On Rokhlin's multiple mixing problem

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January 10, 2019

Abstract

This is a brief survey of Rokhlin's multiple mixing problem.

1 Introduction

In this paper we present a brief survey of Rokhlin's multiple mixing problems. The author used the notes of Terrence Tao [Tao11], [Tao08], the book [Qua09] and the papers [Fer97], [De 06] as a foundation for this survey.

This survey is supposed to be brief, but we also decided to make it self-contained, so we will try to include all the necessary definitions.

2 Strong mixing and examples of strongly mixing systems

Remark. We will be dealing only with probability measures in this paper.

Let us now define the notion of strongly n-mixing systems and then we consider some examples of mixing systems.

Definition 2.1. Let $(X, \mathcal{F}, \mu, T)$ be a measure-preserving system, where $\mu(X) = 1$.

1. The system $(X, \mathcal{F}, \mu, T)$ is *strongly mixing*(*2-fold mixing*, *2-mixing*) if for every measurable $A, B \in \mathcal{F}$ we have
$$\lim_{n\to\infty} \mu(A \cap T^{-n}(B)) = \mu(A)\mu(B).$$
2. The system $(X, \mathcal{F}, \mu, T)$ is *strongly n-mixing*(*n-fold mixing*, *n-mixing*) if for every family of measurable subsets $(A_i)_{1\le i\le n} \in \mathcal{F}$ we have
$$\lim_{a_1,\dots,a_{n-1}\to\infty} \mu(A_1 \cap T^{-a_1}(A_2) \cap \dots \cap T^{-a_1-\dots-a_{n-1}}(A_n)) = \prod_{i=1}^{n} \mu(A_i).$$

Remark. In some texts n-fold mixing systems are called $n-1$-fold mixing systems, and 2-fold mixing is denoted as mixing – this notation goes back to the Rokhlin's paper [Rok49] itself, but, apparently, most authors use the notation we are using in this paper.

Example 2.1.

1. Bernoulli shifts $\sigma : \{0, \dots, n\}^{\mathbb{N}} \to \{0, \dots, n\}^{\mathbb{N}}$ are strongly mixing of all orders.
2. Consider the following map from $S : [0, 1]^2 \to [0, 1]^2$:
$$S(x, y) = \begin{cases} (2x, y/2), & 0 \le x < 1/2, \\ (2x - 2, 1 - y/2), & 1/2 \le x < 1. \end{cases}$$
Then this map, which is called baker's map, defines a strongly mixing system of all orders as well.

3. Any hyperbolic automorphism of $\mathbb{T}^2$ is strongly mixing of all orders.

4. However, no translation of $\mathbb{T}$ is strongly mixing.

As expected, there we can restate this property in terms of functions on X.

Proposition 2.1. TFAE for a measure-preserving system $(X, \mathcal{F}, \mu, T)$ and for every $k \geq 2$:

1. The map T is strongly k-mixing.

2. For every $f_1, \dots, f_k \in L^2(X, \mu)$ we have

$$\lim_{n_1, \dots, n_{k-1} \to \infty} \int_X f_1 (f_2 \circ T^{n_1}) \dots (f_k \circ T^{n_1 + \dots + n_{k-1}}) \mathrm{d}\mu = \prod_{i=1}^{k} \int_X f_i \mathrm{d}\mu.$$

3 Rokhlin's problem and some known results

The following conjecture was introduced by Rokhlin in 1949, see [Rok49].

Conjecture 3.1 (Rokhlin's problem on strongly mixing systems)**.** Any strongly mixing system is strongly 3-mixing.

To this day the Conjecture 3.1 remains unsolved. Moreover, it is not even known whether the Conjecture 3.1 implies that any m-mixing system is n-mixing for some $m < n$ different from 2 and 3.

First of all, let us state a trivial result related to n-fold mixing systems:

Proposition 3.1. Any n-mixing system is m-mixing for $2 \leq m \leq n$.

