

ON THE BOUNDARY OF THE SUPPORT OF SUPER-BROWNIAN MOTION

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We study the density $X(t, x)$ of one-dimensional super-Brownian motion and find the asymptotic behaviour of $P(0 < X(t, x) \leq a)$ as $a \downarrow 0$ as well as the Hausdorff dimension of the boundary of the support of $X(t, \cdot)$. The answers are in terms of the leading eigenvalue of the Ornstein-Uhlenbeck generator with a particular killing term. This work is motivated in part by questions of pathwise uniqueness for associated stochastic partial differential equations.

1. Introduction. We consider the jointly continuous density $X(t, x)$ ($t > 0, x \in \mathbb{R}$) of one-dimensional super-Brownian motion given by the unique in law solution of

$$(1.1) \quad \frac{\partial X(t, x)}{\partial t} = \frac{1}{2} \frac{\partial^2 X(t, x)}{\partial x^2} + \sqrt{X(t, x)} \dot{W}(t, x), \quad X \geq 0.$$

Here \dot{W} is a space-time white noise, X_0 is in the space $M_F(\mathbb{R})$ of finite measures on the line, and $(X_t, t > 0)$ is a continuous process taking values in the space $C_K(\mathbb{R})$ of continuous functions with compact support in \mathbb{R} (see, for example, Section III.4 of Perkins (2002) for these results and the meaning of (1.1)). We abuse notation slightly and also let $X_t(A) = \int_A X(t, x) dx$ denote the continuous $M_F(\mathbb{R})$ -valued process with density $X(t, \cdot)$ for $t > 0$, i.e., the associated super-Brownian motion.

Our goal is to study the boundary of the zero set of X_t , or equivalently the boundary of the support of X_t , given by

$$(1.2) \quad BZ_t = \partial(\{x : X(t, x) = 0\}) = \{x : X(t, x) = 0, \forall \delta > 0 X_t((x-\delta, x+\delta)) > 0\}.$$

The two related questions we consider are:

1. How large is BZ_t ? For example, what is its Hausdorff dimension?
2. What is the asymptotic behaviour of $P(0 < X(t, x) < a)$ as $a \downarrow 0$?

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As super-Brownian motion models a population undergoing random motion and critical reproduction, a detailed understanding of the interface between the population and empty space gives a snapshot of how the population ebbs and flows. Moreover the answers we found are not what was originally expected. Standard estimates show that $X(t, \cdot)$ is locally Hölder continuous of index $1/2 - \varepsilon$ for any $\varepsilon > 0$ (see Proposition 5.6 below). But near the zero set of $X(t, \cdot)$, one can expect more regular behavior as the noise in (1.1) is mollified. In fact Mytnik, Perkins and Sturm (2006) essentially showed that near the zero set of $X(t, \cdot)$ the density is locally Hölder continuous of any index less than one (see Proposition 5.7 below for a precise statement). The increased regularity led to independent conjectures 14 years ago (one by Carl Mueller and Roger Tribe and the other by Ed Perkins and Yongjin Wang) of the following:

Conjecture A. The Hausdorff dimension of BZ_t is zero a.s.

This also was spurred on by wishful thinking as such a result would help prove pathwise uniqueness in equations such as (1.1), as we explain next.

The connection with pathwise uniqueness in stochastic pde's with non-Lipschitz coefficients is one reason for our interest in these questions. Mass moves with a uniform modulus of continuity in equations such as (1.1) and more generally in

$$(1.3) \quad \frac{\partial X(t, x)}{\partial t} = \frac{1}{2} \frac{\partial^2 X(t, x)}{\partial x^2} + X(t, x)^\gamma \dot{W}(t, x), \quad X \geq 0,$$

for $0 < \gamma < 1$ (see Theorem 3.5 of Mueller and Perkins (1992)). This means one can localize the evolution of solutions to (1.3) in space and so if pathwise uniqueness fails then one expects that the solutions X, Y which separate at time T say, will initially separate at points in $BZ_T(X) \cap BZ_T(Y)$, where we have introduced dependence on the particular process. This is because in the interior of the support, say where $X_T \geq \eta > 0$, we have Lipschitz continuous coefficients and so solutions should coincide for a positive time due to the uniform modulus of continuity, and in the interior of the zero set solutions will remain at zero for some positive length of time thanks to the same reasoning (and lack of any immigration terms). As a result one expects that the larger BZ_t is, the easier it is for solutions to separate, and so the less likely pathwise uniqueness is. This reasoning is of course heuristic but here are some precise illustrations of the principle.

THEOREM 1.1. (*Chen (2015)*) *Let $b : \mathbb{R} \rightarrow \mathbb{R}_+$ be a smooth function with support $[0, 1]$. Then pathwise uniqueness fails in*

$$(1.4) \quad \frac{\partial X(t, x)}{\partial t} = \frac{1}{2} \frac{\partial^2 X(t, x)}{\partial x^2} + \sqrt{X(t, x)} \dot{W}(t, x) + b(x).$$

One step in the proof is to show that $BZ_t \cap [0, 1]$ has positive Lebesgue measure with positive probability. This proceeds by first fixing $x \in (0, 1)$ and using a Poisson point process calculation to show that $P(X(t, x) = 0) > 0$. It is then easy to see that the b -immigration forces $X_t((x - \delta, x + \delta)) > 0$ a.s. for each $\delta > 0$. The proof of the theorem then goes on to show that solutions X, Y can separate in $\cup_{t \in [0, 1]} BZ_t(X) \cap BZ_t(Y)$. In short the presence of the immigration b changes the nature of the boundary of the support and makes it possible to establish pathwise non-uniqueness.

On the uniqueness side of things, two of us conjectured that the methods of Mytnik and Perkins (2011) would allow one to establish

$$(1.5) \quad \begin{aligned} &\text{If for some } \varepsilon > 0, P(0 < X(t, x) \leq a) \leq Ca^{1+\varepsilon}, \\ &\text{then pathwise uniqueness would hold in (1.1).} \end{aligned}$$

We never tried to write out a careful proof of the implication in part because we believed the correct answer to Question 2 above was

Conjecture B. $P(0 < X(t, x) \leq a) = O(a)$ (which is consistent with Conjecture A above).

Nonetheless, this would be a rather nice state of affairs as it would suggest that the $\gamma = 1/2$ case of (1.3) is critical and one could expect pathwise uniqueness to hold for $\gamma > 1/2$. It is natural to expect that the $\gamma = 1/2$ case would then require additional work just as in the classical SDE counterpart resolved by Yamada and Watanabe 45 years ago (and unlike the signed case for general SPDE's where $3/4$ -Hölder continuity in the solution variable is critical by Mytnik and Perkins (2011) and Mueller, Mytnik and Perkins (2014)).

Our main results on Questions 1 and 2 will show both Conjectures A and B are in fact false. To describe them, for $\lambda > 0$, let $V(t, x) = V^\lambda(t, x)$ be the unique solution of

$$(1.6) \quad \frac{\partial V}{\partial t} = \frac{1}{2} \frac{\partial^2 V}{\partial x^2} - \frac{1}{2} V^2, \quad V(0, x) = \lambda \delta_0(x),$$

where V is $C^{1,2}$ on $[0, \infty) \times \mathbb{R} \setminus \{(0, 0)\}$. (See Brezis, Peletier and Terman (1986), Mytnik and Perkins (2003) and the references therein.) A simple

scaling argument shows that

$$(1.7) \quad V^{\lambda r}(s, x) = \lambda^2 V^r(\lambda^2 s, \lambda x) \quad \forall r, \lambda, s > 0, x \in \mathbb{R}.$$

If E_μ denotes expectation for X starting at $X_0 = \mu$, then we have (see, for example, Theorem 1.1 in Mytnik and Perkins (2003) and the references there)

$$(1.8) \quad E_{\delta_0}(e^{-\lambda X(t,x)}) = E_{\delta_x}(e^{-\lambda X(t,0)}) = e^{-V^\lambda(t,x)},$$

and by the multiplicative property,

$$(1.9) \quad E_{X_0}(e^{-\lambda X(t,0)}) = \exp\left(-\int V^\lambda(t, y) dX_0(y)\right).$$

So from the above we see that $V^\lambda(t, x) \uparrow V^\infty(t, x)$ as $\lambda \uparrow \infty$, where

$$(1.10) \quad P_{\delta_0}(X(t, x) = 0) = P_{\delta_x}(X(t, 0) = 0) = e^{-V^\infty(t,x)}$$

and so

$$(1.11) \quad V^\infty(t, x) \leq 2/t < \infty$$

because $P_{\delta_0}(X_t \equiv 0) = \exp(-2/t)$ (see, e.g., (II.5.12) in Perkins (2002)). If $r \rightarrow \infty$ in (1.7) with $\lambda^2 = s^{-1}$, we get

$$(1.12) \quad V^\infty(s, x) = s^{-1}F(xs^{-1/2}),$$

where $F(x) = V^\infty(1, x)$. The function F has been studied in the pde literature and can be intrinsically characterized as the solution of an ode. This, and other properties of F , are recalled in Section 3. For now we only need to know that it is a symmetric C^2 function on the line which vanishes at infinity. Let $Lh(x) = \frac{h''}{2} - \frac{x}{2}h'(x)$ be the generator of the Ornstein-Uhlenbeck process, let m be the standard normal distribution on the line, and set $L^F(h) = Lh - Fh$. By standard Sturm-Liouville theory (see Theorem 2.3) there is a complete orthonormal system for $L^2(m)$ consisting of C^2 eigenfunctions for L^F , $\{\psi_n : n \in \mathbb{Z}_+\}$, with corresponding negative eigenvalues $\{-\lambda_n\}$ where $\{\lambda_n\}$ is nondecreasing. The largest eigenvalue λ_0 is simple and satisfies $1/2 < \lambda_0 < 1$. The latter and a bit more is proved in Proposition 3.4.

Here are our answers to the above questions. $\dim(A)$ denotes the Hausdorff dimension of a set $A \subset \mathbb{R}$. In the next two results $X(t, x)$ is the density of super-Brownian motion satisfying (1.1) starting with finite initial measure X_0 and BZ_t is defined as in (1.2).

THEOREM 1.2. (a) For some $C_{1.2}$, for all $a, t > 0$, and $x \in \mathbb{R}$,

$$P_{X_0}(0 < X(t, x) \leq a) \leq C_{1.2} X_0(\mathbb{R}) t^{-(1/2)-\lambda_0} a^{2\lambda_0-1}.$$

(b) For all $K \in \mathbb{N}$ there is a $C(K) > 0$ so that if $X_0(\mathbb{R}) \leq K$, $X_0([-K, K]) \geq K^{-1}$, $t \geq K^{-1}$ and $|x| \leq K$, then

$$P_{X_0}(0 < X(t, x) \leq a) \geq C(K) t^{-(1/2)-\lambda_0} a^{2\lambda_0-1} \quad \text{for all } 0 < a \leq \sqrt{t}.$$

THEOREM 1.3. For all $X_0 \neq 0$ and $t > 0$,

(a) $\dim(BZ_t) \leq 2 - 2\lambda_0$ P_{X_0} -a.s.

(b) $\dim(BZ_t) = 2 - 2\lambda_0$ with positive P_{X_0} -probability.

In the above results both $2\lambda_0 - 1$ and $2 - 2\lambda_0$ are in $(0, 1)$ and so these results do disprove Conjectures A and B. In Theorem 1.3(b) one expects that $\dim(BZ_t) = 2 - 2\lambda_0$ a.s. on $\{X_t \neq 0\}$ but the proof does not show this.

REMARK 1.4. Both the above results extend immediately to solutions of

$$(1.13) \quad \frac{\partial X(t, x)}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 X(t, x)}{\partial x^2} + \sqrt{\gamma X(t, x)} \dot{W}(t, x).$$

where the constants in Theorem 1.2 now may depend on $\gamma, \sigma^2 > 0$. This is clear since a simple scaling result shows that if X is the solution of (1.1), then $\gamma\sigma^{-2}X(\sigma^2t, x)$ has the unique law of any solution of (1.13).

Theorem 1.2 is contained in Theorem 4.8 below. A Tauberian theorem will show that $P_{\delta_0}(0 < X(t, x) \leq a) \sim a^\alpha$ as $a \downarrow 0$ if and only if

$$E_{\delta_0} \left(e^{-\lambda X(t, x)} 1(X(t, x) > 0) \right) \sim \lambda^{-\alpha} \text{ as } \lambda \uparrow \infty.$$

Here \sim means bounded above and below by positive constants and $\alpha = 2\lambda_0 - 1$. If we use (1.9) and (1.10), this becomes

$$e^{-V^\lambda(t, x)} - e^{-V^\infty(t, x)} \sim \lambda^{-\alpha} \quad \text{as } \lambda \uparrow \infty,$$

and using (1.11) this reduces to

$$(1.14) \quad V^\infty(t, x) - V^\lambda(t, x) \sim \lambda^{-\alpha} \quad \text{as } \lambda \uparrow \infty.$$

We have not been careful with dependence on t or x but by Dini's theorem we know that $\lim_{\lambda \rightarrow \infty} V^\lambda(t, x) = V^\infty(t, x)$ uniformly for (t, x) in compact subsets of $[0, \infty) \times \mathbb{R} \setminus \{(0, 0)\}$ —this is Theorem 1 of Kamin and Peletier (1985). Evidently to prove Theorem 1.2 we need a rate of convergence in the Kamin and Peletier result, and Proposition 4.6 will give the following which may be of interest to pde specialists and will be used to complete the proof of Theorem 1.2.

THEOREM 1.5. *There are positive constants \bar{C} and for each $K \geq 1$, $\underline{C}(K)$, such that for all $t > 0$,*

$$(1.15) \quad \sup_x V^\infty(t, x) - V^\lambda(t, x) \leq \bar{C} t^{-\frac{1}{2} - \lambda_0} \lambda^{-(2\lambda_0 - 1)} \quad \forall \lambda > 0,$$

and

$$(1.16) \quad \underline{C}(K) t^{-\frac{1}{2} - \lambda_0} \lambda^{-(2\lambda_0 - 1)} \leq \inf_{|x| \leq K\sqrt{t}} V^\infty(t, x) - V^\lambda(t, x) \quad \forall \lambda \geq t^{-1/2}.$$

The lower bound in Theorem 1.3(b) is established in Section 5.1 (see Theorem 5.5) by first establishing a capacity condition for a set to be non-polar for BZ_t (Corollary 5.3) through a Frostman-type argument, and then taking the above set to be the range of an appropriate subordinator and using the potential theory for the subordinator (a well-known trick). The key to the above capacity condition is a second moment bound (Proposition 5.1) which is established in Section 6. The corresponding upper bound is proved in Theorem 5.9 in Section 5.2 by first modifying the proof of Theorem 1.2 to get a bound on $P_{X_0}(0 < X_t([x, x + \varepsilon]) \leq \varepsilon^2 M)$ (Theorem 5.8 which is proved in Section 7). One then uses this and the improved modulus of continuity for X near its zero set (Proposition 5.7) to carry out a standard covering argument which in fact bounds the box dimension.

Although the failure of Conjectures A and B indicate that our approach may not shed light on the pathwise uniqueness of solutions to (1.1) we remain optimistic that progress can be made on (1.3) for some values of γ in $(1/2, 3/4]$. In fact (1.5) was a special case of the conjecture

$$(1.17) \quad \begin{aligned} &\text{if for some } \alpha > 3 - 4\gamma, P(0 < X(t, x) \leq a) \leq Ca^\alpha, \\ &\text{then pathwise uniqueness would hold in (1.3).} \end{aligned}$$

Note for $\gamma > 3/4$ one can take $\alpha = 0$ but in this case pathwise uniqueness is a special case of the main result in Mytnik and Perkins (2011) whose ideas underly our heuristic proof of (1.17). There is also an exponential dual to solutions of (1.3) for $\gamma \in [1/2, 1)$ (see Mytnik (1998)) and we believe the methods of this paper can be used to resolve the lefthand tail asymptotics for solutions of (1.3) as well. We expect power tail behaviour for all γ , and so by (1.17), for $\gamma < 3/4$ but close to $3/4$, we conjecture that pathwise uniqueness will hold in (1.3). Hence, although the $3/4$ -Hölder condition in Mytnik and Perkins (2011) is sharp for pathwise uniqueness in general (by Mueller, Mytnik and Perkins (2014)), it would not be sharp for the family of non-negative solutions to (1.3).

Convention. We will use E_a^Z to denote expectation for the Markov process Z starting at a point a and (abusing the notation slightly) use E_μ^Z to denote the corresponding expectation where Z_0 now has law μ —sometimes μ will be only a finite measure.

We close this section with a heuristic explanation for the connection between our problem and the largest eigenvalue of L^F . Let $U^\lambda(t, x) = \frac{d}{d\lambda} V^\lambda(t, x)$. Then (1.14) with $\alpha = 2\lambda_0 - 1$ (as is required) would clearly follow from

$$(1.18) \quad U^\lambda(t, x) \sim \lambda^{-2\lambda_0} \text{ as } \lambda \uparrow \infty.$$

A formal differentiation of (1.6) shows that $U = U^\lambda$ satisfies

$$\frac{\partial U}{\partial t} = \frac{1}{2}U'' - VU, \quad U_0 = \delta_0,$$

which has Feynmann-Kac representation

$$(1.19) \quad \begin{aligned} U^\lambda(t, x) &= E_x^B \left(\delta_0(B_t) \exp \left\{ - \int_0^t V^\lambda(t-s, B_s) ds \right\} \right) \\ &= E_0^{(\lambda^2 t, \lambda x)} \left(\exp \left\{ - \int_0^{\lambda^2 t} V^1(s, B_s) ds \right\} \right) p_t(x). \end{aligned}$$

The last line follows by time reversal and the scaling relation (1.7). Here B under $E_0^{(\lambda^2 t, \lambda x)}$ is a Brownian motion starting at 0 conditioned to equal λx at time $\lambda^2 t$. A further use of (1.7) shows that

$$(1.20) \quad H(u, x) = uV^1(u, \sqrt{u}x) = V^{\sqrt{u}}(1, x) \uparrow F(x) \text{ as } u \uparrow \infty.$$

If $Y(u) = B(e^u - 1)e^{-u/2}$, $u \geq 0$, then Y is an Ornstein-Uhlenbeck process with $Y_0 = B_0$. If we ignore the conditioning in (1.19), and use the Markov property for B at $t = 1$, we obtain as $\lambda \rightarrow \infty$ (ignoring dependence on (t, x))

$$\begin{aligned} U^\lambda(t, x) &\sim E_0^B \left(\exp \left\{ - \int_0^{\lambda^2 t} V^1(s, B_s) ds \right\} \right) \\ &\sim E_m \left(\exp \left\{ - \int_0^{\lambda^2 t - 1} (s+1) V^1 \left(s+1, \frac{\sqrt{s+1} B_s}{\sqrt{s+1}} \right) (s+1)^{-1} ds \right\} \right) \\ &= E_m^Y \left(\exp \left\{ - \int_0^{\log(\lambda^2 t)} H(e^u, Y_u) du \right\} \right) \quad (s = e^u - 1) \\ &\sim E_m \left(\exp \left\{ - \int_0^{\log(\lambda^2 t)} F(Y_u) du \right\} \right) \quad (\text{by (1.20) and a bit of work}) \\ &\sim C \exp \{ -\lambda_0 \log(\lambda^2 t) \} \\ &\sim C \lambda^{-2\lambda_0}, \end{aligned}$$

giving us the required (1.18). The next to last line follows by a standard eigenfunction expansion recalled in Section 2. There are of course a number of non-rigorous steps in the above. Some of them (like the use of (1.20)) will be verified in this work, but our basic approach will not follow this plan but rather depend on a Campbell measure formula (see the proof of Lemma 4.1 in Section 4).

2. Eigenfunction Expansions. Let Y_t denote the Ornstein-Uhlenbeck process associated with the infinitesimal generator L . In this section we usually will drop the Y in the notation E_x^Y . Its semigroup is denoted by P_t and its resolvent by R_λ . If $\phi \in C[-\infty, \infty]$ (the space of continuous functions with finite limits at $\pm\infty$) with $\phi \geq 0$, we let $L^\phi h = Lh - \phi h$, the generator associated with the diffusion, Y^ϕ , obtained by killing Y at time $\rho_\phi = \inf\{t : \int_0^t \phi(Y_s) ds > e\}$, where e is an independent exponential variable. We denote its semigroup and resolvent by P_t^ϕ and R_λ^ϕ ($\lambda > 0$), respectively. The above semigroups are strongly continuous contraction semigroups on $L^2(m)$. The contraction part is elementary as m is stationary for Y . For the strong continuity see Lemma 2.1 below for P_t and it is easy to check that $\lim_{t \rightarrow 0} \|P_t f - P_t^\phi f\|_2 = 0$ for all $f \in L^2(m)$. For now we will consider L and L^ϕ defined on $D = \{h \in C^2 \cap L^2(m) : Lh \in L^2(m)\}$.

