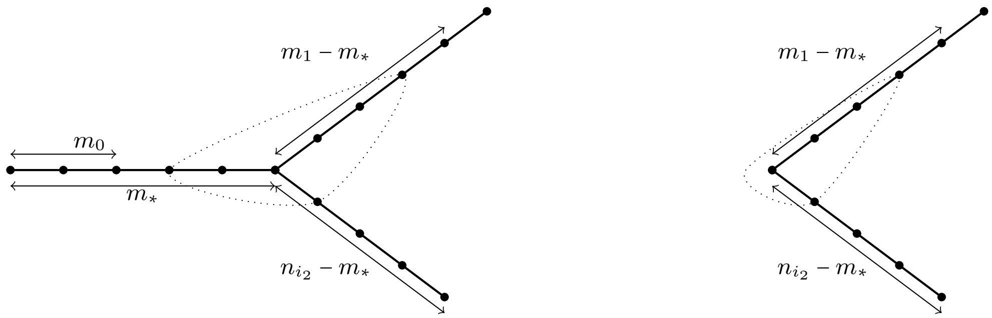

FIG 1. *Two examples of star graphs $\mathbf{S}^3_{\vec{q}}$ with branch lengths $(q_0, q_1, q_2) = (m_*, n_1 - m_*, n_{i_2} - m_*)$ equal to $(5,5,4)$ and $(0,5,4)$ respectively. The dotted line represents where the lace expansion from the branch point has stopped, which defines $m_1 - 1 \geq m_*$, $m_2 - 1 \geq m_*$, and $m_0 + 1 \leq m_*$.*

form

$$\hat{t}^{(3)}_{\vec{n}}(\vec{k}) = \sum_{m_1=1}^{n_1} \sum_{m_2=1}^{n_{i_2}} \Bigg[\hat{\xi}_{m_1,m_2}(\vec{k}) + \mathbb{1}_{\{(m_1 \wedge m_2) \geq 2\}} \sum_{m_0=0}^{(m_1\wedge m_2)-2} \hat{t}^{(2)}_{m_0}(k_1+k_2)\rho \hat{D}(k_1+k_2) \\ \times \hat{\xi}_{m_1-(m_0+1),m_2-(m_0+1)}(\vec{k}) \Bigg] \hat{t}^{(2)}_{n_1-m_1}(k_1)\hat{t}^{(2)}_{n_{i_2}-m_2}(k_2) + \hat{\kappa}^{(3)}_{n_1,n_{i_2}}(\vec{k}),$$

where the error term $\hat{\kappa}^{(3)}_{\vec{n}}(\vec{k})$ contains degenerate cases where the path from the origin to (n_1, x_1) in the tree passes through (n_{i_2}, x_{i_2}) or vice versa, as well as terms corresponding to $m_1 - 1 = n_1$ or $m_2 - 1 = n_{i_2}$. The first term in the sum over m_1 and m_2 corresponds to the case where the expansion reaches all the way to the origin (including situations where the branching time is $m_* = 0$, such as depicted on the right hand side of Figure 1). In such cases there is no $\hat{t}^{(2)}_{m_0}(k_1+k_2)\rho\hat{D}(k_1+k_2)$ term, and we can think of this as corresponding to $m_0 = -1$. Note also that the sum over the branch point is included in the $\hat{\xi}$ terms.

It follows from Proposition 7.1 and (7.53) below (both proved in [34]) that vertex coefficient $\hat{\xi}_{\ell_1,\ell_2}$ is bounded uniformly in $\vec{k}$ as follows (a is as below (3.6)):

$$|\hat{\xi}_{\ell_1,\ell_2}(\vec{k})| \leq C b_{\ell_1,\ell_2}, \tag{7.1}$$

where

$$b_{\ell_1,\ell_2} = ((\ell_1 \vee \ell_2)+1)^{-(a+1)} \tag{7.2}$$

$$+ (\ell_1 \vartriangle \ell_2)^{-a}((\ell_1 \wedge \ell_2)+1)^{-a}. \tag{7.3}$$

So far we have considered only the lattice tree 3-point function. For the cases $r > 3$ we repeat the lace expansion from the first branch point, in the direction of the origin, (n_1, x_1) and (n_{i_2}, x_{i_2}) (where now i_2 need not be equal to 2) but including the additional connections to some (x_i, n_i) for $i \in J_r \setminus \{1, i_2\}$. The extra connections appear in the form of indicator functions for the connections in the above analysis. They add further complexity to the Feynman diagrams, by adding extra paths from some part(s) of the diagrams to (x_i, n_i), and a number of cases (corresponding to the many places in the diagram where the extra paths can be attached) need to be considered. For example, when $r = 5$, each of the two further connections to (x_i, n_i) with $i \in J_5 \setminus \{1, i_2\}$ may originate on the branch to x_1, n_1 prior to ($\leq$) $m_1 - 1$ or after ($>$) $m_1 - 1$ or on the other branch prior to $m_2 - 1$ or after $m_2 - 1$. The superscript in $\hat{\chi}$ indicates one plus the number of these connections that satisfy the former (i.e. being prior to $m_1 - 1$ or $m_2 - 1$ on the respective branches). Indeed, we will see that

$$\hat{\chi}^{(1)}_{m_1,m_2}(\vec{k}) = \mathbb{1}_{\{(m_1 \wedge m_2) \geq 2\}} \sum_{m_0=0}^{(m_1 \wedge m_2)-2} \hat{t}^{(2)}_{m_0}(k_1 + k_2)\rho \hat{D}(k_1 + k_2)\hat{\xi}_{m_1-(m_0+1),m_2-(m_0+1)}(\vec{k}) + \hat{\xi}_{m_1,m_2}(\vec{k}). \tag{7.4}$$

Using (7.1)-(7.3), we get a bound of the form (3.10). The factor $\hat{\chi}^{(2)}_{m_1,m_2;n_*}(\vec{k})$ obeys the related form

$$\hat{\chi}^{(2)}_{m_1,m_2;n_*}(\vec{k}) = \mathbb{1}_{\{(m_1 \wedge m_2) \geq 2\}} \sum_{m_0=0}^{(m_1 \wedge m_2)-2} \hat{t}^{(2)}_{m_0}(k_1 + k_2)\rho \hat{D}(k_1 + k_2)\hat{\xi}_{m_1-(m_0+1),m_2-(m_0+1);n_*}(\vec{k}), + \hat{\xi}_{m_1,m_2;n_*}(\vec{k}) \tag{7.5}$$

where $\hat{\xi}_{\ell_1,\ell_2;n_*}(\vec{k})$ corresponds to $\hat{\xi}_{\ell_1,\ell_2}(\vec{k})$ where an extra line to (x_*, n_*) is added, where x_* is the spatial variable related to n_*. As a result, $\hat{\xi}_{\ell_1,\ell_2;n_*}(\vec{k})$ obeys the same bound as $\hat{\xi}_{\ell_1,\ell_2}(\vec{k})$, apart from being multiplied by a factor $\ell_1 \vee \ell_2$. It is not hard to see that therefore (7.5) is bounded by

$$|\hat{\chi}^{(2)}_{m_1,m_2;n_*}(\vec{k})| \leq C \sum_{m_0=0}^{m_1 \wedge m_2} ((m_1 \vee m_2) - m_0) b_{m_1-m_0,m_2-m_0} \leq C(m_1 \vartriangle m_2)^{-(a-1)}, \tag{7.6}$$

as required. The bound on $\hat{\chi}^{(3)}_{m_1,m_2;n_*,n_{**}}(\vec{k})$ is similar.

We turn our attention to the bound on $\hat{\kappa}^{(5)}_{\vec{n}}(\vec{k})$, which contains the contributions where $m_1 - 1 = n_1$ or $m_2 - 1 = n_{i_2}$. To make the description as easy as possible, suppose that $i_2 = 2$, consider the terms in the lace expansion where $m_1 - 1 = n_1$, and assume that $m_2 - 1$ corresponds to the branch point m_*. This assumption simplifies the situation considerably, because in such terms there is no interaction between the branch of the backbone to (x_{i_2}, n_{i_2}) and the branches to $(o, 0)$ and (x_1, n_1) (all from the branch point). In other words, the intersections between ribs take place on an interval of nodes (containing the branch point) in the backbone from $m_0 + 1$ to $m_1 - 1 = n_1$. The leading contribution from these terms is therefore of the form

$$\sum_{m_0=0}^{(m_1 \wedge m_2)-2} \hat{t}^{(2)}_{m_0}(k_1 + k_2)\rho \hat{D}(k_1 + k_2)\hat{\pi}_{n_1-(m_0+1);n_2,n_3,n_4}(\vec{k}), \tag{7.7}$$

where the expansion coefficient $\hat{\pi}_{n_1-(m_0+1);n_2,n_3,n_4}(\vec{k})$ is due to the expansion coefficient arising in the lace expansion for the two-point function $\hat{t}^{(2)}$, where three extra connections are added. We add the branch to n_2 by connecting it somewhere along the backbone from $(o, 0)$ to n_1, between $m_0 + 1$ and $m_1 - 1 = n_1$ (and this tells us where the branch point is) while the branches n_3 and n_4 must be attached either within the same interval (after the branchpoint), or to the branch from the branch point to (x_{i_2}, n_{i_2}), or to the other added branch. Summing over the number of possible addition/connection locations gives rise to a factor of at most $(n_1 - m_0)$ in the first case and $(\bar{n} - m_0)$ in each of the other two cases. The ordinary two-point function expansion coefficient satisfies [34]

$$|\hat{\pi}_m(\vec{k})| \le C(m+1)^{-(a+1)}, \tag{7.8}$$

and we obtain

$$|\hat{\pi}_{n_1-(m_0+1);n_2,n_3,n_4}(\vec{k})| \le C(\bar{n} - m_0 + 1)^2 \times (n_1 - m_0 + 1)^{-a}, \tag{7.9}$$

and we conclude (see (6.1) below) that this contribution to $\hat{\kappa}^{(5)}_{\vec{n}}(\vec{k})$ is bounded by

$$C \sum_{m_0 \le n_1} C(\bar{n} - m_0 + 1)^2 \times (n_1 - m_0 + 1)^{-a} \le C|\bar{n} - \underline{n}|^2 + C\bar{n}^{3-a}, \tag{7.10}$$

as required. In obtaining this bound we have used $(\bar{n} - m_0 + 1)^2 \le 2(\bar{n} - \underline{n})^2 + 2(n_1 - m_0 + 1)^2$. General contributions to $\hat{\kappa}^{(5)}_{\vec{n}}(\vec{k})$ are more difficult to handle as one needs bounds on more complicated lace expansion coefficients. We will need "diagrammatic bounds" of the form (7.8) and its just noted extension. These are obtained in Propositions 7.1 and 7.3 below.

The above discussion briefly explains how, in the context of lattice trees, the main assumptions in Condition 3.2, that is, (3.9), and (3.10)–(3.12), follow from the bound $|\hat{\pi}_m(\vec{k})| \le C(m+1)^{-(a+1)}$ in (7.8). Despite differing in the details, much of the discussion above applies equally well to the setting of oriented percolation (and to some extent the contact process as well). Indeed, a bound of the form (7.8) is one of the crucial steps in the analysis of the lace expansion for the r-point function for OP, and has been derived in [33, Proposition 2.2(i)], while a bound of the form (7.1) is proved for OP in [33, Proposition 2.3(ii)] (see also [30, Proposition 2.2], where such a statement is proved for CP and reproved for OP). This gives some idea as to how we arrived at Condition 3.2, and why we hope that it can be verified for other models as well, notably OP. Henceforth we return our attention to LT, and make the above steps precise in this context.

7.2. *The expansion.* Using similar notation to that in [34, Section 2.1], we let $\mathbf{S}^3_{\vec{j}} = \mathbf{S}^3_{j_0,j_1,j_2}$ be the abstract tree with one branch point of degree 3 and three legs having j_0, j_1 and j_2 vertices respectively. We refer to the vertices in $\mathbf{S}^3_{\vec{j}}$ as $[i,j]$, where $i \in \{0,1,2\}$ is the label of the branch, and $0 \le j \le j_i$ for $i = 0,1,2$. [1] We refer to the three branches as $\mathbf{S}^1_i = [i,[0,j_i]]$, with the vertices $[i,0]$, $i = 0,1,2$ being identified. If some $j_i = 0$, then the degree of the vertex $[0,0]$ is < 3, and the branch point is a misnomer. In particular, $\mathbf{S}^3_{0,0,0}$ corresponds to a single vertex.

Fix $r \in \{3,4,5\}$. Let $n_0 = 0$ and $x_0 = o$. The quantity $t^{(r)}_{\vec{n}}(\vec{x})$ is the probability that $(x_i, n_i) \in \mathcal{T}$ for each $i \in \{0,\dots,r-1\}$, i.e. for each i, there is a path of length n_i in the lattice tree $\mathcal{T}$ from o leading to x_i. While for BRW there may be several ancestral lines satisfying this restriction, because of the lattice tree restriction, the vertices making up these paths are

[1] In [34], the branches were instead labelled $1,2,3$.

unique. Recall that $t^{(r)}_{\vec{n}}(\vec{x})$ involves a normalising constant ρ^{-1}, where $\rho = \rho_{z_c}(o)$ is bounded above and below $(1 < \rho < C)$ uniformly in L (see e.g. [17]).

For fixed $x \in \mathbb{Z}^d$, let $\mathbb{T}(x)$ be the set of lattice trees containing x (and not necessarily the origin). For fixed $\vec{n}$, $\vec{x}$, let $\mathbb{T}_{\vec{n}}(\vec{x})$ denote the set of lattice trees containing $x_0 = 0, \dots, x_{r-1}$ with tree distances $0, n_1, \dots, n_{r-1}$ from the origin respectively. Each $\mathcal{T} \in \mathbb{T}_{\vec{n}}(\vec{x})$ contains a minimal subtree $\mathcal{M}(\mathcal{T})$, containing $x_0, \dots, x_{r-1}$ and that subtree has the topology of some finite rooted tree $\mathbf{T}$ with labelled leaves $\alpha_0 = 0, \alpha_1, \dots, \alpha_{r-1}$. For each such $\mathbf{T}$ the branch generation $m_* = |\bigwedge_{i=1}^{r-1} \alpha_i|$ (recall from Section 1.2.1 that $\bigwedge_{i=1}^{r-1} \alpha_i$ is the most recent common ancestor of the α_i's) and the index $i_2 = \inf\{i \geq 2 : |\alpha_i \wedge \alpha_1| = m_*\}$ are uniquely defined. For fixed $\vec{n}$, $i_2 \in \{2,3,4,5\}$ and $m_* \leq \underline{n}$, let $\mathbf{T}(\vec{n}, m_*; i_2)$ denote the set of $\mathbf{T}$ with this m_* and i_2. For fixed $\mathbf{T}$ let $\mathbb{T}_{\mathbf{T}}(\vec{x})$ denote the set of lattice trees $\mathcal{T}$ containing $\vec{x}$ with minimal subtree $\mathcal{M}(\mathcal{T})$ having topology $\mathbf{T}$. It follows that

$$t^{(r)}_{\vec{n}}(\vec{x}) = \sum_{i_2=2}^{r-1} \sum_{m_* \leq \underline{n}} \sum_{\mathbf{T} \in \mathbf{T}(\vec{n}, m_*; i_2)} \rho^{-1} \sum_{\mathcal{T} \in \mathbb{T}_{\mathbf{T}}(\vec{x})} W(\mathcal{T}). \tag{7.11}$$

Now each $\mathbf{T} \in \mathbf{T}(\vec{n}, m_*; i_2)$ itself consists of a minimal tree $\mathbf{T}_{12}$ containing $0, \alpha_1, \alpha_{i_2}$ with a branch point $\beta = \alpha_1 \wedge \alpha_{i_2}$ of generation m_*, and $r-3$ branches $\mathbf{T}_{s_j}$ from vertices $\vec{s} = (s_j \in \mathbf{T}_{12} : j \in \{2, \dots, r-1\} \setminus i_2)$ (note that some of the s_j may be equal) that are compatible with $\mathbf{T}_{12}$ and i_2 in the sense that

$$\begin{aligned} \vec{s} \in (\mathbf{T}_{12;i_2})^{r-3} \equiv \Big\{ &(s_2, s_3, \dots, s_{i_2-1}, s_{i_2+1}, \dots, s_{r-1}) \text{ such that each } s_i \in \mathbf{T}_{12}, \\ &|s_i| \geq m_* \text{ for each } i > i_2, \text{ and } |s_i| > m_* \text{ for each } 1 < i < i_2 \Big\}. \end{aligned}$$

Note that, with a relabelling of vertices, $\mathbf{T}_{12}$ has the topology of $\mathbf{S}^3_{\vec{q}}$, where $(q_0, q_1, q_2) = (m_*, n_1 - m_*, n_{i_2} - m_*)$ (see e.g. Figure 1). The point of changing notation from $\mathbf{T}_{12}$ to $\mathbf{S}^3_{\vec{q}}$ is that we want to use the lace expansion from [34] on $\mathbf{S}^3_{\vec{q}}$, expanding from the branch point in the direction of the 3 leaves. The vertices 0, α_1, and α_{i_2} in $\mathbf{T}_{12}$ have the corresponding labels $[0, q_0]$, $[1, q_1]$ and $[2, q_2]$ respectively in $\mathbf{S}^3_{\vec{q}}$ while the branch point $\alpha_1 \wedge \alpha_{i_2}$ has the label $[0,0] = [1,0] = [2,0]$ in $\mathbf{S}^3_{\vec{q}}$. Also the compatible vertices $\vec{s}$ (some of which may be equal) from which to attach branches in the $\mathbf{S}^3_{\vec{q}}$ labelling become

$$\begin{aligned} \vec{s} \in (\mathbf{S}^3_{\vec{q};i_2})^{r-3} \equiv \Big\{ &(s_2, s_3, \dots, s_{i_2-1}, s_{i_2+1}, \dots, s_{r-1}) \text{ such that each } s_i \in \mathbf{S}^3_{\vec{q}}, \\ &s_i \in [1,[0,q_1]] \cup [2,[0,q_2]] \text{ for each } i > i_2, \\ &\text{and } s_i \in [1,[1,q_1]] \cup [2,[1,q_2]] \text{ for each } 1 < i < i_2 \Big\}. \end{aligned} \tag{7.12}$$

Given $\vec{x}, \vec{n}, m_*$, and i_2, let $\Phi = \Phi(\vec{q}, \vec{x}; i_2) = \{\phi : \mathbf{S}^3_{\vec{q}} \to \mathbb{Z}^d : \phi([0, q_0]) = o, \phi([1, q_1]) = x_1, \phi([2, q_2]) = x_{i_2}\}$, and for $\phi \in \Phi$ define

$$\mathbb{T}(\vec{q}, \phi) = \{\vec{\mathcal{R}} = \{\mathcal{R}_s : s \in \mathbf{S}^3_{\vec{q}}\} : \mathcal{R}_s \in \mathbb{T}(\phi(s)) \forall s\}.$$

For $j_1 \in \{1,2\}$, $s = [j_1, j_2] \in \mathbf{S}^3_{\vec{q}}$, $\phi \in \Phi$, and $\mathcal{R}_s$ a lattice tree containing $\phi(s)$, we write $(x, n) \in \mathcal{R}_s$ if $x \in \mathcal{R}_s$ and the tree distance from x to $\phi(s)$ in $\mathcal{R}_s$ is $n - (j_2 + m_*)$ (this implies that the tree distance from x to o in $\mathcal{R}_s \cup \phi(\mathbf{S}^3_{\vec{q}})$ is n if this combined object is a lattice tree). We now write

$$t^{(r)}_{\vec{n}}(\vec{x}) = \sum_{i_2=2}^{r-1} \sum_{m_* \le n_1 \wedge n_{i_2}} t^{(r;i_2)}_{m_*,\vec{n}}(\vec{x}), \tag{7.13}$$

where (recall $\vec{q} = \vec{q}(m_*, n_1, n_{i_2}) = (m_*, n_1 - m_*, n_{i_2} - m_*)$)

$$\begin{aligned} t^{(r;i_2)}_{m_*,\vec{n}}(\vec{x}) = &\sum_{\vec{s} \in (\mathbf{S}^3_{\vec{q};i_2})^{r-3}} \rho^{-1} \sum_{\phi \in \Phi} W(\phi(\mathbf{S}^3_{\vec{q}})) \\ &\times \sum_{\vec{\mathcal{R}} \in \mathbb{T}(\vec{q},\phi)} \Big[\prod_{s \in \mathbf{S}^3_{\vec{q}}} W(\mathcal{R}_s)\Big]\Big[\prod_{s,t \in \mathbf{S}^3_{\vec{q}}} [1 + U_{st}]\Big]\Big[\prod_{j \notin 1, i_2} \mathbb{1}_{\{(x_j,n_j) \in \mathcal{R}_{s_j}\}}\Big], \end{aligned} \tag{7.14}$$

$$W(\phi(\mathbf{S}^3_{\vec{q}})) = \prod_{\alpha\alpha' \in \mathbf{S}^3_{\vec{q}}} z_c D(\phi(\alpha') - \phi(\alpha)),$$

and

$$U_{st} = -\mathbb{1}_{\{\mathcal{R}_s \cap \mathcal{R}_t \neq \varnothing\}}.$$

This is to be understood as follows. A lattice tree $\mathcal{T}$ with given $\vec{n}, \vec{x}, m_*, i_2$ consists of a minimal tree containing x_1, x_{i_2} (that is also equal to an embedding of $\mathbf{S}^3_{\vec{q}}$ for $\vec{q}$ defined by m_*, i_2 and $\vec{n}$) together with some branches, which are lattice trees $\mathcal{R}_s$ associated to each of the vertices $s \in \mathbf{S}^3_{\vec{q}}$. These $\mathcal{R}_s$ must be mutually avoiding for the resulting object to be a lattice tree, and in addition, connections to $(x_j, n_j) : j \in \{2, \dots, r-1\} \smallsetminus \{i_2\}$ must exist within them. The weight of any lattice tree can be written as the product of the weight of its minimal tree and the weights of its branches. The sum over ϕ is over all embeddings of $\mathbf{S}^3_{\vec{q}}$, which are not necessarily 1-1, but the product of $[1 + U_{st}]$ means that we only count 1-1 embeddings (since $\phi(s) \in \mathcal{R}_s$ for each $s \in \mathbf{S}^3_{\vec{q}}$) and that the $\mathcal{R}_s$ are mutually avoiding.

For fixed $\vec{x}$, $\vec{n}$ and i_2, define $\kappa^{(r;1)}_{\vec{n}}(\vec{x})$ to be the contribution to (7.13) from $m_* = n_1 \wedge n_{i_2}$ (in which case $m_* = \underline{n}$). Then

$$\kappa^{(r;1)}_{\vec{n}}(\vec{x}) \equiv \sum_{i_2=2}^{r-1} t^{(r;i_2)}_{n_1 \wedge n_{i_2},\vec{n}}(\vec{x}), \tag{7.15}$$

and by definition,

$$t^{(r)}_{\vec{n}}(\vec{x}) = \sum_{i_2=2}^{r-1} \sum_{m_*=0}^{(n_1 \wedge n_{i_2})-1} t^{(r;i_2)}_{m_*,\vec{n}}(\vec{x}) + \kappa^{(r;1)}_{\vec{n}}(\vec{x}). \tag{7.16}$$

Let us now focus on the term $t^{(r;i_2)}_{m_*,\vec{n}}(\vec{x})$ when $m_* < n_1 \wedge n_{i_2}$. Note that by definition of i_2, $m_* < n_1 \wedge n_{i_2}$ implies that $m_* \le n^*(i_2)$, where $n^*(i_2) = \min_{i \le r-1}\{n_i - \mathbb{1}_{\{i \le i_2\}}\}$.

Given an abstract tree $\mathbf{S}$ and lattice trees $\vec{\mathcal{R}} = (\mathcal{R}_s : s \in \mathbf{S})$, let

$$K[\mathbf{S}] = K[\mathbf{S}](\vec{\mathcal{R}}) = \prod_{s,t \in \mathbf{S}} [1 + U_{st}].$$

Then (7.14) becomes

(7.17)
$$t^{(r;i_2)}_{m_*,\vec{n}}(\vec{x}) = \sum_{\vec{s}\in(\mathbf{S}^3_{\vec{q};i_2})^{r-3}} \rho^{-1} \sum_{\phi\in\Phi} W(\phi(\mathbf{S}^3_{\vec{q}})) \sum_{\vec{\mathcal{R}}\in\mathbb{T}(\vec{q},\phi)} \Big[\prod_{s\in\mathbf{S}^3_{\vec{q}}} W(\mathcal{R}_s)\Big] K[\mathbf{S}^3_{\vec{q}}]\Big[\prod_{j\notin 1,i_2} \mathbb{1}_{\{(x_j,n_j)\in\mathcal{R}_{s_j}\}}\Big].$$

The lace expansion for the three-point function for lattice trees involves expanding the product $K[\mathbf{S}^3_{\vec{q}}]$, and in each term of the expansion searching outwards (from the root $[0,0]$ of $\mathbf{S}^3_{\vec{q}}$, i.e. from the branch point β of $\mathbf{T}$) for the first point at which there is no indicator involving a pair of vertices on either side of this point. This expansion was carried out in [34, (2.17)]. To prepare for this, for fixed i, q_i and M_i define $\mathbf{S}^1_{i+}$ to be the tree with no branch points and $q_i - M_i$ vertices labelled $[i, M_i + 1]$ to $[i, q_i]$. Then the lace expansion [34, (2.17)] yields

$$K[\mathbf{S}^3_{\vec{q}}] = \sum_{\vec{M}\le\vec{q}} J[\mathbf{S}^3_{\vec{M}}] \prod_{i=0}^{2} K[\mathbf{S}^1_{i+}], \tag{7.18}$$

where $J[\mathbf{S}^3_{\vec{M}}]$ is defined below and we use the notation $\vec{M} \le \vec{q}$ (resp. $<$) to mean that the inequality holds componentwise. This arises in the following way. Notice that

$$K[\mathbf{S}] = \prod_{s,t\in\mathbf{S}} [1 + U_{st}] = \sum_{\Gamma\in\mathcal{G}(\mathbf{S})} \prod_{st\in\Gamma} U_{st},$$

where $\mathcal{G}(\mathbf{S})$ denotes the collection of all subsets of "edges" st with $s, t \in \mathbf{S}$ (and $\mathbf{S}$ is $\mathbf{S}^3_{\vec{q}}$ or $\mathbf{S}^1_{i+}$). If both s and t are on the same branch i of $\mathbf{S}^3_{\vec{q}}$, i.e. $s = [i, j]$ and $t = [i, j']$ for some $j < j'$ (or vice versa) then st is said to cover the interval $[i, [j, j']]$. If $s = [i, j]$ and $t = [i', j']$ with $i \ne i'$ and $j, j' \ne 0$ then st is said to cover $[i, [0, j]] \cup [i', [0, j']]$. A graph $\Gamma \in \mathcal{G}(\mathbf{S}^3_{\vec{j}})$ is a *connected* graph if every edge in $\mathbf{S}^3_{\vec{j}}$ is contained within an interval covered by some $st \in \Gamma$. For each $\Gamma \in \mathcal{G}(\mathbf{S}^3_{\vec{q}})$ there exists a unique $\vec{M}(\Gamma)$ such that $\mathbf{S}^3_{\vec{M}}$ is the largest connected subgraph of $\mathbf{S}^3_{\vec{q}}$ containing $[0,0]$ on which $\mathcal{G}$ is connected. Letting

$$J[\mathbf{S}^3_{\vec{M}}] = \sum_{\Gamma\in\mathcal{G}^{\mathrm{conn}}(\mathbf{S}^3_{\vec{M}})} \prod_{st\in\Gamma} U_{st},$$

where $\mathcal{G}^{\mathrm{conn}}$ denotes the subset of $\mathcal{G}$ consisting of connected graphs, gives (7.18). See e.g. [34, Section 2] for more details.