Proof. We know that for any $A_1, \dots, A_n \in \mathcal{F}$ we have

$$\lim_{k_1, \dots, k_{n-1} \to \infty} \mu(A_1 \cap T^{-k_1}(A_2) \cap \dots \cap T^{-k_1 - \dots - k_{n-1}}(A_n)) = \mu(A_1) \dots \mu(A_n).$$

Now set $A_{m+1} = \dots = A_n = X$. □

Now we are going to state the partial results related to the Conjecture 3.1 by Kalikow, Host, Ryzhikov and Pollicott.

3.1 Finite rank measure-preserving transformations

First of all, let us try to define finite rank transformations. We will follow the survey [Fer97], which calls this notion "the lecturer's nightmare", and we are inclined to agree with the author of the paper: the existing definitions of finite rank transformations don't really seem to be intuitive at a first glance.

Definition 3.1. Let $(X, \mathcal{F}, \mu)$ be a measure space, and consider two finite partitions P, P' of X. Then we define the *distance* $|P - P'|$ between P and P' as follows:

$$|P - P'| = \sum_i \mu(P_i \Delta P'_i).$$

Definition 3.2. A measure-preserving system $(X, \mathcal{F}, \mu, T)$ is of *rank one* if for any partition P of X and $\varepsilon > 0$ there exist

- a subset $F \subset X$
- an integer $h \in \mathbb{Z}$

- a partition P' of X

such that

- the subsets $T^k F$ for $0 \le k \le h-1$ are disjoint
- $|P - P'| < \varepsilon$
- the partition P' is refined by a partition formed by $T^k F$ and $X \setminus (\cup T^k F)$.

It is not entirely obvious why such systems even exist, but there is a more explicit definition and some examples, which are described in the same paper.

Theorem 3.1 ([Kal84])**.** A strongly mixing transformation of rank one is strongly k-mixing for all $k \ge 2$.

Now let us define the notion of a finite-rank transformation:

Definition 3.3. A measure-preserving system $(X, \mathcal{F}, \mu, T)$ is of rank no more than r if for every partition P of X and $\varepsilon > 0$ there exist

- r subsets $F_i \subset X$,
- r positive integers h_i,
- A partition P' of X

such that

- The subsets $(T^j F_i)_{\substack{1 \le j \le r \\ 0 \le j \le h_i - 1}}$ are disjoint,
- $|P' - P| < \varepsilon$,
- P' is refined by the partition made of the $(T^j F_i)_{\substack{1 \le j \le r \\ 0 \le j \le h_i - 1}}$ and $X \setminus (\cup_{\substack{1 \le j \le r \\ 0 \le j \le h_i - 1}} T^j F_i)$.

Theorem 3.2 ([Ryz93])**.** A strongly mixing transformation of finite rank is strongly k-mixing for all $k \ge 2$.

3.2 Zero entropy measure-preserving systems

As Thouvenot proved here [Tho95], it is enough to prove Rokhlin's problem for measure-preserving systems with zero entropy.

3.3 Systems with purely singular spectrum

There is a very important observation: for every measure-preserving system $(X, \mathcal{F}, \mu, T)$ you can consider the following isometric operator, which is called the *Koopman operator*:

$$U_T : L^2(X, \mu) \to L^2(X, \mu), \quad U_T(f) = f \circ T.$$

For example, T is ergodic if and only if 1 is a simple eigenvalue of U_T. We can write down the definitions of weak and strong mixing only using inner products in $L^2(X, \mu)$, so these are "spectral" properties as well: they only depend on the spectrum of U_T as an element of a Banach algebra $\mathcal{B}(L^2(X, \mu))$.

So, it is reasonable to expect that if we impose stronger conditions on the spectrum of U_T, we will get stronger properties of T, as well.

Proposition 3.2. Let $(X, \mathcal{F}, \mu, T)$ be a measure-preserving system. Then U_T is an isometric operator. Moreover, if T^{-1} is well-defined and measurable, then U_T is a unitary operator.