LEMMA 2.1. *For all $f \in L^2(m)$, $\lim_{t \downarrow 0} E_m((f(Y_t) - f(Y_0))^2) = 0$.*

PROOF. As Y_t is stationary under P_m , a standard approximation argument allows us to assume f is continuous with compact support. The result now follows by Dominated Convergence. \blacksquare

We let L_0^ϕ and L_0 denote the infinitesimal generators of the $L^2(m) \equiv L^2$ -semigroups P_t^ϕ and P_t , respectively, on their domains $D(L_0^\phi)$ and $D(L_0)$, respectively. So, for example

$$D(L_0) = \{f \in L^2 : \exists L_0 f \in L^2 \text{ such that } \lim_{t \downarrow 0} \|(P_t f - f)/t - L_0 f\|_2 = 0\}.$$

The subscript 0 is a temporary measure to avoid confusion which we address now.

LEMMA 2.2. (a) $D(L_0^\phi) = D(L_0)$ and $L_0^\phi f = L_0 f - \phi f$ for all $f \in D(L_0)$.
 (b) L_0^ϕ is an extension of the differential operator L^ϕ , the latter on D .

PROOF. (a) This is a routine calculation. The fact that $\phi \in C[-\infty, \infty]$ helps here.

(b) By the above we may assume $\phi = 0$. If $f \in D$, then $M_f(t) = f(Y_t) - f(Y_0) - \int_0^t Lf(Y_s) ds$ is a square integrable martingale under P_m . Therefore

$$\begin{aligned} \|(P_t f - f)/t - Lf\|_2^2 &= \int \left(E_x \left(\int_0^t Lf(Y_s) ds \right) / t - Lf(x) \right)^2 dm(x) \\ &\leq \int_0^t E_m((Lf(Y_s) - Lf(Y_0))^2) ds / t. \end{aligned}$$

The last expression approaches 0 as $t \rightarrow 0+$ by the previous Lemma, and the result follows. \blacksquare

Henceforth we will drop the subscript 0's on L_0^ϕ (in view of the above result this should cause no confusion).

Here is the result we will need to describe our main results. In (e), $C([0, \infty), \mathbb{R})$ is the usual space of continuous paths with the topology of uniform convergence on bounded time sets.

THEOREM 2.3. (a) *There is a complete orthonormal system (cons), $\{\psi_n : n \in \mathbb{Z}_+\}$, of C^2 eigenfunctions in $L^2(m)$ for L^ϕ satisfying $L^\phi \psi_n = -\lambda_n \psi_n$, where $\{\lambda_n\}$ is a nondecreasing non-negative sequence diverging to ∞ , $-\lambda_0$ is a simple eigenvalue and $\psi_0 > 0$.*

(b) *R_λ^ϕ is a symmetric Hilbert-Schmidt integral operator on $L^2(m)$. There is a jointly continuous symmetric kernel $G_\lambda^\phi : \mathbb{R}^2 \rightarrow [0, \infty)$ such that*

$$R_\lambda^\phi h(x) = \int G_\lambda^\phi(x, y) h(y) dm(y),$$

and

$$G_\lambda^\phi(x, y) = \sum_{n=0}^{\infty} \frac{1}{\lambda + \lambda_n} \psi_n(x) \psi_n(y),$$

where the series converges in $L^2(m \times m)$ and uniformly absolutely on compacts.

(c) *The killed diffusion Y^ϕ has a jointly (in (t, x, y)) continuous transition density, $q^\phi(t, x, y) \equiv q(t, x, y)$, for $t > 0$, given by*

$$q(t, x, y) = \sum_{n=0}^{\infty} e^{-\lambda_n t} \psi_n(x) \psi_n(y),$$

where the convergence is in $L^2(m \times m)$ and uniformly absolutely for $(t, x, y) \in [\varepsilon, \infty) \times [\varepsilon, \varepsilon^{-1}]^2$ for any $\varepsilon > 0$. Moreover, if $0 < \delta < 1/4$, and $s^* = s^*(\delta) > 0$ satisfies

$$(2.1) \quad 2\delta = \frac{e^{-s^*/2} - e^{-s^*}}{1 - e^{-s^*}}$$

(s^* will increase to ∞ as $\delta \downarrow 0$), then there is a $c(\delta)$ such that

$$(2.2) \quad q(t, x, y) \leq c(\delta)e^{-\lambda_0 t} \exp(\delta(x^2 + y^2)) \text{ for all } t \geq s^*(\delta).$$

(d) If $\theta = \int \psi_0 dm$, then for any $\delta > 0$ there is a $c_\delta > 0$ such that for all $t \geq 0$ and $x \in \mathbb{R}$,

$$(2.3) \quad e^{\lambda_0 t} P_x(\rho_\phi > t) = \theta \psi_0(x) + r(t, x),$$

where

$$(2.4) \quad |r(t, x)| \leq c_\delta e^{\delta x^2} e^{-(\lambda_1 - \lambda_0)t},$$

$$(2.5) \quad \psi_0(x) \leq c_\delta e^{\delta x^2},$$

and for $t \geq s^*(\delta)$,

$$(2.6) \quad |r(t, x)| \leq \sum_1^\infty e^{-(\lambda_n - \lambda_0)t} |\psi_n(x)| \int |\psi_n| dm \leq c_\delta e^{\delta x^2} e^{-(\lambda_1 - \lambda_0)t}.$$

(e) As $T \rightarrow \infty$, $P_x(Y \in \cdot | \rho_\phi > T) \rightarrow P_x^\infty$ weakly on $C([0, \infty), \mathbb{R})$ where P_x^∞ is the law of the diffusion with transition density (with respect to m),

$$(2.7) \quad \tilde{q}(t, x, y) \equiv q(t, x, y) \frac{\psi_0(y)}{\psi_0(x)} e^{\lambda_0 t}.$$

PROOF. Parts (a), (b) and the first equation in (c) follow from standard Sturm-Liouville theory. Here note that multiplication by the square root of the normal density converts L^ϕ into the operator

$$A^\phi g = g''/2 - (x^2/8 - 1/4 + \phi)g,$$

now acting on $L^2(dx)$, and so one can proceed as in Example 2 in Section 9.5 of Coddington and Levinson (1955). One applies a minor variant of Mercer's Theorem on p. 245 of Riesz and Sz.-Nagy (1955) for the uniform convergence; only the continuity of G_λ^ϕ takes a bit of work. The bound (2.2) and part (d) can be proved through minor modifications of the arguments in the proof of Theorem 1.1 of Uchiyama (1980). Details of these arguments may be found in the Appendix of Mueller, Mytnik and Perkins (2015).

(e) Fix $x \in \mathbb{R}$. If $0 \leq t_1 < t_2 < T$ and ϕ_i are bounded measurable functions, then

$$\begin{aligned}
& E_x \left(\prod_{i=1}^2 \phi_i(Y_{t_i}) 1(\rho_\phi > T) \right) / P_x(\rho_\phi > T) \\
&= E_x \left(\prod_{i=1}^2 \phi_i(Y_{t_i}) 1(\rho_\phi > t_2) P_{Y_{t_2}}(\rho_\phi > T - t_2) \right) / P_x(\rho_\phi > T) \\
&= \left[E_x \left(\prod_{i=1}^2 \phi_i(Y_{t_i}) 1(\rho_\phi > t_2) \frac{\theta \psi_0(Y_{t_2}) e^{-\lambda_0(T-t_2)}}{\theta \psi_0(x) e^{-\lambda_0 T}} \right) \right. \\
(2.8) \quad & \left. + E_x \left(\prod_{i=1}^2 \phi_i(Y_{t_i}) 1(\rho_\phi > t_2) \frac{e^{-\lambda_0(T-t_2)} r(T-t_2, Y_{t_2})}{\theta \psi_0(x) e^{-\lambda_0 T}} \right) \right] \\
& \quad \times \frac{\theta \psi_0(x) e^{-\lambda_0 T}}{\theta \psi_0(x) e^{-\lambda_0 T} + r(T, x) e^{-\lambda_0 T}} \\
& \equiv [T_1 + T_2] \times T_3,
\end{aligned}$$

where (2.3) is used in the second equality. By using (2.4) with $\delta = 1/4$ we have

$$\begin{aligned}
|T_2| &\leq \frac{\|\phi_1\|_\infty \|\phi_2\|_\infty e^{\lambda_0 t_2} E_x(c_\delta e^{\delta Y_{t_2}^2}) e^{-(\lambda_1 - \lambda_0)(T-t_2)}}{\theta \psi_0(x)} \\
&\leq c(x) e^{\lambda_0 t_2} e^{-(\lambda_1 - \lambda_0)(T-t_2)} \rightarrow 0 \text{ as } T \rightarrow \infty.
\end{aligned}$$

By (2.4) we also have

$$\begin{aligned}
|T_3 - 1| &\leq \frac{|r(T, x)| e^{-\lambda_0 T}}{\theta \psi_0(x) e^{-\lambda_0 T} - |r(T, x)| e^{-\lambda_0 T}} \\
&\leq \frac{c_\delta e^{\delta x^2} e^{-(\lambda_1 - \lambda_0)T}}{\theta \psi_0(x) - c_\delta e^{\delta x^2} e^{-(\lambda_1 - \lambda_0)T}} \rightarrow 0 \text{ as } T \rightarrow \infty.
\end{aligned}$$

Use these last results in (2.8) to conclude that

$$\begin{aligned}
& \lim_{T \rightarrow \infty} E_x \left(\prod_{i=1}^2 \phi_i(Y_{t_i}) 1(\rho_\phi > T) \right) \\
&= E_x \left(\prod_{i=1}^2 \phi_i(Y_{t_i}) 1(\rho_\phi > t_2) \psi_0(Y_{t_2}) e^{\lambda_0 t_2} \right) / \psi_0(x) \\
&= \int \int q(t_1, x, x_1) q(t_2 - t_1, x_1, x_2) \prod_{i=1}^2 \phi_i(x_i) \frac{\psi_0(x_2)}{\psi_0(x)} \\
& \quad \times e^{\lambda_0 t_1} e^{\lambda_0(t_2 - t_1)} dm(x_1) dm(x_2)
\end{aligned}$$

which by definition equals

$$\int \int \tilde{q}(t_1, x, x_1) \tilde{q}(t_2 - t_1, x_1, x_2) \prod_{i=1}^2 \phi_i(x_i) dm(x_1) dm(x_2).$$

Similar reasoning gives the convergence of the k -dimensional distributions for all k .

It remains to establish tightness. If $0 \leq s < t \leq t_0 < T$ with $t - s \leq 1$, then

$$(2.9) \quad E_x((Y_t - Y_s)^4 | \rho_\phi > T) \leq E_x((Y_t - Y_s)^4 P_{Y_t}(\rho_\phi > T - t)) / P_x(\rho_\phi > T).$$

It is now easy to use (2.3), (2.4) and (2.5), together with Cauchy-Schwarz to bound the right-hand side of (2.9) by $c'(x, t_0)(t - s)^2$ at least for $T > T_0(x, t_0)$. This gives the required tightness and the proof of (e) is complete. ■

3. A Nonlinear Differential Equation and Some Associated Eigenvalues. Recall that $F(x) = V^\infty(1, x)$. We start by recording the convergence results from Theorems 1 and 2 of Kamin and Peletier (1985) which were discussed in Section 1. Part (b) in fact is immediate from (a) and (1.7).

PROPOSITION 3.1. (a) $\lim_{\lambda \rightarrow \infty} V^\lambda(t, x) = V^\infty(t, x)$, where the convergence is uniform on compact subsets of S .

(b) For any $\lambda > 0$ and $a > 0$,

$$\lim_{t \rightarrow \infty} \sup_{|x| \leq a} |tV^\lambda(t, xt^{1/2}) - F(x)| = 0.$$

In the pde literature $F : \mathbb{R} \rightarrow [0, \infty)$ is characterized as the unique solution of the following differential equation:

$$(3.1) \quad \begin{aligned} (i) \quad & \frac{F''(y)}{2} + \frac{y}{2} F'(y) + F(y) - \frac{F^2(y)}{2} = 0, \\ (ii) \quad & F > 0, F \text{ is } C^2 \text{ on } \mathbb{R}, \\ (iii) \quad & F'(0) = 0, F(y) \sim c_0 y e^{-y^2/2} \text{ as } y \rightarrow \infty. \end{aligned}$$

Here \sim means the ratio goes to one as $y \rightarrow \infty$ and $c_0 > 0$. This result follows from Brezis, Peletier and Terman (1986), with $f : [0, \infty) \rightarrow [0, \infty)$ satisfying equations (1.7)-(1.9) of that reference (with $N = 1$ and $p = 2$ in our setting), where $F(y) = 2f(\sqrt{2}y)$ for $y \geq 0$ and we extend F to the line by symmetry. The above ode is then immediate from the Theorem

following (1.9) in Brezis, Peletier and Terman (1986) and the trivial fact that the condition $F'(0) = 0$ and fact that F is C^2 on the positive half-line ensures the symmetric extension is C^2 on the line. In fact (see the the aforementioned Theorem of Brezis et al) uniqueness holds if the strong asymptotic in condition (iii) is replaced with

$$(iii)' \quad F'(0) = 0, \quad \lim_{y \rightarrow \infty} y^2 F(y) = 0.$$

Here are some additional properties of F .

LEMMA 3.2. (a) For all $y \geq x_0 \geq 0$,

$$F'(y) = \exp\{-y^2/2 + x_0^2/2\} F'(x_0) + \int_{x_0}^y \exp\{-y^2/2 + z^2/2\} F(z)(F(z) - 2) dz.$$

(b) $\lim_{y \rightarrow \infty} e^{y^2/2} \frac{F'(y)}{y^2} = -c_0$, where c_0 is as in (3.1)(iii).

(c) $1 < F(0) < 2$.

(d) F is strictly decreasing on $[0, \infty)$.

PROOF. The differential equation (3.1)(i) may be rewritten as

$$(e^{z^2/2} F'(z))' = e^{z^2/2} F(z)(F(z) - 2).$$

(a) follows easily. To derive (b), take x_0 large and then use the asymptotics from (3.1)(iii). For (c), note by (a) with $x_0 = 0$, F is increasing until $F \leq 2$. So if $F(0) > 2$, it can never pass below 2, a contradiction. If $F(0) = 2$, then by uniqueness to the initial value problem, $F \equiv 2$, another contradiction. It now follows from (a) with $x_0 = 0$ that $F' < 0$ for positive values until F hits 2 but evidently this can therefore never happen. This proves (d). It remains to prove $F(0) > 1$. A simple calculation using (3.1)(i) gives $(yF + F')' = F(F - 1)$. Integrating over the line we get $\int_{\mathbb{R}} F^2 dy = \int_{\mathbb{R}} F dy$. If $F(0) \leq 1$, then by (d), $0 < F < 1$ on $(0, \infty)$ which contradicts the above equality of integrals. (Note that the Remark prior to Lemma 11 in Brezis, Peletier and Terman (1986) gives $F(0) \geq 1$.) ■

In the pde literature $V^\infty(t, x)$ given by (1.12) is called a very singular solution of the heat equation with absorption. One can easily check (or see Section 1 of Brezis, Peletier and Terman (1986)) that $V = V^\infty$ is a $C^{1,2}$ (on $S = [0, \infty) \times \mathbb{R} - \{(0, 0)\}$) solution of

$$(3.2) \quad (i) \quad \frac{\partial V}{\partial t} = \frac{1}{2} \frac{\partial^2 V}{\partial x^2} - \frac{1}{2} V^2$$

$$(ii) \quad V(0, x) = 0 \text{ for all } x \neq 0; \quad \lim_{t \rightarrow 0} \int_{\mathbb{R}} V(t, x) dx = \infty.$$

Recall that $X_t(dx) = X(t, x)dx$, where X solves (1.1). Translation invariance and (1.9) imply that

$$(3.3) \quad E_{X_0}^X(e^{-\lambda X(t,x)}) = \exp\left(-\int V^\lambda(t, y-x) dX_0(y)\right).$$

We let \mathbb{N}_x denote the canonical measure associated with X starting at δ_x (see Section II.7 of Perkins (2002)). It is an easy consequence of Theorem II.7.2 of the latter reference that

$$(3.4) \quad \exp\left(-\int (1 - e^{-\lambda X(t,0)}) d\mathbb{N}_x(X)\right) = e^{-V^\lambda(t,x)}.$$

PROPOSITION 3.3. *For all $x \in \mathbb{R}$, $e^{-F(x)} = P_{\delta_x}(X(1,0) = 0)$ and $F(x) = \mathbb{N}_x(X(1,0) > 0)$.*

PROOF. The first equality is immediate from (1.10). Let $t = 1$ and $\lambda \uparrow \infty$ in (3.4) to derive the second. \blacksquare

We recall the general exponential duality which underlies the above (see, e.g., Theorem II.5.11 of Perkins (2002)). For ϕ non-negative, bounded and measurable, let $V(t, x) = V(\phi)(t, x)$ be the unique mild solution of

$$(3.5) \quad \frac{\partial V}{\partial t} = \frac{1}{2} \frac{\partial^2 V}{\partial x^2} - \frac{1}{2} V^2, \quad V_0 = \phi.$$

If $X_t(\phi) = \int \phi dX_t$, then,

$$(3.6) \quad E_{X_0}^X(\exp(-X_t(\phi))) = \exp(-X_0(V_t(\phi))).$$

If $\phi \in C_b^2(\mathbb{R})$ (functions with continuous bounded partials of order up to 2) then $V(t, x)$ has continuous bounded derivatives of order up to 1 in t and 2 in x , and (3.5) holds in the classical (i.e., pointwise) sense.

Now return to the eigenfunction expansions of Section 2 in the case where $\phi = F$ or $F/2$. We denote dependence on ϕ by $\lambda_0(\phi)$ and ψ_0^ϕ , and $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(m)$.

PROPOSITION 3.4. (a) $\lambda_0(F/2) = \frac{1}{2}$ and the corresponding eigenfunction is $\psi_0^{F/2}(x) = c_F e^{x^2/2} F(x)$, where $c_F > 0$ is a normalizing constant.
 (b) $\frac{1}{2} < \lambda_0(F) < 1$. More precisely,

$$\frac{1}{2} + \frac{1}{2} \int F(x) (\psi_0^F)^2(x) dm(x) \leq \lambda_0(F) \leq 1 - \frac{1}{2} \int (c_F e^{x^2/2} F(x))'(x)^2 dm(x).$$

PROOF. Let $\psi(x) = e^{x^2/2}F(x) \in C^2 \cap L^2(m)$ (the latter by (3.1)(iii)). Then

$$\begin{aligned} L^{F/2}\psi &= \frac{1}{2}(\psi'e^{-x^2/2})'e^{x^2/2} - \frac{F}{2}\psi \\ &= e^{x^2/2}\left[\frac{F''}{2} + \frac{xF'}{2} + \frac{F}{2} - \frac{F^2}{2}\right] \\ &= -e^{x^2/2}\left(\frac{F}{2}\right), \end{aligned}$$

the last by (3.1)(i). This shows $\psi \in D \subset D(L)$ and $L^{F/2}\psi = -\frac{1}{2}\psi$. Recall by Theorem 2.3(a), the eigenfunction corresponding to the simple eigenvalue $-\lambda_0(F/2)$ is positive, as is F . By orthogonality of eigenfunctions corresponding to distinct eigenvalues we must therefore have $\lambda_0(F/2) = \frac{1}{2}$ and $\psi_0^{F/2} = c\psi$, for some normalizing constant $c > 0$.

(b) The variational characterization of λ_0 gives

$$(3.7) \quad \lambda_0(F) = \min\{\langle -L^F\psi, \psi \rangle : \psi \in D(L), \|\psi\|_2 = 1\},$$

where the minimum is attained at $\psi = \psi_0^F$. (The latter is clear and to see the former one can set $\psi = R_\lambda^F\phi$ and expand ϕ in terms of the basis ψ_n .) If we set $\psi = \psi_0^{F/2}$ we therefore get

$$\begin{aligned} \lambda_0(F) \leq \langle -L^F\psi, \psi \rangle &= 2\langle -L^{F/2}\psi, \psi \rangle + \langle L\psi, \psi \rangle \\ &= 2\lambda_0(F/2) - \frac{1}{2} \int_{-\infty}^{\infty} \psi'(x)^2 dm \\ &= 1 - \frac{1}{2}\|\psi'\|_2^2 < 1, \end{aligned}$$

where the next to last equality holds by an integration by parts (Lemma 3.2(b) handles the boundary terms), and the last equality holds by (a). Turning next to the lower bound on $\lambda_0(F)$, if $\psi_0 = \psi_0^F$, we have, using the variational characterization of $\lambda_0(F/2)$,

$$\begin{aligned} \lambda_0(F) = \langle -L^F\psi_0, \psi_0 \rangle &= \langle -L^{F/2}\psi_0, \psi_0 \rangle + \frac{1}{2} \int F\psi_0^2 dm \\ &\geq \lambda_0(F/2) + \frac{1}{2} \int F\psi_0^2 dm = \frac{1}{2} + \frac{1}{2} \int F\psi_0^2 dm. \end{aligned}$$

■

Henceforth we will write ψ_0 , λ_n and ρ for ψ_0^F , $\lambda_n(F)$ and ρ_F , respectively.