Note that $\mathbf{S}^3_{\vec{M}}$ has a simpler topology (i.e. no branch point) when some $M_i = 0$ (as is the case when $m_* = 0$ for example), and in fact $\mathbf{S}^3_{0,0,0}$ corresponds to a single vertex. Rearranging the order of summation we obtain

$$t^{(r;i_2)}_{m_*,\vec{n}}(\vec{x}) = \rho^{-1} \sum_{\vec{M}\le\vec{q}} \sum_{\phi\in\Phi} W(\phi(\mathbf{S}^3_{\vec{q}}))$$
$$\times \sum_{\vec{\mathcal{R}}\in\mathbb{T}(\vec{q},\phi)} \Big[\prod_{s\in\mathbf{S}^3_{\vec{q}}} W(\mathcal{R}_s)\Big]\Big[J[\mathbf{S}^3_{\vec{M}}] \prod_{i=0}^{2} K[\mathbf{S}^1_{i+}]\Big]\Big[\sum_{\vec{s}\in(\mathbf{S}^3_{\vec{q};i_2})^{r-3}} \prod_{j\notin 1,i_2} \mathbb{1}_{\{(x_j,n_j)\in\mathcal{R}_{s_j}\}}\Big]. \tag{7.19}$$

Let $\vec{I}(i_2) = (I_0, I_1, I_2)(i_2)$ be a partition of $\{1, \dots, r-1\}$ satisfying all the restrictions in Condition 3.2. Let $\big(\mathbf{S}^3_{\vec{q},\vec{M}}(\vec{I})\big)^{r-3}$ be the subset of $(\mathbf{S}^3_{\vec{q};i_2})^{r-3}$ such that $s_j \in [1, [M_1+1, q_1]]$ for each $j \in I_1 \setminus \{1\}$, $s_j \in [2, [M_2+1, q_2]]$ for each $j \in I_2 \setminus \{i_2\}$, and $s_j \in [1, [0, M_1]] \cup [2, [0, M_2]]$ for $j \in I_0$. Then

$$t^{(r;i_2)}_{m_*,\vec{n}}(\vec{x}) = \rho^{-1} \sum_{\vec{I}(i_2)} \sum_{\vec{M} \le \vec{q}} \sum_{\phi \in \Phi} W(\phi(\mathbf{S}^3_{\vec{q}})) \sum_{\vec{\mathcal{R}} \in \mathbb{T}(\vec{q},\phi)} \Big[\prod_{s \in \mathbf{S}^3_{\vec{q}}} W(\mathcal{R}_s) \Big] \Big[J[\mathbf{S}^3_{\vec{M}}] \prod_{i=0}^{2} K[\mathbf{S}^1_{i+}] \Big]$$
$$\times \Big[\sum_{\vec{s} \in (\mathbf{S}^3_{\vec{q},\vec{M}}(\vec{I}))^{r-3}} \prod_{j \notin 1, i_2} \mathbb{1}_{\{(x_j, n_j) \in \mathcal{R}_{s_j}\}} \Big]. \tag{7.20}$$

We next decompose (7.20) into pieces based on $\vec{M}$. The cases where some $M_i = q_i$ require slightly different treatment, so we first consider the case $\vec{M} < \vec{q}$.

The case $\vec{M} < \vec{q}$: In this case $m_* > 0$ and each $\mathbf{S}^1_{i+}$ is non-empty (although if $M_i + 1 = q_i$ then it consists of a single vertex). Any $\phi : \mathbf{S}^3_{\vec{q}} \to \mathbb{Z}^d$ can be represented as $(\phi_\pi, \phi_0, \phi_1, \phi_2)$, where $\phi_\pi : \mathbf{S}^3_{\vec{M}} \to \mathbb{Z}^d$ and $\phi_i : \mathbf{S}^1_{i+} \to \mathbb{Z}^d$ are the restrictions of ϕ to those subgraphs of $\mathbf{S}^3_{\vec{q}}$. Let us write $v_i = \phi([i, M_i]) = \phi_\pi([i, M_i])$ and $y_i = \phi([i, M_i + 1]) = \phi_i([i, M_i + 1])$ for each $i = 0, 1, 2$. We will sum over $\vec{v}, \vec{y}$. The weight $W(\phi(\mathbf{S}^3_{\vec{q}}))$ factors as

$$W(\phi) = W(\phi_\pi) \prod_{i=0}^{2} W(\phi_i) z_c D(y_i - v_i) \tag{7.21}$$

Let

$$\begin{aligned}
\Phi_\pi =& \Phi_\pi(\vec{v}) = \big\{\phi{:}\, \mathbf{S}^3_{\vec{M}} \to \mathbb{Z}^d \text{ such that } \phi([i, M_i]) = v_i, \text{ for } i = 0, 1, 2\big\}, \\
\Phi^*_\pi =& \Phi^*_\pi(\vec{v}) = \{\phi \in \Phi_\pi : \phi([0, 0]) = o\}, \\
\Phi_i =& \Phi_i(y_i, x'_i) = \big\{\phi : \mathbf{S}^1_{i+} \to \mathbb{Z}^d \text{ such that } \phi([i, M_i + 1]) = y_i, \phi([i, q_i]) = x'_i\big\},
\end{aligned}$$

where $x'_0 = x_0 = o$, $x'_1 = x_1$ and $x'_2 = x_{i_2}$. Note that this forces $y_i = x'_i$ if $M_i + 1 = q_i$.

Let

$$\begin{aligned}
\mathbb{T}_\pi &= \big\{\vec{\mathcal{R}} = \{\mathcal{R}_s{:}\, s \in \mathbf{S}^3_{\vec{M}}\}{:}\, \mathcal{R}_s \in \mathbb{T}(\phi_\pi(s)) \,\forall s\big\}, &&\text{and} \\
\mathbb{T}_i &= \big\{\vec{\mathcal{R}} = \{\mathcal{R}_s{:}\, s \in \mathbf{S}^1_{i+}\}{:}\, \mathcal{R}_s \in \mathbb{T}(\phi_i(s)) \,\forall s\big\}, &&\text{for each } i = 0, 1, 2.
\end{aligned}$$

Also introduce

$$I'_i = \begin{cases} I_1 \setminus \{1\}, & \text{if } i = 1 \\ I_2 \setminus \{i_2\}, & \text{if } i = 2, \end{cases}$$

$$(\mathbf{S}^1_{i+,M_i}(I_i))^{|I_i|-1} = \big\{(s_j)_{j \in I'_i} : s_j \in [i, [(M_i + 1), q_i]] \;\forall j \in I'_i\big\}, \; i = 1, 2, \tag{7.22}$$

and

$$\begin{aligned}
(\mathbf{S}^3_{\vec{M}}(I_0))^{|I_0|} = \big\{(s_j)_{j \in I_0} :& s_j \in [1, [1, M_1]] \cup [2, [1, M_2]] \;\forall j < i_2 \text{ and} \\
& s_j \in [0, [0, M_1]] \cup [2, [0, M_2]] \;\forall j > i_2\big\}.
\end{aligned}$$

Then the contribution to (7.20) from $\vec{M} < \vec{q}$ is

(7.23)
$$\rho^{-1}\sum_{\vec{I}(i_2)}\sum_{\vec{M}<\vec{q}}\sum_{\vec{y}}\Big[\sum_{\phi_0\in\Phi_0}W(\phi_0(\mathbf{S}^1_{0+}))\sum_{\vec{\mathcal{R}}\in\mathbb{T}_0}\big[\prod_{s\in\mathbf{S}^1_{0+}}W(\mathcal{R}_s)\big]K[\mathbf{S}^1_{0+}]\Big]$$

(7.24)
$$\times\prod_{i=1}^{2}\Big[\sum_{\phi_i\in\Phi_i}W(\phi_i(\mathbf{S}^1_{i+}))\sum_{\vec{\mathcal{R}}\in\mathbb{T}_i}\big[\prod_{s\in\mathbf{S}^1_{i+}}W(\mathcal{R}_s)\big]K[\mathbf{S}^1_{i+}]\sum_{\vec{s}\in(\mathbf{S}^1_{i+,M_i}(I_i))^{|I_i|-1}}\prod_{j\in I_i'}\mathbb{1}_{\{(x_j,n_j)\in\mathcal{R}_{s_j}\}}\Big]$$
$$\times\Big[\sum_{\vec{v}}\big[\prod_{i=0}^{2}z_cD(y_i-v_i)\big]\sum_{\phi_\pi\in\Phi_\pi}W(\phi_\pi(\mathbf{S}^3_{\vec{M}}))$$

(7.25)
$$\times\sum_{\vec{\mathcal{R}}\in\mathbb{T}_\pi}\big[\prod_{s\in\mathbf{S}^3_{\vec{M}}}W(\mathcal{R}_s)\big]J[\mathbf{S}^3_{\vec{M}}]\sum_{\vec{s}\in(\mathbf{S}^3_{\vec{M}}(I_0))^{|I_0|}}\prod_{j\in I_0}\mathbb{1}_{\{(x_j,n_j)\in\mathcal{R}_{s_j}\}}\Big].$$

We define the terms A_0, A_i, $i=1,2$ and A_π (which depend on $m_*,\vec{n},\vec{y},\vec{x},\vec{M},\vec{I}$) according to the terms in the large brackets $[]$ above so that (7.23)-(7.25) is equal to

$$\rho^{-1}\sum_{\vec{I}(i_2)}\sum_{\vec{M}<\vec{q}}\sum_{y_0,y_1,y_2}A_0A_1A_2A_\pi.$$

Now letting $m_0 = m_* - M_0 - 1$ and $m_i = m_* + M_i + 1$ for $i = 1,2$, observe that by the symmetry of D and translation invariance (and recalling that $t^{(2)}_0(y) = \mathbb{1}_{\{y=o\}}$)

$$A_0 = \rho t^{(2)}_{m_0}(y_0), \quad \text{and} \quad A_i = \rho t^{(|I_i|+1)}_{\vec{n}_{I_i}-m_i}(\vec{x}_{I_i}-y_i) \text{ for } i=1,2. \tag{7.26}$$

Here we adopt the convention (consistent with (7.11)) that $t^{(r)}_{\vec{n}}(\vec{x}) = 0$ if one of the components of $\vec{n}$ is negative. Let us demonstrate why this holds for example for the term A_1 in the case $I_1 = \{1,2,3\}$. Given $\vec{M}$, and $\vec{q}$ define $\ell_1 = q_1 - M_1 - 1 = n_1 - m_1$ and $\mathbf{S}^1_{\ell_1} = \mathbf{S}^1_{1+}$ (as defined for these values of M_1 and q_1). Now observe that for general $\vec{\ell}$ and $\vec{w}$,

$$t^{(4)}_{\vec{\ell}}(\vec{w}) = \rho^{-1}\sum_{\mathcal{T}\in\mathbb{T}_{\vec{\ell}}(\vec{w})}W(\mathcal{T}) = \rho^{-1}\sum_{\mathcal{T}\in\mathbb{T}_{\ell_1}(w_1)}W(\mathcal{T})\mathbb{1}_{\{(w_2,\ell_2)\in\mathcal{T}\}}\mathbb{1}_{\{(w_3,\ell_3)\in\mathcal{T}\}}. \tag{7.27}$$

Any lattice tree $\mathcal{T}$ containing the points $(o,0)$, and (w_i,ℓ_i) for $i=1,\dots,3$ can be expressed as the union of a backbone $\phi_1(\mathbf{S}^1_{\ell_1})$ (starting at $(o,0)$ and ending at (w_1,ℓ_1)) and mutually avoiding ribs $\mathcal{R}_i$ (themselves lattice trees emanating from each vertex $\phi_1(i)$ along the backbone), such that w_2 and w_3 are vertices within this collection of ribs, of tree distances ℓ_2 and ℓ_3 from the root in the lattice tree $\mathcal{T}$.

It follows that (7.27) can be written as

$$t^{(4)}_{\vec{\ell}}(\vec{w}) = \rho^{-1}\sum_{\phi_1\in\Phi_1(o,w_1)}W(\phi_1(\mathbf{S}^1_{\ell_1}))$$
$$\times\sum_{\vec{\mathcal{R}}\in\mathbb{T}_i}\prod_{i\in\mathbf{S}^1_{\ell_1}}W(\mathcal{R}_i)K[\mathbf{S}^1_{\ell_1}]\sum_{s',s''\in\mathbf{S}^1_{\ell_1}}\mathbb{1}_{\{(w_2,\ell_2)\in\mathcal{R}_{s'}\}}\mathbb{1}_{\{(w_3,\ell_3)\in\mathcal{R}_{s''}\}}. \tag{7.28}$$

On the other hand, by translating by $-y_1$ we have that the term A_1 (for $I_1 = \{1,2,3\}$) is

$$\sum_{\phi_1 \in \Phi_1(o, x_1 - y_1)} W(\phi_1(\mathbf{S}^1_{\ell_1}))$$

$$\times \sum_{\vec{\mathcal{R}} \in \mathbb{T}_i} \Big[\prod_{s \in \mathbf{S}^1_{\ell_1}} W(\mathcal{R}_s) \Big] K[\mathbf{S}^1_{\ell_1}] \sum_{s', s'' \in \mathbf{S}^1_{\ell_1}} \mathbb{1}_{\{(x_2 - y_1, n_2 - m_1) \in \mathcal{R}_{s'}\}} \mathbb{1}_{\{(x_3 - y_1, n_3 - m_1) \in \mathcal{R}_{s''}\}} \tag{7.29}$$

$$= \rho t^{(4)}_{\vec{n}_{I_1} - m_1}(\vec{x}_{I_1} - y_1), \tag{7.30}$$

since $n_1 - m_1 = \ell_1$ and we have set $\ell_i = n_i - m_1$, $i = 2, 3$.

We write

$$t^{(r)}_{\vec{n}}(\vec{x}) - \kappa^{(r;1)}_{\vec{n}}(\vec{x}) = t^{(r)\heartsuit}_{\vec{n}}(\vec{x}) + t^{(r)\spadesuit}_{\vec{n}}(\vec{x}), \tag{7.31}$$

where $t^{(r)\heartsuit}_{\vec{n}}(\vec{x})$ denotes the contribution to $t^{(r)}_{\vec{n}}(\vec{x}) - \kappa^{(r;1)}_{\vec{n}}(\vec{x})$ from terms with $\vec{M} < \vec{q}$. Since also $\sum_{i_2=2}^{r-1} \sum_{\vec{I}(i_2)} = \sum_{\vec{I}}$, we obtain

$$t^{(r)\heartsuit}_{\vec{n}}(\vec{x}) = \sum_{\vec{I}} \sum_{m_1=1}^{n_1} \sum_{m_2=1}^{n_{i_2}} \sum_{\vec{y}} \Big(\prod_{j=1}^{2} t^{(|I_j|+1)}_{\vec{n}_{I_j} - m_j}(\vec{x}_{I_j} - y_j) \Big) \Big[\rho^2 \sum_{m_*=1}^{((m_1 \wedge m_2) - 1) \wedge n^*(i_2)} \sum_{m_0=0}^{m_*-1} t^{(2)}_{m_0}(y_0) A_\pi(\vec{M}) \Big]. \tag{7.32}$$

We continue to identify $A_\pi(\vec{M})$, which is given by the last line in (7.25). To connect up with the calculations in [34] we wish to describe this quantity viewed from the perspective of the branch point, so via a change of coordinates we will identify $\phi_\pi \in \Phi_\pi$ with the branch point y^* and the translated map in Φ^*_π. For $\vec{M} = (M_0, M_1, M_2)$ and $\vec{v} = (v_0, v_1, v_2)$ we let

$$\pi_{\vec{M}}(\vec{v}) = \sum_{\phi_\pi \in \Phi^*_\pi(\vec{v})} W(\phi_\pi(\mathbf{S}^3_{\vec{M}})) \sum_{\vec{\mathcal{R}} \in \mathbb{T}_\pi} \Big[\prod_{s \in \mathbf{S}^3_{\vec{M}}} W(\mathcal{R}_s) \Big] J[\mathbf{S}^3_{\vec{M}}] \tag{7.33}$$

be the same quantity defined in [34, Definition 4.12] and bounded in [34, Section 6]. Recall that for $j_1 \in \{1, 2\}$, $s = [j_1, j_2] \in \mathbf{S}^3_{\vec{q}}$, $\phi \in \Phi$, and $\mathcal{R}_s$ a lattice tree containing $\phi(s)$, we write $(x, n) \in \mathcal{R}_s$ if $x \in \mathcal{R}_s$ and the tree distance from x to $\phi(s)$ in $\mathcal{R}_s$ is $n - (j_2 + m_*)$ (which implies that the tree distance from x to o in $\mathcal{R}_s \cup \phi(\mathbf{S}^3_{\vec{q}})$ is n if this combined object is a lattice tree). We now wish to record the spatial and temporal location of the vertex $x \in \mathcal{R}_s$ relative to the branch point $(y_*, m_* = m_0 + M_0 + 1)$ of $\mathbf{S}^3_{\vec{q}}$. To this end, for $\phi_\pi \in \Phi^*_\pi$, we write

$$(x_*, n_*) \,\hat{\in}\, \mathcal{R}_s \text{ if } x_* \in \mathcal{R}_s \text{ and the tree distance from } x_* \text{ to } \phi_\pi(s) \text{ in } \mathcal{R}_s \text{ is } n_* - j_2.$$

(The latter implies that the tree distance to the branch point of $\mathbf{S}^3_{\vec{q}}$ is $j_2 + n_* - j_2 = n_*$.) We now define

$$\pi_{\vec{M}; n_*}(\vec{v}; x_*) = \sum_{s_* \in (\mathbf{S}^3_{\vec{M}}(I_0))^1} \sum_{\phi_\pi \in \Phi^*_\pi(\vec{v})} W(\phi_\pi(\mathbf{S}^3_{\vec{M}}))$$

$$\times \sum_{\vec{\mathcal{R}} \in \mathbb{T}_\pi} \Big[\prod_{s \in \mathbf{S}^3_{\vec{M}}} W(\mathcal{R}_s) \Big] J[\mathbf{S}^3_{\vec{M}}] \mathbb{1}_{\{(x_*, n_*) \hat{\in} \mathcal{R}_{s_*}\}}, \tag{7.34}$$

and

$$\pi_{\vec{M};n_*,n_{**}}(\vec{v};x_*,x_{**}) = \sum_{\substack{(s_*,s_{**})\in \\ (\mathbf{S}^3_{\vec{M}}(I_0))^2}} \sum_{\phi_\pi\in\Phi^*_\pi(\vec{v})} W(\phi_\pi(\mathbf{S}^3_{\vec{M}})) \times \sum_{\vec{\mathcal{R}}\in\mathbb{T}_\pi} [\prod_{s\in\mathbf{S}^3_{\vec{M}}} W(\mathcal{R}_s)] J[\mathbf{S}^3_{\vec{M}}] \mathbb{1}_{\left\{\substack{(x_*,n_*)\hat{\in}\mathcal{R}_{s_*}, \\ (x_{**},n_{**})\hat{\in}\mathcal{R}_{s_{**}}}\right\}}. \tag{7.35}$$

In terms of this notation, we can identify

$$A_\pi(\vec{M}) = \sum_{y_*}\sum_{v_0,v_1,v_2} \pi_{\vec{M};\vec{n}_{I_0}-m_0-M_0-1}(\vec{v}-y_*;\vec{x}_{I_0}-y_*) z_c^3 \prod_{i=0}^{2} D(y_i-v_i). \tag{7.36}$$

Furthermore, define, using the same conventions, and writing $n^*(i_2,m_1,m_2) = ((m_1 \wedge m_2)-1)\wedge n^*(i_2)$

$$\chi^{(|I_0|+1)\heartsuit}_{m_1,m_2;\vec{n}_{I_0}}(\vec{y};\vec{x}_{I_0}) = \rho^2 \sum_{m_*=1}^{n^*(i_2,m_1,m_2)} \sum_{m_0=0}^{m_*-1} t^{(2)}_{m_0}(y_0) A_\pi(\vec{M}). \tag{7.37}$$

Then, we obtain that

$$t^{(r)\heartsuit}_{\vec{n}}(\vec{x}) = \sum_{\vec{I}}\sum_{m_1=1}^{n_1}\sum_{m_2=1}^{n_{i_2}}\sum_{\vec{y}} \chi^{(|I_0|+1)\heartsuit}_{m_1,m_2;\vec{n}_{I_0}}(\vec{y};\vec{x}_{I_0}) \Big(\prod_{j=1}^{2} t^{(|I_j|+1)}_{\vec{n}_{I_j}-m_j}(\vec{x}_{I_j}-y_j)\Big). \tag{7.38}$$

The case where $M_i = q_i$ for some $i\in\{0,1,2\}$: Let $Q = Q(\vec{M},\vec{q}) = \{i\in\{0,1,2\} : M_i = q_i\}$, and $Q^c = \{0,1,2\}\setminus Q$. Then $\mathbf{S}^1_{i+}$ and $\mathbb{T}_i$ are empty for each $i\in Q$, and there is no sum over ϕ_i. Moreover, $W(\phi(\mathbf{S}^3_{\vec{q}}))$ factors as

$$W(\phi) = W(\phi_\pi)\prod_{i\in Q^c} W(\phi_i) z_c D(y_i-v_i). \tag{7.39}$$

Then the contribution to (7.20) from $\vec{M}\nless\vec{q}$ is given by

$$\rho^{-1}\sum_{\vec{I}(i_2)}\sum_{\substack{\vec{M}\le\vec{q}: \\ \vec{M}\nless\vec{q}}}\sum_{y_0,y_1,y_2} A'_0A'_1A'_2A'_\pi,$$

where for $i=0,1,2$, we define x'_i as in the $\vec{M}<\vec{q}$ case and

$$A'_i = \begin{cases} A_i, & \text{if } i\in Q^c, \\ \mathbb{1}_{\{y_i=x'_i\}}, & \text{if } i\in Q, \end{cases}$$

and

$$A'_\pi = \sum_{\vec{v}}\Big[\prod_{i\in Q}\mathbb{1}_{\{v_i=x'_i\}}\Big]\Big[\prod_{i\in Q^c} z_c D(y_i-v_i)\Big] \times \sum_{\phi_\pi\in\Phi_\pi} W(\phi_\pi(\mathbf{S}^3_{\vec{M}})) \sum_{\vec{\mathcal{R}}\in\mathbb{T}_\pi}\Big[\prod_{s\in\mathbf{S}^3_{\vec{M}}} W(\mathcal{R}_s)\Big] J[\mathbf{S}^3_{\vec{M}}] \sum_{\vec{s}\in(\mathbf{S}^3_{\vec{M}}(I_0))^{|I_0|}}\prod_{j\in I_0}\mathbb{1}_{\{(x_j,n_j)\in\mathcal{R}_{s_j}\}}. \tag{7.40}$$

Setting $m_0 = -1$ if $M_0 = q_0$, $m_1 = n_1 + 1$ if $M_1 = q_1$, and $m_2 = n_{i_2} + 1$ if $M_2 = q_2$ (this is consistent with our notation $m_0 = m_* - M_0 - 1$ and $m_i = m_* + M_i + 1$ for $i = 1,2$ if $M_i < q_i$), we see that the contribution $t_{\vec{n}}^{(r)\spadesuit}(\vec{x})$ to $t_{\vec{n}}^{(r)}(\vec{x}) - \kappa_{\vec{n}}^{(r;1)}(\vec{x})$ from terms with $\vec{M} \nleq \vec{q}$ is given by

$$\begin{aligned}t_{\vec{n}}^{(r)\spadesuit}(\vec{x}) = &\sum_{\vec{I}} \sum_{Q\neq\varnothing} \sum_{m_1\le n_1+\mathbb{1}_{\{1\in Q\}}} \sum_{m_2\le n_{i_2}+\mathbb{1}_{\{2\in Q\}}} \sum_{\vec{y}} \Big(\prod_{j\in Q\setminus\{0\}} \mathbb{1}_{\{y_j=x'_j\}} \mathbb{1}_{\{m_j-1=n_j\}} \Big) \\ &\times \Big(\prod_{j\in Q^c\setminus\{0\}} t_{\vec{n}_{I_j}-m_j}^{(|I_j|+1)}(\vec{x}_{I_j} - y_j) \Big) \\ &\times \Big[\rho^{2-|Q|} \sum_{m_*=0}^{n^*(i_2,m_1,m_2)} \sum_{m_0=-\mathbb{1}_{\{0\in Q\}}}^{m_*-1} \big[\mathbb{1}_{\{0\in Q^c\}} t_{m_0}^{(2)}(y_0) + \mathbb{1}_{\{0\in Q\}} \mathbb{1}_{\{m_0=-1\}} \big] A'_\pi(\vec{M}) \Big]. \end{aligned} \tag{7.41}$$

We let $\kappa_{\vec{n}}^{(r;2)}(\vec{x})$ denote the contribution to $t_{\vec{n}}^{(r)\spadesuit}(\vec{x})$ from $Q \cap \{1,2\} \neq \varnothing$.