In particular, in the invertible case U_T is self-adjoint, so we can apply the Spectral Theorem to this operator.

Remark. The following definition is not explicitly written in any of the references to this survey.

Definition 3.4. Let $(X, \mathcal{F}, \mu, T)$ be a measure-preserving system, where T admits a measurable inverse. Then we call this system purely singular(absolutely continuous) if the spectral measure of U_T is singular(absolutely continuous) with respect to the Lebesgue measure on $\mathbb{T}$.

The reader might get confused here, because spectral measures take values in the set of projections of $L^2(X, \mu)$, but we can still define the notion of a support and check the absolute continuity.

This result was proven by Bernard Host in 1991 in [Hos91]:

Theorem 3.3 ([Hos91], Theorem 1)**.** Let $(X, \mathrm{Bor}(X), \mu, T)$ be a strongly mixing purely singular Borel measure-preserving system, where T admits a measurable inverse. Then it is strongly k-mixing for every $k \geq 2$.

3.4 Multiple mixing for hyperbolic flows

There is a preprint [Pol] of M. Pollicott in which he explores this problem for hyperbolic flows on compact surfaces. We will state some theorems from this paper, which are relevant to our survey.

Theorem 3.4. [KS01], [Pol, Theorem 1.2]. Let $(X, \mathcal{F}, \mu, T)$ be a measure-preserving system, where X is a compact manifold, μ is a Gibbs measure, and T is an Axiom A diffeomorphism. Then this system is strongly 3-mixing, moreover, we can observe the exponential decay: there exists a constant $0 < \theta < 1$ such that for every Hölder functions $f_1, f_2, f_3 \in C^\alpha(X)$ with zero mean we have

$$\int_X (f_1 \circ T^{n_1+n_2})(f_2 \circ T^{n_2}) f_3 \mathrm{d}\mu = O(\theta^{n_1+n_2}) \text{ as } n_1, n_2 \to \infty.$$

A similar observation can be made for hyperbolic flows on compact surfaces:

Theorem 3.5. [Pol, Theorem 1.4]. Let $\phi_t : X \to X$ denote a geodesic flow on a compact surface of negative curvature, and let μ be a Gibbs measure for a Hölder continuous potential. Then there exists a $\varepsilon > 0$ such that for every $f_1, f_2, f_3 \in C^\alpha(X)$ with mean zero we have

$$\int_X (f_1 \circ \varphi_{t_1+t_2})(f_2 \circ \varphi_{t_1}) f_3 = O(e^{-\varepsilon(|t_1|+|t_2|)}) \text{ as } t_1, t_2 \to +\infty.$$

Theorem 3.6. [Pol, Theorem 1.5]. In the setting of the previous theorem let us also assume that the hyperbolic flow ϕ_t satisfies a diophantine condition on the ratio of the lengths of a pair of closed orbits.

Then for every $f_1, f_2, f_3 \in C^\alpha(X)$ with mean zero and every $\beta > 0$ we have

$$\int_X (f_1 \circ \varphi_{t_1+t_2})(f_2 \circ \varphi_{t_1}) f_3 = O((t_1 t_2)^{-\beta}) \text{ as } t_1, t_2 \to \infty.$$

4 Weak mixing

In this section we will explore the notion of weakly mixing systems.

Proposition 4.1. TFAE for a measure-preserving system $(X, \mathcal{F}, \mu, T)$:

1. For any $A, B \in \mathcal{F}$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left| \mu(A \cap T^{-k} B) - \mu(A)\mu(B) \right| = 0. \tag{1}$$

2. For any $f, g \in L^2(X,\mu)$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left| \int f(g \circ T^k) \mathrm{d}\mu - \int_X f \mathrm{d}\mu \int_X g \mathrm{d}\mu \right| = 0.$$

3. For every function $f \in L^2(X,\mu)$ with $\int_X f \mathrm{d}\mu = 0$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left\langle f \circ T^k, f \right\rangle = 0. \tag{2}$$

Functions in $L^2(X,\mu)$ which satisfy (2) are called *weakly mixing*.