4. Asymptotics for Super-Brownian Motion at the Boundary of its Support. Recall that $X(t, x)$ is the density of super-Brownian motion which solves (1.1). Define

$$(4.1) \quad H_u(x) = uV^1(u, \sqrt{u}x) = V^{\sqrt{u}}(1, x) \uparrow V^\infty(1, x) = F(x), \text{ as } u \uparrow \infty,$$

uniformly on compacts, where we have used (1.7) in the second equation and Proposition 3.1 in the convergence. By (1.11) one obtains the elementary inequality

$$(4.2) \quad e^{-V^\lambda(t, x)} - e^{-V^\infty(t, x)} \leq V^\infty(t, x) - V^\lambda(t, x) \leq e^{2/t}(e^{-V^\lambda(t, x)} - e^{-V^\infty(t, x)}).$$

Notation. We let $p(t, x) = p_t(x)$ denote the standard Brownian density.

In the following Lemma recall that Y is the Ornstein-Uhlenbeck process starting at x under P_x .

LEMMA 4.1. *Let $h \geq 0$ be a bounded Borel measurable function on the real line, let B be a standard Brownian motion starting at 0 under P_0^B , and set $T = \log(\lambda^2 t)$. Then for $\lambda^2 t \geq 1$ and any finite initial measure X_0 ,*

$$(4.3) \quad \begin{aligned} & E_{X_0}^X \left(\int e^{-\lambda X(t, x)} h(x) X(t, x) dx \right) \\ &= E_0^B \left(\exp \left(- \int_0^1 V^1(u, B(u)) du \right) E_{B_1}^Y \left(\exp \left(- \int_0^T H_{e^s}(Y_s) ds \right) \right) \right. \\ & \left. \times \int \left[h(w_0 + \sqrt{t}Y_T) \exp \left(- \frac{1}{t} \int H_{e^r}(Y_T + \frac{w_0 - x_0}{\sqrt{t}}) dX_0(x_0) \right) \right] dX_0(w_0) \right). \end{aligned}$$

PROOF. Let W_t be a Brownian motion starting with initial ‘‘law’’ X_0 under the finite measure $E_{X_0}^W$. Apply the Campbell measure formula for X_t , or more specifically use Theorem 4.1.1 and then Theorem 4.1.3 of Dawson and Perkins (1991) with $\beta = 1$ and $\gamma = 1/2$ to see that

$$(4.4) \quad \begin{aligned} & E_{X_0}^X \left(\int e^{-\lambda X(t, x)} h(x) X(t, x) dx \right) \\ &= E_{X_0}^W \times E_{X_0}^X \left(h(W_t) \exp(-\lambda X(t, W_t)) \exp \left(- \int_0^t V^\lambda(t-s, W_s - W_t) ds \right) \right). \end{aligned}$$

In the above we approximate $X(t, x)$ by $\int p_\varepsilon(x-y)X_t(dy)$ and let $\varepsilon \downarrow 0$ in order to apply Theorem 4.1.3 in Dawson and Perkins (1991). This limiting argument is easy to justify; use (3.6) with $\phi^{\varepsilon, \lambda} = \lambda p_\varepsilon$ and the bound

$V(\phi^{\lambda,\varepsilon})(t-s, x) \leq \lambda p_{\varepsilon+t-s}(x) \leq \lambda(t-s)^{-1/2}$ to take the limit through the Lebesgue integral in s . Now use (3.3) and then the scaling (1.7) to conclude that

$$\begin{aligned}
& E_{X_0}^X \left(\int e^{-\lambda X(t,x)} h(x) X(t, x) dx \right) \\
&= E_{X_0}^W \left(h(W_t) \exp \left(- \int_0^t V^\lambda(t-s, W_t - W_s) ds - \int V^\lambda(t, W_t - x_0) dX_0(x_0) \right) \right) \\
&= E_{X_0}^W \left(h(W_t) \exp \left(- \int_0^t V^1(\lambda^2(t-s), \lambda(W_t - W_s)) \lambda^2 ds \right. \right. \\
(4.5) \quad & \left. \left. - \lambda^2 \int V^1(\lambda^2 t, \lambda(W_t - x_0)) dX_0(x_0) \right) \right).
\end{aligned}$$

If $\hat{W}_s = W_t - W_{t-s}$ and $B_u = \lambda \hat{W}_{\lambda^{-2}u}$ for $u \leq \lambda^2 t$, then under $E_{X_0}^W$ and conditional on W_0 , B is a Brownian motion starting from 0. Noting that $W_t = W_0 + \lambda^{-1} B_{\lambda^2 t}$, we may rewrite (4.5) as

$$\begin{aligned}
& E_{X_0}^W \left(h(W_0 + \frac{1}{\lambda} B_{\lambda^2 t}) \exp \left(- \int_0^{\lambda^2 t} V^1(u, B_u) du \right. \right. \\
& \quad \left. \left. - \lambda^2 \int V^1(\lambda^2 t, B_{\lambda^2 t} + \lambda(W_0 - x_0)) dX_0(x_0) \right) \right) \\
&= E_{X_0}^W \left(\exp \left(- \int_0^1 V^1(u, B_u) du \right) E_{B_1}^B \left[h(W_0 + \lambda^{-1} B_{\lambda^2 t-1}) \right. \right. \\
& \quad \times \exp \left(- \int_0^{\lambda^2 t-1} V^1(1+u, B_u) du \right. \\
& \quad \left. \left. - \lambda^2 \int V^1(\lambda^2 t, B_{\lambda^2 t-1} + \lambda(W_0 - x_0)) dX_0(x_0) \right) \right] \right).
\end{aligned}$$

Set

$$Y_s = B(e^s - 1)e^{-s/2},$$

which is an Ornstein-Uhlenbeck process starting at B_1 under $E_{B_1}^B$. Then

(4.5) equals

$$\begin{aligned}
& E_{X_0}^W \left(\exp \left(- \int_0^1 V^1(u, B_u) du \right) E_{B_1}^Y \left[\exp \left(- \int_0^T e^s V^1(e^s, e^{s/2} Y_s) ds \right) \right. \right. \\
& \quad \left. \left. \times h(W_0 + \sqrt{t} Y_T) \exp \left(- \int \frac{e^T}{t} V^1(e^T, e^{T/2} Y_T + \frac{e^{T/2}}{\sqrt{t}} (W_0 - x_0)) dX_0(x_0) \right) \right] \right) \\
& = E_{X_0}^W \left(\exp \left(- \int_0^1 V^1(u, B_u) du \right) E_{B_1}^Y \left[\exp \left(- \int_0^T H_{e^s}(Y_s) ds \right) h(W_0 + \sqrt{t} Y_T) \right. \right. \\
& \quad \left. \left. \times \exp \left(- t^{-1} \int H_{e^T} \left(Y_T + \frac{W_0 - x_0}{\sqrt{t}} \right) dX_0(x_0) \right) \right] \right).
\end{aligned}$$

Recalling that W_0 is independent of B under $E_{X_0}^W$, we see that the above equals the required expression. \blacksquare

We will use the following Lemma which will allow us to apply Lemma 4.1 to first get a preliminary bound on $V^\infty - V^\lambda$ for large λ , and then reapply these results to get exact asymptotics.

LEMMA 4.2. *Assume for some $r > 1$ and $\underline{\lambda} \geq 0$,*

$$E_{\delta_0}^X \left(\int e^{-\lambda X(t,x)} X(t,x) dx \right) \leq C(t) \lambda^{-r} \quad \forall \lambda > \underline{\lambda}.$$

Then

$$\sup_x [V^\infty(t,x) - V^\lambda(t,x)] \leq e^{6/t} t^{-1/2} C(t/2) (r-1)^{-1} \lambda^{1-r} \quad \forall \lambda > \underline{\lambda}.$$

PROOF. Recalling (1.8), we have

$$e^{-V^\lambda(t,x)} - e^{-V^\infty(t,x)} = E_{\delta_0}^X \left(e^{-\lambda X(t,x)} \mathbf{1}(X(t,x) > 0) \right).$$

The lefthand side is Lebesgue integrable in x (e.g., it is bounded by $V^\infty(t,x)$ which is Lebesgue integrable by (1.12) and the asymptotics for F) and so

$$\int e^{-V^\lambda(t,x)} - e^{-V^\infty(t,x)} dx = E_{\delta_0}^X \left(\int e^{-\lambda X(t,x)} \mathbf{1}(X(t,x) > 0) dx \right).$$

It is easy to differentiate with respect to $\lambda > 0$ through the integrals on the righthand side and so conclude for any $\lambda > \underline{\lambda}$,

$$\begin{aligned}
(4.6) \quad & - \frac{d}{d\lambda} \left(\int e^{-V^\lambda(t,x)} - e^{-V^\infty(t,x)} dx \right) = E_{\delta_0}^X \left(\int e^{-\lambda X(t,x)} X(t,x) dx \right) \\
& \leq C(t) \lambda^{-r}.
\end{aligned}$$

For $\lambda > \underline{\lambda}$ integrate the above over $[\lambda, \infty)$ and so deduce from (4.2) that for $\lambda > \underline{\lambda}$,

$$(4.7) \quad \begin{aligned} \int V^\infty(t, x) - V^\lambda(t, x) dx &\leq e^{2/t} \int e^{-V^\lambda(t, x)} - e^{-V^\infty(t, x)} dx \\ &\leq \frac{e^{2/t} C(t)}{r-1} \lambda^{1-r}. \end{aligned}$$

Next use the Markov property of X to see that for $\lambda > \underline{\lambda}$,

$$(4.8) \quad \begin{aligned} e^{-V^\lambda(t, x)} - e^{-V^\infty(t, x)} &= E_{\delta_x}^X \left(e^{-\lambda X(t, 0)} \mathbf{1}(X(t, 0) > 0) \right) \\ &= E_{\delta_x}^X \left(E_{X(t/2)}^X \left(e^{-\lambda X(t/2, 0)} \mathbf{1}(X(t/2, 0) > 0) \right) \right) \\ &= E_{\delta_x}^X \left(e^{-X_{t/2}(V_{t/2}^\lambda)} - e^{-X_{t/2}(V_{t/2}^\infty)} \right) \\ &\leq E_{\delta_x}^X (X_{t/2}(V_{t/2}^\infty - V_{t/2}^\lambda)) \\ &= \int p(t/2, y - x) (V^\infty(t/2, y) - V^\lambda(t/2, y)) dy \\ &\leq t^{-1/2} \frac{e^{4/t} C(t/2)}{r-1} \lambda^{1-r}, \end{aligned}$$

where we used (4.7) in the last line. Finally use (4.2) again to obtain the required bound. \blacksquare

The critical term in (4.3) is $\exp\left(-\int_0^T H_{e^s}(Y_s) ds\right)$. To estimate its decay rate, we introduce

$$(4.9) \quad \begin{aligned} Z_T = Z_T(Y) &= \exp\left(\int_0^T F(Y_s) - H_{e^s}(Y_s) ds\right) \\ &= \exp\left(\int_0^T V^\infty(1, Y_s) - V^{e^{s/2}}(1, Y_s) ds\right) \uparrow Z_\infty(Y) \leq \infty \text{ as } T \rightarrow \infty. \end{aligned}$$

Let $\lambda_0 = \lambda_0(F) \in (\frac{1}{2}, 1)$ be as in Proposition 3.4. Choose $\varepsilon \in (0, 2\lambda_0 - 1)$ and set

$$(4.10) \quad \delta_{(4.10)} = 2\lambda_0 - \varepsilon > 1.$$

LEMMA 4.3. (a) For all $t > 0$ there is a $C(t)$, non-increasing in t , such that $\sup_x V^\infty(t, x) - V^\lambda(t, x) \leq C(t) \lambda^{1-\delta_{(4.10)}}$ for all $\lambda > 0$.

(b) There is a constant C so that $Z_\infty - Z_T \leq C e^{-T(\delta_{(4.10)}-1)/2}$ for all $T \geq 0$. In particular Z_∞ is uniformly bounded by some constant C_Z .

PROOF. By Lemma 3.2(b) and (4.1) we may first choose K and then T_0 so that

$$(4.11) \quad \sup_{|x| \geq K} F(x) < \varepsilon/2, \quad \sup_{s \geq T_0} \sup_{|x| \leq K} F(x) - H_{e^s}(x) < \varepsilon/2,$$

which in turn implies

$$(4.12) \quad \sup_{s \geq T_0} \sup_x F(x) - H_{e^s}(x) < \varepsilon/2.$$

Now take $h \equiv 1$ in Lemma 4.1, recall that m is the invariant law for Y and use the Markov property at T_0 to conclude that for $T \equiv \log(\lambda^2 t) \geq T_0$,

$$(4.13) \quad \begin{aligned} & E_{X_0}^X \left(\int e^{-\lambda X(t,x)} X(t,x) dx \right) / X_0(1) \\ & \leq E_m^Y \left(\exp \left(- \int_0^T H_{e^s}(Y_s) ds \right) \right) \\ & \leq E_m^Y \left(E_{Y_{T_0}}^Y \left(\exp \left(- \int_0^{T-T_0} H_{e^{s+T_0}}(Y_s) ds \right) \right) \right) \\ & \leq E_m^Y \left(\exp \left(- \int_0^{T-T_0} F(Y_s) ds \right) \right) \exp((\varepsilon/2)(T - T_0)) \quad \text{by (4.12)} \\ & \leq (t\lambda^2)^{\varepsilon/2} P_m^Y(\rho_F > T - T_0), \end{aligned}$$

where we recall that ρ_F is the lifetime of the killed Ornstein-Uhlenbeck process Y^F . Now use Theorem 2.3(d) to see that for $\lambda \geq \lambda'(\varepsilon)$, (4.13) is at most

$$(t\lambda^2)^{\varepsilon/2} e^{-\lambda_0(T-T_0)} [\theta \int \psi_0 dm + c' e^{-(\lambda_1 - \lambda_0)(T-T_0)}] \leq c \cdot (\sqrt{t}\lambda)^{-\delta_{(4.10)}},$$

for some universal constant c . We now may apply Lemma 4.2 to conclude that

$$\sup_x V^\infty(t,x) - V^\lambda(t,x) \leq C(t) \lambda^{1-\delta_{(4.10)}},$$

first for $\lambda > \lambda(\varepsilon)$, and then for all $\lambda > 0$, the latter using (1.11) and by increasing $C(t)$. It is easy to use the explicit form for the constant in Lemma 4.2 to see that we may take $C(t)$ to be non-increasing in t .

Turning next to (b), we have from (4.9) and (a),

$$Z_T \leq \exp \left(C(1) \int_0^\infty e^{-(\delta_{(4.10)} - 1)s/2} ds \right) \equiv c_0.$$

This bound and (a) imply that

$$\begin{aligned} Z_\infty - Z_T &\leq Z_\infty \left[1 - \exp\left(-\int_T^\infty V^\infty(1, Y_s) - V^{e^{s/2}}(1, Y_s) ds\right) \right] \\ &\leq c_0 C(1) \int_T^\infty e^{-(\delta_{(4.10)}-1)s/2} ds \\ &= \frac{2c_0 C(1)}{\delta_{(4.10)} - 1} e^{-(\delta_{(4.10)}-1)T/2}. \end{aligned}$$

■

Recall the law P_x^∞ from Theorem 2.3(e). We need a slight extension of the latter result.

LEMMA 4.4. *For any $\phi : \mathbb{R} \rightarrow \mathbb{R}$ bounded and measurable,*

$$\lim_{T \rightarrow \infty} E_x^Y(Z_T \phi(Y_T) | \rho > T) = E_x^\infty(Z_\infty) \int \phi \psi_0 dm / \theta.$$

PROOF. If $T > T_1 > 0$, then the monotonicity of $T \rightarrow Z_T$ and Lemma 4.3(b) imply that

$$(4.14) \quad E_x^Y(|Z_T - Z_{T_1}| |\phi(Y_T)| | \rho > T) \leq C \|\phi\|_\infty e^{-T_1(\delta_{(4.10)}-1)/2}.$$

So using this and the bound in Lemma 4.3(b), it clearly suffices to show that for each $T_1 > 0$,

$$(4.15) \quad \lim_{T \rightarrow \infty} E_x^Y(Z_{T_1} \phi(Y_T) | \rho > T) = E_x^\infty(Z_{T_1}) \int \phi \psi_0 dm / \theta.$$

By the Markov property and the eigenfunction expansion in Theorem 2.3(c), the lefthand side of the above is

$$\begin{aligned} &\lim_{T \rightarrow \infty} E_x^Y(Z_{T_1} 1(\rho > T_1) \int q(T - T_1, Y_{T_1}, z) \phi(z) dm(z) / P_x^Y(\rho > T)) \\ &= \lim_{T \rightarrow \infty} E_x^Y(Z_{T_1} 1(\rho > T_1) e^{-\lambda_0(T-T_1)} \int \phi \psi_0 dm \psi_0(Y_{T_1})) / P_x^Y(\rho > T) \\ (4.16) \quad &+ \lim_{T \rightarrow \infty} \delta(x, T), \end{aligned}$$

where (recall that Z_∞ is uniformly bounded)

$$|\delta(x, T)| \leq C \|\phi\|_\infty E_x^Y \left(\int \sum_{n=1}^\infty |\psi_n(Y_{T_1})| e^{-\lambda_n(T-T_1)} |\psi_n(z)| dm(z) \right) / P_x^Y(\rho > T).$$

Now use the second inequality in (2.6) with $\delta = 1/8$ to deduce that for $T - T_1 \geq s^*(1/8)$,

$$\begin{aligned} |\delta(x, T)| &\leq C \|\phi\|_\infty e^{-\lambda_1 T} c_\delta E_x^Y \left(e^{Y_{T_1}^2/8} \right) / P_x^Y(\rho > T) \\ &\leq C'(x) e^{-\lambda_1 T} / P_x^Y(\rho > T) \rightarrow 0 \text{ as } T \rightarrow \infty, \end{aligned}$$

the last convergence by (2.3) and (2.4). (2.3) also shows that the first term in (4.16) is

$$\begin{aligned} &\lim_{T \rightarrow \infty} E_x^Y(Z_{T_1} \psi_0(Y_{T_1}) 1(\rho > T_1)) \frac{e^{\lambda_0 T_1}}{\psi_0(x) \theta + r(T, x)} \int \phi \psi_0 dm \\ &= E_x^Y \left(Z_{T_1} \frac{\psi_0(Y_{T_1})}{\psi_0(x)} 1(\rho > T_1) \right) e^{\lambda_0 T_1} \int \phi \psi_0 dm / \theta \quad (\text{by (2.4)}) \\ &= E_x^\infty(Z_{T_1}) \int \phi \psi_0 dm / \theta. \end{aligned}$$

This establishes (4.15) and so completes the proof. \blacksquare

PROPOSITION 4.5. *Assume $h \geq 0$ is a bounded Borel function on the line.*

(a) *There is a universal constant $C_{4.5} > 0$ such that for any $t > 0$,*

$$\begin{aligned} &\lim_{\lambda \rightarrow \infty} (\lambda^2 t)^{\lambda_0} E_{X_0}^X \left(\int e^{-\lambda X(t, x)} h(x) X(t, x) dx \right) \\ &= C_{4.5} \int \int h(w_0 + \sqrt{t}z) \exp \left(-t^{-1} \int F(z + t^{-1/2}(w_0 - x_0)) dX_0(x_0) \right) \\ &\quad \times \psi_0(z) dm(z) dX_0(w_0). \end{aligned}$$

(b) *There is a constant C such that for all $\lambda, t > 0$,*

$$(\lambda^2 t)^{\lambda_0} E_{X_0}^X \left(\int e^{-\lambda X(t, x)} h(x) X(t, x) dx \right) \leq C \|h\|_\infty X_0(1).$$

PROOF. Set $T = \log(\lambda^2 t)$. To simplify the expression obtained in Lemma 4.1, introduce

$$\begin{aligned} I_T(w_0, y) &= \exp \left(-t^{-1} \int H_{e^T}(y + t^{-1/2}(w_0 - x_0)) dX_0(x_0) \right), \\ I_\infty(w_0, y) &= \exp \left(-t^{-1} \int F(y + t^{-1/2}(w_0 - x_0)) dX_0(x_0) \right), \end{aligned}$$

and

$$\Psi(w_0, x, T) = E_x^Y \left(\exp \left(- \int_0^T F(Y_s) ds \right) Z_T h(w_0 + \sqrt{t} Y_T) I_T(w_0, Y_T) \right).$$

then Lemma 4.1 states that for $\lambda^2 t \geq 1$,

$$(4.17) \quad \begin{aligned} & E_{X_0}^X \left(\int e^{-\lambda X(t,x)} h(x) X(t,x) dx \right) \\ &= E_0^B \left(\exp \left(- \int_0^1 V^1(u, B(u)) du \right) \int \Psi(w_0, B_1, T) dX_0(w_0) \right). \end{aligned}$$

It follows from (4.1) and Lemma 4.3(a) that

$$0 \leq F(x) - H_u(x) \leq c(1)u^{(1-\delta_{(4.10)})/2} \quad \forall u > 0, x \in \mathbb{R},$$

and so

$$(4.18) \quad 0 \leq (I_T - I_\infty)(w_0, Y_T) \leq \frac{X_0(1)}{t} c(1) e^{T(1-\delta_{(4.10)})/2} = c(1) X_0(1) t^{-(1+\delta_{(4.10)})/2} \lambda^{1-\delta_{(4.10)}}.$$