The remaining contribution is when $Q = \{0\}$, whence $M_0 = m_*$ (in particular this includes all remaining cases where $m_* = 0$) and

$$\begin{aligned}t_{\vec{n}}^{(r)\spadesuit}(\vec{x}) = &\kappa_{\vec{n}}^{(r;2)}(\vec{x}) + \sum_{\vec{I}} \sum_{m_1\le n_1} \sum_{m_2\le n_{i_2}} \sum_{\vec{y}} \Big(\prod_{j=1}^{2} t_{\vec{n}_{I_j}-m_j}^{(|I_j|+1)}(\vec{x}_{I_j} - y_j) \Big) \\ &\times \Big[\rho \mathbb{1}_{\{y_0=o\}} \sum_{m_*=0}^{n^*(i_2,m_1,m_2)} A'_\pi(m_*, M_1, M_2) \Big]. \end{aligned} \tag{7.42}$$

Define

$$\chi_{m_1,m_2;\vec{n}_{I_0}}^{(|I_0|+1)\spadesuit}(\vec{y}, \vec{x}_{I_0}) = \rho \mathbb{1}_{\{y_0=o\}} \sum_{m_*=0}^{n^*(i_2,m_1,m_2)} A'_\pi(m_*, M_1, M_2). \tag{7.43}$$

Then, we obtain that

$$t_{\vec{n}}^{(r)\spadesuit}(\vec{x}) = \kappa_{\vec{n}}^{(r;2)}(\vec{x}) + \sum_{\vec{I}} \sum_{m_1=1}^{n_1} \sum_{m_2=1}^{n_{i_2}} \sum_{\vec{y}} \chi_{m_1,m_2;\vec{n}_{I_0}}^{(|I_0|+1)\spadesuit}(\vec{y}; \vec{x}_{I_0}) \Big(\prod_{j=1}^{2} t_{\vec{n}_{I_j}-m_j}^{(|I_j|+1)}(\vec{x}_{I_j} - y_j) \Big). \tag{7.44}$$

We now combine the above results. Letting

$$\chi_{m_1,m_2;\vec{n}_{I_0}}^{(|I_0|+1)}(\vec{y}, \vec{x}_{I_0}) = \chi_{m_1,m_2;\vec{n}_{I_0}}^{(|I_0|+1)\heartsuit}(\vec{y}, \vec{x}_{I_0}) + \chi_{m_1,m_2;\vec{n}_{I_0}}^{(|I_0|+1)\spadesuit}(\vec{y}, \vec{x}_{I_0}),$$

we have arrived at

$$t_{\vec{n}}^{(r)}(\vec{x}) = \kappa_{\vec{n}}^{(r;1)}(\vec{x}) + \kappa_{\vec{n}}^{(r;2)}(\vec{x}) + \sum_{\vec{I}} \sum_{m_1\le n_1} \sum_{m_2\le n_{i_2}} \sum_{\vec{y}} \chi_{m_1,m_2;\vec{n}_{I_0}}^{(|I_0|+1)}(\vec{y}, \vec{x}_{I_0}) \Big(\prod_{j=1}^{2} t_{\vec{n}_{I_j}-m_j}^{(|I_j|+1)}(\vec{x}_{I_j} - y_j) \Big). \tag{7.45}$$

Given $\vec{k} \in \mathbb{R}^{d\times(r-1)}$ let $\vec{k}_{I_i} = (k_j : j \in I_i)$. Multiplying (7.45) by $e^{i\vec{k}\cdot\vec{x}}$, and summing over $\vec{x}$, we obtain

$$\begin{aligned}\hat{t}^{(r)}_{\vec{n}}(\vec{k}) =& \hat{\kappa}^{(r;1)}_{\vec{n}}(\vec{k}) + \hat{\kappa}^{(r;2)}_{\vec{n}}(\vec{k}) + \sum_{\vec{I}} \sum_{m_1\le n_1} \sum_{m_2\le n_{i_2}} \sum_{y_0,y_1,y_2} \\ & \sum_{\vec{x}_{I_1}} \mathrm{e}^{\mathrm{i}\sum_{j_1\in I_1} k_{j_1}\cdot(x_{j_1}-y_1)} \mathrm{e}^{\mathrm{i}\sum_{j_1\in I_1} k_{j_1}\cdot y_1} t^{(|I_1|+1)}_{\vec{n}_{I_1}-m_1}(\vec{x}_{I_1}-y_1) \\ & \times \sum_{\vec{x}_{I_2}} \mathrm{e}^{\mathrm{i}\sum_{j_2\in I_2} k_{j_2}\cdot(x_{j_2}-y_2)} \mathrm{e}^{\mathrm{i}\sum_{j_2\in I_2} k_{j_2}\cdot y_2} t^{(|I_2|+1)}_{\vec{n}_{I_2}-m_2}(\vec{x}_{I_2}-y_2) \\ & \times \sum_{\vec{x}_{I_0}} \Big(\prod_{i\in I_0} \mathrm{e}^{\mathrm{i}k_i\cdot x_i}\Big) \chi^{(|I_0|+1)}_{m_1,m_2;\vec{n}_{I_0}}(\vec{y},\vec{x}_{I_0}).\end{aligned} \tag{7.46}$$

This is equal to

$$\hat{\kappa}^{(r;1)}_{\vec{n}}(\vec{k}) + \hat{\kappa}^{(r;2)}_{\vec{n}}(\vec{k}) + \sum_{\vec{I}} \sum_{m_1\le n_1} \sum_{m_2\le n_{i_2}} \hat{t}^{(|I_1|+1)}_{\vec{n}_{I_1}-m_1}(\vec{k}_{I_1}) \hat{t}^{(|I_2|+1)}_{\vec{n}_{I_2}-m_2}(\vec{k}_{I_2}) \hat{\chi}^{(|I_0|+1)}_{m_1,m_2;\vec{n}_{I_0}}(\vec{k}),$$

where we define

$$\hat{\chi}^{(|I_0|+1)}_{m_1,m_2;\vec{n}_{I_0}}(\vec{k}) = \sum_{\vec{y}} \sum_{\vec{x}_{I_0}} \Big(\prod_{t=1}^{2} \mathrm{e}^{\mathrm{i}\sum_{j_t\in I_t} k_{j_t}\cdot y_t} \prod_{i\in I_0} \mathrm{e}^{\mathrm{i}k_i\cdot x_i}\Big) \chi^{(|I_0|+1)}_{m_1,m_2;\vec{n}_{I_0}}(\vec{y},\vec{x}_{I_0}). \tag{7.47}$$

7.3. *The bounds on $\hat{\chi}^{(r)}$ assuming diagrammatic bounds.* We continue to bound the coefficients arising in the lace expansion. We start with $|I_0| = 0$, and see that from (7.37) and using (7.36),

$$\chi^{(1)\heartsuit}_{m_1,m_2}(\vec{y}) = \rho^2 \sum_{y_*} \sum_{\vec{v}} \sum_{\substack{m_0,M_0\ge 0:\\ M_0+m_0\le n^*(i_2,m_1,m_2)-1}} t^{(2)}_{m_0}(y_0) \pi_{\vec{M}}(\vec{v}-y_*) \prod_{i=0}^{2} z_c D(y_i - v_i), \tag{7.48}$$

where $(M_0, M_1, M_2) = (M_0, m_1 - m_0 - M_0 - 2, m_2 - m_0 - M_0 - 2)$. We note that in (7.48), the spatial location of the vertex at time m_1 is equal to $y_0+(v_0-y_0)+(y_*-v_0)+(v_1-y_*)+(y_1-v_1) = y_1$, as required. Here we recall the spatial location of the branch point is y_*.

Similarly, from (7.43) with $I_0 = \varnothing$,

$$\chi^{(1)\spadesuit}_{m_1,m_2}(\vec{y}) = \rho \mathbb{1}_{\{y_o=o\}} \sum_{y_*} \sum_{m_*\le n^*(i_2,m_1,m_2)} \sum_{\vec{v}} \mathbb{1}_{\{v_0=o\}} \pi_{(m_*,M_1,M_2)}(\vec{v}-y_*) \prod_{i=1}^{2} z_c D(y_i - v_i). \tag{7.49}$$

We are aiming for the bound (3.10). Note that from (7.48)

$$|\hat{\chi}^{(1)\heartsuit}_{m_1,m_2}(\vec{k})| \le C \sum_{\substack{m_0,M_0\ge 0:\\ M_0+m_0\le (m_1\wedge m_2)-2}} \sum_{\vec{v}} \sum_{y_*} \sum_{\vec{y}} t^{(2)}_{m_0}(y_0) |\pi_{\vec{M}}(\vec{v}-y_*)| \prod_{i=0}^{2} z_c D(y_i - v_i). \tag{7.50}$$

Using that $\sum_{y_i} D(x_i - y_i) = 1$ and also that $\sup_m \sum_x t_m(x) \le K$, we get that

$$\begin{aligned}|\hat{\chi}^{(1)\heartsuit}_{m_1,m_2}(\vec{k})| &\le C \sum_{m_0,M_0} \sum_{\vec{v}} \sum_{y_0} \sum_{y_*} t^{(2)}_{m_0}(y_0) |\pi_{\vec{M}}(\vec{v}-y_*)| D(y_0 - v_0) \\ &= C \sum_{m_0,M_0} \sum_{\vec{u}} \sum_{y_0} \sum_{y_*} t^{(2)}_{m_0}(y_0) |\pi_{\vec{M}}(\vec{u})| D(y_0 - u_0 - y_*) \\ &\le C \sum_{m_0,M_0: M_0+m_0\le n^*(i_2,m_1,m_2)-1} \sum_{\vec{u}} |\pi_{\vec{M}}(\vec{u})|. \end{aligned} \tag{7.51}$$

Similarly,

$$|\hat{\chi}^{(1)\spadesuit}_{m_1,m_2}(\vec{k})| \leq C \sum_{m_0,M_0:M_0+m_0\leq n^*(i_2,m_1,m_2)-1} \mathbb{1}_{\{m_0=-1\}} \sum_{\vec{u}} |\pi_{\vec{M}}(\vec{u})|. \tag{7.52}$$

Since all of the bounds that we will obtain for $\chi^{(|I_0|+1)\heartsuit}$ also apply to $\chi^{(|I_0|+1)\spadesuit}$ (they are in fact easier in this case), henceforth we will only derive explicit bounds on $\chi^{(|I_0|+1)\heartsuit}$. Let

$$B(\vec{M}) = \sum_{\ell=0}^{2} \frac{1}{(M_\ell+1)^{(d-6)/2}} \frac{1}{(M_i+M_j+1)^{(d-4)/2}}, \tag{7.53}$$

where $\{i,j\} = \{0,1,2\} \smallsetminus \{\ell\}$. The following bound is proved in [34, Section 6]:

PROPOSITION 7.1 (Bound on π [34]). *There exists $C > 0$ such that for sufficiently spread-out lattice trees above 8 dimensions,*

$$\sum_{\vec{u}} |\pi_{\vec{M}}(\vec{u})| \leq CB(\vec{M}). \tag{7.54}$$

PROOF. See [34], where bounds are proved for the various contributions to $\pi_{\vec{M}}$, corresponding to laces of various types. All these bounds are of the form as given in $B(\vec{M})$. As in [34, Definition 2.8], a lace L on $\mathbf{S}^3_{\vec{M}}$ (with all $M_i > 0$) is *acyclic* if there is at least one branch $\mathbf{S}^1_i$, $i \in \{0,1,2\}$ (called a special branch) such that there is exactly one bond, $st \in L$, covering the branch point of $\mathbf{S}^3_{\vec{M}}$ that has an endpoint strictly on branch $\mathbf{S}^1_i$. A lace that is not acyclic is called cyclic. Cyclic laces have 3 edges covering the branchpoint, while acyclic laces may have two or three. In each case there can be many edges that do not cover the branchpoint.

The contribution from acyclic laces with two edges covering the branchpoint comes from [34, (6.3-6.5)].[2] The contribution from acyclic laces with three edges covering the branchpoint comes from [34, (6.6-6.9)] and [34, end of Section 6.2], together with simplifications (note that in [34], the quantities m_i do not represent the same thing that they do in this paper). For example note that [34, third line of (6.6)] (ignoring the constants in the numerator) is bounded by

$$\begin{aligned} &\frac{M_1}{(M_1+M_2+1)^{d/2}} \sum_{m_2\in[M_2/2,M_2]} \frac{1}{(M_2-m_2+1)^{(d-8)/2}} \sum_{m_1\leq M_1} \frac{1}{(m_1+M_3+1)^{(d-4)/2}} \\ &\leq \frac{M_1M_2}{(M_1+M_2+1)^{d/2}} \sum_{m\geq M_3} \frac{1}{(m+1)^{(d-4)/2}} \leq \frac{1}{(M_1+M_2+1)^{(d-4)/2}} \frac{C}{(M_3+1)^{(d-6)/2}}. \end{aligned}$$

The contribution from cyclic laces comes from [34, (6.10), and end of Section 6.3] together with simplifications. ∎

Recall that $(M_0,M_1,M_2) = (M_0, m_1-m_0-M_0-2, m_2-m_0-M_0-2)$, where $m_1,m_2 \geq 2$. By changing variables to $m_1' = m_1-2$ and $m_2' = m_2-2$, the following lemma is sufficient to verify (3.10).

[2] Note the typo in (6.3) in the reference, where the very last exponent should be $(d-4)/2$ instead of $(d-6)/2$.

LEMMA 7.2 (Bound on $\hat\chi^{(1)}_{m_1,m_2}$). *For $d>8$, with $a=(d-6)/2>1$, and for $m_1,m_2\geq 0$,*

$$\sum_{m_0,M_0: m_0+M_0\leq (m_1\wedge m_2)} B(M_0, m_1-m_0-M_0, m_2-m_0-M_0) \leq C(m_1 \vartriangle m_2)^{-a}. \tag{7.55}$$

Consequently, for sufficiently spread-out lattice trees above 8 dimensions, uniformly in $\vec k$,

$$|\hat\chi^{(1)}_{m_1,m_2}(\vec k)| \leq C(m_1 \vartriangle m_2)^{-a}. \tag{7.56}$$

PROOF. We start with (7.55). By symmetry in m_1,m_2, there are two terms to consider. Firstly, when $\ell=0, \{i,j\}=\{1,2\}$ in (7.53) and with $m=m_0+M_0$,

$$\begin{aligned}\sum_{m_0,M_0: m_0+M_0\leq (m_1\wedge m_2)} &(M_0+1)^{-(d-6)/2}(m_1+m_2-2m_0-2M_0+1)^{-(d-4)/2}\\ &= \sum_{m\leq (m_1\wedge m_2)}\sum_{M_0\leq m}(M_0+1)^{-(d-6)/2}(m_1+m_2-2m+1)^{-(d-4)/2}\\ &\leq C\sum_{m\leq (m_1\wedge m_2)}(m_1+m_2-2m+1)^{-(d-4)/2}\\ &\leq C(m_1\vartriangle m_2)^{-(d-6)/2}.\end{aligned} \tag{7.57}$$

Similarly, when $\ell=2, \{i,j\}=\{0,2\}$ in (7.53) and again with $m=m_0+M_0$,

$$\begin{aligned}\sum_{m_0,M_0: m_0+M_0\leq (m_1\wedge m_2)} &(m_2-m_0-M_0+1)^{-(d-6)/2}(m_1-m_0+1)^{-(d-4)/2}\\ &= \sum_{m\leq (m_1\wedge m_2)}(m_2-m+1)^{-(d-6)/2}\sum_{m_0\leq m}(m_1-m_0+1)^{-(d-4)/2}\\ &\leq C\sum_{m\leq (m_1\wedge m_2)}(m_2-m+1)^{-(d-6)/2}(m_1-m+1)^{-(d-6)/2}\\ &\leq C(m_1\vartriangle m_2)^{-(d-6)/2}.\end{aligned} \tag{7.58}$$

As $\ell=1$ is similar, (7.56) now follows immediately by (7.51), (7.52) and Proposition 7.1. ∎

We continue our analysis of $\hat\chi^{(|I_0|+1)\heartsuit}$ for $|I_0|>0$ by defining some notation. Recall (7.33), (7.34) and (7.35). Then for $|I_0|\leq 2$ and with $(M_0,M_1,M_2)=(M_0, m_1-m_0-M_0-2, m_2-m_0-M_0-2)$, and using that the temporal variable of the branch point equals $m_*=m_0+M_0+1$, we have

$$\begin{aligned}\chi^{(|I_0|+1)\heartsuit}_{m_1,m_2;\vec n_{I_0}}(\vec y;\vec x_{I_0}) = \rho^2\sum_{y_*}\sum_{\vec v}\sum_{m_0,M_0: M_0+m_0\leq n^*(i_2,m_1,m_2)-1} &t^{(2)}_{m_0}(y_0)\left(\prod_{i=0}^{2} z_c D(y_i-v_i)\right)\\ &\times \pi_{\vec M;\vec n_{I_0}-m_0-M_0-1}(\vec v-y_*;\vec x_{I_0}-y_*).\end{aligned} \tag{7.59}$$

We are aiming for the bounds (3.11) and (3.12), and we proceed as in (7.48)-(7.51). Note that from (7.47),

$$
\begin{aligned}
&|\hat{\chi}^{(|I_0|+1)\heartsuit}_{m_1,m_2;\vec{n}_{I_0}}(\vec{k})| \\
&\le C \sum_{\substack{M_0,m_0:\\ M_0+m_0<n^*(i_2,m_1,m_2)}} \sum_{y_0}\sum_{\vec{x}_{I_0}}\sum_{y_*}\sum_{\vec{v}} t^{(2)}_{m_0}(y_0)|\pi_{\vec{M};\vec{n}_{I_0}-m_0-M_0-1}(\vec{v}-y_*;\vec{x}_{I_0}-y_*)|D(y_0-v_0) \qquad (7.60)\\
&\le C \sum_{\substack{M_0,m_0:\\ M_0+m_0<n^*(i_2,m_1,m_2)}} \sum_{\vec{u}}\sum_{\vec{x}_{I_0}} |\pi_{\vec{M};\vec{n}_{I_0}-m_0-M_0-1}(\vec{u};\vec{x}_{I_0})|. \qquad (7.61)
\end{aligned}
$$

Denote the canonical basis vectors in $\mathbb{R}^3$ by $\vec{e}_1,\vec{e}_2,\vec{e}_3$. To bound $\chi^{(|I_0|+1)}$ for $|I_0|=1,2$, we will make use of the following version of Proposition 7.1 with extra arms attached, in which $p=(10-d)/2$ when $d<10$, $p\in(0,\frac{1}{2})$ for $d=10$ and $p=0$ for $d>10$.

Proposition 7.3 (Bound on π with extra arms). *For lattice trees in dimensions $d>8$ with L sufficiently large,*

$$
\sum_{\vec{u},x_*}|\pi_{\vec{M};n_*}(\vec{u};x_*)| \le C\Big[n_*^p+\sum_{i=0}^{2}M_i\Big]B(\vec{M})+C\sum_{0\le t\le n_*}\Big[\sum_{j=1}^{3}B(\vec{M}+t\vec{e}_j)\Big] \qquad (7.62)
$$

and, recalling that $(M\vee n)_{**}=(M_1\vee M_2\vee n_*\vee n_{**})$,

$$
\begin{aligned}
\sum_{\vec{u},x_*,x_{**}}|\pi_{\vec{M};n_*,n_{**}}(\vec{u};x_*,x_{**})| \le\,& C(M\vee n)_{**}\Big[(n_*\vee n_{**})^p+\sum_{i=0}^{2}M_i\Big]B(\vec{M}) \\
&+C(M\vee n)_{**}\sum_{0\le t\le n_*\vee n_{**}}\Big[\sum_{j=1}^{3}B(\vec{M}+t\vec{e}_j)\Big]. \qquad (7.63)
\end{aligned}
$$

The proof of Proposition 7.3 is deferred to Appendix A where the arguments in [34] which led to Propostion 7.1 are suitably modified. Let us now prove the required bounds (3.11) and (3.12) assuming Proposition 7.3 with $p=(10-d)/2$ when $d<10$, $p\in(0,\frac{1}{2})$ for $d=10$ and $p=0$ for $d>10$.

PROPOSITION 7.4 (Bound on $\hat\chi^{(r)}_{m_1,m_2;n}$ with $r=2,3$). *For $d>8$ and $a=(d-6)/2$, and for $m_1,m_2\geq 0$,*

$$\sum_{\substack{m_0,M_0:\\ m_0+M_0\leq(m_1\wedge m_2)}}(m_1-m_0-M_0)B(M_0,m_1-m_0-M_0,m_2-m_0-M_0)\leq C(m_1\vartriangle m_2)^{-a}[(m_1\vartriangle m_2)+m_1^p], \tag{7.64}$$

$$\sum_{\substack{m_0,M_0:\\ m_0+M_0\leq(m_1\wedge m_2)}}M_0B(M_0,m_1-m_0-M_0,m_2-m_0-M_0)\leq C(m_1\vartriangle m_2)^{-a}[(m_1\vartriangle m_2)+m_1^p], \tag{7.65}$$

$$\sum_{0\leq t\leq n}\sum_{\substack{m_0,M_0:\\ m_0+M_0\leq(m_1\wedge m_2)}}B\Big((M_0,m_1-m_0-M_0,m_2-m_0-M_0)+t\vec e_j\Big)\leq C(m_1\vartriangle m_2)^{-a}[(m_1\vartriangle m_2)+(m_1\vee m_2)^p]. \tag{7.66}$$

Consequently, for sufficiently large L, above 8 dimensions, and uniformly in $\vec k$,

$$|\hat\chi^{(2)}_{m_1,m_2;n_*}(\vec k)|\leq C(m_1\vartriangle m_2)^{-a}[(m_1\vartriangle m_2)+(m_1\vee m_2\vee n_*)^p], \tag{7.67}$$

and (recall that $(m\vee n)_{**}=m_1\vee m_2\vee n_*\vee n_{**}$)

$$|\hat\chi^{(3)}_{m_1,m_2;n_*,n_{**}}(\vec k)|\leq C(m\vee n)_{**}(m_1\vartriangle m_2)^{-a}[(m_1\vartriangle m_2)+(m\vee n)^p_{**}], \tag{7.68}$$

PROOF. We start by proving (7.64). From (7.53) there are three terms to consider. Firstly, when $\ell=0,\{i,j\}=\{1,2\}$ in (7.53) and with $m=m_0+M_0$,

$$\begin{aligned}\sum_{m_0+M_0\leq(m_1\wedge m_2)}&(m_1-m_0-M_0)(M_0+1)^{-(d-6)/2}(m_1+m_2-2m_0-2M_0+1)^{-(d-4)/2}\\ &\leq\sum_{m\leq(m_1\wedge m_2)}\sum_{M_0\leq m}(M_0+1)^{-(d-6)/2}(m_1+m_2-2m+1)^{-(d-6)/2}\\ &\leq C\sum_{m\leq(m_1\wedge m_2)}(m_1+m_2-2m+1)^{-(d-6)/2}\\ &\leq C(m_1\vartriangle m_2)^{-(d-8)/2}.\end{aligned} \tag{7.69}$$

Next, when $\ell=1,\{i,j\}=\{0,2\}$ in (7.53) and again with $m=m_0+M_0$,

$$\begin{aligned}\sum_{m_0+M_0\leq(m_1\wedge m_2)}&(m_1-m_0-M_0)(m_1-m_0-M_0+1)^{-(d-6)/2}(m_2-m_0+1)^{-(d-4)/2}\\ &\leq\sum_{m\leq(m_1\wedge m_2)}\sum_{m_0\leq m}(m_1-m+1)^{-(d-8)/2}(m_2-m_0+1)^{-(d-4)/2}\\ &\leq C\sum_{m\leq(m_1\wedge m_2)}(m_1-m+1)^{-(d-8)/2}(m_2-m+1)^{-(d-6)/2}\\ &\leq C(m_1\vartriangle m_2)^{-(d-8)/2}\sum_{m\leq m_2}(m_2-m+1)^{-(d-6)/2}\\ &\quad+C(m_1\vartriangle m_2)^{-(d-6)/2}\sum_{m\leq m_1}(m_1-m+1)^{-(d-8)/2}\end{aligned} \tag{7.70}$$

$$\leq C(m_1\vartriangle m_2)^{-(d-6)/2}\left[(m_1\vartriangle m_2)+m_1^p\right], \tag{7.71}$$

where p is as described above. Finally, when $\ell = 2, \{i,j\} = \{0,1\}$ in (7.53) and again with $m = m_0 + M_0$,

$$\begin{aligned}
\sum_{m_0+M_0\leq(m_1\wedge m_2)} & (m_1-m_0-M_0)(m_1-m_0+1)^{-(d-4)/2}(m_2-m_0-M_0+1)^{-(d-6)/2} \\
&\leq \sum_{m\leq(m_1\wedge m_2)}(m_2-m+1)^{-(d-6)/2}\sum_{m_0\leq m}(m_1-m_0+1)^{-(d-6)/2} \\
&\leq C\sum_{m\leq(m_1\wedge m_2)}(m_1-m+1)^{-(d-8)/2}(m_2-m+1)^{-(d-6)/2} \\
&\leq C(m_1\vartriangle m_2)^{-(d-6)/2}[(m_1\vartriangle m_2)+m_1^p],
\end{aligned} \tag{7.72}$$

as in the derivation of (7.71). This proves that all contributions in (7.64) obey the required bound.