If any of the conditions 1-3 are satisfied, then the system is called *weakly mixing*.

Remark. Do not confuse the condition (1) with a slightly weaker one: for every $A, B \in \mathcal{F}$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(A \cap T^{-k}(B)) = \mu(A)\mu(B).$$

This propery is, actually, equivalent to ergodicity. There is a following chain of strict inclusions:

$$\{\text{ergodic}\} \supsetneq \{\text{weak-mixing}\} \supsetneq \{\text{strong-mixing}\}.$$

For an example of a strongly mixing but not weak-mixing system see [Cha69] or [Par81].

4.1 Weak mixing implies "weak mixing of all orders"

Due to H. Furstenberg, we know that the weakly mixing version of the Conjecture 3.1 holds true.

Theorem 4.1 ([Fur80])**.** Let $(X, \mathcal{F}, \mu, T)$ be a weakly mixing system. Then for every $k > 1$, distinct $a_1, \ldots, a_k \in \mathbb{Z}$ and for every $f_1, \ldots, f_k \in L^2(X,\mu)$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} \int (f_1 \circ T^{a_1 m})(f_2 \circ T^{a_2 m}) \ldots (f_k \circ T^{a_k m}) \mathrm{d}\mu = \prod_{i=1}^{k} \int_X f_i \mathrm{d}\mu \tag{3}$$

Here we will present a sketch of a proof of this theorem, presented in [Tao08], which relies on the van der Corput's lemma about asymptotically orthogonal vectors in Hilbert spaces. To the author's understanding, this proof closely follows the original proof of Furstenberg in [Fur80].

Sketch of the proof. **Remark.** For simplicity we assume our vector spaces to be real-valued.
First of all, let us state this lemma explicitly:

Lemma 4.1 (van der Corput's lemma)**.** Let (v_k) be a bounded sequence of elements in a Hilbert space H. If

$$\lim_{h\to\infty} \frac{1}{h} \sum_{n=0}^{h-1} \left(\limsup_{n\to\infty} \left| \frac{1}{n} \sum_{k=0}^{n-1} \langle v_k, v_{k+h} \rangle \right| \right) = 0,$$

then the following Cesáro limit is zero:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} v_k = 0.$$

Using this lemma, we can prove the following corollary:

Corollary 4.1. Let $(X, \mathcal{F}, \mu, T)$ be a mps and $f \in L^2(X, \mu)$ be a weakly mixing function. Then for every $g \in L^2(X, \mu)$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |\langle f \circ T^n, g\rangle| = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |\langle f, g \circ T^n\rangle| = 0.$$

Proof. Let us use the fact that for bounded non-negative sequences c_k we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} c_k = 0 \Leftrightarrow \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} c_k^2 = 0.$$

Now we apply this observation to the sequence $c_k = \left|\left\langle f \circ T^k, g\right\rangle\right|$, so it suffices to show that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left\langle f \circ T^k, g\right\rangle^2 = 0.$$

This is where we need our Hilbert space to be real-valued: we want $\left\langle f \circ T^k, g\right\rangle^2 = \left|\left\langle f \circ T^k, g\right\rangle\right|^2$.

The Cauchy-Schwartz inequality gives us the following estimate:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left\langle f \circ T^k, g\right\rangle^2 = \lim_{n\to\infty} \left\langle \frac{1}{n} \sum_{k=0}^{n-1} \left\langle g, f \circ T^k\right\rangle (f \circ T^k), g \right\rangle \le ||g|| \lim_{n\to\infty} \left\| \frac{1}{n} \sum_{k=0}^{n-1} \left\langle g, f \circ T^k\right\rangle (f \circ T^k) \right\|.$$