Therefore

$$(4.19) \quad \begin{aligned} \frac{\Psi(w_0, x, T)}{P_x^Y(\rho > T)} &= E_x^Y (Z_T(Y) h(w_0 + \sqrt{t} Y_T) I_T(w_0, Y_T) | \rho > T) \\ &= E_x^Y (Z_T(Y) h(w_0 + \sqrt{t} Y_T) I_\infty(w_0, Y_T) | \rho > T) + \delta(T, x, w_0), \end{aligned}$$

where by (4.18),

$$(4.20) \quad |\delta(T, x, w_0)| \leq \|h\|_\infty C_Z c(1) X_0(1) t^{-(1+\delta_{(4.10)})/2} \lambda^{1-\delta_{(4.10)}} \rightarrow 0 \text{ as } \lambda \rightarrow \infty.$$

Lemma 4.4, together with (4.19) and (4.20), show that

$$(4.21) \quad \lim_{\lambda \rightarrow \infty} \frac{\Psi(w_0, x, T)}{P_x^Y(\rho > T)} = E_x^\infty(Z_\infty) \int h(w_0 + \sqrt{t} z) I_\infty(w_0, z) \psi_0(z) dm(z) / \theta,$$

and the first line in (4.19) implies

$$(4.22) \quad \frac{|\Psi(w_0, x, T)|}{P_x^Y(\rho > T)} \leq C_Z \|h\|_\infty.$$

Apply (2.3) and (2.4) with $\delta = 1/8$ to conclude

$$(4.23) \quad |e^{\lambda_0 T} P_x^Y(\rho > T) - \theta \psi_0(x)| \leq c_\delta e^{\delta x^2} e^{-(\lambda_1 - \lambda_0)T}.$$

Returning to (4.17) we have for $\lambda^2 t \geq 1$ (assumed henceforth)

$$\begin{aligned}
& (\lambda^2 t)^{\lambda_0} E_{X_0}^X \left(\int e^{-\lambda X(t,x)} X(t,x) dx \right) \\
&= E_0^B \left[\exp \left(- \int_0^1 V^1(u, B(u)) du \right) \int \frac{\Psi(w_0, B_1, T)}{P_{B_1}^Y(\rho > T)} dX(w_0) e^{\lambda_0 T} P_{B_1}^Y(\rho > T) \right] \\
&\rightarrow E_0^B \left(\exp \left(- \int_0^1 V^1(u, B(u)) du \right) E_{B_1}^\infty(Z_\infty) \right. \\
(4.24) \quad & \left. \times \int h(w_0 + \sqrt{t}z) I_\infty(w_0, z) \psi_0(z) dm(z) dX_0(w_0) \psi_0(B_1) \right)
\end{aligned}$$

as $\lambda \rightarrow \infty$, where (4.21) is used in the last and Dominated Convergence may be applied thanks to (4.22), (4.23) and (2.5). This gives (a) with

$$(4.25) \quad C_{4.5} = E_0^B \left(\exp \left(- \int_0^1 V^1(u, B_u) du \right) E_{B_1}^\infty(Z_\infty(Y)) \psi_0(B_1) \right).$$

For (b), if $\lambda^2 t \geq 1$, use (4.22), (4.23) and (2.5) to bound the second line in the display (4.24) by

$$c \|h\|_\infty X_0(1) E_0^B(e^{B(1)^2/8}) \leq c \|h\|_\infty X_0(1).$$

If $\lambda^2 t < 1$, the expression to be bounded is at most $\|h\|_\infty E_{X_0}^X(X_t(1)) = \|h\|_\infty X_0(1)$. \blacksquare

Here is the promised refinement of Lemma 4.3(a) giving the exact rate of convergence in Proposition 3.1(a).

PROPOSITION 4.6. (a) *There is a constant $\bar{C}_{4.6}$ such that*

$$\sup_x V^\infty(t, x) - V^\lambda(t, x) \leq \bar{C}_{4.6} t^{-\frac{1}{2} - \lambda_0} \lambda^{-(2\lambda_0 - 1)} \quad \forall \lambda > 0.$$

(b) *For any $K \geq 1$ there is a $\underline{C}_{4.6}(K) > 0$ which is non-increasing in K such that for any $t > 0$,*

$$(4.26) \quad \inf_{|x| \leq K\sqrt{t}} V^\infty(t, x) - V^\lambda(t, x) \geq \underline{C}_{4.6}(K) t^{-\frac{1}{2} - \lambda_0} \lambda^{-(2\lambda_0 - 1)} \quad \forall \lambda \geq t^{-1/2},$$

and

$$(4.27) \quad \inf_{|x| \leq K\sqrt{t}} V^\infty(t, x) - V^\lambda(t, x) \geq \underline{C}_{4.6}(K) (t^{-1} \wedge t^{-\frac{1}{2} - \lambda_0}) \lambda^{-(2\lambda_0 - 1)} \quad \forall \lambda \geq 1.$$

PROOF. (a) Apply Proposition 4.5(b) and Lemma 4.2 to see that for $\lambda^2 t \geq 1$,

$$\begin{aligned} \sup_x V^\infty(t, x) - V^\lambda(t, x) &\leq e^{6/t} t^{-1/2 - \lambda_0} C(2\lambda_0 - 1)^{-1} \lambda^{-(2\lambda_0 - 1)} \\ &= C' e^{6/t} t^{-1} (\sqrt{t}\lambda)^{-(2\lambda_0 - 1)}. \end{aligned}$$

For $\lambda^2 t < 1$, by (1.11) the lefthand side of the above is at most

$$\sup_x V^\infty(t, x) \leq 2t^{-1} \leq 2e^{6/t} t^{-1} (\sqrt{t}\lambda)^{-(2\lambda_0 - 1)}.$$

This proves (a) but with an additional factor of $e^{6/t}$. This can be removed by applying the scaling result (1.7) and the above bound for $t = 1$ to conclude that for all $\lambda > 0$,

(4.28)

$$V^\infty(t, x) - V^\lambda(t, x) = t^{-1} (V^\infty - V^{\sqrt{t}\lambda})(1, x/\sqrt{t}) \leq \bar{C} t^{-1} (\sqrt{t}\lambda)^{-(2\lambda_0 - 1)}.$$

(b) Set $h = 1_{[-K, K]}$ in Proposition 4.5 with $X_0 = \delta_0$ and argue as in the first line of (4.6) to see that for $\lambda \geq \underline{\lambda}(t, K)$ and a universal positive constant c ,

$$\begin{aligned} -\frac{d}{d\lambda} \int_{-K}^K e^{-V^\lambda(t, x)} - e^{-V^\infty(t, x)} dx &= E_{\delta_0} \left(\int e^{-\lambda X(t, x)} h(x) X(t, x) dx \right) \\ &\geq c \int_{-K/\sqrt{t}}^{K/\sqrt{t}} \exp(-t^{-1} F(z)) \psi_0(z) dm(z) t^{-\lambda_0} \lambda^{-2\lambda_0} \\ (4.29) \qquad \qquad \qquad &= c_0(t, K) \lambda^{-2\lambda_0}. \end{aligned}$$

Integrate out λ to conclude that for $\lambda \geq \underline{\lambda}(t, K)$,

$$\begin{aligned} \int_{-K}^K V^\infty(t, x) - V^\lambda(t, x) dx &\geq \int_{-K}^K e^{-V^\lambda(t, x)} - e^{-V^\infty(t, x)} dx \\ (4.30) \qquad \qquad \qquad &\geq \frac{c_0(t, K)}{2\lambda_0 - 1} \lambda^{1 - 2\lambda_0}. \end{aligned}$$

If $0 < \lambda < \underline{\lambda}(t, K)$, then

(4.31)

$$\int_{-K}^K (V^\infty - V^\lambda)(t, x) dx \geq \int_{-K}^K (V^\infty - V^{\underline{\lambda}(t, K)})(t, x) dx \equiv c_1(t, K) > 0.$$

The last inequality holds since the first line of (4.8) shows strict positivity of $V^\infty(t, x) - V^\lambda(t, x)$ for all $t > 0$ and x . (4.30) and (4.31) imply that for

some $c_2(t, K) > 0$,

$$(4.32) \quad \int_{-K}^K (V^\infty - V^\lambda)(t, x) dx \geq c_2(t, K) \lambda^{1-2\lambda_0} \quad \forall \lambda \geq 1.$$

From the third line of (4.8) we have

$$\begin{aligned} e^{-V^\lambda(t, x)} - e^{-V^\infty(t, x)} &= E_{\delta_x} \left(e^{-X_{t/2}(V_{t/2}^\lambda)} - e^{-X_{t/2}(V_{t/2}^\infty)} \right) \\ &\geq E_{\delta_x} \left(e^{-X_{t/2}(V_{t/2}^\infty)} X_{t/2}(V_{t/2}^\infty - V_{t/2}^\lambda) \right) \\ &\geq E_{\delta_x} \left(e^{-(4/t)X_{t/2}(1)} X_{t/2}(V_{t/2}^\infty - V_{t/2}^\lambda) \right) \quad (\text{by (1.11)}). \end{aligned}$$

So if $r = r(t) = 4/t$ and $G = G(t) = V_{t/2}^\infty - V_{t/2}^\lambda$, we seek a lower bound on

$$(4.33) \quad E_{\delta_x}^X(e^{-rX_{t/2}(1)} X_{t/2}(G)).$$

If $V_t(\phi)$ denotes the nonlinear semigroup associated with X (see Section II.5 of Perkins (2002)), we may use the Campbell measure formula for X_t as in (4.4) along with $V_s(r) = \frac{2r}{2+rs}$ (ie. the non-linear semigroup with $\phi \equiv r$ constant) to see that if W is a Brownian motion starting at x , then for $|x| \leq K$, (4.33) equals

$$\begin{aligned} &E_x^W \times E_{\delta_x}^X \left(e^{-rX_{t/2}(1)} \exp\left(-\int_0^{t/2} \frac{2r}{2+rs} ds\right) G(W_{t/2}) \right) \\ &= \exp\left(\frac{-2r}{2+(rt/2)}\right) \left(1 + \frac{rt}{4}\right)^{-2} \int p_{t/2}(y-x) G(y) dy \\ &\geq c(t, K) \int_{-K}^K G(y) dy \\ (4.34) \quad &\geq c'(t, K) \lambda^{-(2\lambda_0-1)}, \quad \text{for all } \lambda \geq 1, \end{aligned}$$

the last by (4.32). Now use the first inequality in (4.2) and the above to derive

$$\inf_{|x| \leq K} V^\infty(t, x) - V^\lambda(t, x) \geq c'(t, K) \lambda^{-(2\lambda_0-1)} \quad \text{for all } \lambda \geq 1,$$

and where we may assume $c'(t, K) > 0$ is non-increasing in K for each t .

The scaling relation in (4.28) and the above bound for $t = 1$ shows that for all $\lambda \geq t^{-1/2}$,

$$\begin{aligned} \inf_{|x| \leq K\sqrt{t}} V^\infty(t, x) - V^\lambda(t, x) &= \inf_{|x| \leq K} t^{-1} (V^\infty - V^{\sqrt{t}\lambda})(1, x) \\ (4.35) \quad &\geq c'(1, K) t^{-\frac{1}{2}-\lambda_0} \lambda^{-(2\lambda_0-1)}. \end{aligned}$$

This gives (4.26) and it remains to prove (4.27). By (4.26) we may assume $t < 1$ and $1 \leq \lambda \leq t^{-1/2}$. Then $c(K) \equiv \inf_{|x| \leq K} V^\infty(1, x) - V^1(1, x) > 0$, where the last inequality holds by the strict positivity of the difference for each x as noted above. By scaling, as in the above display, the lefthand side of (4.35) is at least $t^{-1}c(K)$ which implies (4.27). \blacksquare

The following Tauberian Theorem is implicit in Theorem 1 of de Haan and Stadtmüller (1985) (see especially p. 350). The explicit constants below are not given there but follow from an elementary argument which can be found in an Appendix of Mueller, Mytnik and Perkins (2015).

LEMMA 4.7. *Let U be the distribution function of a sub-probability on $(0, \infty)$, set $\hat{U}(\lambda) = \int_0^\infty e^{-\lambda x} dU(x)$ and let $p > 0$.*

(a) *Assume for some $C_2 > 0$,*

$$(4.36) \quad \hat{U}(\lambda) \leq C_2 \lambda^{-p} \text{ for all } \lambda > 0.$$

Then $U(a) \leq eC_2 a^p$ for all $a > 0$.

(b) *Assume (4.36) and for some $C_1 > 0$, $\underline{\lambda} \geq 0$,*

$$(4.37) \quad \hat{U}(\lambda) \geq C_1 \lambda^{-p} \text{ for all } \lambda > \underline{\lambda}.$$

If $d_1 = \frac{C_1}{2} \left(2 \log \left(\left(\frac{2p}{e} \right)^p \frac{4eC_2}{C_1} \right) \vee 2p \vee \underline{\lambda} \right)^{-p}$, then

$$(4.38) \quad U(a) \geq d_1 a^p \text{ for all } a \in [0, 1].$$

In particular if $p \leq 1$ and $\underline{\lambda} \leq 4$, then

$$(4.39) \quad U(a) \geq \frac{C_1}{4} \left(\log \left(\frac{4eC_2}{C_1} \right) \right)^{-1} a^p \text{ for all } a \in [0, 1].$$

THEOREM 4.8. *Let $X(t, x)$ be the density of super-Brownian motion satisfying (1.1) with finite initial measure X_0 .*

(a) $P_{X_0}^X(0 < X(t, x) \leq a) \leq e\bar{C}_{4.6} X_0(1) t^{-(1/2)-\lambda_0} a^{2\lambda_0-1} \forall a, t > 0, x \in \mathbb{R}$.

(b) *For all $K \geq 1$ there is a $\underline{C}_{4.8}(K) > 0$ such that if $X_0(1) \leq Kt$ and $X_0([x - K\sqrt{t}, x + K\sqrt{t}])/X_0(1) \geq K^{-2}$, then*

$$(4.40) \quad P_{X_0}^X(0 < X(t, x) \leq a) \geq \underline{C}_{4.8}(K) X_0(1) t^{-(1/2)-\lambda_0} a^{2\lambda_0-1} \quad \forall 0 \leq a \leq \sqrt{t}.$$

In particular if $|x - x_0| \leq K\sqrt{t}$ and $t \geq K^{-1}$, then

$$(4.41) \quad P_{\delta_{x_0}}^X(0 < X(t, x) \leq a) \geq \underline{C}_{4.8}(K) t^{-(1/2)-\lambda_0} a^{2\lambda_0-1} \quad \forall 0 \leq a \leq \sqrt{t},$$

and if $X_0(1) \leq K$, $X_0([-K, K]) \geq K^{-1}$, $t \geq K^{-1}$, and $|x| \leq K$, then

$$(4.42) \quad P_{X_0}^X(0 < X(t, x) \leq a) \geq \underline{C}_{4.8}(K^2) K^{-1} t^{-(1/2)-\lambda_0} a^{2\lambda_0-1} \quad \forall 0 \leq a \leq \sqrt{t}.$$

PROOF. We will apply Lemma 4.7 to $U(a) = P(0 < X(t, x) \leq a)$. By (1.9), (1.10), and translation invariance,

$$\begin{aligned} \hat{U}(\lambda) &= E_{X_0}^X(e^{-\lambda X(t, x)} \mathbf{1}(X(t, x) > 0)) \\ (4.43) \quad &= \exp\left(-\int V^\lambda(t, y-x) dX_0(y)\right) - \exp\left(-\int V^\infty(t, y-x) dX_0(y)\right). \end{aligned}$$

(a) Use Proposition 4.6(a) to see that for $\lambda > 0$,

$$(4.44) \quad \hat{U}(\lambda) \leq \int (V^\infty - V^\lambda)(t, y-x) dX_0(y) \leq \bar{C}_{4.6} X_0(1) t^{-(1/2)-\lambda_0} \lambda^{-(2\lambda_0-1)},$$

and so (a) is immediate from Lemma 4.7(a).

(b) Consider first $t = 1$. By (4.43), (1.11) and (4.26) for $\lambda \geq 1$,

$$\begin{aligned} \hat{U}(\lambda) &\geq \exp\left(-\int V^\infty(1, y-x) dX_0(y)\right) \int (V^\infty - V^\lambda)(1, y-x) dX_0(y) \\ &\geq \exp(-2X_0(1)) \underline{C}_{4.6}(K) X_0([x-K, x+K]) \lambda^{-(2\lambda_0-1)} \\ &\equiv C_1(K, x, X_0) \lambda^{-(2\lambda_0-1)}. \end{aligned}$$

So by this and (4.44) we may use (4.39) in Lemma 4.7 (with $p = 2\lambda_0 - 1 < 1$) to see that

$$P_{X_0}^X(0 < X(1, x) \leq a) \geq \frac{C_1(K, x, X_0)}{4} \left(\log\left(\frac{4e\bar{C}_{4.6}X_0(1)}{C_1(K, x, X_0)}\right)\right)^{-1} a^{2\lambda_0-1} \quad \forall a \in [0, 1].$$

For general t we may use the scaling relation (4.28) and (4.43) to see that if $X_0^t(A) = t^{-1}X_0(\sqrt{t}A)$, then by the above for $0 \leq a \leq \sqrt{t}$,

$$(4.45) \quad \begin{aligned} &P_{X_0}^X(0 < X(t, x) \leq a) \\ &= P_{X_0^t}^X(0 < X(1, x/\sqrt{t}) \leq a/\sqrt{t}) \\ &\geq \frac{C_1(K, x/\sqrt{t}, X_0^t)}{4} \left(\log\left(\frac{4e\bar{C}_{4.6}X_0(1)/t}{C_1(K, x/\sqrt{t}, X_0^t)}\right)\right)^{-1} (a/\sqrt{t})^{2\lambda_0-1}. \end{aligned}$$

A simple calculation, using the definition of C_1 , shows that there is a $C_0(K) > 0$ so that if X_0 is as in (b), then

$$C_1(K, x/\sqrt{t}, X_0^t) \geq C_0(K) X_0(1)/t,$$

and so

$$\log\left(\frac{4e\bar{C}_{4.6}X_0(1)/t}{C_1(K, x, X_0)}\right) \leq C_3(K).$$

Use the above in (4.45) to conclude that

$$P_{X_0}^X(0 < X(t, x) \leq a) \geq C(K)X_0(1)t^{-1}(a/\sqrt{t})^{2\lambda_0-1}.$$

This proves (4.40), and the last two assertions follow by elementary reasoning. (The last follows first for $K \geq 4$, and hence for all $K \geq 1$.) \blacksquare

5. Proof of Theorem 1.3.

5.1. *Lower Bound on the Hausdorff Dimension.* Recall that X is as in (1.1) and the boundary of the zero set is

$$(5.1) \quad BZ_t = \partial(\{x : X(t, x) = 0\}) = \{x : X(t, x) = 0, \forall \delta > 0 X_t((x-\delta, x+\delta)) > 0\}.$$

The key step in our lower bound on the Hausdorff dimension of the boundary of the zero set, $\dim(BZ_t)$, is the following second moment bound. The bound will depend on a diffusion parameter $\sigma_0^2 > 1$, whose exact value is not important (but $\sigma_0^2 = 6$ will work). For a finite initial measure X_0 and $t > 0$, define $X_0 p_u(x) = \int p_u(x - w_0)X_0(dw_0)$ and

$$(5.2) \quad \begin{aligned} h_{t, X_0}(z_1, z_2) &= t^{-2\lambda_0} \prod_{i=1}^2 X_0 p_{\sigma_0^2 t}(z_i) \\ &+ t^{-\lambda_0} \int_0^t (t-s)^{-\lambda_0} p_{8\sigma_0^2(t-s)}(z_1 - z_2) X_0 p_{4\sigma_0^2 t}(z_1) ds. \end{aligned}$$

PROPOSITION 5.1. *There is a constant $C_{5.1}$ such that for all $\lambda^2 \geq (9/t)$, and all z_1, z_2 ,*

$$(a) \quad \lambda^{4\lambda_0} E_{X_0}^X(X_t(z_1)X_t(z_2)e^{-\lambda X_t(z_1) - \lambda X_t(z_2)}) \leq C_{5.1} h_{t, X_0}(z_1, z_2).$$

$$(b) \quad h_{t, X_0}(z_1, z_2) \leq C_{5.1}(t^{-\lambda_0-1/2}X_0(1)|z_1 - z_2|^{1-2\lambda_0} + (t^{-\lambda_0-1/2}X_0(1))^2).$$

We will prove this result in Section 6 below, but first show how it can be used to obtain lower bounds on $\dim(BZ_t)$. The lack of symmetry between z_1 and z_2 in the definition of h_{t, X_0} indicates that our bound is not optimal, but it is the negative power along the diagonal which will be important for us, and our results suggest that this is optimal.

If we introduce random measures

$$(5.3) \quad L_t^\lambda(\phi) = (\lambda^2 t)^{\lambda_0} \int \phi(x) X(t, x) e^{-\lambda X(t, x)} dx,$$

then Proposition 4.5 shows that for some finite measure ℓ_t and any bounded Borel function ϕ ,

$$\lim_{\lambda \rightarrow \infty} E(L_t^\lambda(\phi)) = \ell_t(\phi),$$

and the above result shows that $E(L_t^\lambda(1)^2)$ remains bounded as $\lambda \rightarrow \infty$.