We continue by proving (7.65). There are three cases to consider. Firstly, when $\ell = 0, \{i,j\} = \{1,2\}$ in (7.53) and with $m = m_0 + M_0$,

$$\begin{aligned}
\sum_{\substack{m_0,M_0:\\ m_0+M_0\leq(m_1\wedge m_2)}} & M_0(M_0+1)^{-(d-6)/2}(m_1+m_2-2m_0-2M_0+1)^{-(d-4)/2} \\
&\leq \sum_{m\leq m_1\wedge m_2}\sum_{M_0\leq m_1\wedge m_2}(M_0+1)^{-(d-8)/2}(m_1+m_2-2m+1)^{-(d-4)/2} \\
&\leq C(m_1\wedge m_2)^p(m_1\vartriangle m_2)^{-(d-6)/2},
\end{aligned}$$

as required. Secondly, when $\ell = 1, \{i,j\} = \{0,2\}$ in (7.53) and if $m_1 \leq m_2$ then

$$\begin{aligned}
\sum_{\substack{m_0,M_0:\\ m_0+M_0\leq(m_1\wedge m_2)}} & M_0(m_2-m_0+1)^{-(d-4)/2}(m_1-m_0-M_0+1)^{-(d-6)/2} \\
&\leq \sum_{\substack{m_0,M_0:\\ m_0+M_0\leq m_1}}(m_2-m_0+1)^{-(d-6)/2}(m_1-m_0-M_0+1)^{-(d-6)/2} \\
&\leq C(m_1\vartriangle m_2)^{-(d-6)/2}\sum_{m_0\leq m_1}\sum_{M_0\leq m_1-m_0}(m_1-m_0+1)^{-(d-6)/2} \\
&\leq C(m_1\vartriangle m_2)^{-(d-6)/2}\sum_{m_0\leq m_1}(m_1-m_0+1)^{-(d-8)/2} \\
&\leq C(m_1\vartriangle m_2)^{-(d-6)/2}(m_1\wedge m_2)^p.
\end{aligned}$$

Finally, when $\ell = 1, \{i,j\} = \{0,2\}$ in (7.53) and if $m_2 \leq m_1$ then

$$\begin{aligned}
\sum_{\substack{m_0,M_0:\\ m_0+M_0\leq(m_1\wedge m_2)}} & M_0(m_2-m_0+1)^{-(d-4)/2}(m_1-m_0-M_0+1)^{-(d-6)/2} \\
&\leq \sum_{m_0\leq m_2}(m_2-m_0+1)^{-(d-6)/2}\sum_{M_0\leq m_2-m_0}(m_1-m_0-M_0+1)^{-(d-6)/2} \\
&\leq C\sum_{m_0\leq m_2}(m_2-m_0+1)^{-(d-6)/2}(m_1\vartriangle m_2)^{-(d-8)/2} \\
&\leq C(m_1\vartriangle m_2)^{-(d-8)/2} = C(m_1\vartriangle m_2)^{-(d-6)/2}(m_1\vartriangle m_2).
\end{aligned}$$

This proves that all contributions in (7.65) obey the required bound.

We continue by proving (7.66). By (7.53) the $+t$ term appears as $(\bullet+t)^{-b}$, where $b = (d-6)/2$ or $(d-4)/2$. Summing this term over $t<\infty$ gives at most $C(\bullet)^{1-b} = C(\bullet)^1(\bullet)^{-b}$, from which we obtain the desired bounds from (7.64) and (7.65).

We next prove (7.67). By (7.61), combined with (7.62) in Proposition 7.3,

$$
(7.73) \qquad |\hat{\chi}^{(2)\heartsuit}_{m_1,m_2;n}(\vec{k})| \le C \sum_{\substack{m_0,M_0:\\ m_0+M_0\le(m_1\wedge m_2)-2}} \Big[n^p + \sum_{i=0}^{2} M_i\Big] B(\vec{M}) + C \sum_{\substack{m_0,M_0:\\ m_0+M_0\le(m_1\wedge m_2)-2}} \sum_{t\le n} \Big[\sum_{j=1}^{3} B(\vec{M}+t\vec{e}_j)\Big],
$$

and the bound on $|\hat{\chi}^{(2)\spadesuit}|$ is the same. Expressing the M_i in terms of m_i, the claim now follows from (7.64)–(7.66) together with (7.55). The proof of (7.68) is identical once we notice that the extra factor of $(m\vee n)_{**}$ arises from (7.63) and the fact that $M_i\le m_i$, $i=1,2$. ∎

7.4. *The bounds on $\hat{\kappa}^{(r)}$ assuming Proposition 7.3.* In this section we prove that (3.9) holds for lattice trees, via the following proposition:

PROPOSITION 7.5. *For lattice trees with $d>8$, and L sufficiently large,* (3.9) *holds with $a = 2\wedge((3d-20)/(d-4))>1$.*

PROOF. Recall the notation of Section 7.2. By (7.14) and (7.12), $t^{(r;i_2)}_{n_1\wedge n_{i_2},\vec{n}}(\vec{x}) = 0$ unless $n_1\wedge n_{i_2} = \underline{n}$. From (7.15),

$$
(7.74) \qquad \left|\hat{\kappa}^{(r;1)}_{\vec{n}}(\vec{k})\right| = \Big|\sum_{\vec{x}} e^{i\vec{k}\cdot\vec{x}} \sum_{i_2=2}^{r-1} t^{(r;i_2)}_{n_1\wedge n_{i_2},\vec{n}}(\vec{x})\Big| \le \sum_{\vec{x}}\sum_{i_2=2}^{r-1} t^{(r;i_2)}_{n_1\wedge n_{i_2},\vec{n}}(\vec{x}) = \sum_{i_2=2}^{r-1} \hat{t}^{(r;i_2)}_{\underline{n},\vec{n}}(\vec{0}).
$$

From (7.14) if $\underline{n}>0$, then $q_0\ge 1$ and proceeding as Section 7.2,

$$
\begin{aligned}
t^{(r;i_2)}_{\underline{n},\vec{n}}(\vec{x}) &= \sum_{\vec{s}\in(\mathbf{S}^3_{\vec{q};i_2})^{r-3}} \rho^{-1}\sum_{\phi\in\Phi} W(\phi(\mathbf{S}^3_{\vec{q}})) \sum_{\vec{\mathcal{R}}\in\mathbb{T}(\vec{q},\phi)} \Big[\prod_{s\in\mathbf{S}^3_{\vec{q}}} W(\mathcal{R}_s)\Big]\Big[\prod_{j\notin 1,i_2} \mathbb{1}_{\{(x_j,n_j)\in\mathcal{R}_{s_j}\}}\Big] \\
&\qquad \times \Big[\prod_{s,t\in\mathbf{S}^3_{\vec{q}}} [1+U_{st}]\Big] \\
&\le \sum_{\vec{s}\in(\mathbf{S}^3_{\vec{q};i_2})^{r-3}} \rho^{-1}\sum_{\phi\in\Phi} W(\phi(\mathbf{S}^3_{\vec{q}})) \sum_{\vec{\mathcal{R}}\in\mathbb{T}(\vec{q},\phi)} \Big[\prod_{s\in\mathbf{S}^3_{\vec{q}}} W(\mathcal{R}_s)\Big]\Big[\prod_{j\notin 1,i_2} \mathbb{1}_{\{(x_j,n_j)\in\mathcal{R}_{s_j}\}}\Big] \\
&\qquad \times \Big[\prod_{s,t\in\mathbf{S}^1_{0,[1,q_0]}} [1+U_{st}] \prod_{s',t'\in\mathbf{S}^3_{\vec{q}}\setminus\mathbf{S}^1_{0,[1,q_0]}} [1+U_{s't'}]\Big]
\end{aligned}
$$

$$
(7.75) \qquad \le \rho\Big(t^{(2)}_{\underline{n}-1} * D * t^{(r;i_2)}_{0,\vec{n}-\underline{n}}\Big)(\vec{x}).
$$

Note that in deriving the bound (7.75) we have used the fact (as in Section 7.2) that $W(\phi(\mathbf{S}^3_{\vec{q}}))$ factors as $W(\phi(\mathbf{S}^1_{0,[1,j_0]}))W(\phi(\mathbf{S}^1_{0,[0,1]}))W(\phi(\mathbf{S}^3_{\vec{q}}\setminus\mathbf{S}^1_{0,[1,j_0]}))$. At least one component, say j, of $\vec{n}-\underline{n}$ is equal to 0. In this case $t^{(r;i_2)}_{\vec{n}-\underline{n}}(\vec{y}) \equiv t^{(r;i_2)}_{0,\vec{n}-\underline{n}}(\vec{x}) = 0$ unless $y_j=0$ and so $t^{(r;i_2)}_{\vec{n}-\underline{n}}(\vec{y})$ is bounded by the $r-1$-point function of the remaining coordinates. Hence by (7.75) for each i_2,

$$\hat{t}^{(r;i_2)}_{\underline{n},\vec{n}}(\vec{0}) \le \rho \hat{t}^{(2)}_{\underline{n}-1}(0)\hat{D}(0)\hat{t}^{(r;i_2)}_{0,\vec{n}-\underline{n}}(\vec{0}) \le \rho \hat{t}^{(2)}_{\underline{n}-1}(0)\hat{D}(0)\hat{t}^{(r-1)}_{\vec{n}-\underline{n}}(\vec{0}) \le C|\bar{n}-\underline{n}|^{r-3},$$

where we have abused the $r-1$-point notation slightly and used the fact (see (2.5) and (3.4), or [34, Theorem 1.9]) that $\hat{t}^{(r)}_{\vec{n}}(\vec{0}) \le C\overline{n}^{r-2}$. Similarly,

$$\hat{t}^{(r;i_2)}_{\underline{n},\vec{n}}(\vec{0}) \le \hat{t}^{(r-1)}_{\vec{n}-\underline{n}}(\vec{0}) \le C|\bar{n}-\underline{n}|^{r-3}, \qquad \text{if } \underline{n}=0.$$

Multiplying the constant by a factor of $r-1$ arising from summing over $i_2 \le r-1$ in (7.74) gives the required bound on $|\hat{\kappa}^{(r;1)}_{\vec{n}}(\vec{x})|$.

Recall now that $\kappa^{(r;2)}_{\vec{n}}(\vec{x})$ is the contribution to (7.41) due to $Q \cap \{1,2\} \ne \varnothing$, so that $m_1-1=n_1$ or $m_2-1=n_{i_2}$ (and with $m_* < \underline{n}$). We concentrate on the contribution due to $Q=\{1\}$ only (where $m_1-1=n_1$ and $n^*(i_2,m_1,m_2)=n^*(i_2,m_2)\equiv(m_2-1)\wedge n^*(i_2))$, as the other contributions are similar. From (7.40) , (7.34) and (7.35), and as in (7.59), for $|I_0|\le 2$ and with $(M_0,M_1,M_2)=(M_0,n_1-m_0-M_0-1,m_2-m_0-M_0-2)$ this is equal to

$$\begin{aligned}&\sum_{\vec{I}}\sum_{m_2\le n_{i_2}}\sum_{y_0,y_2,v_0,v_2,y_*} t^{(|I_2|+1)}_{\vec{n}_{I_2}-m_2}(\vec{x}_{I_2}-y_2)\\ &\quad\times\Big[\rho\sum_{\substack{m_0,M_0:\\ m_0+M_0\le n^*(i_2,m_2)-1}} t^{(2)}_{m_0}(y_0)\pi_{\vec{M};\vec{n}_{I_0}-m_0-M_0-1}(\vec{v}-y_*;\vec{x}_{I_0}-y_*)\Big]\Big(\prod_{i\in\{0,2\}} z_cD(y_i-v_i)\Big).\end{aligned}\tag{7.76}$$

By taking the Fourier transform, using $\sum_{\vec{x}} t^{(|I_2|+1)}_{\vec{n}-m_2}(\vec{x}) \le (\bar{n}-m_2+1)^{|I_2|-1}$, and proceeding as in (7.51) and (7.61), we have

$$\Big|\hat{\kappa}^{(r;2)}_{\vec{n}}(\vec{k})\Big| \le C\sum_{\vec{I}}\sum_{m_2\le n_{i_2}}(\bar{n}-m_2+1)^{|I_2|-1}\sum_{\substack{m_0,M_0:\\ m_0+M_0\le n^*(i_2,m_2)-1}}\sum_{\vec{u}}\sum_{\vec{x}_{I_0}}\Big|\pi_{\vec{M};\vec{n}_{I_0}-m_0-M_0-1}(\vec{u};\vec{x}_{I_0})\Big|,\tag{7.77}$$

where it is understood that $\pi_{\vec{M};\vec{n}_{I_0}}(\vec{u};\vec{x}_{I_0})=\pi_{\vec{M}}(\vec{u})$ if $I_0=\varnothing$.

The $|I_0|=0$ contribution to (7.77) is bounded, as in Proposition 7.1 and Lemma 7.2, by

$$\begin{aligned}C\sum_{\vec{I}:|I_0|=0}\sum_{m_2\le n_{i_2}}(\bar{n}-m_2+1)^{|I_2|-1}&\sum_{\substack{m_0,M_0:\\ m_0+M_0\le(n_1\wedge m_2)}}\sum_{\vec{u}}\Big|\pi_{\vec{M}}(\vec{u})\Big|\\ &\le C\sum_{\vec{I}:|I_0|=0}\sum_{m_2\le n_{i_2}}(\bar{n}-m_2+1)^{|I_2|-1}(n_1\vartriangle m_2)^{-b},\end{aligned}\tag{7.78}$$

with $b=(d-6)/2$ and $|I_2|\le r-2$. We take a power $\varepsilon=\varepsilon(r)\in(0,1)$ to be determined later on, and split depending on whether $m_2 \le n_1-\bar{n}^{\varepsilon}$ or not. When $m_2 > n_1-\bar{n}^{\varepsilon}$, then we can bound

$$(\bar{n}-m_2+1)^{|I_2|-1}=(\bar{n}-n_1+n_1-m_2+1)^{|I_2|-1}\le C(\bar{n}-\underline{n})^{r-3}+C\bar{n}^{(r-3)\varepsilon},\tag{7.79}$$

and summing $(n_1 \vartriangle m_2)^{-b}$ over m_2 in (7.78) gives a constant. This gives a bound of

$$C\big[(\bar{n}-\underline{n})^{r-3}+\bar{n}^{(r-3)\varepsilon}\big]. \tag{7.80}$$

When, instead, $m_2 \le n_1 - \bar{n}^{\varepsilon}$, then we can bound $(\bar{n}-m_2+1)^{|I_2|-1} \le \bar{n}^{r-3}$, and use that

$$\sum_{m_2 \le n_1-\bar{n}^{\varepsilon}} (n_1 \vartriangle m_2)^{-b} \le C\bar{n}^{-(d-8)\varepsilon/2}. \tag{7.81}$$

We now verify (3.9) with $a = 2 \wedge ((3d-20)/(d-4) > 1$ and p as in the previous section but chosen in $(0,1/3)$ if $d = 10$. Combining the above bounds yields that the contribution to $\hat{\kappa}^{(r;2)}$ due to $|I_0| = 0$ is bounded by

$$C\big[(\bar{n}-\underline{n})^{r-3}+\bar{n}^{(r-3)\varepsilon}+\bar{n}^{r-3}\bar{n}^{-(d-8)\varepsilon/2}\big]. \tag{7.82}$$

When $r = 3$ or 4, $\varepsilon = 1/2$ will prove the claim. When $r = 5$, we optimize over ε which gives $\varepsilon = 4/(d-4)$, yielding a bound of the form $C\big[(\bar{n}-\underline{n})^2+\bar{n}^{8/(d-4)}\big]$. We complete the proof by writing $8/(d-4) = 3-(3d-20)/(d-4) \le 3-a$ so that the bounds (3.9) hold.

The contribution to (7.77) from $|I_0| = 1$ is 0 if $r = 3$. For $r \ge 4$ it is bounded using Lemma 7.2 and Propositions 7.3 and 7.4 (and using the trivial bound $(\bar{n}-m_2+1)^{|I_2|-1} \le \bar{n}^{r-4}$ since $|I_2| \le r-3$) by

$$\begin{aligned}
&\sum_{\vec{I}:|I_0|=1}\sum_{m_2\le n_{i_2}}(\bar{n}-m_2+1)^{|I_2|-1}\sum_{\substack{m_0,M_0:\\ m_0+M_0\le n^*(i_2,m_2)-1}}\sum_{\vec{u}}\sum_{x_{I_0}}\left|\pi_{\vec{M};n_{I_0}-m_0-M_0-1}(\vec{u};x_{I_0})\right|\\
&\quad\le C\bar{n}^{r-4}\sum_{m_2\le n_{i_2}}(n_1 \vartriangle m_2)^{-b}\big[(n_1 \vartriangle m_2)+(n_1\vee m_2)^p+n_{I_0}^p\big]\\
&\quad\le C\bar{n}^{r-4}\Big[\sum_{m_2\le n_{i_2}}(n_1 \vartriangle m_2)^{1-b}+\bar{n}^p\Big]\le \bar{n}^{p+r-4}.
\end{aligned} \tag{7.83}$$

This satisfies (3.9) when $r = 4$ since $p < 1$ and when $r = 5$ provided $3-a \ge p+1$, which holds for our choice of a and p ($p < 1/3$ is used here).

The contribution to (7.77) from $|I_0| = 2$ is 0 if $r = 3, 4$. For $r = 5$, since $|I_2|-1 = 0$ the contribution is bounded, again using Lemma 7.2 and Propositions 7.3 and 7.4, by

$$C\sum_{\vec{I}:|I_0|=2}\sum_{m_2\le n_{i_2}}\sum_{\substack{m_0,M_0:\\ m_0+M_0\le n^*(i_2,m_2)-1}}\sum_{\vec{u}}\sum_{\vec{x}_{I_0}}\left|\pi_{\vec{M};\vec{n}_{I_0}-m_0-M_0-1}(\vec{u};\vec{x}_{I_0})\right| \tag{7.84}$$

$$\le C\bar{n}\sum_{m_2\le n_{i_2}}(n_1 \vartriangle m_2)^{-b}\big[(n_1 \vartriangle m_2)+(n_1\vee m_2)^p+\overline{n_{I_0}}^p\big]. \tag{7.85}$$

As in (7.83) and the previous argument, this is at most $\bar{n}^{1+p} \le \bar{n}^{3-a}$, and so satisfies (3.9) with $r = 5$. ■

Appendix A - Proof of Proposition 7.3. Recall that Proposition 7.3 states that for lattice trees in dimensions $d > 8$ with L sufficiently large,

$$\sum_{\vec{u},x_*}|\pi_{\vec{M};n_*}(\vec{u};x_*)| \le \Big[n_*^p+\sum_{j=0}^{2}M_j\Big]B(\vec{M})+\sum_{0\le t\le n_*}\sum_{j=1}^{3}B(\vec{M}+te_j) \tag{A.1}$$

and, recalling that $(M \vee n)_{**} = (M_1 \vee M_2 \vee n_* \vee n_{**})$,

$$\sum_{\vec{u},x_*,x_{**}} |\pi_{\vec{M};n_*,n_{**}}(\vec{u};x_*,x_{**})| \leq (M \vee n)_{**} \left[[\sum_{j=0}^{2} M_j + n_*^p] B(\vec{M}) + \sum_{t \leq n_*} \sum_{j=1}^{3} B(\vec{M} + te_j) \right] \tag{A.2}$$

In this section we prove these results by making simple but important modifications to the diagrammatic bounds derived in [34] that were used there to prove Proposition 7.1. In that work the derivation of the diagrams and subsequent bounding totalled about 30 pages. We will not repeat the arguments to the same level of detail here, but will instead focus on the modifications required. We start in the next section by giving an overview of the proof.

A.1 Overview of the proof. Recall first how diagrams arise. By putting absolute values around $J[\mathbf{S}^3_{\vec{M}}]$ in (7.34), (7.35), and (7.33),

$$|\pi_{\vec{M};n_*}(\vec{v};x_*)| \leq \sum_{s_* \in (\mathbf{S}^3_{\vec{M}}(I_0))^1} \sum_{\phi_\pi \in \Phi^*_\pi(\vec{v})} W(\phi_\pi(\mathbf{S}^3_{\vec{M}})) \sum_{\vec{\mathcal{R}} \in \mathbb{T}_\pi} [\prod_{s \in \mathbf{S}^3_{\vec{M}}} W(\mathcal{R}_s)] |J[\mathbf{S}^3_{\vec{M}}]| \mathbb{1}_{\{(x_*,n_*) \hat{\in} \mathcal{R}_{s_*}\}}, \tag{A.3}$$

and

$$\begin{aligned} |\pi_{\vec{M};n_*,n_{**}}(\vec{v};x_*,x_{**})| \leq & \sum_{\substack{(s_*,s_{**}) \in \\ (\mathbf{S}^3_{\vec{M}}(I_0))^2}} \sum_{\phi_\pi \in \Phi^*_\pi(\vec{v})} W(\phi_\pi(\mathbf{S}^3_{\vec{M}})) \\ & \times \sum_{\vec{\mathcal{R}} \in \mathbb{T}_\pi} [\prod_{s \in \mathbf{S}^3_{\vec{M}}} W(\mathcal{R}_s)] |J[\mathbf{S}^3_{\vec{M}}]| \mathbb{1}_{\{(x_*,n_*) \hat{\in} \mathcal{R}_{s_*}\}} \mathbb{1}_{\{(x_{**},n_{**}) \hat{\in} \mathcal{R}_{s_{**}}\}}. \end{aligned} \tag{A.4}$$

If the indicators $\mathbb{1}_{\{(x_*,n_*) \hat{\in} \mathcal{R}_{s_*}\}}$ and $\mathbb{1}_{\{(x_{**},n_{**}) \hat{\in} \mathcal{R}_{s_{**}}\}}$ would be absent, then we would simply obtain $|\pi_{\vec{M}}(\vec{v})|$, and in this appendix we study these extra indicators.

Recall that $J[\mathbf{S}^3_{\vec{M}}] = \sum_{\Gamma \in \mathcal{G}^{\rm conn}(\mathbf{S}^3_{\vec{M}})} \prod_{st \in \Gamma} U_{st}$. There are many connected graphs to sum over, so it is convenient to restructure the sum in terms of minimally (except possibly for an extra edge covering the branch point) connected graphs, which are called *laces*. Here we call a connected graph *minimal* when removing any of its edges disconnect the graph. Indeed, for each connected graph on $\mathbf{S}^3_{\vec{M}}$ one can construct an associated lace. Any given lace L has a corresponding set of compatible edges $\mathcal{C}(L)$ such that if we combine any subset of $\mathcal{C}(L)$ with L, we obtain a connected graph on $\mathbf{S}^3_{\vec{M}}$ whose associated lace is again L. In particular, any edge $s't'$ that is completely covered by some edge $st \in L$ is compatible with L. See [34, Section 2] for precise definitions. Then [34, (2.10)] states that

$$J[\mathbf{S}^3_{\vec{M}}] = \sum_{N=1}^{\infty} \sum_{L \in \mathcal{L}^{(N)}(\mathbf{S}^3_{\vec{M}})} \prod_{st \in L} U_{st} \prod_{s't' \in \mathcal{C}(L)} [1 + U_{s't'}], \tag{A.5}$$

where $\mathcal{L}^{(N)}(\mathbf{S}^3_{\vec{M}})$ is the set of laces on $\mathbf{S}^3_{\vec{M}}$ consisting of exactly N edges. We define quantities $\pi^{(N)}$ to be the contributions to the corresponding π quantities from laces consisting of exactly N edges. Note that the sum over N is in fact finite as for N large (depending on $\vec{M}$) all summands are 0 (finite $\vec{M}$ limits the number of possible edges in a lace).

Since $U_{st} \in \{0, -1\}$, this is only non-zero when all $U_{st} = -1$ for all $st \in L$, in which case the terms in the sum over N are alternating in sign. Clearly,

$$|J[\mathbf{S}^3_{\vec{M}}]| \leq \sum_{N=1}^{\infty} \sum_{L \in \mathcal{L}^{(N)}(\mathbf{S}^3_{\vec{M}})} \prod_{st \in L} [-U_{st}] \prod_{s't' \in \mathcal{C}(L)} [1 + U_{s't'}], \tag{A.6}$$

and so

(A.7)

$$\sum_{\vec{\mathcal{R}} \in \mathbb{T}_\pi} [\prod_{s \in \mathbf{S}^3_{\vec{M}}} W(\mathcal{R}_s)] |J[\mathbf{S}^3_{\vec{M}}]| \leq \sum_{N=1}^{\infty} \sum_{L \in \mathcal{L}^{(N)}(\mathbf{S}^3_{\vec{M}})} \sum_{\vec{\mathcal{R}} \in \mathbb{T}_\pi} [\prod_{s \in \mathbf{S}^3_{\vec{M}}} W(\mathcal{R}_s)] \prod_{st \in L} [-U_{st}] \prod_{s't' \in \mathcal{C}(L)} [1 + U_{s't'}].$$

Of course, since each $[1 + U_{s't'}]$ is either 0 or 1, we obtain a further upper bound by taking a product over any convenient subset $\mathcal{C} \subset \mathcal{C}(L)$ on the right-hand side of (A.6). Note that [34, page 687, after (2.11)] shows that we need only consider minimal laces. Henceforth (as in [34]), it will be assumed that all of our laces are minimal (recall that our definition allowed for an extra edge covering the branch point).

We start with (A.3). We note that the effect due to the extra indicator $\mathbb{1}_{\{(x_*, n_*) \hat{\in} \mathcal{R}_{s_*}\}}$ is to add a path to the corresponding vertex coming from $s_* \in \mathbf{S}^3_{\vec{M}}$ within the existing diagram. There are different cases, depending on whether $s_* \in \mathbf{S}^3_{\vec{M}}$ is part of an edge in the lace or not. When it is part of an edge of the lace, it can in fact be part of *several* edges, with a maximum of three. We bound these cases separately.

When $s_* \in \mathbf{S}^3_{\vec{M}}$ is not part of an edge in the lace, we can use self-repellence (essentially dropping some factors of $[1 + U_{s't'}]$ in (A.7)) to upper bound this effect on $\pi_{\vec{M}; n_*}$ by multiplying the bound by the number of possible $s_* \in \mathbf{S}^3_{\vec{M}}$, where s_* cannot lie on the branch $(0, [0, M_0])$, and multiplying by $\hat{t}^{(2)}_{n_* - m_{s_*}}(0)$ where m_{s_*} is the temporal coordinate of $s_* \in \mathbf{S}^3_{\vec{M}}$. Now, $|\hat{t}^{(2)}_m(0)| \leq K$ uniformly in m, and thus we just multiply our bound by $M_1 + M_2 + 1$, which is an upper bound on the number of possible ribs that are not involved in an edge in the lace. This leads to the $\sum_{i=0}^2 M_i$ contribution to the right side of (7.62).