Conjecture. There is a random finite non-trivial measure L_t on \mathbb{R} such that for any bounded continuous ϕ ,

$$L_t^\lambda(\phi) \rightarrow L_t(\phi) \text{ in } L^2 \text{ as } \lambda \rightarrow \infty.$$

Assuming this, it is then not hard to show that for some sequence $\lambda_n \uparrow \infty$, $L_t^{\lambda_n}$ approaches L_t weakly on the space of measures a.s. and that L_t is supported by BZ_t . We further conjecture that $L_t(1) > 0$ a.s. on $\{X_t(1) > 0\}$.

If $g_\beta(r) = r^{-\beta}$ for $\beta > 0$, and μ is a finite measure on \mathbb{R} , and A is an analytic subset of \mathbb{R} , let

$$\langle \mu, \mu \rangle_{g_\beta} = \int \int g_\beta(|x - y|) d\mu(x) d\mu(y),$$

$$I(g_\beta)(A) = \inf\{\langle \mu, \mu \rangle_{g_\beta} : \mu \text{ a probability supported by } A\}.$$

Then the g_β -capacity of A is $C(g_\beta)(A) = I(g_\beta)(A)^{-1}$ (see, e.g. Hawkes (1979), Section 3).

THEOREM 5.2. *For every $K \geq 1$ there is a positive constant $C_{5.2}(K)$, non-increasing in K , so that for any analytic subset A of $[-K, K]$, initial measure, X_0 , satisfying $X_0(1) \leq K$ and $X_0([-K, K]) \geq 1/K$, and $t \in [K^{-1}, K]$,*

$$P_{X_0}^X(A \cap BZ_t \neq \emptyset) \geq C_{5.2}(K)C(g_{2\lambda_0-1})(A).$$

PROOF. Let $0 < \delta_0 < e^{-1}$, and let $0 < k_1 < 1 < k_2$ be the solutions of $\delta_0 = k_i e^{-k_i}$. We will choose δ_0 small enough below, noting that as $\delta \downarrow 0$, $k_1(\delta) \downarrow 0$ and $k_2(\delta) \uparrow \infty$. We approximate BZ_t by

$$BZ(\varepsilon) \equiv \{x : X(t, x)e^{-X(t, x)/\varepsilon} \geq \delta_0 \varepsilon\} = \{x : k_1 \varepsilon \leq X(t, x) \leq k_2 \varepsilon\},$$

where the second equality is by an elementary calculus argument and we have suppressed dependence on $t > 0$. Now fix $K \geq 1$ and assume X_0 and t are as in the statement of the Theorem. Let F be a compact subset of $[-K, K]$. If $I(A) = I(g_{2\lambda_0-1})(A)$ and $C(A) = C(g_{2\lambda_0-1})(A)$, we may choose $\{x_i^N : 1 \leq i \leq N\} \subset F$ so that (suppressing the superscript N) as $N \rightarrow \infty$,

$$(5.4) \quad I_N \equiv \frac{1}{N(N-1)} \sum_i \sum_{j \neq i} |x_i - x_j|^{-(2\lambda_0-1)} \rightarrow I(F) = 1/C(F).$$

(See, for example, Lemma A of Taylor (1961).) By Theorem 4.8 there are constants $\bar{C}(K) > \underline{C}(K) > 0$ so that for $0 < \varepsilon < \varepsilon_0(K)$,

$$\begin{aligned} P(k_1\varepsilon \leq X(t, x_i) \leq k_2\varepsilon) \\ &\geq [\underline{C}(K)k_2^{2\lambda_0-1} - \bar{C}(K)k_1^{2\lambda_0-1}]\varepsilon^{2\lambda_0-1} \\ &\geq \varepsilon^{2\lambda_0-1}, \end{aligned}$$

where in the last line we have chosen $\delta_0 = \delta_0(K)$ sufficiently small so that k_2 is very large and k_1 is close to 0. Therefore by inclusion-exclusion and Proposition 5.1, for $\varepsilon < \varepsilon_0(K)$,

$$\begin{aligned} P_{X_0}^X(F \cap BZ(\varepsilon) \neq \emptyset) &\geq \sum_{i=1}^N P_{X_0}^X(x_j \in BZ(\varepsilon)) - \sum \sum_{i \neq j} P_{X_0}^X(x_i, x_j \in BZ(\varepsilon)) \\ &\geq N\varepsilon^{2\lambda_0-1} - \sum_{i \neq j} \frac{E_{X_0}^X(X(t, x_i)X(t, x_j)e^{-X(t, x_i)/\varepsilon - X(t, x_j)/\varepsilon})}{(\delta_0\varepsilon)^2} \\ &\geq N\varepsilon^{2\lambda_0-1} - c'(K) \sum_{i \neq j} (1 + |x_i - x_j|^{1-2\lambda_0})\varepsilon^{4\lambda_0-2} \\ &\geq N\varepsilon^{2\lambda_0-1} - c(K)[N\varepsilon^{2\lambda_0-1}]^2 I_N. \end{aligned}$$

Now choose $\varepsilon_N \rightarrow 0$ so that $N\varepsilon_N^{2\lambda_0-1} = \frac{1}{2I_N c(K)}$. Therefore

$$P_{X_0}^X(F \cap BZ(\varepsilon_N) \neq \emptyset) \geq \frac{1}{4c(K)I_N} \rightarrow (4c(K))^{-1}C(F) \text{ as } N \rightarrow \infty.$$

This implies that

$$P_{X_0}^X(F \cap BZ(\varepsilon_N) \neq \emptyset \text{ infinitely often}) \geq (4c(K))^{-1}C(F).$$

An elementary argument shows that the event on the left-hand side implies that $F \cap BZ_t \neq \emptyset$ and so the proof is complete for $A = F$ compact. Use the inner regularity of capacity to now extend the result to analytic subsets of $[-K, K]$. \blacksquare

COROLLARY 5.3. *Let A be an analytic set such that $C(g_{2\lambda_0-1})(A) > 0$. Then for any non-zero X_0 and any $t > 0$, $P_{X_0}^X(A \cap BZ_t \neq \emptyset) > 0$.*

PROOF. Choose K large enough so that $C(g_{2\lambda_0-1})(A \cap [-K, K]) > 0$ (inner regularity of capacity), $X_0([-K, K]) \geq 1/K$, and $K > t \vee t^{-1} \vee X_0(1)$. The result is then immediate from the above theorem. \blacksquare

LEMMA 5.4. *Let $\alpha = 2\lambda_0 - 1$ and let Z be the subordinator starting at zero with Lévy measure ν where*

$$H(x) = \nu([x, \infty)) = x^{-\alpha}(\log((1/x) + 1))^2.$$

Then

$$(5.5) \quad C(g_\alpha)(\{Z_s : s \in (0, 1)\}) > 0 \text{ a.s.}$$

and

$$(5.6) \quad \begin{aligned} &\text{any analytic set } A \text{ satisfying } \dim(A) < 2 - 2\lambda_0 \text{ is polar for } Z, \\ &\text{i.e., } P(Z_t \in A \text{ for some } t > 0) = 0. \end{aligned}$$

PROOF. If

$$g(\lambda) = \int_0^\infty 1 - e^{-\lambda u} d\nu(u) = \lambda \int_0^\infty H(u)e^{-\lambda u} du,$$

then Karamata's Abelian-Tauberian theorem and a short calculation shows that for some $c > 0$, $\lim_{\lambda \rightarrow \infty} g(\lambda)/(\lambda^\alpha(\log \lambda)^2) = c$. If

$$f(r) = r^\alpha(\log(1/r))^{-2}(\log \log(1/r))^{1-\alpha}$$

and $f - m(A)$ is the f -Hausdorff measure of A , then Fristedt and Pruitt (1971) (see also Lemma 2.1 of Hawkes (1975)) shows that

$$f - m(\{Z_s : s \leq t\}) = c_H t \text{ for some positive } c_H.$$

By Taylor (1961), this implies (5.5). Note that $\lim_{t \rightarrow 0^+} \frac{tg(1/t)}{t^{1-\alpha} \log(1/t)^2} = c$, and so by Theorem 4.4(iii) of Hawkes (1975), (5.6) also holds. (A short calculus argument shows that $(\log H)'' \geq 0$ and so by Theorem 2.1 of Hawkes (1975), the condition B of Theorem 4.4(iii) is valid.) \blacksquare

Recall that $\dim(B)$ is the Hausdorff dimension of a set $B \subset \mathbb{R}$.

THEOREM 5.5. *If $X_0 \neq 0$ and $t > 0$, then $P_{X_0}^X(\dim(BZ_t) \geq 2 - 2\lambda_0) > 0$.*

PROOF. Let Z_t be as in Lemma 5.4 and set $F = \{Z_s : s \in (0, 1)\}$. We assume Z is independent of the super-Brownian motion X and so work on the product space $(\Omega, \mathcal{F}, P) = (\Omega_X, \mathcal{F}_X, P_{X_0}^X) \times (\Omega_Z, \mathcal{F}_Z, P_0^Z)$. By (5.5) and Corollary 5.3 we have $P(BZ_t(\omega_1) \cap F(\omega_2) \neq \emptyset) > 0$. This implies that,

$$P_{X_0}^X(\{\omega_1 : P_0^Z(\{\omega_2 : F(\omega_2) \cap BZ_t(\omega_1) \neq \emptyset\}) > 0\}) > 0.$$

By (5.6), this in turn implies that $P_{X_0}^X(\dim(BZ_t) \geq 2 - 2\lambda_0) > 0$, as required. \blacksquare

5.2. *Upper Bound on the Hausdorff Dimension.* We begin with a classical result on the modulus of continuity of the density $X(t, x)$, $t > 0$, of super-Brownian motion, the solution of (1.1), where the initial condition is an arbitrary finite measure X_0 .

Notation. If $(t_i, x_i) \in \mathbb{R}_+ \times \mathbb{R}$ for $i = 1, 2$, let

$$d((t_1, x_1), (t_2, x_2)) = \sqrt{|t_1 - t_2|} + |x_1 - x_2|.$$

The following is an easy consequence of Theorem II.4.2 (and its proof) of Perkins (2002) and standard consequences of Kolmogorov's continuity criteria. (One should use the decomposition (III.4.11) in Perkins (2002) for $t \geq 2t_0$.)

PROPOSITION 5.6. *If $\xi \in (0, 1/2)$, then for any $K \in \mathbb{N}$ there is a $\rho(K, \xi, \omega) > 0$ a.s. such that*

$$\forall (t, x) \in [K^{-1}, K] \times [-K, K], \forall (t', x') \in [0, \infty) \times \mathbb{R}, d((t, x), (t', x')) \leq \rho \quad (5.7)$$

implies $|X(t', x') - X(t, x)| \leq d((t', x'), (t, x))^\xi$.

Moreover there is a $\delta_{5.6} > 0$, depending only on ξ , and a constant $C(X_0(1), K, \xi)$ so that

$$(5.8) \quad P_{X_0}^X(\rho(K, \xi) \leq r) \leq C(X_0(1), K, \xi)r^{\delta_{5.6}} \text{ for all } r > 0.$$

Near the zero set $Z = \{(t, x) : X(t, x) = 0\}$ one can improve the above modulus since the noise term in (1.1) will be mollified. This idea plays a central role in the pathwise uniqueness arguments in Mytnik, Perkins and Sturm (2006) and Mytnik and Perkins (2011). The following result can be derived using the same proof as that of Theorem 2.3 in Mytnik and Perkins (2011) (see also Corollary 4.2 of Mytnik, Perkins and Sturm (2006)). There are a few minor changes as these references study the difference of two solutions as opposed to the solution itself. The minor changes that are required are outlined in an Appendix of Mueller, Mytnik and Perkins (2015).

THEOREM 5.7. *If $\xi \in (0, 1)$, then for any $K \in \mathbb{N}$ there is a $\rho_{5.7}(K, \xi, \omega) > 0$ a.s. such that*

$$\forall (t, x) \in Z \text{ such that } t \geq K^{-1}, \forall (t', x') \in [0, \infty) \times \mathbb{R}, d((t, x), (t', x')) \leq \rho_{5.7} \quad (5.9)$$

implies $X(t', x') \leq d((t', x'), (t, x))^\xi$.

In order to get a good cover of BZ_t we need a version of our low density bound Theorem 4.8(a) for small intervals. (Set $M = 1$ and $\varepsilon = a$ in the following result to compare.)

THEOREM 5.8. *There is a $C_{5.8} > 0$ so that for all $t > 0$, $M \geq 1$, $0 < \varepsilon \leq \sqrt{t}$, $x \in \mathbb{R}$, and X_0 ,*

$$P_{X_0}(0 < X_t([x, x + \varepsilon]) \leq \varepsilon^2 M) \leq C_{5.8} X_0(1) t^{-(1/2) - \lambda_0} M^{19} \varepsilon^{2\lambda_0 - 1}.$$

This will be proved in Section 7 below. We now show how it gives an upper bound on $\dim(BZ_t)$.

THEOREM 5.9. *For all $t > 0$, $\dim(BZ_t) \leq 2 - 2\lambda_0 P_{X_0}^X$ -a.s.*

PROOF. By scaling we may take $t = 1$. By translation invariance, it suffices to show

$$(5.10) \quad \dim(BZ_1 \cap [0, 1]) \leq 2 - 2\lambda_0 P_{X_0}^X - \text{a.s.}$$

Fix $\delta > 0$ and choose $\xi \in (0, 1)$ so that $19(1 - \xi) < \delta$. Let $\rho_{5.7}(\xi, \omega)$ be as in Theorem 5.7 with $K = 2$. Then by Theorems 5.8 and 5.7 for any $x \in [0, 1]$ and $0 < \varepsilon \leq 1$,

$$\begin{aligned} P_{X_0}^X([x, x + \varepsilon] \cap BZ_1 \neq \emptyset, 3\varepsilon < \rho_{5.7}) &\leq P_{X_0}^X(0 < X_1((x - \varepsilon, x + 2\varepsilon)) \leq 6\varepsilon^{\xi+1}) \\ &\leq C X_0(1) \varepsilon^{2\lambda_0 - 1 + 19(\xi - 1)} \\ &\leq C X_0(1) \varepsilon^{2\lambda_0 - 1 - \delta}. \end{aligned}$$

A standard covering argument using intervals of the form $[i/N, (i + 1)/N]$ now gives $\dim(BZ_1 \cap [0, 1]) \leq 2 - 2\lambda_0 + \delta$ and (5.10) follows. \blacksquare

Theorem 1.3 is now immediate from Theorems 5.5 and 5.9.

6. Proof of Proposition 5.1. We now consider the proof of Proposition 5.1. As before, Y is the Ornstein-Uhlenbeck process starting with law μ under P_μ^Y and we enlarge this space to include an independent random variable W_0 with “law” X_0 . The same convention is in place on the space carrying a standard Brownian motion starting at 0 under P_0^B , and P_t^B denotes the Brownian semigroup. We fix a pair of bounded non-negative continuous functions on the line, ϕ_i , $i = 1, 2$, and define

$$(6.1) \quad \psi^\lambda(u, x) = E_0^B \left(\phi_2(\lambda^{-1} B_{u\lambda^2} + x) \exp \left\{ - \int_0^{\lambda^2 u} V^1(r, B_r) dr \right\} \right).$$

Recall from the end of Section 3 that $\rho = \rho_F$. We note that the parameters $\lambda_i \geq 0$, $i = 1, 2$, in the following result are not the eigenvalues $\lambda_i(F)$ from Section 3.

LEMMA 6.1. *There is a constant $C_{6.1}$ so that for all $\lambda_i \geq 0$, $\lambda_i^2 t \geq 1$, if $T^i = \log(\lambda_i^2 t)$, then*

$$\begin{aligned}
& E_{X_0}^X \left(\int \int \phi_1(x_1) \phi(x_2) \exp(-\lambda_1 X_t(x_1) - \lambda_2 X_t(x_2)) dX_t(x_1) dX_t(x_2) \right) \\
& \leq C_{6.1} \prod_{i=1}^2 E_m^Y(\phi_i(W_0 + \sqrt{t} Y_{T^i}) 1(\rho > T^i)) \\
& \quad + \int_0^t E_0^B \left(\int \phi_1(W_0 + B_t) \exp\left(-\int_0^t V_{t-r}^{\lambda_1}(B_t - B_r) dr\right) \right. \\
(6.2) \quad & \left. \times \psi^{\lambda_2}(t-s, W_0 + B_s) \right) ds.
\end{aligned}$$

PROOF. For $\lambda_i \geq 0$, $x_i \in \mathbb{R}$ ($i=1,2$) and $\delta > 0$ let $\vec{\lambda} = (\lambda_1, \lambda_2)$, $\vec{x} = (x_1, x_2)$ and $V_t(x) = V_t^{\delta, \vec{\lambda}, \vec{x}}(x)$ be the unique smooth solution of

$$(6.3) \quad \frac{\partial V}{\partial t} = \frac{\Delta V}{2} - \frac{V^2}{2}, \quad V_0(\cdot) = \lambda_1 p_\delta(\cdot - x_1) + \lambda_2 p_\delta(\cdot - x_2),$$

so that

$$(6.4) \quad E_{X_0}^X(e^{-\lambda_1 P'_\delta X_t(x_1) - \lambda_2 P'_\delta X_t(x_2)}) = \exp(-X_0(V_t))$$

(recall (3.5) and (3.6)). Let $U_t^{(j)}(x) \equiv U_t^{(j), \delta, \vec{\lambda}, \vec{x}}(x)$ denote the unique solution of

$$(6.5) \quad \frac{\partial U_t^{(j)}}{\partial t} = \frac{\Delta U_t^{(j)}}{2} - V_t^{\delta, \vec{\lambda}, \vec{x}} U_t^{(j)}, \quad U_0^{(j)}(\cdot) = p_\delta(\cdot - x_j),$$

so that by Feynmann-Kac (see p. 268 of Karatzas and Shreve (1991)),

$$(6.6) \quad U_t^{(j)}(x) = E_x^B \left(p_\delta(B_t - x_j) \exp\left(-\int_0^t V_{t-s}^{\delta, \vec{\lambda}, \vec{x}}(B_s) ds\right) \right).$$

Next define

$$(6.7) \quad \tilde{V}_t^{\delta, \vec{\lambda}, \vec{x}}(x) = V_t^{\delta, 0, \lambda_2, \vec{x}}(x) + \int_0^{\lambda_1} U_t^{(1), \delta, \lambda, \lambda_2, \vec{x}}(x) d\lambda.$$

In the above it is easy to justify differentiation with respect to t and x through the integral and so by (6.5) we have

$$(6.8) \quad \frac{\partial \tilde{V}}{\partial t} - \frac{\Delta \tilde{V}}{2} = -\frac{(V_t^{\delta,0,\lambda_2,\vec{x}})^2}{2} - \int_0^{\lambda_1} V_t^{\delta,\lambda,\lambda_2,\vec{x}} U_t^{(1),\delta,\lambda,\lambda_2,\vec{x}} d\lambda,$$

and by integration by parts,

$$(6.9) \quad \frac{(\tilde{V}_t^{\delta,\vec{\lambda},\vec{x}})^2}{2} = \frac{(V_t^{\delta,0,\lambda_2,\vec{x}})^2}{2} + \int_0^{\lambda_1} V_t^{\delta,\lambda,\lambda_2,\vec{x}} U_t^{(1),\delta,\lambda,\lambda_2,\vec{x}} d\lambda.$$

A quick check of the initial condition at $t = 0$ and a comparison of (6.8) and (6.9) show that \tilde{V} satisfies (6.3) and so $\tilde{V} = V$. Continuity of V in $\vec{\lambda}$ is clear from (6.4) and hence continuity of $U^{(j)}$ in $\vec{\lambda}$ follows from (6.6). This allows us to differentiate (6.7) with respect to λ_1 and conclude

$$(6.10) \quad \frac{\partial V^{\delta,\vec{\lambda},\vec{x}}}{\partial \lambda_1} = U^{(1),\delta,\vec{\lambda},\vec{x}}, \text{ and symmetrically, } \frac{\partial V^{\delta,\vec{\lambda},\vec{x}}}{\partial \lambda_2} = U^{(2),\delta,\vec{\lambda},\vec{x}}.$$

We can differentiate the left-hand side of (6.4) (set $X_0 = \delta_x$) with respect to λ_1 and then λ_2 under the integral. This shows that $U_t^{\delta,\vec{\lambda},\vec{x}}(x) = \frac{\partial^2}{\partial \lambda_2 \partial \lambda_1} (V^{\delta,\vec{\lambda},\vec{x}})$ exists and is continuous in $\vec{\lambda}$, and the differentiation yields (use (6.10))

$$(6.11) \quad \begin{aligned} E_{X_0}^X \left(\prod_{i=1}^2 (P'_\delta X_t(x_i) e^{-\lambda_i P'_\delta X_t(x_i)}) \right) \\ = \exp(-X_0(V_t^{\delta,\vec{\lambda},\vec{x}})) [X_0(U^{(1),\delta,\vec{\lambda},\vec{x}}) X_0(U^{(2),\delta,\vec{\lambda},\vec{x}}) - X_0(U^{\delta,\vec{\lambda},\vec{x}})] \\ \equiv T_1(\delta, \vec{\lambda}, \vec{x}) - T_2(\delta, \vec{\lambda}, \vec{x}). \end{aligned}$$

It is clear from (3.6) that if V^{δ,λ_i,x_i} is the solution to (6.3) with initial condition $V_0^{\delta,\lambda_i,x_i}(\cdot) = \lambda_i p_\delta(\cdot - x_i)$, then

$$(6.12) \quad V^{\delta,\vec{\lambda},\vec{x}} \geq V^{\delta,\lambda_i,x_i} \quad \text{for } i = 1, 2.$$

Using the above and (6.6), we see that

$$\begin{aligned} & \int \int \phi_1(x_1) \phi_2(x_2) T_1(\delta, \vec{\lambda}, x_1, x_2) dx_1 dx_2 \\ & \leq \int \int \prod_{i=1}^2 \phi_i(x_i) E_0^B \left(p_\delta(W_0 + B_t - x_i) \right. \\ & \quad \left. \times \exp\left(-\int_0^t V_{t-s}^{\delta,\lambda_i,x_i}(W_0 + B_s) ds\right) \right) dx_1 dx_2 \end{aligned}$$

The above equals

$$\int \int \prod_{i=1}^2 \phi_i(x_i) E_0^B \left(p_\delta(W_0 + B_t - x_i) \right. \\ \left. \times \exp \left(- \int_0^t V_{t-s}^{\delta, \lambda_i, 0}(W_0 + B_s - x_i) ds \right) \right) dx_1 dx_2$$

(6.13)

$$= \prod_{i=1}^2 E_0^B \left(\phi_i(W_0 + B_{t+\delta}) \exp \left(- \int_0^t V_{t-s}^{\delta, \lambda_i, 0}(B_s - B_{t+\delta}) ds \right) \right).$$

An elementary argument using (3.6) shows that

$$(6.14) \quad \lim_{\delta \downarrow 0} \exp(-V_{t-s}^{\delta, \lambda_i, 0}(B_s - B_{t-\delta})) = \exp(-V_{t-s}^{\lambda_i}(B_s - B_t)).$$

The elementary bound $V(\phi)(t, x) \leq P'_t(\phi)(x)$ for ϕ non-negative, bounded and measurable shows that

$$(6.15) \quad V_{t-s}^{\delta, \lambda_i, 0}(B_s - B_{t+\delta}) \leq P'_{t-s}(\lambda_i p_\delta)(B_s - B_{t+\delta}) \leq \lambda_i(t-s)^{-1/2}.$$

The above two results allow us to apply Dominated Convergence in (6.13) and conclude that

$$\begin{aligned} & \limsup_{\delta \downarrow 0} \int \int \phi_1(x_1) \phi_2(x_2) T_1(\delta, \vec{\lambda}, x_1, x_2) dx_1 dx_2 \\ & \leq \prod_{i=1}^2 E_0^B \left(\phi_i(W_0 + B_t) \exp \left(- \int_0^t V^{\lambda_i}(t-s, B_t - B_s) ds \right) \right) \\ & = \prod_{i=1}^2 E_0^B \left(\phi_i(W_0 + B_t) \exp \left(- \int_0^t V^{\lambda_i}(s, B_s) ds \right) \right) \text{ (time reversal)} \\ & = \prod_{i=1}^2 E_0^B \left(\phi_i(W_0 + \lambda_i^{-1} B_{\lambda_i^2 t}) \exp \left(- \int_0^{\lambda_i^2 t} V^1(u, B_u) du \right) \right) \\ & \quad \text{(by the scaling (1.7))} \\ & \leq \prod_{i=1}^2 E_m^B \left(\phi_i(W_0 + \lambda_i^{-1} B_{\lambda_i^2 t-1}) \right. \\ & \quad \left. \times \exp \left(- \int_0^{\lambda_i^2 t-1} (1+u) V^1(1+u, B_u) (1+u)^{-1} du \right) \right), \end{aligned}$$

where we have used the Markov property at $t = 1$ in the last line. Now proceed as in the proof of Proposition 4.5, using Lemma 4.3(b), to conclude that

$$(6.16) \quad \begin{aligned} & \limsup_{\delta \downarrow 0} \int \int \phi_1(x_1) \phi_2(x_2) T_1(\delta, \vec{\lambda}, x_1, x_2) dx_1 dx_2 \\ & \leq C \prod_{i=1}^2 E_m^Y(\phi_i(W_0 + \sqrt{t} Y_{T^i}) 1(\rho > T^i)). \end{aligned}$$

Consider next the contribution from T_2 in (6.11). By (6.10) and (6.6) we have

$$\frac{\partial}{\partial \lambda_2} V_{t-s}^{\delta, \vec{\lambda}, \vec{x}}(x) \leq E_x^B(p_\delta(B_{t-s} - x_2)) \leq (t-s)^{-1/2}.$$

As the above bound is Lebesgue integrable, the Dominated Convergence Theorem and (6.10) give

$$\frac{\partial}{\partial \lambda_2} \int_0^t V_{t-s}^{\delta, \vec{\lambda}, \vec{x}}(B_s) ds = \int_0^t U_{t-s}^{(2), \delta, \vec{\lambda}, \vec{x}}(B_s) ds.$$

This in turn allows us to differentiate (6.6) (with $j = 1$) with respect to λ_2 and conclude

$$\begin{aligned} U_t^{\delta, \vec{\lambda}, \vec{x}}(x) &= -E_x^B \left(p_\delta(B_t - x_1) \exp\left(-\int_0^t V_{t-r}^{\delta, \vec{\lambda}, \vec{x}}(B_r) dr\right) \right. \\ & \quad \left. \times \int_0^t U_{t-s}^{(2), \delta, \vec{\lambda}, \vec{x}}(B_s) ds \right). \end{aligned}$$

Therefore $-T_2(\delta, \vec{\lambda}, \vec{x}) \geq 0$ and

$$\begin{aligned} & - \int \int T_2(\delta, \vec{\lambda}, \vec{x}) \phi_1(x_1) \phi_2(x_2) dx_1 dx_2 \\ & \leq \int \int E_0^B \left(p_\delta(W_0 + B_t - x_1) \exp\left(-\int_0^t V_{t-r}^{\delta, \vec{\lambda}, \vec{x}}(W_0 + B_r) dr\right) \phi_1(x_1) \right. \\ & \quad \left. \times \int_0^t U_{t-s}^{(2), \delta, \vec{\lambda}, \vec{x}}(W_0 + B_s) ds \right) \phi_2(x_2) dx_1 dx_2 \\ & \leq \int_0^t E_0^B \left(\int p_\delta(W_0 + B_t - x_1) \exp\left(-\int_0^t V_{t-r}^{\delta, \lambda_1, x_1}(W_0 + B_r) dr\right) \phi_1(x_1) \right. \\ & \quad \left. \times \int E_{\hat{B}(s)+W_0}^{\hat{B}} \left(p_\delta(\hat{B}_{t-s} - x_2) \exp\left(-\int_0^{t-s} V_{t-s-u}^{\delta, \lambda_2, x_2}(\hat{B}_u) du\right) \right) \phi_2(x_2) dx_2 dx_1 \right) ds. \end{aligned}$$

In the last line \hat{B} is a Brownian motion and we have used (6.12) and (6.6). The above equals

$$\begin{aligned} & \int_0^t E_0^B \left(\phi_1(W_0 + B_{t+\delta}) \exp\left(-\int_0^t V_{t-r}^{\delta, \lambda_1, 0}(B_r - B_{t+\delta}) dr\right) \right. \\ & \quad \left. \times E_{\hat{B}_s + W_0}^{\hat{B}} \left(\phi_2(\hat{B}_{t-s+\delta}) \exp\left(-\int_0^{t-s} V_{t-s-u}^{\delta, \lambda_2, 0}(\hat{B}_u - \hat{B}_{t-s}) du\right) \right) \right) ds \\ & \rightarrow \int_0^t E_0^B \left(\phi_1(W_0 + B_t) \exp\left(-\int_0^t V_{t-r}^{\lambda_1}(B_t - B_r) dr\right) \right. \\ & \quad \left. \times E_{\hat{B}_s + W_0}^{\hat{B}} \left(\phi_2(\hat{B}_{t-s}) \exp\left(-\int_0^{t-s} V_{t-s-u}^{\lambda_2}(\hat{B}_{t-s} - \hat{B}_u) du\right) \right) \right) ds, \end{aligned}$$

as $\delta \downarrow 0$. Here we used Dominated Convergence and (6.15) as in (6.14). If $B'_u = \hat{B}_{t-s} - \hat{B}_{t-s-u}$ (a Brownian motion starting at 0), then the above shows that

$$\begin{aligned} & \limsup_{\delta \downarrow 0} \left[-\int \int T_2(\delta, \vec{\lambda}, \vec{x}) \phi_1(x_1) \phi_2(x_2) dx_1 dx_2 \right] \\ & \leq \int_0^t E_0^B \left(\phi_1(W_0 + B_t) \exp\left(-\int_0^t V_{t-r}^{\lambda_1}(B_t - B_r) dr\right) \right. \\ & \quad \left. \times E_0^{B'} \left(\phi_2(W_0 + B_s + B'_{t-s}) \exp\left(-\int_0^{t-s} V_r^{\lambda_2}(B'_r) dr\right) \right) \right) ds \\ & = \int_0^t E_0^B \left(\phi_1(W_0 + B_t) \exp\left(-\int_0^t V^{\lambda_1}(t-r, B_t - B_r) dr\right) \right. \\ & \quad \left. \times \psi^{\lambda_2}(t-s, W_0 + B_s) \right) ds. \end{aligned} \tag{6.17}$$

The last line is an easy consequence of the scaling relation (1.7). Now use (6.11) and Fatou's lemma to see that

$$\begin{aligned} & E_{X_0}^X \left(\int \int \phi_1(x_1) \phi_2(x_2) X_t(x_1) X_t(x_2) e^{-\lambda_1 X_t(x_1) - \lambda_2 X_t(x_2)} dx_1 dx_2 \right) \\ & \leq \liminf_{\delta \downarrow 0} \left[\int \int \prod_{i=1}^2 \phi_i(x_i) T_1(\delta, \vec{\lambda}, \vec{x}) dx_1 dx_2 \right. \\ & \quad \left. + \int \int \prod_{i=1}^2 \phi_i(x_i) (-T_2(\delta, \vec{\lambda}, \vec{x})) dx_1 dx_2 \right]. \end{aligned}$$

Finally apply (6.16) and (6.17) to bound the above by the required expression. \blacksquare

Proof of Proposition 5.1. Let T_1 and T_2 denote the first and second terms, respectively on the right-hand side of (6.2), where t and $\lambda = \lambda_1 = \lambda_2$ are fixed as in Proposition 5.1, and let $T = \log(\lambda^2 t)$. Consider first the much easier T_1 . Recall from (2.3)-(2.5) we have for any $\delta > 0$,

$$(6.18) \quad P_x^Y(\rho > T) \leq c_\delta e^{\delta x^2} e^{-\lambda_0 T}.$$

Therefore

$$\begin{aligned} & \lambda^{2\lambda_0} E_m^Y(\phi_i(W_0 + \sqrt{t}Y_T)1(\rho > T)) \\ &= \lambda^{2\lambda_0} \int \int \int \phi_i(w_0 + \sqrt{t}y) q_T(y_0, y) dm(y_0) dm(y) dX_0(w_0) \\ &\leq \int \int \phi_i(w_0 + \sqrt{t}y) \lambda^{2\lambda_0} P_y^Y(\rho > T) dm(y) dX_0(w_0) \\ &\leq \int \int \phi_i(w_0 + \sqrt{t}y) t^{-\lambda_0} c_\delta e^{\delta y^2} dm(y) dX_0(w_0), \end{aligned}$$

the last by (6.18). Now let $\sigma^2 = (1 - 2\delta)^{-1}$. A simple substitution now shows the above is at most

$$ct^{-\lambda_0} \int \phi_i(x_i) X_0 p_{t\sigma^2}(x_i) dx_i.$$

This in turn implies

$$(6.19) \quad \lambda^{4\lambda_0} T_1 \leq c_{\sigma^2} t^{-2\lambda_0} \int \int \prod_{i=1}^2 [\phi_i(x_i) X_0 p_{t\sigma^2}(x_i)] dx_1 dx_2,$$

where here σ^2 is any number greater than 1. Below we will choose a convenient value of σ^2 when doing the T_2 bound.

Turning now to T_2 , we can write $T_2 = \int_0^t T_2(s) ds$, where $T_2(s)$ is the integrand on the right-hand side of (6.2). We may replace the Brownian motion B_t with $\lambda^{-1} B_{t\lambda^2}$ (the new B is still a Brownian motion starting at 0) and use the scaling relation (1.7) to conclude after a short and familiar argument that

$$(6.20) \quad \begin{aligned} T_2(s) &= E_0^B \left(\phi_1(W_0 + \lambda^{-1} B_{t\lambda^2}) \psi^\lambda(t-s, W_0 + \lambda^{-1}(B_{\lambda^2 t} - B_{\lambda^2(t-s)})) \right. \\ &\quad \left. \times \exp\left(-\int_0^{\lambda^2 t} V^1(r, B_r) dr\right) \right). \end{aligned}$$

Case 1. Assume $\lambda^2(t-s) > 1$.

Apply the Markov property of B at time 1 to the right-hand side of (6.20) and conclude that

$$(6.21) \quad \begin{aligned} T_2(s) &\leq E_m^B \left(\phi_1(W_0 + \lambda^{-1} B_{t\lambda^2-1}) \psi^\lambda(t-s, \lambda^{-1}(B_{t\lambda^2-1} - B_{\lambda^2(t-s)-1})) \right. \\ &\quad \left. \times \exp\left(-\int_0^{\lambda^2 t-1} V^1(r+1, B_r) dr\right) \right), \end{aligned}$$

where B remains independent of W_0 . As before, $Y(u) = B(e^u - 1)e^{-u/2}$ is a stationary Ornstein-Uhlenbeck process. If $U = \log(\lambda^2(t-s))$, then arguing as in the proof of Lemma 4.1, we may re-express (6.21) as

$$(6.22) \quad \begin{aligned} T_2(s) &\leq E_m^Y \left(\phi_1(W_0 + \sqrt{t}Y_T) \psi^\lambda(t-s, W_0 + \sqrt{t}Y_T - \sqrt{t-s}Y_U) \right. \\ &\quad \left. \times \exp\left(-\int_0^T H_{e^u}(Y_u) du\right) \right) \\ &\leq e^{Cz} E_m^Y \left(\phi_1(W_0 + \sqrt{t}Y_T) \psi^\lambda(t-s, W_0 + \sqrt{t}Y_T - \sqrt{t-s}Y_U) \right. \\ &\quad \left. \times \exp\left(-\int_0^T F(Y_u) du\right) \right), \end{aligned}$$

where we used Lemma 4.3 in the last inequality. The above equals

$$(6.23) \quad \begin{aligned} &e^{Cz} \int \phi_1(w_0 + \sqrt{t}y_2) \psi^\lambda(t-s, w_0 + \sqrt{t}y_2 - \sqrt{t-s}y_1) q_U(y_0, y_1) \\ &\quad \times q_{T-U}(y_1, y_2) dm(y_0) dm(y_1) dm(y_2) dX_0(w_0) \\ &\leq C_{\delta_1} \int \phi_1(w_0 + \sqrt{t}y_2) \psi^\lambda(t-s, w_0 + \sqrt{t}y_2 - \sqrt{t-s}y_1) \\ &\quad \times e^{\delta_1 y_1^2} (t-s)^{-\lambda_0} \lambda^{-2\lambda_0} q_{T-U}(y_1, y_2) dm(y_1) dm(y_2) dX_0(w_0), \end{aligned}$$

where in the last line, $1/2 > \delta_1 > 0$, and we used (6.18) and the symmetry of q_U in integrating out y_0 .

At this point we take a break from the long proof and obtain a bound on ψ^λ .

LEMMA 6.2. *For any $\sigma_0^2 > 1$ there is a $C_{6.2}(\sigma_0^2)$ such that for all x and all $0 \leq s < t$,*

$$\psi^\lambda(t-s, x) \leq C_{6.2} \lambda^{-2\lambda_0} (t-s)^{-\lambda_0} P'_{\sigma_0^2(t-s)} \phi_2(x).$$

PROOF. If $\lambda^2(t-s) \leq 1$, we can drop the negative exponential in the definition of ψ^λ and note that $\lambda^{-2\lambda_0}(t-s)^{-\lambda_0} \geq 1$ to conclude that

$$\psi^\lambda(t-s, x) \leq \lambda^{-2\lambda_0}(t-s)^{-\lambda_0} P'_{t-s} \phi_2(x),$$

from which the required bound follows easily.

Assume now that $\lambda^2(t-s) > 1$. If Y and U are as above, then the same reasoning leading to (6.22) and (6.23) above (compare the definition of ψ^λ with the right-hand side of (6.20) without the ψ^λ) leads to (for any $0 < \delta < 1/2$)

$$\begin{aligned} \psi^\lambda(t-s, x) &\leq CE_m^Y \left(\phi_2(\sqrt{t-s}Y_U + x) \exp\left(-\int_0^U F(Y_u) du\right) \right) \\ (6.24) \quad &\leq C_\delta (t-s)^{-\lambda_0} \lambda^{-2\lambda_0} \int \phi_2(x + \sqrt{t-s}y) e^{\delta y^2} dm(y). \end{aligned}$$

An elementary argument now gives the required bound with $\sigma_0^2 = (1-2\delta)^{-1}$.

■

Returning to the proof of Proposition 5.1 in Case 1, we use the above lemma in (6.23) and then the substitution $z_1 = w_0 + \sqrt{t}y_2$, to obtain

$$\begin{aligned} T_2(s) &\leq C(\sigma_0^2, \delta_1) \lambda^{-4\lambda_0} (t-s)^{-2\lambda_0} \\ &\quad \times \int \int \int \phi_1(w_0 + \sqrt{t}y_2) P'_{\sigma_0^2(t-s)} \phi_2(w_0 + \sqrt{t}y_2 - \sqrt{t-s}y_1) e^{\delta_1 y_1^2} \\ &\quad \quad \quad \times q_{T-U}(y_1, y_2) dm(y_1) dm(y_2) dX_0(w_0) \\ &\leq C(\sigma_0^2, \delta_1) \lambda^{-4\lambda_0} (t-s)^{-2\lambda_0} \int \int \phi_1(z_1) \phi_2(z_2) \\ &\quad \times \left[\int \int p_{\sigma_0^2(t-s)}(z_2 - z_1 + \sqrt{t-s}y_1) q_{\log(t/t-s)}(y_1, (z_1 - w_0)t^{-1/2}) \right. \\ (6.25) \quad &\quad \left. \times e^{\delta_1 y_1^2} p_t(z_1 - w_0) dm(y_1) dX_0(w_0) \right] dz_1 dz_2. \end{aligned}$$

Case 1a Assume also $t/2 \leq s$.

Let $\delta = \frac{\sqrt{2}-1}{2}$ for which $e^{-s^*(\delta)} = 1/2$ (recall that $s^*(\delta)$ is as in (2.1)). Therefore $\log(t/t-s) \geq \log 2 = s^*(\delta)$, and so by (2.2),

$$q_{\log(t/t-s)}(y_1, (z_1 - w_0)t^{-1/2}) \leq ce^{-\lambda_0 \log(t/t-s)} \exp(\delta(y_1^2 + (z_1 - w_0)^2 t^{-1})).$$

If $\delta_1 = \delta$ and $\sigma_0^2 = (1 - 4\delta)^{-1} = (3 - 2\sqrt{2})^{-1} \leq 6$, this implies that the expression in square brackets in (6.25) is at most

$$\begin{aligned}
& ct^{-\lambda_0}(t-s)^{\lambda_0} \int \int p_{\sigma_0^2(t-s)}(z_2 - z_1 + \sqrt{t-s}y_1)e^{2\delta y_1^2} dm(y_1) \\
& \quad \times \exp(\delta(z_1 - w_0)^2/t)p_t(z_1 - w_0)dX_0(w_0) \\
& \leq ct^{-\lambda_0}(t-s)^{\lambda_0} \int \int p_{\sigma_0^2(t-s)}(z_2 - z_1 + w) \exp(-(1-4\delta)w^2/(2(t-s))) \\
& \quad \times (2\pi(t-s))^{-1/2} dw \exp(-(1-2\delta)(z_1 - w_0)^2/2t)(2\pi t)^{-1/2} dX_0(w_0) \\
& \leq ct^{-\lambda_0}(t-s)^{\lambda_0} \int p_{\sigma_0^2(t-s)}(z_2 - z_1 + w)p_{\sigma_0^2(t-s)}(w) dw X_0 p_{\sigma_0^2 t}(z_1) \\
& = ct^{-\lambda_0}(t-s)^{\lambda_0} p_{2\sigma_0^2(t-s)}(z_2 - z_1) X_0 p_{\sigma_0^2 t}(z_1).
\end{aligned}$$

Use the above in (6.25) to conclude that in Case 1a,

$$\begin{aligned}
(6.26) \quad T_2(s) & \leq C\lambda^{-4\lambda_0}(t-s)^{-\lambda_0}t^{-\lambda_0} \\
& \quad \times \int \phi_1(z_1)\phi_2(z_2)p_{2\sigma_0^2(t-s)}(z_2 - z_1)X_0 p_{\sigma_0^2 t}(z_1) dz_1 dz_2.
\end{aligned}$$

Case 1b. Assume $0 \leq s < t/2 (\leq t - \lambda^{-2})$.