The situation where s_* is part of (at least one) edge of the lace is much more complicated, so let us first recall how to derive the diagrammatic bounds without any extra arms. For this purpose, define $\rho(x) = \rho_{z_c}(x)$, which is a two-point function for paths of unrestricted length.

If $s \in \mathbf{S}^3_{\vec{M}}$ is part of (at least one) edge of the lace, then (A.6) contains a factor $-U_{st}$ (or $-U_{ts}$) for every $t \in \mathbf{S}^3_{\vec{M}}$ for which $st \in L$. For $s \in \mathbf{S}^3_{\vec{M}}$, let y_s denote the spatial location of the root of the rib $\mathcal{R}_s$. A factor $-U_{st}$ or $-U_{ts}$, which are the indicators that $\mathcal{R}_s$ intersects $\mathcal{R}_t$, implies that both y_s and y_t send out an arm to a vertex z_{st} that is in the intersection of $\mathcal{R}_s$ and $\mathcal{R}_t$, but imposes no restriction on the length of that arm. Thus, while the backbone consists of paths of some fixed lengths m for an appropriate m, the factors U_{st} give rise to paths of an unrestricted length, and hence factors of the form $\rho(z_{st} - \cdot)$. When $s \in \mathbf{S}^3_{\vec{M}}$ is part of precisely one edge in the lace, we get a factor $\rho(z_{st} - y_s)$ that originates from the factor U_{st} at the vertex s (together with a similar term at $t \in \mathbf{S}^3_{\vec{M}}$, depending on the number of other $U_{s't}$ in the lace). When $s \in \mathbf{S}^3_{\vec{M}}$ is part of precisely two edges in the lace, this gives rise to a factor $\rho^{(3)}(z_{st_1}-y_s, z_{st_2}-y_s)$ that originates from the factors U_{st_1} and U_{st_2} at the vertex s, where for $\vec{z}\in\mathbb{Z}^{d(r-1)}$, $\rho^{(r)}(\vec{z})=\sum_{\mathcal{T}\ni o,z_1,\dots,z_{r-1}} W(\mathcal{T})$. In particular,

$$\rho^{(3)}(z_1,z_2)=\sum_{\mathcal{T}\ni o,z_1,z_2} W(\mathcal{T})=\rho\sum_{n,n'} t^{(3)}_{n,n'}(z_1,z_2) \tag{A.8}$$

is the (unnormalized) three-point function summed out over the backbone lengths. This can easily be bounded by

$$\rho^{(3)}(z_1,z_2)\le\sum_{w\in\mathbb{Z}^d}\rho(w)\rho(z_1-w)\rho(z_2-w). \tag{A.9}$$

Intuitively this comes from ignoring the mutual-avoidance between the three trees emanating from the branch point w. More formally, note that the right hand side of (A.9) can be expressed as

$$\sum_w\sum_{\mathcal{T}'\ni 0,w} W(\mathcal{T}')\sum_{\mathcal{T}''\ni w,z_1} W(\mathcal{T}'')\sum_{\mathcal{T}'''\ni w,z_2} W(\mathcal{T}''') \tag{A.10}$$

and (A.9) follows since every $\mathcal{T}\ni 0,z_1,z_2$ appears as at least one triple $\mathcal{T}'\cup\mathcal{T}''\cup\mathcal{T}'''$ for some w.

When $s\in\mathbf{S}^3_{\vec{M}}$ is part of precisely three edges in the lace, this gives rise to a factor $\rho^{(4)}(z_{st_1}-y_s,z_{st_2}-y_s,z_{st_3}-y_s)$ that originates from the factors U_{st_1}, U_{st_2} and U_{st_3} at the vertex $s\in\mathbf{S}^3_{\vec{M}}$, where now

$$\rho^{(4)}(z_1,z_2,z_3)=\rho\sum_{n,n',n''} t^{(4)}_{n,n',n''}(z_1,z_2,z_3) \tag{A.11}$$

is the (unnormalized) four-point function summed out over the backbone lengths. This can again easily be bounded in a similar way as in (A.9), but things are a bit more difficult since now there are multiple possible topologies for the subtree leading to the points (z_1,n), (z_2,n') and (z_3,n'').

We now begin describing the effect of an addition of the indicator $\mathbb{1}_{\{(x_*,n_*)\hat{\in}\mathcal{R}_{s_*}\}}$. When $s_*\in\mathbf{S}^3_{\vec{M}}$ is part of precisely one edge in the lace, the factor $\rho(z_{s_*t}-y_{s_*})$ is replaced with $\rho^{(3)}_{n_*-m_{s_*}}(z_{s_*t}-y_{s_*},x_*-y_{s_*})$, where now

$$\rho^{(3)}_m(z_1,z_2)=\rho\sum_n t^{(3)}_{n,m}(z_1,z_2). \tag{A.12}$$

is the (unnormalized) three-point function for which part of the tree has fixed length and part has a length that is being summed out over. Moreover, arguing as for (A.9),

$$\rho^{(3)}_m(z_1,z_2)\le\rho^2\sum_{w\in\mathbb{Z}^d}\sum_{m'\le m} t^{(2)}_{m'}(w)t^{(2)}_{m-m'}(z_2-w)\rho(z_1-w). \tag{A.13}$$

Comparing (A.9) to (A.13) we see that (apart from normalization constants) the effect of the indicator $\mathbb{1}_{\{(x_*,n_*)\hat{\in}\mathcal{R}_{s_*}\}}$ is (a) to add a line to (x_*,n_*) of time length $n_*-m_{s_*}$ and displacement x_*-y_s; and (b) to restrict the time length of parts of the paths in the three-point function.

When $s_* \in \mathbf{S}^3_{\vec{M}}$ is part of precisely two edges in the lace, the factor $\rho^{(3)}(z_{s_*t_1}-y_{s_*}, z_{s_*t_2}-y_{s_*})$ is replaced with a factor $\rho^{(4)}_{n_*-m_{s_*}}(z_{s_*t_1}-y_{s_*}, z_{s_*t_2}-y_{s_*}, x_*-y_{s_*})$, where now

$$\rho^{(4)}_m(z_1,z_2,z_3) = \rho \sum_{m',m''\le m} t^{(4)}_{m',m'',m}(z_1,z_2,z_3). \tag{A.14}$$

In bounding this quantity in a similar way to (A.13) one must consider different possible connection topologies, but again the effect of the indicator $\mathbb{1}_{\{(x_*,n_*)\hat{\in}\mathcal{R}_{s_*}\}}$ is (a) to add a line to (x_*,n_*) of time length $n_*-m_{s_*}$ and displacement $x_*-y_{s_*}$; and (b) to restrict the time length of parts of the paths in the four-point function.

Finally, when $s_* \in \mathbf{S}^3_{\vec{M}}$ is part of precisely three edges in the lace, the factor $\rho^{(4)}(z_{s_*t_1}-y_{s_*}, z_{s_*t_2}-y_{s_*} z_{s_*t_3}-y_{s_*})$ is replaced with a factor $\rho^{(5)}_{n_*-m_{s_*}}(z_{s_*t_1}-y_{s_*}, z_{s_*t_2}-y_{s_*}, z_{s_*t_3}-y_{s_*}, x_*-y_{s_*})$, where now

$$\rho^{(5)}_m(z_1,z_2,z_3,z_4) = \rho \sum_{m',m'',m'''\le m} t^{(4)}_{m',m'',m''',m}(z_1,z_2,z_3,z_4), \tag{A.15}$$

and again the effect of the indicator is (a) and (b) above.

A similar analysis can be performed when we have the two indicators $\mathbb{1}_{\{(x_*,n_*)\hat{\in}\mathcal{R}_{s_*}\}}$ and $\mathbb{1}_{\{(x_{**},n_{**})\hat{\in}\mathcal{R}_{s_{**}}\}}$. The situation where $s_*=s_{**}$ adds further complexity to the situations described above in that *another* extra line to a point $(x_{**}-y_{s_*}, n_{**}-m_{s_*})$ is added. We refrain from giving more details here.

Let us now describe how we handle these modified diagrams. The bounds on the diagrams are obtained by bounding combinations of two-point functions, using the bounds $\|t^{(2)}_m\|_1 \le K$, $\|t^{(2)}_m\|_\infty \le K/(m+1)^{d/2}$, as well as the x-space bound $\rho(x) \le K(\|x\|_2+1)^{-(d-2)}$. The latter x-space bound may be found in Theorem 1.2 of [17]. Essentially, [34, Lemma 5.4] (restated below as Lemma A.1) states that for every $\ell \ge 1$ and all $m_i \ge 1$,

$$\sup_x (t^{(2)}_{m_1} * \cdots * t^{(2)}_{m_\ell} * \rho^{*k})(x) \le C(m_1+\cdots+m_\ell+1)^{-(d-2k)/2}, \tag{A.16}$$

and also gives bounds on the L^1 norm which imply the above bound on $\|t^{(2)}_m\|_1$. (In fact, in [34] the fixed-length 2-point functions are unnormalized and moreover in [34, Lemma 5.4] one is dealing with convolutions of ρ^{*k} and fixed length functions of the form $h_m = z_cD * \rho t_{m-2} * z_cD$ instead of t_m, but the same bounds hold up to constants.) The bound (A.16) is used repeatedly to bound the diagrams in [34], and so it is important to understand how this kind of bound can be used on our modified diagrams.

In Proposition 7.3 we will, for example, wish to bound (A.13) summed over z_2, instead of $\rho^{(2)}(z_1)$. We may use the fact that $\sum_z t^{(2)}_l(z-w) \le K$ uniformly in l, to obtain a bound on this sum of $C\sum_{m'\le m}(t^{(2)}_{m'} * \rho)(z_1)$ (instead of $\rho(z_1)$). The extra line in $C\sum_{m'\le m}(t^{(2)}_{m'} * \rho)(z_1)$ compared to $\rho(z_1)$ will be carried along in this bounding scheme, and, since it is always next to a line in the backbone (i.e. a line of fixed length), it will amount to replacing a line $t^{(2)}_{m_\ell}(v-u)$ for some v,u by $C\sum_{m'\le m}(t^{(2)}_{m_\ell} * t^{(2)}_{m'})(v-u)$. Thus, instead of the left hand side of (A.16) we would have something of the form

$$C\sum_{m'\le m} \sup_x (t^{(2)}_{m_1} * \cdots * t^{(2)}_{m_\ell} * t^{(2)}_{m'} * \rho^{*k})(x) \le C\sum_{m'\le m}(m_1+\cdots+m_\ell+m'+1)^{-(d-2k)/2}, \tag{A.17}$$

since the bound (A.16) still applies to the left hand side of (A.17) (which has an extra path of length m'). As explained in more detail below, following the bounds in [34, Section 6] then implies a bound of the form of the second term in (7.62). We use similar ideas for the bound on $\pi_{\vec{M}}$ with *two* extra arms. As explained in more detail below, this extra arm gives rise to another factor of $\bar{n}$ in our bound.

The remainder of the proof is organised as follows. In Section A.2, we start by investigating the contribution to $\pi_{\vec{M};n}$ from laces on an interval, starting with the single-edge lace in Section A.2.1, and considering first the effect of adding the indicator $\mathbb{1}_{\{(x_*,n_*)\hat{\in}\mathcal{R}_{s_*}\}}$, followed by the combined effect of the indicators $\mathbb{1}_{\{(x_*,n_*)\hat{\in}\mathcal{R}_{s_*}\}}$ and $\mathbb{1}_{\{(x_{**},n_{**})\hat{\in}\mathcal{R}_{s_{**}}\}}$. We continue the analysis with two-edge laces on an interval in Section A.2.2, followed by the general case of laces on an interval with more than two edges in Section A.2.3. Finally in Section A.3 we study general laces on $\mathbf{S}^3_{\vec{M}}$.

A.2 Laces on an interval. Recall Section 1.2.2. Define $h_m(u)$ by

$$h_m(u) = \begin{cases} z_c^2(D * \rho t^{(2)}_{m-2} * D)(u) & \text{if } m \geq 2, \\ z_c D(u) & \text{if } m = 1, \\ \mathbb{1}_{\{u=o\}} & \text{if } m = 0, \end{cases} \tag{A.18}$$

where $\rho t^{(2)}_0(u) = \rho \mathbb{1}_{\{u=o\}}$. For $l \geq 1$ and given $\vec{m} \in \mathbb{Z}^l_+$, we define $h^{(*l)}_{\vec{m}}(x) = (h_{m_1} * \cdots * h_{m_l})(x)$ to be the l-fold spatial convolution of the h_{m_i}, and similarly for $l \in \{1,2,3,4\}$ we define $\rho^{(*l)}(x)$ to be the l-fold spatial convolution of $\rho(\cdot)$ with itself, with $\rho^{(*0)}(x) \equiv \mathbb{1}_{\{x=o\}}$.

Define

$$\rho'(x) = \sum_{m=0}^{\infty} h_m(x) = \mathbb{1}_{\{x=o\}} + z_c D(x) + z_c^2(D * \rho * D)(x). \tag{A.19}$$

It is easy to show that for some $\nu > 0$, $\rho'(x) \leq C_\nu \left(\mathbb{1}_{\{x=o\}} + \mathbb{1}_{\{x\neq o\}}[L^{2-\nu}(|x|+1)^{d-2}]^{-1} \right)$, for each x assuming that $\rho(x)$ satisfies this bound (but with a different constant). In other words, $\rho'(x)$ satisfies [34, (1.4)] with a different constant C.

In this section we consider the case where exactly one of the M_j's, say $M_i \equiv M$, in $\mathbf{S}^3_{\vec{M}}$ is non-zero. This corresponds to $\mathbf{S} = [i,[0,M_i]] = [i,[0,M]]$, and we will denote vertices $[i,\ell]$ in $[i,[0,M]]$ by ℓ.

A.2.1 The single-edge lace. Consider the further special case of the unique lace L on $\mathbf{S}$ consisting of one edge (i.e., the lace $L = \{0M\}$). Thus, we study $\pi^{(1)}_{0,M,0}(v_1)$, where we recall that $\pi^{(1)}_{\vec{M}}(\vec{v})$ denotes the contribution to $\pi_{\vec{M}}(\vec{v})$ from laces containing only one edge. Every other edge on $\mathbf{S}$ is compatible with this lace. In particular, for this L,

$$\prod_{st\in L}[-U_{st}] \prod_{s't'\in\mathcal{C}(L)}[1+U_{s't'}] \leq [-U_{0M}]K[1,M-1]. \tag{A.20}$$

Also $-U_{0M} = \mathbb{1}_{\{\mathcal{R}_0\cap\mathcal{R}_M\neq\varnothing\}} \leq \sum_{z_1\in\mathbb{Z}^d} \mathbb{1}_{\{z_1\in\mathcal{R}_0\}}\mathbb{1}_{\{z_1\in\mathcal{R}_M\}}$. Of course by definition, $o \in \mathcal{R}_0$ and $v_1 \in \mathcal{R}_M$.

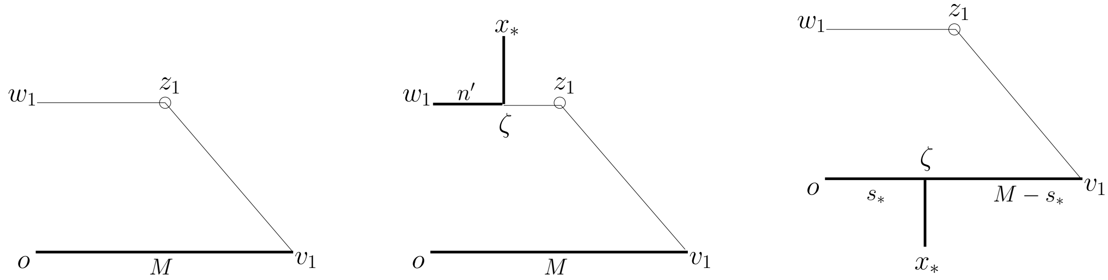

FIG 2. *The diagram arising from a lace containing only one edge, and two of the diagrams arising from adding an extra arm; at $s_* = 0$ and $0 < s_* < M$ respectively. Dark lines correspond to h terms and light lines to ρ terms.*

Let $\mathbf{S}- = \{1, \dots, M-1\}$ and $\Phi^*_{\mathbf{S}-}(\zeta_1, \zeta_2) = \{\phi' : \mathbf{S}- \to \mathbb{Z}^d : \phi'(1) = \zeta_1, \phi'(M-1) = \zeta_2\}$. From (7.33), (A.7), and (A.20), and assuming that $M > 2$, we can (for example) bound $\sum_{\vec{v}} |\pi^{(1)}_{(0,M,0)}(\vec{v})|$ by

$$\sum_{\vec{v}} \mathbb{1}_{\{v_0=v_2=o\}} \sum_{\phi_\pi \in \Phi^*_{\mathbf{S}}(v_1)} W(\phi_\pi(\mathbf{S})) \sum_{\vec{\mathcal{R}} \in \mathbb{T}_\pi} [\prod_{s \in \mathbf{S}^3_{\vec{M}}} W(\mathcal{R}_s)][-U_{0M}] K[1, M-1]$$

(A.21)
$$\le \sum_{z_1} \sum_{\vec{v}} \mathbb{1}_{\{v_0=v_2=o\}} \sum_{\phi_\pi \in \Phi^*_{\mathbf{S}}(v_1)} W(\phi_\pi(\mathbf{S})) \sum_{\vec{\mathcal{R}} \in \mathbb{T}_\pi} [\prod_{s \in \mathbf{S}} W(\mathcal{R}_s)] \mathbb{1}_{\{z_1 \in \mathcal{R}_0\}} \mathbb{1}_{\{z_1 \in \mathcal{R}_M\}} K[1, M-1]$$

$$= \sum_{z_1} \sum_{v_1} \Big[\sum_{\mathcal{R}_0 \ni o, z_1} W(\mathcal{R}_0) \Big] \Big[\sum_{\mathcal{R}_M \ni v_1, z_1} W(\mathcal{R}_M) \Big] \sum_{\zeta_1, \zeta_2} z_c D(\zeta_1) z_c D(v_1 - \zeta_2)$$

(A.22)
$$\times \Big[\sum_{\phi' \in \Phi^*_{\mathbf{S}-}(\zeta_1, \zeta_2)} W(\phi'(\mathbf{S}-)) \sum_{\mathcal{R}_s \in \mathbb{T}(\phi'(s)), s \in \mathbf{S}-} [\prod_{s \in \mathbf{S}-} W(\mathcal{R}_s)] K[1, M-1] \Big].$$

As for A_i in (7.26) this is equal to

(A.23)
$$\sum_{z_1} \sum_{v_1} \rho(z_1 - o) \rho(z_1 - v_1) \sum_{\zeta_1, \zeta_2} z_c^2 D(\zeta_1) \rho t^{(2)}_{M-2}(\zeta_2 - \zeta_1) D(v_1 - \zeta_2)$$

$$\le \sum_{v_1} h_M(v_1) \rho^{(*2)}(v_1 - o)$$

(A.24)
$$\le \sup_{w_1} \sum_{v_1} h_M(v_1) \rho^{(*2)}(v_1 - (o + w_1))$$

(A.25)
$$= \sup_{w} h_M * \rho^{(*2)}(w).$$

The supremum over w_1 may seem odd in the above but this kind of bound will be useful when we will need to bound multiple edge laces. See the first diagram in Figure 2. The same is true when $M \le 2$.

Notation. We will use C_β to denote positive constants of the form $C\beta(L)$ where $\beta(L)$ approaches 0 as $L \to \infty$ and C is a constant which may depend on d but not L, and may change from line to line.

Applying the following lemma (see [34, Lemma 5.4]) with $l = 1$ and $m_1 = M$, we see that the quantity in (A.25) is bounded above by $C_\beta(M+1)^{-(d-4)/2}$.

LEMMA A.1. *The following bounds hold for all $\ell \geq 1$ and $k \in \{1,\dots 4\}$ with $t \equiv \sum_{i=1}^{l} t_i$,*

$$\|h_{\vec{t}}^{(*l)} * \rho^{(*k)}\|_\infty \leq \frac{C_\beta}{(t+1)^{(d-2k)/2}}, \qquad \textit{and} \quad \|h_{\vec{t}}^{(*l)}\|_1 \leq C_l. \tag{A.26}$$

Note that to prove Lemma A.1, the only fact about $\rho(\cdot)$ required (apart from symmetry and translation invariance) is the bound $\rho(x) \leq K(\|x\|_2+1)^{-(d-2)}$ (see [34, (1.4), Lemma 5.8, Prop. 5.9, Lemma 5.10, Sec.5.4.1]). Since $\rho'(\cdot)$ also satisfies this bound it follows that Lemma A.1 remains valid when any ρ is replaced by ρ'.

Recall that $\pi^{(N)}_{(0,M,0);n_*}(\vec{v};x_*)$ denotes the contribution to $\pi_{(0,M,0);n_*}(\vec{v};x_*)$ from laces containing only N edges. As in (A.21), $\sum_{\vec{v},x_*}|\pi^{(1)}_{(0,M,0);n_*}(\vec{v};x_*)|$ is bounded by

(A.27)
$$\sum_{s_*\in\mathbf{S}}\sum_{x_*}\sum_{z_1}\sum_{v_1}\sum_{\phi_\pi\in\Phi^*_{\mathbf{S}}(v_1)} W(\phi_\pi(\mathbf{S}))$$
$$\times \sum_{\vec{\mathcal{R}}\in\mathbb{T}_\pi}\Big[\prod_{s\in\mathbf{S}} W(\mathcal{R}_s)\Big]\mathbb{1}_{\{z_1\in\mathcal{R}_0\cap\mathcal{R}_M\}}K[1,s_*-1]K[s_*+1,M-1]\mathbb{1}_{\{(x_*,n_*)\hat{\in}\mathcal{R}_{s_*}\}},$$

where if $s_* = 0$ or $s_* = M$, $K[1,s_*-1]K[s_*+1,M-1] \equiv K[1,M-1]$.

Proceeding as above, the contribution to the above sum from $s_* = 0$ is

$$\sum_{x_*}\sum_{z_1}\sum_{v_1}\Big[\sum_{\mathcal{R}_0\ni o,z_1} W(\mathcal{R}_0)\mathbb{1}_{\{(x_*,n_*)\in\mathcal{R}_0\}}\Big]\rho(v_1-z_1)h_M(v_1). \tag{A.28}$$

Let ζ denote the "branch point" in $\mathcal{R}_0$ for the connection from o to z_1 and x_* (it could be o, x_*, z_1 or some other point). The connection from o to x_* in $\mathcal{R}_0$ is of length n_*, and the branch point ζ is connected to o by some path of length $n' \leq n_*$ in $\mathcal{R}_0$. Proceeding as above, (A.28) is bounded by

$$\sum_{x_*}\sum_{z_1}\sum_{v_1}\sum_{n'\leq n_*}\sum_{\zeta} h_{n'}(\zeta)h_{n_*-n'}(x_*-\zeta)\rho(\zeta-z_1)\rho(v_1-z_1)h_M(v_1)$$
$$\leq \sum_{z_1}\sum_{v_1}\sum_{n'\leq n_*}\sum_{\zeta} h_{n'}(\zeta)\rho(\zeta-z_1)\Big[\sum_{x_*} h_{n_*-n'}(x_*-\zeta)\Big]\rho(v_1-z_1)h_M(v_1)$$
$$\leq C_1\sum_{n'\leq n_*}\sup_w h_M * \rho^{(*2)} * h_{n'}(w) \leq C'_\beta \sum_{n'\leq n_*}(M+n'+1)^{-(d-4)/2}, \tag{A.29}$$

where we have used Lemma A.1 to bound both the sum over x_* and the supremum over w. See the second diagram in Figure 2. The contribution from $s_* = M$ gives the same bound. We are left to bound

$$\sum_{0<s_*<M}\sum_{x_*}\sum_{z_1}\sum_{v_1}\sum_{\phi_\pi\in\Phi^*_{\mathbf{S}}(v_1)} W(\phi_\pi(\mathbf{S}))$$
$$\times \sum_{\vec{\mathcal{R}}\in\mathbb{T}_\pi}\Big[\prod_{s\in\mathbf{S}} W(\mathcal{R}_s)\Big]\mathbb{1}_{\{z_1\in\mathcal{R}_0\cap\mathcal{R}_M\}}K[1,s_*-1]K[s_*+1,M-1]\mathbb{1}_{\{(x_*,n_*)\hat{\in}\mathcal{R}_{s_*}\}}. \tag{A.30}$$

This is bounded by

$$\sum_{0<s_*<M}\sum_{x_*}\sum_{z_1}\sum_{v_1}\sum_{\zeta} h_{s_*}(\zeta)h_{M-s_*}(v_1-\zeta)h_{n_*-s_*}(x_*-\zeta)\rho(v_1-z_1)\rho(z_1-o)$$

(A.31)

$$= \sum_{0<s_*<M}\sum_{v_1}\sum_{\zeta} h_{s_*}(\zeta)h_{M-s_*}(v_1-\zeta)\left[\sum_{x_*} h_{n_*-s_*}(x_*-\zeta)\right]\rho^{(*2)}(v_1)$$
$$\le C_1 \sum_{0<s_*<M} h^{(*2)}_{(s_*,M-s_*)} * \rho^{(*2)}(o) \le C_\beta M(M+1)^{-(d-4)/2} \le C_\beta M(M+1)^{-(d-4)/2},$$

where we have again used Lemma A.1 to bound the sum over x_*. See the third diagram in Figure 2.

So what we have seen in this section is that the bound on $\pi^{(1)}$ was obtained by using Lemma A.1 and takes the form $C_\beta(M+1)^{-(d-4)/2}$. When we add an extra arm to $\pi^{(1)}$, we again use Lemma A.1, but the bound becomes

$$C_\beta \sum_{n'\le n_*}(M+n'+1)^{-(d-4)/2} + C_\beta M(M+1)^{-(d-4)/2}, \tag{A.32}$$

with the first term arising when the extra arm is added at a vertex of $\mathbf{S}$ coinciding with one of the endpoints of edges of the lace, and the second term coming from the rest (when the extra arm does not coincide with the lace edge), as in the second and third diagrams of Figure 2.