The last inequality is immediate by our hypothesis that $\lambda^2 t \geq 9 > 2$. Return to (6.25) with the choices of δ_1 and σ_0^2 made in the previous case and let $R = \log(t/t-s)$. Bounding the transition density (with respect to Lebesgue measure) of the killed Ornstein-Uhlenbeck process starting at y_1 by the un-killed process and noting the latter has a normal density with mean $y_1 e^{-R/2}$ and variance $1 - e^{-R}$, we get

$$q_R(y_1, z)e^{-z^2/2}(2\pi)^{-1/2} \leq \exp\left(\frac{-(z - y_1 e^{-R/2})^2}{2(1 - e^{-R})}\right)(2\pi)^{-1/2}(1 - e^{-R})^{-1/2}.$$

Setting $z = \frac{z_1 - w_0}{\sqrt{t}}$ and simplifying, this becomes

$$q_R(y_1, (z_1 - w_0)/\sqrt{t})p_t(z_1 - w_0) \leq p_s(z_1 - w_0 - \sqrt{t-s}y_1).$$

Now use this in (6.25) to conclude

$$\begin{aligned}
(6.27) \quad T_2(s) & \leq C\lambda^{-4\lambda_0}(t-s)^{-2\lambda_0} \int \int \phi_1(z_1)\phi_2(z_2) \\
& \quad \times \int \left[\int p_{\sigma_0^2(t-s)}(z_2 - z_1 + \sqrt{t-s}y_1)e^{\delta_1 y_1^2} \right. \\
& \quad \left. \times p_s(z_1 - w_0 - \sqrt{t-s}y_1) dm(y_1) \right] dX_0(w_0) dz_1 dz_2.
\end{aligned}$$

The fact that $\sigma_0^2 = (1 - 4\delta_1)^{-1} > (1 - 2\delta_1)^{-1} > 1$ and a simple substitution shows that the term in square brackets is at most

(6.28)

$$f_{s,t}(z_1, z_2, w_0) = \int p_{\sigma_0^2(t-s)}(z_2 - z_1 + w) p_{\sigma_0^2(t-s)}(w) p_{\sigma_0^2 s}(z_1 - w_0 - w) dw.$$

We claim that

$$(6.29) \quad f_{s,t}(z_1, z_2, w_0) \leq c p_{8\sigma_0^2(t-s)}(z_2 - z_1) p_{4\sigma_0^2 t}(z_1 - w_0).$$

By scaling it suffices to obtain the above for $\sigma_0 = 1$. Set $a = z_1 - z_2$ and $b = z_1 - w_0$, so that

$$f_{s,t}(z_1, s_2, w_0) = \int p_{t-s}(w - a) p_{t-s}(w) p_s(b - w) dw.$$

A simple calculation (complete the square) shows that

$$f_{s,t}(z_1, s_2, w_0) = (2\pi)^{-1} (t^2 - s^2)^{-1/2} \exp\left(-\frac{a^2 t + 2b^2(t-s) - 2ab(t-s)}{2(t^2 - s^2)}\right).$$

Now use $ab \leq \alpha a^2 + (4\alpha)^{-1} b^2$ with $\alpha = 5/16$ to see that

$$\begin{aligned} f_{s,t}(z_1, s_2, w_0) &\leq (2\pi)^{-1} (t^2 - s^2)^{-1/2} \exp\left(-\frac{a^2(1-2\alpha)}{2(t+s)}\right) \exp\left(-\frac{b^2(2-(2\alpha)^{-1})}{2(t+s)}\right) \\ &\leq (2\pi)^{-1} (t^2 - s^2)^{-1/2} \exp\left(-\frac{a^2(1-2\alpha)}{6(t-s)}\right) \exp\left(-\frac{b^2(2-(2\alpha)^{-1})}{3t}\right) \\ &\leq (2\pi)^{-1} (t-s)^{-1/2} \exp\left(-\frac{a^2}{16(t-s)}\right) t^{-1/2} \exp\left(-\frac{b^2}{8t}\right), \end{aligned}$$

where we use $s < t/2$ in the next to last line and the value of α in the last line. This completes the proof of (6.29).

Now insert (6.29) into (6.27), noting that $(t-s)^{-\lambda_0} \leq ct^{-\lambda_0}$, to conclude that in Case 1b,

$$(6.30) \quad \begin{aligned} T_2(s) &\leq C \lambda^{-4\lambda_0} (t-s)^{-\lambda_0} t^{-\lambda_0} \\ &\quad \times \int \int \phi_1(z_1) \phi_2(z_2) p_{8\sigma_0^2(t-s)}(z_2 - z_1) X_0 p_{4\sigma_0^2 t}(z_1) dz_1 dz_2. \end{aligned}$$

Case 2. Assume $\lambda^2(t-s) \leq 1$.

Use (6.20), the Markov property at time 1 and then argue as in (6.23) to

see that

$$\begin{aligned}
T_2(s) &\leq E_0^B \left(E_{B_1(\omega)}^B \left(\phi_1(W_0 + \lambda^{-1} B_{\lambda^2 t-1}) \psi^\lambda(t-s, W_0 + \lambda^{-1} (B_{\lambda^2 t-1} - B_{\lambda^2(t-s)}(\omega))) \right. \right. \\
&\quad \left. \left. \times \exp \left(- \int_0^{\lambda^2 t-1} V^1(r+1, B_r) dr \right) \right) \right) \\
&\leq C \int \int p_{\lambda^2(t-s)}(x_0) p_{1-\lambda^2(t-s)}(x_1 - x_0) \\
&\quad \times E_{x_1}^Y \left(\phi_1(W_0 + \sqrt{t} Y_T) \psi^\lambda(t-s, W_0 + \sqrt{t} Y_T - x_0 \lambda^{-1}) \right. \\
&\quad \left. \times \exp \left(- \int_0^T F(Y_u) du \right) \right) dx_1 dx_0.
\end{aligned}$$

By definition, $\psi^\lambda(t-s, x) \leq P'_{t-s} \phi_2(x)$, and so arguing as in the derivation of (6.25) we get

$$\begin{aligned}
(6.31) \quad T_2(s) &\leq C \int \int \phi_1(z_1) \phi_2(z_2) \left[\int \int \int p_{t-s}(z_2 - z_1 + \frac{x_0}{\lambda}) p_{\lambda^2(t-s)}(x_0) \right. \\
&\quad \left. \times p_{1-\lambda^2(t-s)}(x_1 - x_0) q_T(x_1, (z_1 - w_0) t^{-1/2}) \right. \\
&\quad \left. \times p_t(z_1 - w_0) dx_1 dx_0 dX_0(w_0) \right] dz_1 dz_2.
\end{aligned}$$

Let $g_{s,t}(z_1, z_2)$ denote the expression in square brackets. A simple calculation shows that our condition $\lambda^2 t \geq 9$ implies $T \geq s^*(1/8)$, and so by (2.2),

$$q_T(x_1, (z_1 - w_0) t^{-1/2}) \leq C (\lambda^2 t)^{-\lambda_0} \exp \left(\frac{1}{8} \left(x_1^2 + \frac{(z_1 - x_0)^2}{t} \right) \right).$$

First use this in (6.31), and then set $\sigma_1^2 = 4/3$ and use $1 - \lambda^2(t-s) \leq 1$ and an easy calculation to obtain

$$\begin{aligned}
g_{s,t}(z_1, z_2) &\leq C \lambda^{-2\lambda_0} t^{-\lambda_0} \int \int p_{t-s}(z_2 - z_1 + (x_0/\lambda)) p_{\lambda^2(t-s)}(x_0) \\
&\quad \times p_{1-\lambda^2(t-s)}(x_1 - x_0) e^{x_1^2/8} dx_1 dx_0 p_{\sigma_1^2 t}(z_1 - w_0) dX_0(w_0) \\
&\leq C \lambda^{-2\lambda_0} t^{-\lambda_0} \int p_{t-s}(z_2 - z_1 + (x_0/\lambda)) p_{\lambda^2(t-s)}(x_0) \\
&\quad \times e^{x_0^2/4} dx_0 X_0 p_{\sigma_1^2 t}(z_1).
\end{aligned}$$

If $\sigma_2^2 = \frac{\lambda^2(t-s)}{1-\lambda^2(t-s)/2}$, then

$$(6.32) \quad \lambda^2(t-s) \leq \sigma_2^2 \leq 2\lambda^2(t-s)$$

and so

$$e^{x_0^2/4} p_{\lambda^2(t-s)}(x_0) = p_{\sigma_2^2}(x_0) \sqrt{\frac{\sigma_2^2}{\lambda^2(t-s)}} \leq \sqrt{2} p_{\sigma_2^2}(x_0),$$

which in turn implies

$$\begin{aligned} g_{s,t}(z_1, z_2) &\leq C \lambda^{-2\lambda_0} t^{-\lambda_0} \int p_{t-s}(z_2 - z_1 + (x_0/\lambda)) p_{\sigma_2^2}(x_0) dx_0 X_0 p_{\sigma_1^2}(z_1) \\ &= C \lambda^{-2\lambda_0} t^{-\lambda_0} p_{(\sigma_2^2/\lambda^2)+(t-s)}(z_2 - z_1) X_0 p_{\sigma_1^2}(z_1) \\ &\leq C \lambda^{-2\lambda_0} t^{-\lambda_0} p_{3(t-s)}(z_2 - z_1) X_0 p_{\sigma_1^2}(z_1), \end{aligned}$$

the last by (6.32). Use this in (6.31) to see that in Case 2,

$$\begin{aligned} T_2(s) &\leq C \lambda^{-2\lambda_0} t^{-\lambda_0} \int \int \phi_1(z_1) \phi_2(z_2) p_{3(t-s)}(z_2 - z_1) X_0 p_{\sigma_1^2}(z_1) dz_1 dz_2 \\ (6.33) \quad &\leq C \lambda^{-2\lambda_0} t^{-\lambda_0} \int \int \phi_1(z_1) \phi_2(z_2) p_{2\sigma_0^2(t-s)}(z_2 - z_1) X_0 p_{\sigma_0^2}(z_1) dz_1 dz_2, \end{aligned}$$

the last using $3 < 2\sigma_0^2$ and $\sigma_1^2 < \sigma_0^2$.

Now combine (6.26), (6.30), and (6.33) to see that (6.30) holds for all $0 < s \leq t$ and conclude that

$$\begin{aligned} T_2 &\leq \int \int \phi_1(z_1) \phi_2(z_2) \\ &\quad \times \left[C \lambda^{-4\lambda_0} t^{-\lambda_0} \int_0^t (t-s)^{-\lambda_0} p_{8\sigma_0^2(t-s)}(z_1 - z_2) X_0 p_{4\sigma_0^2}(z_1) ds \right] dz_1 dz_2. \end{aligned}$$

Combine this with (6.19) to see that if

$$f_{t,\lambda}(z_1, z_2) = \lambda^{4\lambda_0} E(X_t(z_1) X_t(z_2) e^{-\lambda X_t(z_1) - \lambda X_t(z_2)}),$$

then for $\lambda^2 t \geq 9$, and σ_0^2 as above,

$$\int \int \phi_1(z_1) \phi_2(z_2) f_{t,\lambda}(z_1, z_2) dz_1 dz_2 \leq \int \int \phi_1(z_1) \phi_2(z_2) Ch_{t,X_0}(z_1, z_2) dz_1 dz_2.$$

This implies that $f_{t,\lambda}(z) \leq Ch_{t,X_0}(z)$ for Lebesgue a.a. $z = (z_1, z_2)$. Note that Fatou's lemma shows that $h_{t,X_0}(\cdot)$ is lower semicontinuous while $f_{t,\lambda}(\cdot)$ is continuous by Dominated Convergence. Therefore

$$f_{t,\lambda}(z) \leq Ch_{t,X_0}(z) \text{ for all } z,$$

proving (a).

The first term in the definition of $h_{t,X_0}(z)$ is trivially bounded by $Ct^{-2\lambda_0-1}X_0(1)^2$. A simple substitution in the integral shows that the second term is bounded by

$$Ct^{-\lambda_0-1/2}X_0(1)|z_1 - z_2|^{1-2\lambda_0}.$$

This proves (b), and so the proof of Proposition 5.1 is complete \blacksquare

7. Proof of Theorem 5.8. Let $\tilde{V}^{b,\varepsilon} = V(b1_{[0,\varepsilon]})$ and let $\tilde{V}^{\infty,\varepsilon} = \lim_{b \rightarrow \infty} \tilde{V}^{b,\varepsilon}$, where the pointwise finite limit exists by monotonicity (from (3.6)) and the bound

$$(7.1) \quad \tilde{V}^{b,\varepsilon}(t, x) \leq V(b)(t, x) = \frac{b}{1 + (bt/2)} \leq \frac{2}{t}.$$

The scaling relation for $\tilde{V}^{b,\varepsilon}$, is

$$(7.2) \quad \text{for each } r > 0, \tilde{V}^{b,\varepsilon}(s, x) = r^2 \tilde{V}^{br^{-2}, \varepsilon r}(r^2 s, rx).$$

We state two lemmas which will be used to prove Theorem 5.8 and prove them after establishing the theorem.

LEMMA 7.1. *Let μ_1 denote the uniform law on $[0, 1]$. For any $\delta_0 \in (0, 1)$ there is a $C_{7.1}(\delta_0) > 1$ so that for all $\lambda > 0$,*

$$E_{\mu_1}^B \left(\exp \left(- \int_0^1 \tilde{V}^{\lambda, 1}(u, B_u) du \right) \right) \leq C_{7.1} \lambda^{-((1/2)+\lambda_0)+\delta_0}.$$

The analogue of H_u in (4.1) is (for any $b > 0$)

$$(7.3) \quad H^b(u, x) = u \tilde{V}^{b, 1}(u, \sqrt{u}x) = \tilde{V}^{bu, u^{-1/2}}(1, x),$$

where we used (7.2) with $r = u^{-1/2}$. The latter expression suggests that $\lim_{u \rightarrow \infty} H^b(u, x) = V^\infty(1, x) = F(x)$. Let $Z_T^b = \exp \left(\int_0^T F(Y_s) - H^b(e^s, Y_s) ds \right)$. With Lemma 4.3(b) in mind we have:

LEMMA 7.2. *There is a constant $C_{7.2}$ such that for all $b, T > 0$,*

$$Z_T^b \leq C_{7.2}(b \wedge 1)^{-20}.$$

Proof of Theorem 5.8. Let t, ε and M be as in the Theorem and set $\lambda = (\varepsilon M)^{-1}$. An elementary argument, partitioning $[x, x + \varepsilon]$ into two non-overlapping intervals of length $\varepsilon/2$ and replacing M by $4M$, will in fact allow us to assume

$$(7.4) \quad \varepsilon \leq \sqrt{t}/2 < \sqrt{t/2}.$$

By (3.6) and translation invariance,

$$\begin{aligned}
& P_{X_0}^X(0 < X_t([x, x + \varepsilon])/\varepsilon \leq \varepsilon M) \\
& \leq e E_{X_0}^X \left(1(X_t([x, x + \varepsilon]) > 0) \exp(-\lambda X_t([x, x + \varepsilon])/\varepsilon) \right) \\
(7.5) \quad & = e \left[\exp\left(-\int \tilde{V}^{\lambda/\varepsilon, \varepsilon}(t, y - x) dX_0(y)\right) - \exp\left(-\int \tilde{V}^{\infty, \varepsilon}(t, y - x) dX_0(y)\right) \right].
\end{aligned}$$

Differentiate the last inequality (with $x = 0$ and X_0 a point mass) with respect to λ , using Dominated Convergence to take the derivative through the integral, to conclude for $\lambda' > 0$,

$$\begin{aligned}
& \frac{d}{d\lambda'} \left(\exp(-\tilde{V}^{(\lambda'/\varepsilon), \varepsilon}(t, x)) - \exp(-\tilde{V}^{\infty, \varepsilon}(t, x)) \right) \\
& = -\frac{d}{d\lambda'} E_{\delta_x}^X \left(\exp(-(\lambda'/\varepsilon) X_t([0, \varepsilon])) 1(X_t([0, \varepsilon]) > 0) \right) \\
& = -E_{\delta_x}^X \left(\exp(-(\lambda'/\varepsilon) X_t([0, \varepsilon])) X_t([0, \varepsilon])/\varepsilon \right).
\end{aligned}$$

Integrate both sides from λ to ∞ and then integrate out x to conclude (by Fubini)

$$\begin{aligned}
& \int e^{-\tilde{V}^{\lambda/\varepsilon, \varepsilon}(t, x)} - e^{-\tilde{V}^{\infty, \varepsilon}(t, x)} dx \\
& = \int_{\lambda}^{\infty} \int E_{\delta_x}^X \left(e^{-(\lambda'/\varepsilon) X_t([0, \varepsilon])} X_t([0, \varepsilon])/\varepsilon \right) dx d\lambda'.
\end{aligned}$$

The bound (7.1) and the above show that

$$\begin{aligned}
& \int (\tilde{V}^{\infty, \varepsilon} - \tilde{V}^{\lambda/\varepsilon, \varepsilon})(t, x) dx \\
& \leq e^{2/t} \int e^{-\tilde{V}^{\lambda/\varepsilon, \varepsilon}(t, x)} - e^{-\tilde{V}^{\infty, \varepsilon}(t, x)} dx \\
& \leq e^{2/t} \int_{\lambda}^{\infty} \int E_{\delta_x}^X \left(e^{-(\lambda'/\varepsilon) X_t([0, \varepsilon])} X_t([0, \varepsilon])/\varepsilon \right) dx d\lambda' \\
(7.6) \quad & = e^{2/t} \int_{\lambda}^{\infty} \int E_{\delta_0}^X \left(e^{-(\lambda'/\varepsilon) X_t([x, x + \varepsilon])} X_t([x, x + \varepsilon])/\varepsilon \right) dx d\lambda'.
\end{aligned}$$

Now use the Markov property at time $t/2$ just as in (4.8) and then apply

the above to see that

$$\begin{aligned} & e^{-\tilde{V}^{\lambda/\varepsilon,\varepsilon}(t,x)} - e^{-\tilde{V}^{\infty,\varepsilon}(t,x)} \\ & \leq \int p_{t/2}(y-x) (\tilde{V}^{\infty,\varepsilon}(t/2,y) - \tilde{V}^{\lambda/\varepsilon,\varepsilon}(t/2,y)) dy \\ & \leq e^{4/t} t^{-1/2} \int_{\lambda}^{\infty} \int E_{\delta_0}^X \left(e^{-(\lambda'/\varepsilon)X_{t/2}([y,y+\varepsilon])} X_{t/2}([y,y+\varepsilon])/\varepsilon \right) dy d\lambda', \end{aligned}$$

and so by (7.1), if $f_{\varepsilon,t}(\lambda') = \int E_{\delta_0}^X \left(e^{-(\lambda'/\varepsilon)X_t([y,y+\varepsilon])} X_t([y,y+\varepsilon])/\varepsilon \right) dy$, then

$$\sup_x [\tilde{V}^{\infty,\varepsilon} - \tilde{V}^{\lambda/\varepsilon,\varepsilon}](t,x) \leq c(t) \int_{\lambda}^{\infty} f_{\varepsilon,t/2}(\lambda') d\lambda'.$$

Returning to (7.5) we therefore see that (recall $\lambda = (\varepsilon M)^{-1}$)

$$\begin{aligned} P_{X_0}^X(0 < X_t([x,x+\varepsilon])/\varepsilon \leq \varepsilon M) & \leq eX_0(1) \sup_y [\tilde{V}^{\infty,\varepsilon} - \tilde{V}^{\lambda/\varepsilon,\varepsilon}](t,y) \\ (7.7) \qquad \qquad \qquad & \leq c(t)X_0(1) \int_{(\varepsilon M)^{-1}}^{\infty} f_{\varepsilon,t/2}(\lambda') d\lambda'. \end{aligned}$$

Let μ_{ε} denote the uniform distribution on $[0, \varepsilon]$. Argue as in the derivation of (4.4) to see that

$$\begin{aligned} f_{\varepsilon,t}(\lambda') & = E_{\delta_0}^X \left(\int e^{-(\lambda'/\varepsilon)X_t([y,y+\varepsilon])} \int 1_{[0,\varepsilon]}(x-y) X(t,x) dx dy / \varepsilon \right) \\ & = \varepsilon^{-1} \int_0^{\varepsilon} E_{\delta_0}^X \left(\int X(t,x) \exp(-(\lambda'/\varepsilon)X_t(x-z+[0,\varepsilon])) dx \right) dz \\ & \leq \varepsilon^{-1} \int_0^{\varepsilon} E_0^B \left(\exp\left(-\int_0^t \tilde{V}^{\lambda'/\varepsilon,\varepsilon}(t-s, B_s - B_t + z) ds\right) \right) dz \\ (7.8) \qquad \qquad \qquad & = E_{\mu_{\varepsilon}}^B \left(\exp\left(-\int_0^t \tilde{V}^{\lambda'/\varepsilon,\varepsilon}(s, B_s) ds\right) \right). \end{aligned}$$

Now apply the scaling relation (7.2) with $r = \varepsilon^{-1}$ to see that for $\varepsilon^{-2}t \geq 1$,

$$\begin{aligned} f_{\varepsilon,t}(\lambda') & \leq E_{\mu_{\varepsilon}}^B \left(\exp\left(-\int_0^t \tilde{V}^{\lambda'\varepsilon,1}(\varepsilon^{-2}s, \varepsilon^{-1}B_s)\varepsilon^{-2} ds\right) \right) \\ & = E_{\mu_1}^B \left(\exp\left(-\int_0^{t\varepsilon^{-2}} \tilde{V}^{\lambda'\varepsilon,1}(u, B_u) du\right) \right) \\ (7.9) \qquad \qquad \qquad & = E_{\mu_1}^B \left(\exp\left(-\int_0^1 \tilde{V}^{\lambda'\varepsilon,1}(u, B_u) du\right) \psi(B_1) \right), \end{aligned}$$

where

$$\psi(x) = E_x^B \left(\exp \left(- \int_0^{t\varepsilon^{-2}-1} \tilde{V}^{\lambda'\varepsilon,1}(u+1, B_u) du \right) \right),$$

and we have used $\varepsilon^{-2}t \geq 1$. So if Y_s is the Ornstein-Uhlenbeck process $B(e^s-1)e^{-s/2}$, $T = \log(t\varepsilon^{-2})$ and $b = \lambda'\varepsilon$, then, as in the proof of Lemma 4.1,

$$\psi(x) = E_x^Y \left(\exp \left(- \int_0^T H^b(e^s, Y_s) ds \right) \right) = E_x^Y \left(Z_T^b \exp \left(- \int_0^T F(Y_s) ds \right) \right).$$

By Lemma 7.2 and then (6.18),

$$(7.10) \quad \psi(x) \leq C_{7.2}(b \wedge 1)^{-20} P_x^Y(\rho > T) \leq C_{7.2}(b \wedge 1)^{-20} c_\delta e^{\delta x^2} \varepsilon^{2\lambda_0} t^{-\lambda_0}.$$

Take $0 < \delta < 1/4$ and use the above in (7.9) and then Hölder's inequality with $q = (4\delta)^{-1}$ and $p = (1 - 4\delta)^{-1}$, to get (with $b = \lambda'\varepsilon$)

$$f_{\varepsilon,t}(\lambda') \leq c_\delta (b \wedge 1)^{-20} \varepsilon^{2\lambda_0} t^{-\lambda_0} E_{\mu_1}^B \left(\exp \left(- \int_0^1 \tilde{V}^{b,1}(u, B_u) du \right) \right)^{1-4\delta}.$$

So recalling (7.7), applying the above found with $t/2$ in place of t (by (7.4) we have $\varepsilon^{-2} \frac{t}{2} \geq 1$) and using Lemma 7.1, we have

$$\begin{aligned} & P_{X_0}^X(0 < X_t([x, x + \varepsilon]) \leq \varepsilon^2 M) \\ & \leq c(t) X_0(1) \varepsilon^{2\lambda_0} c'_\delta \int_{(\varepsilon M)^{-1}}^\infty (\varepsilon \lambda' \wedge 1)^{-20} \\ & \quad \times \left[E_{\mu_1}^B \left(\exp \left(- \int_0^1 \tilde{V}^{\lambda'\varepsilon,1}(u, B_u) du \right) \right) \right]^{1-4\delta} d\lambda' \\ & \leq c(t) X_0(1) \varepsilon^{2\lambda_0-1} c'_\delta \left[\int_{M^{-1}}^1 w^{-20} dw \right. \\ & \quad \left. + \int_1^\infty \left[E_{\mu_1}^B \left(\exp \left(- \int_0^1 \tilde{V}^{w,1}(u, B_u) du \right) \right) \right]^{1-4\delta} dw \right] \\ & \leq c(t) X_0(1) \varepsilon^{2\lambda_0-1} c'_\delta \left[M^{19} + \int_1^\infty C_{7.1}(\delta_0) w^{-(\frac{1}{2} + \lambda_0) + \delta_0(1-4\delta)} dw \right] \\ & \leq c(t) X_0(1) \varepsilon^{2\lambda_0-1} M^{19}, \end{aligned}$$

by choosing δ and δ_0 sufficiently small since $\lambda_0 > 1/2$. This gives the required result for $t = 1$. By scaling (see, e.g., Exercise II.5.5 in Perkins (2002)) it follows for general $t > 0$. \blacksquare

Proof of Lemma 7.2. Lemma 3.2(c) and (7.1) show that $F, H^b \leq 2$ and so by (3.6) and (1.8),

$$\begin{aligned}
(7.11) \quad F(x) - H^b(u, x) &\leq e^2 \left[e^{-H^b(u, x)} - e^{-F(x)} \right] \\
&\leq e^2 \left[E_{\delta_x}^X \left(\exp \left(-bu \int_0^{u^{-1/2}} X(1, y) dy \right) - 1(X(1, 0) = 0) \right) \right] \\
&\leq e^2 \left[E_{\delta_x}^X \left(1(0 < X(1, 0) < u^{-1/8}) \right) \right. \\
&\quad \left. + E_{\delta_x}^X \left(1(X(1, 0) \geq u^{-1/8}) \exp \left(-bu^{1/2} \frac{\int_0^{u^{-1/2}} X(1, y) dy}{u^{-1/2}} \right) \right) \right] \\
&\equiv e^2(T_1 + T_2).
\end{aligned}$$

Theorem 4.8(a) implies that

$$(7.12) \quad T_1 \leq Cu^{-(2\lambda_0-1)/8}.$$

We apply the modulus of continuity in Proposition 5.6 with $\xi = 3/8$ and $K = 2$ and so set $\rho_0(\omega) = \rho(2, 3/8, \omega)$. Choose u_0 so that

$$(7.13) \quad u^{-((\xi/2)-(1/8))} \leq 1/2 \text{ for all } u \geq u_0.$$

If $u \geq u_0$, $\rho_0(\omega) \geq u^{-1/2}$ and $y \in [0, u^{-1/2}]$, then on $\{X(1, 0) \geq u^{-1/8}\}$,

$$X(1, y) \geq X(1, 0) - |y|^\xi \geq u^{-1/8} - u^{-\xi/2} \geq u^{-1/8}/2 \text{ (by (7.13)).}$$

Therefore for $u \geq u_0$,

$$\begin{aligned}
(7.14) \quad T_2 &\leq P_{X_0}^X(\rho_0 < u^{-1/2}) + \exp\left(-\frac{b}{2}u^{(1/2)-(1/8)}\right) \\
&\leq Cu^{-\delta_{5.6}/2} + \exp\left(-\frac{b}{2}u^{3/8}\right).
\end{aligned}$$

Combining (7.12) and (7.14) we have

$$F(x) - H^b(u, x) \leq C(u^{-(2\lambda_0-1)/8} + u^{-\delta_{5.6}/2}) + e^2 \exp\left(-\frac{b}{2}u^{3/8}\right),$$

first for $u \geq u_0$, and then for all $u \geq 1$ by increasing the constant C .

Therefore,

$$\begin{aligned}
Z_T^b &\leq C \exp\left(\int_0^T e^2 \exp\left(-\frac{b}{2} e^{\frac{3s}{8}}\right) ds\right) \\
&= C \exp\left(\frac{8e^2}{3} \int_{b/2}^{(b/2)e^{3T/8}} e^{-w} w^{-1} dw\right) \\
&\leq C \exp\left(\frac{8e^2}{3} \int_{(b/2)\wedge 1}^1 w^{-1} dw\right) \leq C[b \wedge 1]^{-20}. \quad \blacksquare
\end{aligned}$$

Proof of Lemma 7.1. Set $\gamma = \frac{1}{2} + \lambda_0 \in (1, \frac{3}{2})$ and $\beta = \frac{1}{2} - \frac{\gamma}{4} \in (\frac{1}{8}, \frac{1}{4})$. The exponent on λ (in the statement of the Lemma) is negative and so we may assume without loss of generality that $\lambda \geq 1$. For $\lambda \geq 1$ use an obvious symmetry to see that

$$\begin{aligned}
\psi(\lambda) &\equiv E_{\mu_1}^B\left(\exp\left(-\int_0^1 \tilde{V}^{\lambda,1}(s, B_s) ds\right)\right) \\
&= E_{\mu_1}^B\left(1(B_0 \in [\lambda^{-\beta}, 1 - \lambda^{-\beta}])E_{B_0}^B\left(\exp\left(-\int_0^1 \tilde{V}^{\lambda,1}(s, B_s) ds\right)\right)\right) \\
&\quad + 2E_{\mu_1}^B\left(1(B_0 \in [0, \lambda^{-\beta}])E_{B_0}^B\left(\exp\left(-\int_0^1 \tilde{V}^{\lambda,1}(s, B_s) ds\right)\right)\right) \\
(7.15) \quad &\equiv T_1 + 2T_2.
\end{aligned}$$

By Feynman-Kac we have

$$(7.16) \quad \tilde{V}^{\lambda,1}(t, x) = E_x^B\left(\lambda 1_{[0,1]}(B_t) \exp\left(-\int_0^t \frac{\tilde{V}^{\lambda,1}(t-s, B_s)}{2} ds\right)\right).$$

If $\tilde{V}_0^{\lambda,1}$ was non-negative continuous, this would follow from Karatzas and Shreve (1991) (p. 268) and for general non-negative Borel initial conditions it follows by taking bounded pointwise limits (use (3.6)). Now bound $\tilde{V}^{\lambda,1}(u, x)$ above by $V(\lambda)(u, x) = \frac{2\lambda}{2+\lambda u}$ and use this bound in the exponent in (7.16) to conclude that

$$\begin{aligned}
\tilde{V}^{\lambda,1}(t, x) &\geq \lambda P_x^B(B_t \in [0, 1]) \exp\left(-\int_0^t \frac{2\lambda}{2(2+\lambda s)} ds\right) \\
(7.17) \quad &= \frac{2\lambda}{2+\lambda t} P_x^B(B_t \in [0, 1]).
\end{aligned}$$

If $x \in [\lambda^{-\beta}/2, 1 - (\lambda^{-\beta}/2)]$ and $t \leq \lambda^{-2\beta(1+\varepsilon)}$, for some $\varepsilon > 0$, then by (7.17),

$$\begin{aligned}\tilde{V}^{\lambda,1}(t, x) &\geq \frac{2\lambda}{2 + \lambda t} (1 - P_x^B(B_t \notin [0, 1])) \\ &\geq \frac{\lambda}{1 + (\lambda t/2)} (1 - \eta_\lambda),\end{aligned}$$

where $\eta_\lambda = \exp(-\lambda^{2\beta\varepsilon/8})$. Therefore

$$\begin{aligned}T_1 &\leq \int_{\lambda^{-\beta}}^{1-\lambda^{-\beta}} E_x^B \left(\exp \left(-2 \int_0^{\lambda^{-2\beta(1+\varepsilon)}} \frac{(1-\eta_\lambda)\lambda}{(1+(\lambda s/2))^2} ds \right) \right. \\ &\quad \times \mathbf{1} \left(\sup_{s \leq \lambda^{-2\beta(1+\varepsilon)}} |B_s - x| \leq \frac{1}{2} \lambda^{-\beta} \right) dx \\ &\quad + P_0^B \left(\sup_{s \leq \lambda^{-2\beta(1+\varepsilon)}} |B_s| > \frac{1}{2} \lambda^{-\beta} \right) \\ &\leq \left[1 + \frac{\lambda^{1-2\beta(1+\varepsilon)}}{2} \right]^{-2(1-\eta_\lambda)} + C \exp(-\lambda^{2\beta\varepsilon/8}) \\ (7.18) \quad &\leq C_\varepsilon \lambda^{-(1-2\beta(1+\varepsilon))2(1-\eta_\lambda)} \leq C_\varepsilon \lambda^{-2(1-2\beta(1+2\varepsilon))},\end{aligned}$$

where the last inequality holds first for $\lambda \geq \lambda(\varepsilon)$, and then for all $\lambda \geq 1$ by increasing C_ε .

For T_2 we use the scaling relation (7.2) with $r = \lambda^\beta$ to see that

$$\tilde{V}^{\lambda,1}(s, x) = \lambda^{2\beta} \tilde{V}^{\lambda^{1-2\beta}, \lambda^\beta}(\lambda^{2\beta} s, \lambda^\beta x),$$

and so

$$\begin{aligned}T_2 &= \lambda^{-\beta} E_{\mu_{\lambda^{-\beta}}}^B \left(\exp \left(- \int_0^1 \tilde{V}^{\lambda^{1-2\beta}, \lambda^\beta}(\lambda^{2\beta} s, \lambda^\beta B_s) \lambda^{2\beta} ds \right) \right) \\ &= \lambda^{-\beta} E_{\mu_1}^B \left(\exp \left(- \int_0^{\lambda^{2\beta}} \tilde{V}^{\lambda^{1-2\beta}, \lambda^\beta}(u, B_u) du \right) \right) \\ (7.19) \quad &\leq \lambda^{-\beta} E_{\mu_1}^B \left(\exp \left(- \int_0^1 \tilde{V}^{\lambda^{1-2\beta}, 1}(u, B_u) du \right) \hat{\psi}(B_1) \right),\end{aligned}$$

where

$$\hat{\psi}(x) = E_x^B \left(\exp \left(- \int_0^{\lambda^{2\beta}-1} \tilde{V}^{\lambda^{1-2\beta}, 1}(s+1, B_s) ds \right) \right).$$

By (7.10) with $b = \lambda^{1-2\beta}$ and $\lambda^{2\beta}$ in place of $t\varepsilon^{-2}$, for any $\delta > 0$,

$$\hat{\psi}(x) \leq C_\delta e^{\delta x^2} \lambda^{-2\beta\lambda_0}.$$

Use this in (7.19), and then for any $p > 1$ choose $\delta > 0$ so that $1 - 2\delta > p^{-1}$, and apply Hölder's inequality to see that

$$(7.20) \quad T_2 \leq c_p \lambda^{-\beta(1+2\lambda_0)} E_{\mu_1}^B \left(\exp \left(- \int_0^1 \tilde{V}^{\lambda^{1-2\beta,1}}(u, B_u) dy \right) \right)^{1/p}.$$

Combine (7.15), (7.18) and (7.20) to conclude that for any $\varepsilon > 0$ and $p > 1$ there are constants C_ε and c_p so that

$$(7.21) \quad \begin{aligned} \psi(\lambda) &\leq C_\varepsilon \lambda^{-2(1-2\beta(1+2\varepsilon))} + c_p \lambda^{-\beta(1+2\lambda_0)} \psi(\lambda^{1-2\beta})^{1/p} \\ &\leq C_\varepsilon \lambda^{-\gamma+2\varepsilon} + c_p \lambda^{-2\beta\gamma} \psi(\lambda^{\gamma/2})^{1/p}, \end{aligned}$$

where the last line uses $\gamma > 1$. Now fix $r \in (1, \gamma)$, choose $\varepsilon > 0$ so that

$$\gamma - 2\varepsilon > r,$$

and then $p > 1$ so that

$$2\beta\gamma + \frac{\gamma r}{2p} > r.$$

The latter is easily seen to be possible by the choice of r and a bit of arithmetic. With these choices of ε and p we then may choose $\lambda' = \lambda'(r)$ sufficiently large so that (by our choice of ε) for C_ε and c_p as in (7.21)

$$\lambda^{-\gamma+2\varepsilon} \leq \frac{1}{2C_\varepsilon} \lambda^{-r} \text{ for } \lambda \geq \lambda'$$

and (by our choice of p),

$$\lambda^{-2\beta\gamma} \lambda^{-\gamma r/(2p)} \leq \frac{1}{2c_p} \lambda^{-r} \text{ for } \lambda \geq \lambda'.$$

Using the two bounds above we see that (7.21) becomes

$$(7.22) \quad \psi(\lambda) \leq \frac{\lambda^{-r}}{2} + \frac{\lambda^{-r}}{2} \lambda^{\gamma r/(2p)} \psi(\lambda^{\gamma/2})^{1/p} \text{ for } \lambda \geq \lambda'.$$

Let $N = N(r)$ be the minimal natural number such that $2^{(2/\gamma)^N} \geq \lambda'$. Now choose $C_0 = C_0(r) \geq 1$ so that for all $1 \leq \lambda \leq 2^{(2/\gamma)^N}$,

$$(7.23) \quad \psi(\lambda) \leq C_0 \lambda^{-r}.$$

We now prove by induction on $n \geq N$ that (7.23) holds for $1 \leq \lambda \leq 2^{(2/\gamma)^n}$. It holds for $n = N$ by our choice of C_0 , so assume it holds for $1 \leq \lambda \leq 2^{(2/\gamma)^n}$ ($n \geq N$), and let $\lambda \in [2^{(2/\gamma)^n}, 2^{(2/\gamma)^{n+1}}]$. Then $\lambda \geq \lambda'$ and

$$(7.24) \quad \lambda^{\gamma/2} \leq 2^{(2/\gamma)^n}.$$

Therefore by (7.22), and our induction hypothesis,

$$\begin{aligned}\psi(\lambda) &\leq \frac{\lambda^{-r}}{2} + \frac{\lambda^{-r}}{2} \lambda^{\gamma r/(2p)} C_0^{1/p} \lambda^{-\gamma r/(2p)} \\ &\leq C_0 \lambda^{-r}.\end{aligned}$$

This completes the induction and so for any $r < \frac{1}{2} + \lambda_0$, $\psi(\lambda) \leq C_0(r) \lambda^{-r}$ for all $\lambda \geq 1$. \blacksquare

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References.

- BREZIS, H., PELETIER, L.A. and TERMAN, D. (1986). A very singular solution of the heat equation with absorption. *Arch. Rat. Mech. Anal.* **95** 185-209.
- CHEN, Y. (2015). Pathwise uniqueness for the SPDE's of some super-Brownian motion with immigration. To appear in *Ann. Probab.*
- CODDINGTON, E. and LEVINSON, N. (1955). *Theory of Ordinary Differential Equations*. McGraw-Hill, New York, NY.
- DAWSON, D.A. and PERKINS, E.A. (1991). Historical Processes. *Mem. Amer. Math.Soc.* **93** 179p.p..
- EVANS, N. and PERKINS, E. (1991). Absolute continuity results for superprocesses with some applications. *Trans. Amer. Math. Soc.* **325** 661-681.
- FRISTEDT, B. and PRUITT, W.E. (1971). Lower functions for increasing random walks and subordinators. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **18** 167-182.
- DE HAAN, L. and STADTMÜLLER, U. (1985). Dominated variation and related concepts and Tauberian theorems for Laplace transforms. *J. Math. Anal. Appl.* **108** 344-365.
- HAWKES, J. (1975). On the potential theory of subordinators. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **33** 113-132.
- HAWKES, J. (1979). Potential theory of Lévy processes. *Proc. London Math. Soc. (3)* **38** 335-352.
- KAMIN, S. and PELETIER, L.A. (1985). Singular solutions of the heat equation with absorption. *Proc. Amer. Math.Soc.* **95** 205-210.
- KARATZAS, I. and SHREVE, S. E. (1979). *Brownian Motion and Stochastic Calculus* Springer-Verlag, New York, Heidelberg, Berlin.
- KNIGHT, F. (1981). *Essentials of Brownian Motion and Diffusion. Mathematical Surveys, 18*. American Mathematical Society, Providence, Rhode Island.
- LAX, P. (2002). *Functional Analysis*. Wiley-Interscience, New York, NY
- MUELLER, C. MYTNIK, L. and PERKINS, E. (2014). Nonuniqueness for a parabolic SPDE with $\frac{3}{4} - \varepsilon$ -Hölder diffusion coefficients. *Ann. Probab.* **42** 2032-2112.
- MUELLER, C. MYTNIK, L. and PERKINS, E. (2015). On the boundary of the support of super-Brownian motion; with appendices. *Math ArXiv*. **no. 1512.08610**
- MUELLER, C. and PERKINS, E. (2011). The compact support property for solutions to the heat equation with noise. *Probab. Theory Related Fields* **93** 325-358.
- MYTNIK, L. (1998). Weak uniqueness for the heat equation with noise. *Ann. Probab.* **26** 968-984.
- MYTNIK, L. and PERKINS, E. (2003). Regularity and irregularity of $(1 + \beta)$ -stable super-Brownian motion. *Ann. Probab.* **31** 1413-1440.

- MYTNIK, L., PERKINS, E. and STURM, A. (2006). On pathwise uniqueness for stochastic heat equations with non-Lipschitz coefficients. *Ann. Probab.* **34** 1910–1959.
- MYTNIK, L. and PERKINS, E. (2011). Pathwise uniqueness for stochastic heat equations with Hölder continuous coefficients: the white noise case. *Probab. Theory Related Fields* **149** 1–96.
- PERKINS, E. (2002). Dawson-Watanabe superprocesses and measure-valued diffusions. In *Lectures on probability theory and statistics (Saint-Flour, 1999)*. *Lecture Notes in Math.* **1781** 125–324. Springer, Berlin.
- RIESZ, F. and SZ.-NAGY, B. (1955). *Functional Analysis* Frderick Ungar, New York.
- TAYLOR, S. J. (1961). On the connexion between Hausdorff measures and generalized capacity. *Proc. Cambridge Philos. Soc.* **57** 524–531.
- UCHIYAMA, K. (1980). Two contrasting properties of solutions for one-dimensional stochastic partial differential equations. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **54** 75–116.

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