Now consider what happens when we add a second arm, where the two arms arise from indicators $\mathbb{1}_{\{(x_*,n_*)\hat{\in}\mathcal{R}_{s_*}\}}$ and $\mathbb{1}_{\{(x_{**},n_{**})\hat{\in}\mathcal{R}_{s_{**}}\}}$ respectively. There are contributions from $s_{**}=s_*$ and $s_*\ne s_{**}$. In the latter case, without loss of generality assume that $s_*<s_{**}$. If $0<s_*<s_{**}<M$, then (see the first diagram of Figure 3) $h_M(v_1)$ in (A.24) is replaced with

$$\sum_{\zeta_1}\sum_{\zeta_2}\sum_{n'\le n_*\wedge M}\sum_{n''\le(M\wedge n_{**})-n'} h_{n'}(\zeta_1)\sum_{x_*}h_{n_*-n'}(x_*-\zeta_1)h_{n''}(\zeta_2-\zeta_1)$$
$$\times\sum_{x_{**}}h_{n_{**}-n'-n''}(x_{**}-\zeta_2)h_{M-n'-n''}(v_1-\zeta_2)$$
$$\le C\sum_{n''\le n_{**}-n'}[\sum_{n'\le n_*\wedge M} h_{n'} * h_{n''} * h_{M-n'-n''}(v_1)].$$

Note that the sum over n' and n'' replaces the sum over s_*, s_{**} here.

We can now proceed as in (A.31) (performing the "extra" sum over n'' at the very last step) with $h^{(*2)}_{(s_*,M-s_*)}(v_1)$ replaced by $h^{(*3)}_{(n',n'',M-n'-n'')}(v_1)$. These two objects satisfy the same bounds in Lemma A.1, so there is no change to the bounds until we perform the last sum to get an additional factor $\sum_{n''\le n_{**}-n'}1\le n_{**}$ at the end. When $0=s_*<s_{**}<M$, the additional effect of the second arm is to replace $h_M(v_1)$ in (A.28) with

(A.33)

$$\sum_{\zeta_2}\sum_{n''\le n_{**}\wedge M} h_{n''}(\zeta_2)\sum_{x_{**}}h_{n_{**}-n''}(x_{**}-\zeta_2)h_{M-n''}(v_1-\zeta_2) \le C\sum_{n''\le n_{**}\wedge M}[h^{(*2)}_{(n'',M-n'')}(v_1)],$$

and again we get no change to the bounds in (A.29) until performing the final sum over n'' to get an extra factor n_{**}. Similarly when $0=s_*<s_{**}=M$, for the convolution $\sum_{z_1}\rho(z_1-$

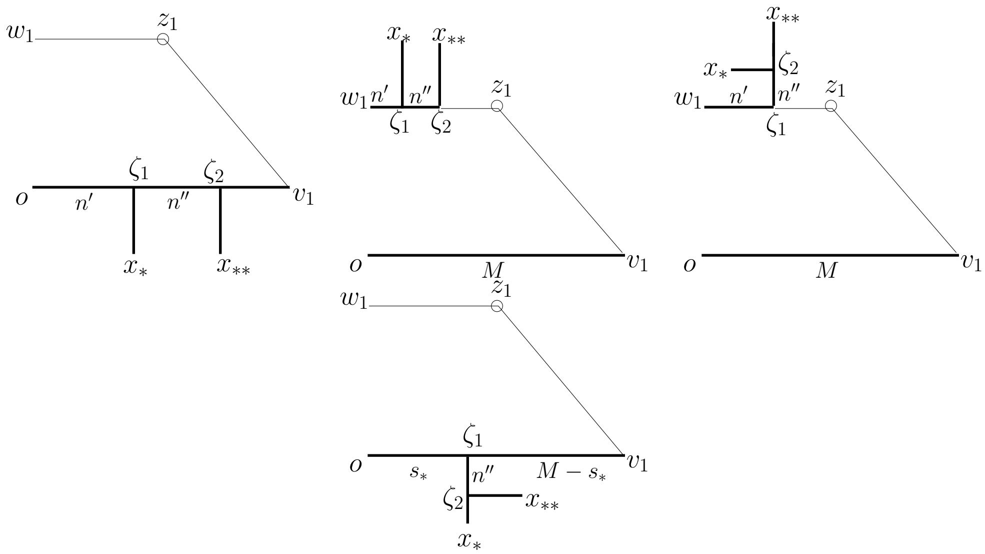

FIG 3. *Some of the diagrams arising from a lace containing only one edge, with two extra arms added at s_* and s_{**}. From left to right the figures correspond to the situations where $0 < s_* < s_{**} < M$, two cases of $s_* = s_{**} = 0$, and $0 < s_* = s_{**} < M$ respectively.*

$w_1)\rho(v_1 - z_1)$ appearing in (A.24), in addition to replacing $\rho(z_1 - w_1)$ by (coming from the first added arm)

$$\sum_{\zeta_1}\sum_{n'} h_{n'}(\zeta_1 - w_1)\sum_{x_*} h_{n_*-n'}(x_* - \zeta_1)\rho(z_1 - \zeta_1), \tag{A.34}$$

we replace $\rho(v_1 - z_1)$ with

(A.35)
$$\sum_{n''\leq n_{**}-M}[\sum_{\zeta_2} h_{n''}(\zeta_2 - v_1)\sum_{x_*} h_{n_*-n'}(x_* - \zeta_1)\rho(z_1 - \zeta_1)] \leq C \sum_{n''\leq n_{**}-M}[h_{n''} * \rho(z_1 - v_1)].$$

This allows us to again proceed as above, performing the sum over n'' last to get an extra factor of n_{**}.

It remains to consider the case $s_* = s_{**}$. If $s_* = s_{**} = 0$ then there is a tree rooted at w_1 that branches (the branching could be degenerate) to the points x_* and x_{**} (and z_1). Either z_1 branches off after the x_*, x_{**} branch point or before (see the second and third diagrams of Figure 3). According to these cases, either $\rho(w_1 - z_1)$ is replaced with

$$\begin{aligned}&\sum_{n''\leq n_{**}}\Big[\sum_{n'\leq n_*}\sum_{\zeta_1} h_{n'}(w_1 - \zeta_1)\sum_{x^*} h_{n_*-n'}(x_* - \zeta_1)\sum_{\zeta_2} h_{n''}(\zeta_2 - \zeta_1)\sum_{x_{**}} h_{n_{**}-n''}(x_{**} - \zeta_2)\rho(z_1 - \zeta_2)\Big]\\ &\quad\leq C\sum_{n''\leq n_{**}}\Big[\sum_{n'\leq n_*} h^{(*2)}_{(n',n'')} * \rho(w_1 - z_1)\Big],\end{aligned}$$

which we can bound as before (getting an extra n_{**} from the sum over n'' at the end), or

$\rho(w_1 - z_1)$ is replaced with

$$\sum_{n''\le n_{**}} \Big[\sum_{n'\le(n_*\wedge n_{**})-n''} \sum_{\zeta_1} h_{n'}(w_1-\zeta_1) \sum_{\zeta_2} h_{n''-n'}(\zeta_2-\zeta_1) \sum_{x_{**}} h_{n_{**}-n''-n'}(x_{**}-\zeta_2) \times \sum_{x_*} h_{n_*-n''-n'}(x^*-\zeta_2)\rho(z_1-\zeta_1)\Big]$$

$$\le C \sum_{n''\le n_{**}} \Big[\sum_{n'\le n_*} \sum_{\zeta_1} h_{n'}(w_1-\zeta_1)\rho(z_1-\zeta_1)\Big],$$

where we have first performed the sums over x_* and x_{**}, and then the sum over ζ_2. We can now proceed as before, summing over n'' at the very last step to get an additional factor of n_{**}. The cases where $s_* = s_{**} = M$ or (see the last diagram in Figure 3) $0 < s_* = s_{**} < M$ can be handled similarly, giving the same extra factor of n_{**}.

In the more general/complicated diagrams to be considered later, the addition of a second arm is handled in the same way, thanks to Lemma A.1 and the fact that we decompose the diagrams *without regard to* s_* *and* s_{**}. We can always perform the final sum over $n'' \le n_{**}$ at the very last step to give the extra factor n_{**}. For this reason, hereafter we explain only the effect on the bounds of adding the first arm.

A.2.2 Two-edge laces on an interval. Consider now the contribution from laces on the interval $\mathbf{S}$ to the sums in Proposition 7.3, or their armless analogues consisting of exactly $N = 2$ edges. These are laces of the form $L = \{0(t_1+t_2), t_1M\}$ with $0 < t_1 < M$ and $t_1 \le (t_1+t_2) < M$, and a sum over all such L is equivalent to summing over t_1 and t_2. Thus,

$$\begin{aligned}
&\sum_{L\in\mathcal{L}^{(2)}(\mathbf{S})} \prod_{st\in L}[-U_{st}] \prod_{s't'\in\mathcal{C}(L)}[1+U_{s't'}] \\
&\quad= \sum_{t_1=1}^{M-1}\sum_{t_2=0}^{M-1-t_1} U_{0(t_1+t_2)}U_{t_1M} \prod_{s't'\in\mathcal{C}(L)}[1+U_{s't'}] \\
&\quad\le \sum_{t_1=1}^{M-1}\sum_{t_2=1}^{M-1-t_1} U_{0(t_1+t_2)}U_{t_1M} K[1,t_1-1]K[t_1+1,t_1+t_2-1]K[t_1+t_2+1,M-1] \\
&\qquad+ \sum_{t_1=1}^{M-1} U_{0t_1}U_{t_1M}K[1,t_1-1]K[t_1+1,M-1],
\end{aligned} \tag{A.36}$$

since each of the intervals $[1,t_1-1]$, $[t_1+1,t_1+t_2-1]$, $[t_1+t_2+1,M-1]$ is covered by one of the edges in the lace (so any edge in these intervals is compatible with the lace). As above $-U_{0t} \le \sum_{z_1\in\mathbb{Z}^d} \mathbb{1}_{\{z_1\in\mathcal{R}_0\}}\mathbb{1}_{\{z_1\in\mathcal{R}_t\}}$, and similarly $-U_{tM} \le \sum_{z_2\in\mathbb{Z}^d} \mathbb{1}_{\{z_2\in\mathcal{R}_t\}}\mathbb{1}_{\{z_2\in\mathcal{R}_M\}}$.

Now let $t_3 = M-(t_1+t_2)$ and use the fact that $W(\phi_\pi(\mathbf{S}))$ factors into the weights of the embeddings of the intervals $[0,t_1]$, $[t_1,t_1+t_2]$, $[t_1+t_2,M]$, and proceed as in Section A.2.1

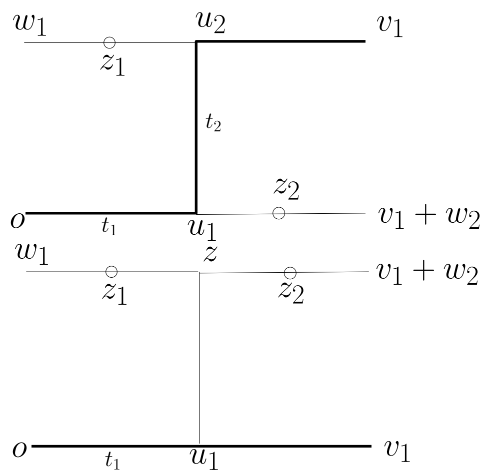

FIG 4. *The opened diagrams corresponding to 2-edge laces on a single branch, with $t_2 > 0$ and $t_2 = 0$ respectively. Dark lines correspond to h terms and light lines to ρ terms.*

to get a bound on $\sum_{\vec{v}} |\pi^{(2)}_{(0,M,0)}(\vec{v})|$ of the form (see (7.33) and (A.7))

$$\sum_{t_1=1}^{M-1} \sum_{t_2=1}^{M-1-t_1} \sum_{v_1,u_1,u_2} \sum_{z_1,z_2} h_{t_1}(u_1) h_{t_2}(u_2-u_1) h_{t_3}(v_1-u_2) \rho(z_1-u_2)\rho(z_1-o)\rho(z_2-u_1)\rho(v_1-z_2) \tag{A.37}$$

$$+ \sum_{t_1=1}^{M-1} \sum_{t_2} \mathbb{1}_{\{t_2=0\}} \sum_{v_1,u_1,z} \sum_{z_1,z_2} h_{t_1}(u_1) h_{t_3}(v_1-u_1) \rho(z_1-z)\rho(z_1-o)\rho(z-u_1)\rho(v_1-z_2)\rho(z_2-z) \tag{A.38}$$

$$\leq \sum_{t_1=1}^{M-1} \sum_{t_2=1}^{M-1-t_1} \sup_{w_1,w_2} \sum_{v_1,u_1,u_2} h_{t_1}(u_1) h_{t_2}(u_2-u_1) h_{t_3}(v_1-u_2) \rho^{(*2)}(u_2-(o+w_1))\rho^{(*2)}(v_1+w_2-u_1) \tag{A.39}$$

$$+ \mathbb{1}_{\{t_2=0\}} \sum_{t_1=1}^{M-1} \sup_{w_1,w_2} \sum_{v_1} \sum_{u_1,z} h_{t_1}(u_1) h_{t_3}(v_1-u_1) \rho^{(*2)}(z-w_1)\rho^{(*2)}(v_1+w_2-z)\rho(z-u_1), \tag{A.40}$$

where in (A.38) we have used the fact that if $z_1 \in \mathcal{R}_{t_1}$ and $z_2 \in \mathcal{R}_{t_1}$ then there exists some branch point z from which these connections in $\mathcal{R}_{t_1}$ take place. The quantities (A.39) and (A.40) can be represented in terms of diagrams as in Figure 4.

We decompose the diagram according to which of t_1, t_2, t_3 are comparable to M. In particular, at least one of t_1, t_2, t_3 needs to be at least $M/3$, and we decompose depending on which it is. We start by bounding (A.40), which corresponds to the case where $t_2 = 0$. We then continue to deal with the general term in (A.39), which corresponds to general $t_2 > 0$.

The case (A.40) with $t_2 = 0$ is easiest. The contribution to (A.40) from $t_3 = M - t_1 \geq t_1$,

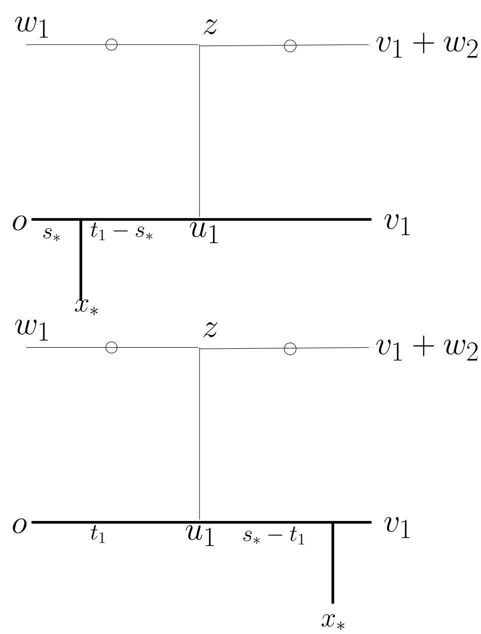

FIG 5. *Two diagrams corresponding to a 2-edge lace (with $t_2 = 0$) on a single branch, with an extra arm added at s_*, where $0 < s_* < t_1$ and $t_1 < s_* < M$ respectively.*

so that $t_1 \le M/2$, is

$$\begin{aligned}&\sum_{t_1 \le M/2} \sup_{w_1,w_2} \sum_{u_1,z} h_{t_1}(u_1)\rho^{(*2)}(z-w_1)\rho(z-u_1)\sum_{v_1} h_{t_3}(v_1-u_1)\rho^{(*2)}(v_1+w_2-(u_1+z-u_1))\\ &\le \sum_{t_1 \le M/2} \sup_{w_1,w_2} \sum_{u_1,z} h_{t_1}(u_1)\rho^{(*2)}(z-w_1)\rho(z-u_1)\\ &\qquad\qquad \times \sup_{z'} \sum_{v_1} h_{t_3}(v_1-u_1)\rho^{(*2)}(v_1+w_2-(u_1+z')).\end{aligned} \tag{A.41}$$

We first bound the sup over z' by $C_\beta(t_3+1)^{-(d-4)/2} \le C'_\beta(M+1)^{-(d-4)/2}$ using Lemma A.1. Then using Lemma A.1 again we write

$$\sum_{t_1 \le M/2} \sup_{w_1} \sum_{u_1,z} h_{t_1}(u_1)\rho^{(*2)}(z-w_1)\rho(z-u_1) \le \sum_{t_1 \le M/2} C'_\beta(t_1+1)^{-(d-6)/2} \le C'_\beta. \tag{A.42}$$

Thus (A.41) is bounded by $(C'_\beta)^2(M+1)^{-(d-4)/2}$. By translating the diagram by $-v_1$ (so shifting the origin to the position of the old v_1) we can obtain the same bound for the contribution to (A.40) from $t_1 \ge t_3$. This completes the bounds on (A.40) with $t_2 = 0$.

We proceed to adapt the bounds on (A.40) with $t_2 = 0$ due to the contribution where we have an extra indicator $\mathbb{1}_{\{(x_*,n_*)\hat{\in}\mathcal{R}_{s_*}\}}$. We refer to this as 'adding an extra arm'. Let $E \equiv \{0, t_1, M\}$ denote the vertices that are part of at least one edge. When we add an extra arm, the quantity corresponding to (A.40) includes terms where $s_* \in E$ and where $s_* \notin E$.

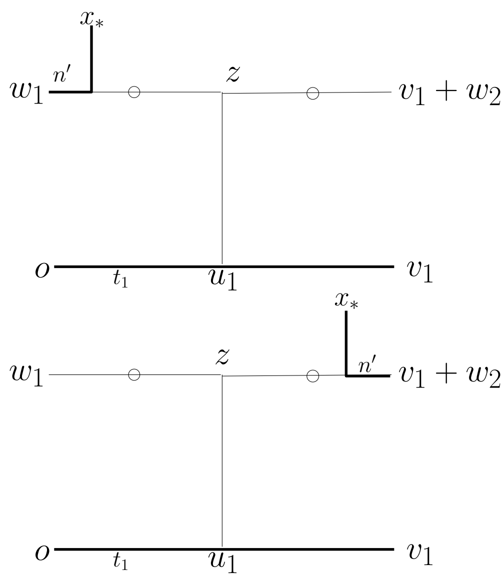

FIG 6. *Two diagrams corresponding to a 2-edge lace with $t_2 = 0$ on a single branch, with an extra arm added at $s_* = 0$ or $s_* = M$ respectively.*

For $s_* \notin E$, and $t_2 = 0$, see Figure 5. After again taking the summation over x_* (e.g. as in (A.31)) we are left with terms (summed over $s_* \notin E$) of the form

(A.43)
$$\sum_{t_1=1}^{M-1} \sup_{w_1,w_2} \sum_{v_1,u_1,z} h^{(*2)}_{(s_*,t_1-s_*)}(u_1) h_{t_3}(v_1-u_1)\rho^{(*2)}(z-w_1)\rho^{(*2)}(v_1+w_2-z)\rho(z-u_1), \quad \text{or,}$$

(A.44)
$$\sum_{t_1=1}^{M-1} \sup_{w_1,w_2} \sum_{v_1,u_1,z} h_{t_1}(u_1) h^{(*2)}_{(s_*-t_1,t_3)}(v_1-u_1)\rho^{(*2)}(z-w_1)\rho^{(*2)}(v_1+w_2-z)\rho(z-u_1),$$

depending on whether $s_* < t_1$ or $s_* > t_1$. Decomposing the diagrams exactly as we did without the extra arm (i.e. ignoring the fact that we have $h^{(*2)}$ instead of h) gives us a bound of

$$\sum_{s_*\notin E} (C'_\beta)^2 (M+1)^{-(d-4)/2} \leq (C'_\beta)^2 \sum_{s_*=1}^{M} (M+1)^{-(d-4)/2} \leq (C'_\beta)^2 (M+1)^{-(d-6)/2}. \tag{A.45}$$

This demonstrates a theme that will appear more generally. When we consider diagrams with an extra arm added, we will always decompose the diagram as in [34] (i.e., ignoring the extra arm). When the extra arm does not fall at an endpoint of an edge of the lace (i.e. when $s_* \notin E$), our job will be rather easy, as above.

A bit more difficult are the cases where $s_* \in E$. This situation gives rise to 1+1+3 terms

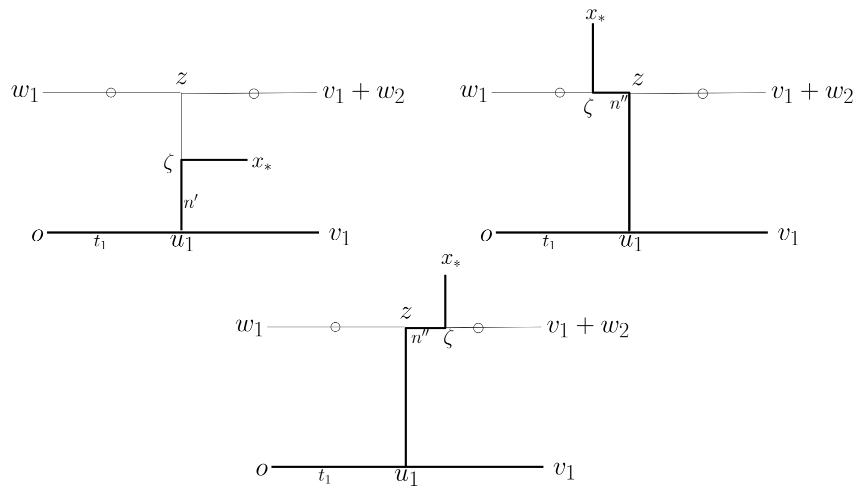

Fig 7. *Three diagrams corresponding to a 2-edge lace with $t_2 = 0$ on a single branch, with an extra arm added at $s_* = t_1$.*

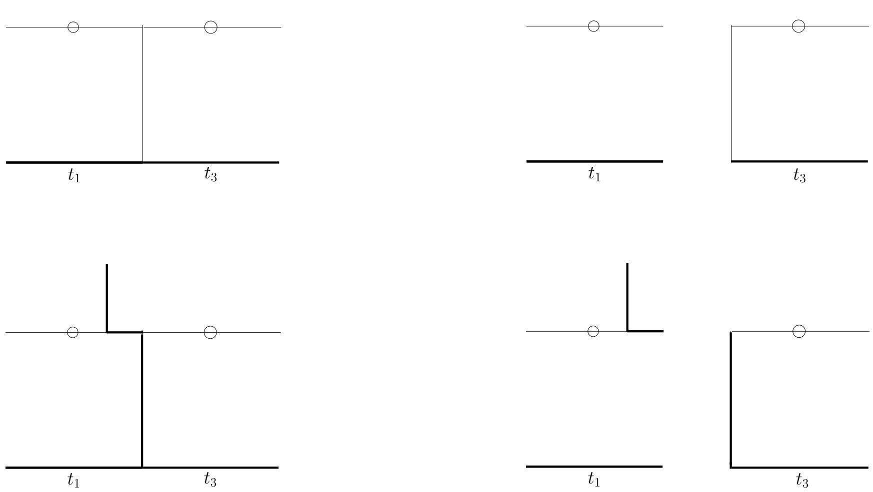

Fig 8. *On the left are two diagrams (see the rightmost diagrams of Figures 4 and 7) corresponding to a 2-edge lace without and with an extra arm added. The same decomposition is used in each case. The decomposition when $t_1 \geq M - t_1$ appears on the right.*

(corresponding to $s_* = 0, M, t_1$ respectively, see e.g. Figures 6 and 7), giving

$$\sum_{n'\le n_*}\sum_{t_1=1}^{M-1}\sup_{w_1,w_2}\sum_{v_1}\sum_{u_1,z} h_{t_1}(u_1)h_{t_3}(v_1-u_1)\rho^{(*2)}(v_1+w_2-z)\rho(z-u_1) \tag{A.46}$$

$$\times\sum_{\zeta}\rho^{(*2)}(z-\zeta)h_{n'}(w_1-\zeta)\sum_{x_*}h_{n_*-n'}(x_*-\zeta) \tag{A.47}$$

$$+\sum_{n'\le n_*-M}\sum_{t_1=1}^{M-1}\sup_{w_1,w_2}\sum_{v_1}\sum_{u_1,z} h_{t_1}(u_1)h_{t_3}(v_1-u_1)\rho^{(*2)}(w_1-z)\rho(z-u_1) \tag{A.48}$$

$$\times\sum_{\zeta}\rho^{(*2)}(z-\zeta)h_{n'}(\zeta-v_1-w_2)\sum_{x_*}h_{n_*-M-n'}(x_*-\zeta) \tag{A.49}$$

$$+\sum_{t_1=1}^{M-1}\sum_{n'\le n_*-t_1}\sup_{w_1,w_2}\sum_{v_1}\sum_{u_1,z} h_{t_1}(u_1)h_{t_3}(v_1-u_1)\rho^{(*2)}(z-w_1)\rho^{(*2)}(v_1+w_2-z) \tag{A.50}$$

$$\times\sum_{\zeta}h_{n'}(\zeta-u_1)\rho(z-\zeta)\sum_{x_*}h_{n_*-n'-t_1}(x_*-\zeta) \tag{A.51}$$

$$+\sum_{t_1=1}^{M-1}\sup_{w_1,w_2}\sum_{v_1}\sum_{u_1} h_{t_1}(u_1)h_{t_3}(v_1-u_1)\sum_{\zeta}\rho^{(*2)}(\zeta-w_1) \tag{A.52}$$

$$\times\sum_{z}\sum_{n'\le n_*-t_1}\sum_{n''\le n_*-t_1-n'} h_{n'}(z-u_1)\,\rho^{(*2)}(v_1+w_2-z)h_{n''}(\zeta-z)\sum_{x_*}h_{n_*-n'-n''-t_1}(x_*-\zeta) \tag{A.53}$$

$$+\sum_{t_1=1}^{M-1}\sup_{w_1,w_2}\sum_{v_1}\sum_{u_1} h_{t_1}(u_1)h_{t_3}(v_1-u_1)\sum_{\zeta}\rho^{(*2)}(v_1+w_2-\zeta) \tag{A.54}$$

$$\times\sum_{z}\sum_{n'\le n_*-t_1}\sum_{n''\le n_*-t_1-n'} h_{n'}(z-u_1)\,\rho^{(*2)}(z-w_1)h_{n''}(\zeta-z)\sum_{x_*}h_{n_*-n'-n''-t_1}(x_*-\zeta). \tag{A.55}$$

Here the terms (A.50)-(A.55) arise according to where the branch point for the connection from u_1 to z_1, z_2 is relative to the branch point for the connection to x_*. In each case above, the sum over x_* can be replaced by C_l using Lemma A.1. We then proceed to decompose the diagrams exactly as in [34], depending on whether $t_1 \ge M/2$ or $t_1 \le M/2$. See Figure 8 for an example in the case of (A.54)-(A.55) when $t_1 \ge M/2$.

We can bound (A.46)-(A.47) as follows. Firstly (after performing the sum over x_*), the contribution from small t_1 is at most

$$\sum_{n'\le n_*}\sum_{t_1\le M/2}\sup_{w_1,w_2}\sum_{u_1,z}h_{t_1}(u_1)\rho(z-u_1)\sum_{\zeta}\rho^{(*2)}(z-\zeta)h_{n'}(w_1-\zeta) \tag{A.56}$$

$$\times\sum_{v_1}h_{t_3}(v_1-u_1)\rho^{(*2)}(v_1+w_2-z) \tag{A.57}$$

Now note that (A.57) is equal to

$$h_{t_3}*\rho^{(*2)}(z-w_2-u_1)\le\frac{C_\beta}{(t_3+1)^{(d-4)/2}}, \tag{A.58}$$

by Lemma A.1. Again using Lemma A.1 the sup in (A.56) is in fact

$$\sup_{w_1}\sum_{u_1,z}h_{t_1}*h_{n'}*\rho^{(*3)}(w_1)\le\frac{C_\beta}{(t_1+n'+1)^{(d-6)/2}}. \tag{A.59}$$

When $t_1>M/2$ we translate the entire diagram by $-v_1$ and make corresponding changes of variables in the sums, then repeat the above analysis to obtain a bound on (A.46)-(A.47) of

(A.60)

$$\sum_{n'\le n_*}\Big[\sum_{M>t_1>M/2}\frac{C_\beta}{(t_1+n'+1)^{(d-4)/2}}\frac{C_\beta}{(t_3+1)^{(d-6)/2}}+\sum_{t_1\le M/2}\frac{C_\beta}{(t_1+n'+1)^{(d-6)/2}}\frac{C_\beta}{(t_3+1)^{(d-4)/2}}\Big]$$

$$\le\sum_{n'\le n_*}\Big[\frac{C_\beta}{(M+n'+1)^{(d-4)/2}}\sum_{t_1\ge M/2}\frac{C_\beta}{(M-t_1+1)^{(d-6)/2}}+\frac{C_\beta}{(M+1)^{(d-4)/2}}\sum_{t_1\le M/2}\frac{C_\beta}{(t_1+n'+1)^{(d-6)/2}}\Big]$$

$$\le\sum_{n'\le n_*}\frac{(C_\beta)^2}{(M+n'+1)^{(d-4)/2}}+\frac{C_\beta}{(M+1)^{(d-4)/2}}\sum_{n'\le n_*}\frac{C_\beta}{(n'+1)^{(d-8)/2}}$$

(A.61)

$$\le\sum_{n'\le n_*}\frac{(C_\beta)^2}{(M+n'+1)^{(d-4)/2}}+\frac{(C_\beta)^2}{(M+1)^{(d-4)/2}}n_*^p.$$

The second and third terms are similar and lead to identical bounds on (A.48)-(A.51).

Using the fact that $\sum_{n'\le K}h_{n'}(x)\le\rho'(x)$ and the above translation and changes of variables for $t_1>M/2$, we see that the bound on (A.52)-(A.53) is

$$\sum_{n''\le n_*}\sum_{t_1<M}\sup_{w_1,w_2}\sum_{u_1,z}h_{t_1}(u_1)h_{t_3}*\rho^{(*2)}(z-u_1-w_2)\rho'(z-u_1)h_{n''}*\rho^{(*2)}(z-w_1)$$

$$\le\sum_{n''\le n_*}\sum_{M>t_1>M/2}\frac{C_\beta}{(n''+t_1+1)^{(d-4)/2}}\frac{C_\beta}{(t_3+1)^{(d-6)/2}}+\sum_{t_1\le M/2}\frac{C_\beta}{(t_3+1)^{(d-4)/2}}\frac{C_\beta}{(n''+t_1+1)^{(d-6)/2}}$$

$$\le\sum_{n''\le n_*}\sum_{t\le M}\Bigg[\frac{C_\beta}{(M+n''+1)^{(d-4)/2}}\frac{C_\beta}{(t+1)^{(d-6)/2}}+\frac{C_\beta}{(M+1)^{(d-4)/2}}\frac{C_\beta}{(t+n''+1)^{(d-6)/2}}\Bigg]$$

$$\le\sum_{n''\le n_*}\frac{(C_\beta)^2}{(M+n''+1)^{(d-4)/2}}+\frac{(C_\beta)^2}{(M+1)^{(d-4)/2}}n_*^p,$$

so also satisfies the bound (A.61). Similarly, (A.54)-(A.55) also satisfies (A.61).

Let us now turn to the case (A.39), which corresponds to general $t_2 > 0$. We again start by analyzing the situation without the extra arms caused by the indicators $\mathbb{1}_{\{(x_*,n_*)\hat{\in}\mathcal{R}_{s_*}\}}$ and $\mathbb{1}_{\{(x_{**},n_{**})\hat{\in}\mathcal{R}_{s_{**}}\}}$. There is complete symmetry between t_1 and t_3 so we shall only consider $t_1 \le t_3$. The sums over t_1, t_2 can be broken up into regions $E^{(2)} = \{\vec{t} : t_1 + t_2 \le 2M/3\}$, $F^{(2)} = \{\vec{t} : t_2 + t_3 \le 2M/3\}$ and $G^{(2)} = \{\vec{t} : t_2 \ge t_1 \vee t_3 = t_3\}$. Note that these regions have considerable overlap, so we get an upper bound when we add up the contributions from sums over t_1, t_2 in these regions. With $t_3 = M - t_1 - t_2$, the contribution to (A.39) from $\vec{t} \in E^{(2)}$ is bounded by

$$\begin{aligned}
&\sum_{t_1=1}^{M-1}\sum_{t_2=1}^{M-1-t_1} \mathbb{1}_{\{t_3\ge M/3\}} \sup_{w_1} \sum_{u_1,u_2} h_{t_1}(u_1)h_{t_2}(u_2-u_1)\rho^{(*2)}(u_2-(o+w_1))\\
&\qquad\qquad \times \sup_{u,u',w_2}\sum_{v_1} h_{t_3}(v_1-(u+u'))\rho^{(*2)}(v_1+w_2-u)\\
&\quad\le \sum_{t_1=1}^{M-1}\sum_{t_2=1}^{M-1-t_1} \mathbb{1}_{\{t_3\ge M/3\}} \frac{C_\beta}{(t_1+t_2+1)^{(d-4)/2}}\frac{C_\beta}{(t_3+1)^{(d-4)/2}}\\
&\quad\le \frac{C_\beta}{(M+1)^{(d-4)/2}}\sum_{t_1=1}^{M-1}\sum_{t_2=1}^{M-1-t_1}\frac{C_\beta}{(t_1+t_2+1)^{(d-4)/2}} \le \frac{(C_\beta)^2}{(M+1)^{(d-4)/2}}.
\end{aligned}\tag{A.62}$$

The same bound can be obtained for $\vec{t} \in F^{(2)}$ by symmetry and translation invariance (indeed, we can translate by $-v_1$ and the condition $t_1 \le t_3$ has not been used yet). For $\vec{t} \in G^{(2)}$ we do use the condition $t_3 \ge t_1$ to get a bound of

$$\begin{aligned}
&\sum_{t_2\ge M/3}\sum_{t_1: t_1\le t_3}\frac{C_\beta}{(t_1+t_2+1)^{(d-4)/2}}\frac{C_\beta}{(t_3+1)^{(d-4)/2}} \le \frac{C_\beta}{(M+1)^{(d-4)/2}}\sum_{t_3\le M}\sum_{t_1\le t_3}\frac{C_\beta}{(t_3+1)^{(d-4)/2}}\\
&\le \frac{C_\beta}{(M+1)^{(d-4)/2}}\sum_{t_3\le M}\frac{C_\beta}{(t_3+1)^{(d-6)/2}} \le \frac{(C_\beta)^2}{(M+1)^{(d-4)/2}}.
\end{aligned}\tag{A.63}$$

We note that the case $G^{(2)}$ is not needed in our induction (in N) argument carried out in Section A.2.3 below for $N > 2$ and extra arms. This will simplify that argument a bit.

Now consider adding an extra arm to (A.39) due to the indicator $\mathbb{1}_{\{(x_*,n_*)\hat{\in}\mathcal{R}_{s_*}\}}$. If the arm is added at $s_* \notin E = \{0, t_1, t_1+t_2, M\}$ we can perform exactly the same decompositions as above, then sum over s_* to obtain the bound (A.45). Consider now the contribution from $s_* \in E$. Again by symmetry we may assume $t_3 \ge t_1$ throughout. If $s_* = 0$, the contribution is at most (we suppress the restrictions $t_i \ge 1$ and $t_3 \ge t_1$)

$$\begin{aligned}
&\sum_{t_1+t_2<M}\sum_{n'\le n_*}\sup_{w_1}\sum_{u_1,u_2,\zeta} h_{t_1}(u_1)h_{n'}(w_1-\zeta)\sum_{x_*}h_{n_*-n'}(x_*-\zeta)h_{t_2}(u_2-u_1)\rho^{(*2)}(\zeta-u_2)\\
&\qquad \times \sup_{w_2}\sum_{v_1} h_{t_3}(v_1-u_2)\rho^{(*2)}(v_1+w_2-u_1)\\
&\quad \le \sum_{t_1+t_2<M}\sum_{n'\le n_*} C_\beta(t_1+t_2+n'+1)^{(d-4)/2}(t_3+1)^{-(d-4)/2}.
\end{aligned}\tag{A.64}$$

If $s_* = t_1 + t_2$ a similar argument gives the same bound. If $s_* = t_1$ we get at most

$$\sum_{t_1+t_2<M} \sum_{n'\le n_*-t_1} \sup_{w_1} \sum_{u_1,u_2} h_{t_1}(u_1)h_{t_2}(u_2-u_1)\rho^{(*2)}(u_2-w_1)$$
$$\times \sup_{w_2} \sum_{\zeta,v_1} h_{n'}(\zeta-u_1)h_{t_3}(v_1-u_2)\rho^{(*2)}(v_1+w_2-\zeta)\Big(\sum_{x_*} h_{n_*-n'-t_1}(x_*-\zeta)\Big)$$
$$\le \sum_{t_1+t_2<M} \sum_{n'\le n_*} C_\beta (n'+t_3+1)^{-(d-4)/2}(t_1+t_2+1)^{(d-4)/2}. \tag{A.65}$$

If $s_* = M$ a similar argument gives the same bound. So the total contribution we need to bound is (consider the cases $t_1 + t_2 \ge M/2$ and $t_3 \ge M/2$ separately)

$$C_\beta\Big[\sum_{t_1+t_2<M:t_3\ge t_1} \sum_{n'\le n_*} (n'+t_3+1)^{-(d-4)/2}(t_1+t_2+1)^{-(d-4)/2}$$
$$+ (t_3+1)^{-(d-4)/2}(t_1+t_2+n'+1)^{-(d-4)/2}\Big]$$
$$\le C_\beta\Big[(M+1)^{-(d-4)/2} \sum_{t_3<M} \sum_{n'\le n_*} (n'+t_3+1)^{-(d-6)/2}$$
$$+ \sum_{n'\le n_*} (n'+M+1)^{-(d-4)/2} \sum_{t_1+t_2<M} (t_1+t_2+1)^{-(d-4)/2}$$
$$+ (M+1)^{-(d-4)/2} \sum_{t_1+t_2<M} \sum_{n'\le n_*} (t_1+t_2+n'+1)^{-(d-4)/2}$$
$$+ \sum_{n'\le n_*} (n'+M+1)^{-(d-4)/2} \sum_{t_3<M} (t_3+1)^{-(d-6)/2}\Big]$$
$$\le C_\beta\Big[(M+1)^{-(d-4)/2} n_*^p + \sum_{n'\le n_*} (M+n'+1)^{-(d-4)/2}\Big],$$

again leading to the bound in (A.61). This proves (7.62) in Proposition 7.3 for laces on an interval in the case where $N = 2$. As we have discussed at the end of Section A.2.1, adding a second extra arm can be handled relatively easily, and gives (7.63) for laces on an interval in the case where $N = 2$.

A.2.3 Laces on an interval with more than two edges. Let

$$\mathcal{H}_M^{(N)} = \Big\{\vec{t} \in \mathbb{Z}_+^{2N-1} : \sum_{i=1}^{2N-1} t_i = M,\ t_{2j} \ge 0, t_{2j-1} > 0\Big\}. \tag{A.66}$$

Then [34, Lemma 5.7] shows that

$$\sum_{\vec{v}} \pi^{(N)}_{(0,M,0)}(\vec{v}) \le \sum_{v_1} \sum_{\vec{t}\in\mathcal{H}_M^{(N)}} M_{\vec{t}}^{(N)}(0,0,v_1,0), \tag{A.67}$$

where $M_{\vec{t}}^{(N)}(a,b,x,y)$ is defined recursively as follows. Firstly,

$$M_{\vec{t}}^{(1)}(a,b,x,y) \equiv h_t(x-a)\rho^{(*2)}\big((x+y)-(a+b)\big), \tag{A.68}$$

and

$$A_{t_1,t_2}(a,b,x,y) \equiv \begin{cases} h_{t_1}(x+y-a)h_{t_2}\big(x-(x+y)\big)\rho^{(*2)}\big((a+b)-x\big), & t_2 \neq 0, \\ h_{t_1}(x-a)\rho((x+y)-x)\rho^{(*2)}\big((a+b)-(x+y)\big), & t_2 = 0. \end{cases} \tag{A.69}$$

Here we have corrected a misprint in the corresponding (5.6) and (5.7) in [34]. Then, for $N > 1$, define

$$M^{(N)}_{\vec{t}}(a,b,x,y) \equiv \sum_{u,v} A_{t_1,t_2}(a,b,u,v) M^{(N-1)}_{(t_3,\dots,t_{2N-1})}(u,v,x,y). \tag{A.70}$$

It is shown in [34, Lemma 5.6] that also

$$M^{(N)}_{\vec{t}}(a,b,x,y) = \sum_{u,v} M^{(N-1)}_{(t_1,\dots,t_{2N-3})}(a,b,u,v) A_{t_{2N-1},t_{2N-2}}(x,y,u,v) \tag{A.71}$$

This enables a proof in [34, Section 5.1, Case 1] by induction on N that

$$\sum_{\vec{t}\in\mathcal{H}^{(N)}_M} \sup_{a,b,y} \sum_x M^{(N)}_{\vec{t}}(a,b,x,y) \leq \frac{(C_\beta)^N}{(M+1)^{(d-4)/2}}. \tag{A.72}$$

We now explain how to carry out the induction. We start by initializing the induction hypothesis. We have seen that this is true for $N = 1$, by Lemma A.1 and for $N = 2$ by decomposing the diagrams and using Lemma A.1. The induction argument involves dividing $\mathcal{H}^{(N)}_M$ up into cases $E^{(N)} = \{\vec{t} : t_1 + t_2 \leq M/2\}$ and $F^{(N)} = \{\vec{t} : t_{2N-2} + t_{2N-1} \leq M/2\}$, which contain all possible $\vec{t}$ when $N \geq 3$. In the former case, we decompose the diagram by breaking off the first piece, and using the fact that the remainder has backbone length at least $M/2$, to obtain bounds

$$\begin{aligned} &\frac{(C_\beta)^{N-1}}{((M/2)+1)^{(d-4)/2}} \left[\sum_{t_1} \sum_{t_2>1} \frac{C_\beta}{(t_1+t_2+1)^{(d-4)/2}} + \sum_{t_1} \frac{C_\beta}{(t_1+1)^{(d-6)/2}} \right] \\ &\qquad \leq \frac{(C_\beta)^N}{(M+1)^{(d-4)/2}}, \end{aligned} \tag{A.73}$$

and the second case gives the same bound by a symmetric argument. This advances the induction hypothesis, and thus completes the proof of (A.72).

When we add an extra arm we again make the same decompositions. Either the extra arm is on the piece that is broken off, or on the remainder of the diagram. We first describe how the induction works when the added arm is not at the endpoint of any edge in the lace.

For $t_1 \geq 2$ and $1 \leq s' \leq t_1 - 1$, define

$$M^{(1);1}_{t_1;s'}(a,b,x,y) \equiv h_{s'} * h_{t_1-s'}(x-a)\rho^{(*2)}\big((x+y)-(a+b)\big), \tag{A.74}$$

and

$$\begin{aligned} &A_{t_1,t_2;1,s'}(a,b,x,y) \\ &\quad = \begin{cases} h_{s'} * h_{t_1-s'}(x+y-a)h_{t_2}\big(x-(x+y)\big)\rho^{(*2)}\big((a+b)-x\big), & t_2 \neq 0, \\ h_{s'} * h_{t_1-s'}(x-a)\rho((x+y)-x)\rho^{(*2)}\big((a+b)-(x+y)\big), & t_2 = 0. \end{cases} \end{aligned} \tag{A.75}$$

Similarly if $t_2 \geq 2$, for $1 \leq s' \leq t_2 - 1$ define

$$\begin{aligned}(A.76)\qquad & A_{t_1,t_2;2,s'}(a,b,x,y)\\ & \equiv h_{t_1}(x+y-a)h_{s'} * h_{t_2-s'}\big(x-(x+y)\big)\rho^{(*2)}\big((a+b)-x\big), \quad t_2 \neq 0.\end{aligned}$$

Note that $A_{;1}$ (resp. $A_{;2}$) corresponds to adding a vertex on an odd (respectively, even) piece of the backbone. Now define for $N > 1$ and $\ell = 1, 2$,

$$(A.77)\qquad M^{(N);\ell}_{\vec{t};s'}(a,b,x,y) \equiv \sum_{u,v} A_{t_1,t_2;\ell,s'}(a,b,u,v) M^{(N-1)}_{(t_3,\dots,t_{2N-1})}(u,v,x,y),$$

and for $N > 1$ and $1 < j < N$ define $M^{(N);2j-1}_{\vec{t};s'}$ and $M^{(N);2j-2}_{\vec{t};s'}$ recursively by

$$(A.78)\qquad M^{(N);2j-1}_{\vec{t};s'}(a,b,x,y) \equiv \sum_{u,v} A_{t_1,t_2}(a,b,u,v) M^{(N-1);2j-3}_{(t_3,\dots,t_{2N-1});s'}(u,v,x,y)$$

$$(A.79)\qquad M^{(N);2j}_{\vec{t};s'}(a,b,x,y) \equiv \sum_{u,v} A_{t_1,t_2}(a,b,u,v) M^{(N-1);2j-2}_{(t_3,\dots,t_{2N-1});s'}(u,v,x,y).$$

Here the extra superscript indicates where on the backbone an extra vertex is added. Modifying the derivation of (A.67), one can readily show that

$$(A.80)\qquad \sum_{\vec{v},x_*} |\pi^{(N),\mathrm{int}}_{(0,M,0);n_*}(\vec{v};x_*)| \leq \sum_{v_1} \sum_{\vec{t}\in\mathcal{H}^{(N)}_M} \sum_{\ell=1}^{2N-1} \sum_{s'=1}^{t_\ell-1} M^{(N),\ell}_{\vec{t};s'}(0,0,v_1,0),$$

where $\pi^{(N),\mathrm{int}}_{(0,M,0);n_*}(\vec{v};x_*)$ is the contribution to $\pi^{(N)}_{(0,M,0);n_*}(\vec{v};x_*)$ from the extra arm being attached at a point which is not the endpoint of any edge in the lace. We prove by induction that

$$\sum_{\vec{t}\in\mathcal{H}^{(N)}_M} \sup_{a,b,y} \sum_x \sum_{\ell=1}^{2N-1} \sum_{s'=1}^{t_\ell-1} M^{(N);\ell}_{\vec{t};s'}(a,b,x,y) \leq \frac{NM(C_\beta)^N}{(M+1)^{(d-4)/2}},$$

or equivalently, absorbing the factor N into the $(C_\beta)^N$,

$$(A.81)\qquad \sum_{\vec{t}\in\mathcal{H}^{(N)}_M} \sup_{a,b,y} \sum_x \sum_{\ell=1}^{2N-1} \sum_{s'=1}^{t_\ell-1} M^{(N);\ell}_{\vec{t};s'}(a,b,x,y) \leq \frac{M(C_\beta)^N}{(M+1)^{(d-4)/2}}.$$

To verify (A.81), first prove the result for $N = 1, 2$, and for $N > 2$ note that the left hand side is bounded by

$$(A.82)\qquad \sum_{\vec{t}\in E^{(N)}} \sup_{a,b,y} \sum_{\ell=1}^{2} \sum_x \sum_{s'=1}^{t_\ell-1} M^{(N);\ell}_{\vec{t}}(a,b,x,y)$$

$$(A.83)\qquad + \sum_{\vec{t}\in E^{(N)}} \sup_{a,b,y} \sum_{\ell=3}^{2N-1} \sum_x \sum_{s'=1}^{t_\ell-1} M^{(N);\ell}_{\vec{t}}(a,b,x,y)$$

$$(A.84)\qquad + \sum_{\vec{t}\in F^{(N)}} \sup_{a,b,y} \sum_{\ell=2N-2}^{2N-1} \sum_x \sum_{s'=1}^{t_\ell-1} M^{(N);\ell}_{\vec{t}}(a,b,x,y)$$

$$(A.85)\qquad + \sum_{\vec{t}\in F^{(N)}} \sup_{a,b,y} \sum_{\ell=1}^{2N-3} \sum_x \sum_{s'=1}^{t_\ell-1} M^{(N);\ell}_{\vec{t}}(a,b,x,y).$$

To bound the term (A.82), use (A.77) and proceed as in the no arm case, applying our bounds on A and the (already proved) bound on the ordinary $M^{(N-1)}$. This bounds (A.82) by

$$\Big[\sum_{\ell=1}^{2}\sum_{t_1,t_2\le M/2}\sum_{1\le s'\le t_\ell}\frac{C_\beta}{(t_1+t_2+1)^{(d-4)/2}}+\sum_{t_1\le M/2}\sum_{s'\le t_1}\frac{C_\beta}{(s'+t_1+1)^{(d-6)/2}}\Big]$$
$$\times\frac{(C_\beta)^{(N-1)}}{((M/2)+1)^{(d-4)/2}}$$
$$\le\frac{M(C_\beta)^N}{((M/2)+1)^{(d-4)/2}}\le\frac{(C_\beta)^N}{(M+1)^{(d-6)/2}},$$

as required. For the term (A.83), for $\ell\ne 1,2$ we use

$$M^{(N);\ell}_{\vec t;s'}(a,b,x,y)\equiv\sum_{u,v}A_{t_1,t_2}(a,b,u,v)M^{(N-1);\ell-2}_{(t_3,\dots,t_{2N-1});s'}(u,v,x,y).$$

Proceeding as in the no-arms case, but now applying the current induction hypothesis we obtain a bound on (A.83) of

$$\sum_{t_1,t_2\le M/2}\sum_{1\le s'\le t_\ell}\frac{C_\beta}{(t_1+t_2+1)^{(d-4)/2}}\frac{M(C_\beta)^{(N-1)}}{((M/2)+1)^{(d-4)/2}}\le\frac{M(C_\beta)^N}{((M/2)+1)^{(d-4)/2}},$$

as required. By symmetry the same bounds apply to (A.84) and (A.85). Having completed the inductive proof of (A.81), we may use (A.80) and conclude that

$$\sum_{\vec v,x_*}|\pi^{(N),\mathrm{int}}_{(0,M,0);n_*}(\vec v;x_*)|\le\frac{(C_\beta)^N}{(M+1)^{(d-6)/2}}.\tag{A.86}$$

A modification of the above inductive proof can be used to establish, for example, the same bounds with k extra factors of M if we add k vertices on the backbone. It can also be used to show that if we replace any ρ in the diagram by ρ' we don't change the bounds. It can also be used to show that if any $\rho(z)$ is replaced with $\sum_{n'\le n_*}h_{n'}(z)$ we get a modification to the bound on the whole diagram that is the same as the modification when $N=1$.

Let $\pi^{(N),\mathrm{end}}_{\vec M;n_*}(\vec v;x_*)$ denote the contribution to $\pi^{(N)}_{\vec M;n_*}(\vec v;x_*)$ from adding the extra arm at an endpoint of some edge of a lace. We proceed less formally now with the inductive argument and focus on the multiplicative factors that must be pulled out in the induction. In place of the upper bound in (A.81) our induction hypothesis now has an upper bound of the form:

$$\sum_{n'\le n_*}\frac{(C_\beta)^N}{(M+n'+1)^{(d-4)/2}}+\frac{(C_\beta)^N}{(M+1)^{(d-4)/2}}n_*^p.\tag{A.87}$$

As before, when we add an extra arm either it is on the piece that is broken off, or on the remainder of the diagram. Without loss of generality, we may consider only $\vec t\in E^{(N)}$.

Consider first $t_2 > 0$. The contribution when the extra arm is added at $s_* = 0$ or $s_* = t_1$ (i.e., is on the piece broken off) may be found using the reasoning in (A.64) and (A.65) and the upper bound without arms in (A.72). This leads to a bound of

$$\frac{(C_\beta)^{N-1}}{(M+1)^{(d-4)/2}} \sum_{t_1,t_2} \left[\sum_{n' \le n_*} \frac{C_\beta}{(t_1+t_2+n'+1)^{(d-4)/2}} \right] \le \frac{(C_\beta)^N}{(M+1)^{(d-4)/2}} n_*^p. \tag{A.88}$$

The remaining contribution in this case is from the extra arm being on some piece of the remaining diagram. There are $N-1$ such pieces where the extra arm can be added, leading to a factor of N which can be absorbed into the exponentially small factor $(C_\beta)^N$. Therefore, using the induction hypothesis (A.87), and (A.72) with $M = t_1 + t_2$, we may bound this contribution by

$$\left[\sum_{n' \le n_*} \frac{(C_\beta)^{N-1}}{(M+n'+1)^{(d-4)/2}} + \frac{(C_\beta)^{N-1}}{(M+1)^{(d-4)/2}} n_*^p \right] \sum_{t_1,t_2} \frac{C_\beta}{(t_1+t_2+1)^{(d-4)/2}},$$

which, in turn, is bounded by (A.87).

Consider now the case $t_2 = 0$ where the arm is added at $s_* = 0$ or t_1. We consider only the latter case here, as the former is relatively straightforward. In this case, as in Figure 7 we get 3 terms incorporating different local topologies of the connections in $\mathcal{R}_{t_1}$ to x_* and other vertices (as in (A.50)-(A.55)). This gives bounds

$$\begin{aligned} &\sum_{n'' \le n_*} \sum_{t_1} \frac{C_\beta}{(t_1+n''+1)^{(d-6)/2}} \frac{(C_\beta)^{N-1}}{(M+1)^{(d-4)/2}} \\ &\quad + \sum_{t_1} \frac{C_\beta}{(t_1+1)^{(d-6)/2}} \left[\sum_{n'' \le n_*} \frac{(C_\beta)^{N-1}}{(M+n''+1)^{(d-4)/2}} + \frac{(C_\beta)^{N-1}}{(M+1)^{(d-4)/2}} n_*^p \right] \\ &\quad + \sum_{n' \le n_*} \sum_{t_1} \frac{C_\beta}{(t_1+n'+1)^{(d-6)/2}} \frac{(C_\beta)^{N-1}}{(M+1)^{(d-4)/2}} \\ &\le \frac{(C_\beta)^N}{(M+1)^{(d-4)/2}} n_*^p + \sum_{n'' \le n_*} \frac{(C_\beta)^N}{(M+n''+1)^{(d-4)/2}} \end{aligned} \tag{A.89}$$

i.e., we again recover the bound (A.87). As for the case $t_2 > 0$, if the arm is added at some other endpoint of a lace edge, then we use the induction hypothesis to get a bound of

$$\left[\sum_{n' \le n_*} \frac{(C_\beta)^{N-1}}{(M+n'+1)^{(d-4)/2}} + \frac{(C_\beta)^{N-1}}{(M+1)^{(d-4)/2}} n_*^p \right] \sum_{t_1} \frac{C_\beta}{(t_1+1)^{(d-6)/2}}.$$

This shows that

$$\sum_{\vec{v},x_*} |\pi^{(N),\mathrm{end}}_{(0,M,0);n_*}(\vec{v};x_*)| \le \sum_{n' \le n_*} \frac{(C_\beta)^N}{(M+n'+1)^{(d-4)/2}} + \frac{(C_\beta)^N}{(M+1)^{(d-4)/2}} n_*^p. \tag{A.90}$$

A.2.4 Summary of the bounds for laces on an interval. Let us summarize what we have shown so far in this section:

We restricted our attention to $\mathbf{S} = [i, [0, M]]$, which corresponds to laces on an interval. We have recalled that, by using Lemma A.1 and decomposing the diagrams, the terms $\sum_{\vec{v}} |\pi^{(N)}_{(0,M,0)}(\vec{v})|$ are bounded by $(C_\beta)^N (M+1)^{-(d-4)/2}$. We have then proved using the same lemma and decomposition that when we add an extra arm, the bound becomes

$$\sum_{\vec{v}, x_*} |\pi^{(N)}_{(0,M,0);n_*}(\vec{v}; x_*)| \le (C_\beta)^N (M+1)^{-(d-4)/2}[M + n_*^p] + \sum_{n' \le n_*} (C_\beta)^N (M + n' + 1)^{-(d-4)/2}. \tag{A.91}$$

The same bound holds with $M = M_i + M_j$ when $\mathbf{S} = [i, [0, M_i]] \cup [j, [0, M_j]]$, since these all correspond to laces on an interval. Thus we have verified (7.62) of Proposition 7.3 when some $M_i = 0$. As we discussed at the end of Section A.2.1, adding a second extra arm can be handled relatively easily, and gives (7.63) for laces on an interval.

A.3 Laces on $\mathbf{S}^3_{\vec{M}}$ where all $M_i > 0$. In [34, Section 6], Proposition 7.1 is proved by establishing

$$\sum_{\vec{u}} |\pi^{(N)}_{\vec{M}}(\vec{u})| \le (C_\beta)^N B(\vec{M}), \tag{A.92}$$

where $B(\vec{M})$ is defined in (7.53) and $\pi^{(N)}$ is the contribution to π from laces containing exactly N edges.

There are many different diagrams that arise, depending on the various types of laces. The laces are first characterised as acyclic or cyclic (see the definition in the proof of Proposition 7.1), and the former case is further characterised by the number of edges in the lace that cover the branch point. In each case the bound is achieved by decomposing the resulting diagrams into components for which (A.72) applies directly (i.e. the components are some of the diagrams that arise in the case of the lace expansion on an interval), or small perturbations of such components that can be bounded by induction on N_i (which roughly speaking is the number of edges with endpoints on branch i). We will first illustrate the general approach and some of the additional difficulties via one particular example of an acyclic lace with two edges covering the branch point. We will complete the proof by describing the basic diagrammatic bounds in general and the modifications to those bounds when we add an extra arm.

Consider the set of acyclic laces on $\mathbf{S}^3_{\vec{M}}$ with two edges covering the branchpoint, and a single additional edge on branch 1, such that all 3 edges have a common endvertex on branch 1. Note that this situation (3 lace edges meeting at a common vertex) did not arise when we considered laces on an interval. Letting s denote the location along branch 1 (see e.g. Figure 9) where these edges meet we can bound the contribution to $\sum_{\vec{u}} |\pi^{(3)}_{\vec{M}}(\vec{u})|$ from such laces by

$$\sum_{\vec{u}} \sum_{v} \sum_{z_1, z_2} h_{M_0}(u_0) h_{M_2}(u_2) \sum_{0 \le s < M_1} h_s(v) h_{M_1 - s}(u_1 - v) \rho(z_1 - v) \rho(z_2 - z_1) \times$$

$$\Big[\rho^{(*2)}(u_1 - z_1)\rho^{(*2)}(u_0 - z_2)\rho^{(*2)}(u_2 - z_2) \tag{A.93}$$

$$+ \rho^{(*2)}(u_1 - z_2)\rho^{(*2)}(u_0 - z_2)\rho^{(*2)}(u_2 - z_1) \tag{A.94}$$

$$+ \rho^{(*2)}(u_1 - z_2)\rho^{(*2)}(u_0 - z_1)\rho^{(*2)}(u_2 - z_2)\Big], \tag{A.95}$$

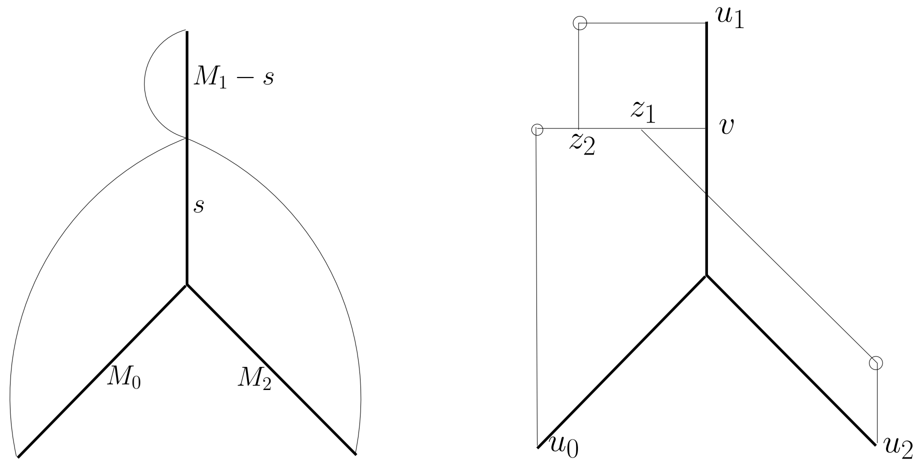

FIG 9. *A lace with 3 edges meeting at a common point and 1 of the 3 diagrams arising from that lace.*

where the three terms (A.93)-(A.95) arise from the three possible topologies of a tree connecting from v to trees from each u_i. Figure 9 corresponds to the contribution from (A.94).

These diagrams are decomposed depending on the relative sizes of M_0 and M_2. Suppose without loss of generality that $M_2 \geq M_0$. Then in each case above we rearrange the terms and take the sum over u_2 inside all other sums and use Lemma A.1 in the form

$$\sup_z \sum_{u_2} h_{M_2}(u_2)\rho^{(*2)}(u_2 - z) \leq \frac{C_\beta}{(M_2+1)^{(d-4)/2}}, \tag{A.96}$$

to get a bound of $C_\beta(M_2 + M_0 + 1)^{-(d-4)/2}$ multiplied by

$$\sum_{u_0,u_1}\sum_v \sum_{z_1,z_2} h_{M_0}(u_0) \sum_{0\leq s<M_1} h_s(v)h_{M_1-s}(u_1 - v)\rho(z_1 - v)\rho(z_2 - z_1)\times$$

$$\Big[\rho^{(*2)}(u_1 - z_1)\rho^{(*2)}(u_0 - z_2) \tag{A.97}$$

$$+ \rho^{(*2)}(u_1 - z_2)\rho^{(*2)}(u_0 - z_2) \tag{A.98}$$

$$+ \rho^{(*2)}(u_1 - z_2)\rho^{(*2)}(u_0 - z_1)\Big]. \tag{A.99}$$

Comparing this with the diagram arising from a lace on an interval containing two edges that share a common endvertex we see that the above diagrams are all the same as that on an interval except that some ρ has been replaced with $\rho^{(*2)}$. Similarly to the discussion below (A.86) we can prove by induction (first proving the $N = 1, N = 2$ cases that diagrams arising from laces (with N edges) on an interval of length M ($= M_0 + M_1$ in this case), and with a single ρ term replaced with $\rho^{(*2)}$, are bounded by $C_\beta^N(M+1)^{-(d-6)/2}$. It follows that the contribution to $\sum_{\vec{u}} |\pi^{(3)}_{\vec{M}}(\vec{u})|$ by acyclic laces of this kind is at most $C_\beta^3(M_0 + M_1 + 1)^{-(d-6)/2}(M_0+M_2+1)^{-(d-4)/2}$. It is easy to see that if we have a lace with the same structure of edges covering the branch point, but with extra edges not covering the branchpoint, we can decompose in the same way. Under this decomposition we first extract the part of the diagram involving M_2 and u_2, which corresponds to a diagram for laces on an interval of length M_2, which we have already shown to be bounded by $(C_\beta)^{N_2}(M_2+1)^{-(d-4)/2} \le (C_\beta)^{N_2}(M_2+M_0+1)^{-(d-4)/2}$ (here we are still assuming that $M_2 \ge M_0$, and C_β is changing from place to place), where N_2-1 is the number of edges contained strictly on this branch. As above, the remainder of the diagram corresponds to those of laces on an interval of length M_0+M_1, with some ρ replaced with $\rho^{(*2)}$, so is bounded by $C_\beta^{N-N_2}(M_0+1)^{-(d-6)/2}$. This achieves the desired bounds with no arms added, for laces with this kind of arrangement of edges covering the branchpoint.

Still considering the same kinds of laces, when we add an extra arm to some (x_*, n_*), it is either added at some backbone vertex that is not the endpoint of any edge of the lace or it is added at an endvertex of some lace edge. In each case, after using $\sum_{x_*} h_\bullet(x_*-\zeta) \le K$ we perform the same decomposition, with the same kinds of caveats as for laces on an interval. When the arm is added at a vertex that is not the endpoint of a lace edge, it is either added on the M_2 branch or the M_0, M_1 branches. In the former case, we can use our result for laces on an interval with an extra arm added, and the above bound for diagrams with a ρ replaced with $\rho^{(*2)}$ to obtain bounds $M_2(C_\beta)^{N_2}(M_2+1)^{-(d-4)/2} \le M_2(C_\beta)^{N_2}(M_2+M_0+1)^{-(d-4)/2}$ and $(C_\beta)^{N-N_2}(M_0+M_1+1)^{-(d-6)/2}$ as required. When the arm is added on the M_0, M_1 branches (not at the endpoint of some lace edge), we first prove by induction that diagrams for N-edge laces on an interval of length M ($=M_0+M_1$ in this case) with some ρ replaced with $\rho^{(*2)}$ and an extra vertex on the backbone are bounded by $M(C_\beta)^N(M+1)^{-(d-6)/2}$ and apply this to the interval of length M_0+M_1 with $N-N_2$ edges on it to get bounds $(M_0+M_1)(C_\beta)^{N-N_2}(M_0+M_1+1)^{-(d-6)/2}$ and (as above) $C_\beta^{N_2}(M_2+M_0+1)^{-(d-4)/2}$.

Suppose instead that the extra arm to some (x_*, n_*) is added at the endvertex of some lace edge, after using $\sum_{x_*} h_\bullet(x_*-\zeta) \le K$ the effect of this is to replace at most two ρ terms in the diagram with ρ', and at least one ρ term in the diagram with $\sum_{n'\le n_*} h_{n'} * \rho$. See for example Figure 10. If the added arm is on branch 2, then we can use the previous section, where we bounded diagrams for laces on an interval with an extra arm added (at an endvertex of a lace edge) to get a bound $(C_\beta)^{N_2}[n_*^p(M_2+1)^{-(d-4)/2} + \sum_{n'\le n_*}(M_2+n'+1)^{-(d-4)/2}] \le (C_\beta)^{N_2}[n_*^p(M_0+M_2+1)^{-(d-4)/2} + \sum_{n'\le n_*}(M_1+M_2+n'+1)^{-(d-4)/2}]$ from branch 2 and as usual $(C_\beta)^{N-N_2}(M_0+M_1+1)^{-(d-6)/2}$ for the remainder of the diagram. If the added arm is on one of the endpoints of a lace edge on branch 0 or 1, then we prove by induction that diagrams arising from laces with N edges on an interval of length M with some ρ replaced with $\rho^{(*2)}$ followed by at most two ρ terms with ρ' terms and at least one ρ term in the diagram with $\sum_{n'\le n_*} h_{n'} * \rho$, are bounded by $(C_\beta)^N[n_*^p(M+1)^{-(d-6)/2} + \sum_{n'\le n_*}(M+n'+1)^{-(d-6)/2}]$. Then we perform the same decomposition, and apply the induction result above with $N-N_2$ edges on an interval of length M_0+M_1 and (getting a bound on the removed branch 2 of $(C_\beta)^{N_2}(M_2+M_0+1)^{-(d-4)/2}$ as usual) to obtain the bound

(A.100)
$$(C_\beta)^N(M_2+M_0+1)^{-(d-4)/2}[n_*^p(M_0+M_1+1)^{-(d-6)/2} + \sum_{n'\le n_*}(M_0+M_1+n'+1)^{-(d-6)/2}].$$

This verifies the bound (7.62) in Proposition 7.3, for acyclic laces with 2 edges covering the branchpoint that each meet another edge (on branch 1) at a common vertex.

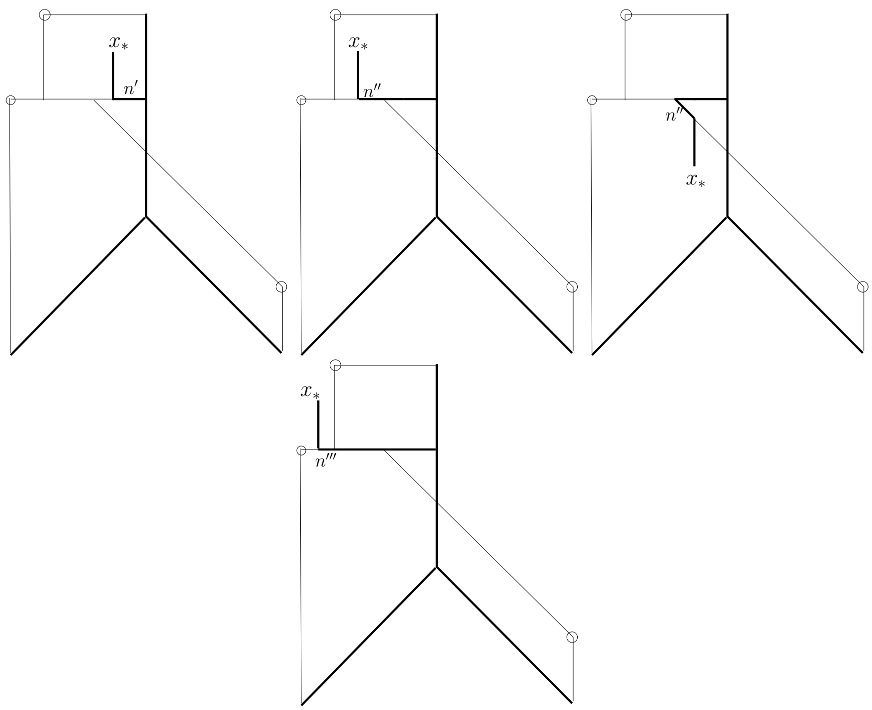

Fig 10. *Some adjusted diagrams from adding an arm at the coincidence point of three endpoints of edges, for the diagram in Figure 9.*

A.3.1 The general case with no added arms. In the above, the diagrams with no arms added were bounded above by $(C_\beta)^N(M_0+M_1+1)^{-(d-6)/2}(M_0+M_2+1)^{-(d-6)/2}$, and the diagrams with an extra arm added satisfied modified bounds that we are now quite familiar with: multiplying by $\sum_{i=0}^2 M_i$; or multiplying by a factor n_*^p; or replacing $(M_0+M_1+1)^{-(d-6)/2}$ or $(M_0+M_2+1)^{-(d-6)/2}$ with $\sum_{n'\le n_*}(M_0+M_1+n'+1)^{-(d-6)/2}$ or $\sum_{n'\le n_*}(M_0+M_2+n'+1)^{-(d-6)/2}$ respectively. We turn now to the general setting of minimal laces on a star shape of degree 3, for which the basic diagrams (with no arms added) are bounded in [34, Section 6]. In [34, Section 6], each diagram is decomposed according to the relative lengths of pieces of the diagram (in particular the M_i and the temporal lengths from the branch point to the first endpoint of some lace edge on each branch). Ignoring the factors $(C_\beta)^N$, their bounds take one of the forms described below, where $[M]\equiv M+1$ for any $M\in\mathbb{N}$, M,M',M'' are the values of M_i, $\overline{M}=M\vee M'\vee M''\approx M+M'+M''$, and $b_i=(d-i)/2$. In addition, s and s' etc. represent lengths of certain subintervals of branches (typically from the branch point to the first endpoint of some lace edge on a branch) of length M or M' respectively.

For laces with 2 edges covering the branchpoint (see [34, (6.1)-(6.5)]), the bounds take the form

$$\sum_{s<M}[s+M'']^{-b_4}[M'+(M-s)]^{-b_4}, \tag{A.101}$$

$$[M'+M'']^{-b_4}[M+M'']^{-b_6}, \tag{A.102}$$

$$[M'+M'']^{-b_4}\sum_{s\le M}[M+M'']^{-b_4}. \tag{A.103}$$

For acyclic laces with 3 edges covering the branch point (see [34, (6.6)-(6.9)]), the bounds take the form

$$[\overline{M}]^{-b_4}\sum_{s\le M}[M+M']^{-b_4}, \tag{A.104}$$

$$\begin{aligned}&[\overline{M}]^{-b_4}\sum_{s'\le M'}\sum_{s''\le M''}\Big[\sum_{s<M}[s+M'']^{-b_4}[M'+(M-s)]^{-b_4}\\ &\quad+[M'+M'']^{-b_4}[M+M'']^{-b_6}+[M'+M'']^{-b_4}\sum_{s\le M}[M+M'']^{-b_4}\Big],\end{aligned} \tag{A.105}$$

$$[\overline{M}]^{-b_0}\sum_{s'\le M'}\sum_{\substack{s,s'':\\ s+s''\le M}}[s+M'']^{-b_4}[M'-s'+M-s-s'']^{-b_8}, \tag{A.106}$$

and

$$[\overline{M}]^{-b_4}[M+M']^{-b_6} \tag{A.107}$$

$$[\overline{M}]^{-b_4}[M]^{-b_6}\sum_{s'\le M'}\sum_{s''\le M''}[M'+s'']^{-b_4}, \tag{A.108}$$

$$[\overline{M}]^{-b_0}[M]^{-b_6}\sum_{s'\le M'}\sum_{s\le M}[M''+s]^{-b_4}[M'-s'+M-s]^{-b_8}, \tag{A.109}$$

$$[\overline{M}]^{-b_2}[M'+M'']^{-b_8}\sum_{s:M-s\ge M''/3}\ \sum_{s'\le M'}[M-s]^{-b_4}, \tag{A.110}$$

$$[\overline{M}]^{-b_4}[M'']^{-b_4}\sum_{s:M-s\le M''/3}\ \sum_{s'\le M'}[M'-s']^{-b_6}, \tag{A.111}$$

and

$$[\overline{M}]^{-b_4}\sum_{s\le M}[M-s]^{-b_6}[M''+s]^{-b_4}, \tag{A.112}$$

$$[M+M']^{-b_4}[M']^{-b_6}[M'']^{-b_6}. \tag{A.113}$$

The last term here arises from the second bound in [34, (6.9)], and is a bound on the 5th and 6th diagrams of [34, Figure 24]. Although the bound presented here is not exactly as it appears in [34, (6.9)], note that it can be obtained from the fact that in the 5th diagram the branches 1 and 3 are topologically equivalent, while in the 6th diagram 2 and 3 are equivalent. So in each case we choose the decomposition presented in [34], depending on which of two equivalent M's is larger.

No new bounds (i.e., none in addition to the above bounds) arise for cyclic laces.

A.3.2 The general case with an extra arm. When an extra arm is added to the diagram, it can be attached at a location that is the endpoint of 0,1,2, or 3 edges of the lace. For a fixed lace, let E_j denote the vertices in $\mathbf{S}^3_{\vec{M}}$ that are the endpoints of j edges of the lace, $j=0,1,2,3$. Although the case of 3 endpoints of lace edges coinciding (see e.g. Figure 9) did not arise in the case of laces on an interval, a particular case has already been treated in this Section.

Regardless of where the extra arm is located, perform the same decompositions of diagrams as in [34] that gave rise to the bounds of the previous section. Ignoring factors of N which can be absorbed by the exponentially small $(C_\beta)^N$ when we sum over N, what is the effect on the bounds when we add the extra arm at s_*? Depending on s_* we have the following contributions. (In what follows, $\bullet$ is always bounded above by $\sum_{j=0}^2 M_j$.)

- The contribution from $s_*\in E_0$ is to multiply the bound by $\sum_{i=0}^2 M_i$.
- The contribution from $s_*\in E_1$ is to replace some $[\bullet]^{-b_i}$ (with $i\in\{2,4,6,8\}$) with $\sum_{n'\le n_*}[\bullet+n']^{-b_i}$.
- The contribution from $s_*\in E_2$ may involve some ρ being replaced by ρ' (which does not affect the bounds), while some $[\bullet]^{-b_i}$ (with $i\in\{2,4,6,8\}$) is replaced with $\sum_{n'\le n_*}[\bullet+n']^{-b_i}$ (to be consistent with our diagrams for laces on an interval the summation variable could be n'' instead of n').
- The contribution from $s_*\in E_3$ may involve one or two ρ being replaced by ρ' (which does not affect the bounds), while some $[\bullet]^{-b_i}$ (with $i\in\{4,6\}$) is replaced with $\sum_{n'\le n_*}[\bullet+n']^{-b_i}$ (to be consistent with our diagrams for laces on an interval the summation variable could be n'' or n''' instead of n').

In the no arms case we then had to transform these bounds into the form of $B(\vec{M})$, so what remains is to consider the effect of these modifications (i.e. of a single replacement of some term $[\bullet]^{-b_i}$ with $\sum_{t\le n_*}[\bullet+t]^{-b_i}$, and t is n', n'' or n''') upon the aforementioned transformation into the form of $B(\vec{M})$. If $i\le 6$ then we can simply use the bound

$$\sum_{t\le n_*}[\bullet+t]^{-b_i}\le[\bullet]^{1-b_i}\le\sum_{j=0}^{2}M_j[\bullet]^{-b_i} \tag{A.114}$$

thus the resulting quantity is bounded by $\sum_{j=0}^{2} M_j$ multiplied by the original bound with no extra arms attached. One can then just use the original transformation, but now including the extra factor $\sum_{j=0}^{2} M_j$.

It therefore remains to consider the case where $i = 8$. There are two situations, corresponding to (A.106) and (A.109) respectively, i.e.

$$[\overline{M}]^{-b_0} \sum_{s' \le M'} \sum_{\substack{s,s'': \\ s+s'' \le M}} [s + M'']^{-b_4} \sum_{t \le n_*} [M' - s' + M - s - s'' + t]^{-b_8}, \quad \text{and} \tag{A.115}$$

$$[\overline{M}]^{-b_0} [M]^{-b_6} \sum_{s' \le M'} \sum_{s \le M} [M'' + s]^{-b_4} \sum_{t \le n_*} [M' - s' + M - s + t]^{-b_8}. \tag{A.116}$$

Here, (A.116) is smaller than (A.115), so we need only bound the former. First perform the sum over t to get a bound of

$$n_*^p [\overline{M}]^{-b_0} \sum_{s' \le M'} \sum_{\substack{s,s'': \\ s+s'' \le M}} [s + M'']^{-b_4} \le n_*^p [\overline{M}]^{-b_0} \sum_{s' \le M'} \sum_{s'' \le M} [M'']^{-b_6} \tag{A.117}$$

$$\le n_*^p [\overline{M}]^{-b_4} [M'']^{-b_6} \le n_*^p B(\vec{M}). \tag{A.118}$$

This establishes the claim (7.62). As we discussed at the end of Section A.2.1, adding a second extra arm can be handled relatively easily, and gives (7.63). ∎

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