So it is enough to show that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left\langle g, f \circ T^k\right\rangle (f \circ T^k),$$

but the coefficients $\left\langle g, f \circ T^k\right\rangle$ are bounded due to the fact that $||f \circ T^k||$ are bounded by $||f||$, so it is equivalent to showing that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k = 0.$$

And that is where we apply the van der Corput's lemma. We get that

$$\lim_{h\to\infty} \frac{1}{h} \sum_{n=0}^{h-1} \left(\limsup_{n\to\infty} \left| \frac{1}{n} \sum_{k=0}^{n-1} \left\langle f \circ T^k, f \circ T^{k+h}\right\rangle \right| \right) = 0$$

due to the fact that f was a weakly mixing function. □

Proposition 4.2. Let $(X, \mathcal{F}, \mu, T)$ be a weakly mixing system. Also let $k > 1$ and consider any distinct non-zero numbers $a_1, \dots, a_k \in \mathbb{Z}$. Now let $f_1, \dots, f_k \in L^\infty(X, \mu)$ such that at least one of f_i has mean 0. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} \int (f_1 \circ T^{a_1 m})(f_2 \circ T^{a_2 m}) \dots (f_k \circ T^{a_k m}) \mathrm{d}\mu = 0.$$

Proof. This proposition can be proven by induction. The base case $k = 1$ is equivalent to the Birkhoff's ergodic theorem, and we know that any weakly mixing system is ergodic.

Now suppose that the statement of this proposition holds true for $k - 1$. Also we can assume that f_1 has mean zero. We aim to apply the van der Corput's lemma to this sequence of vectors:

$$w_n = (f_1 \circ T^{a_1 n}) \dots (f_k \circ T^{a_k n}).$$

Now we use the Cauchy-Schwartz inequality to estimate the inner products $\langle w_{n+h}, w_n \rangle$:

$$\langle w_{n+h}, w_n \rangle = \int_X f_{1,h} \circ T^{(a_1-a_k)n} \dots f_{k-1,h} \circ T^{(a_{k-1}-a_k)n} f_{k,h} \mathrm{d}\mu \leq$$
$$\leq \left\| f_{1,h} \circ T^{(a_1-a_k)h} \dots f_{k-1,h} \circ T^{(a_{k-1}-a_k)h} \right\| \| f_{k,h} \| ,$$

where $f_{j,h} := (f_j \circ T^{a_j h}) f_j$. Now recall that T was weakly mixing, therefore, f_1 is a weakly mixing function. This and the previous Corollary imply that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{h=0}^{n-1} \int_X f_{1,h} \mathrm{d}\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{h=0}^{n-1} \left\langle f_j, f_j \circ T^{a_j h} \right\rangle = 0.$$

Therefore, the Cesáro limit in the statement of van der Corput's lemma will not change if we replace $f_{1,h}$ with $f_{1,h} - \int_X f_{1,h} \mathrm{d}\mu$. And the norms $||f_{k,h}||$ are bounded, so we can safely use the induction hypothesis to prove the proposition.

□

Notice that this is almost what we need, then we use the Cauchy-Schwartz inequality to get the desired result without the mean zero assumption. □

4.2 A remark about the notion of weak k-mixing for $k > 2$

It makes sense to think about the condition (3) as of weak mixing of order k. So, this theorem, basically, says that any weakly mixing system is "weakly mixing of all orders".

Nevertheless, there seems to be an ambiguity in defining a weakly k-mixing system for $k > 2$. For example, in the case $k = 3$ consider the following two Cesáro limits:

$$\lim_{n_1,n_2\to\infty} \frac{1}{n_1 n_2} \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} \left| \mu(A \cap T^{-n_1} B \cap T^{-n_1-n_2} C) - \mu(A)\mu(B)\mu(C) \right|$$

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{1}{n} \left| \mu(A \cap T^{-n} B \cap T^{-2n} C) - \mu(A)\mu(B)\mu(C) \right|$$

The author is not aware whether the definitions of weak 3-mixing, corresponding to these limits are equivalent.

5 Strong mixing for G-systems

Here we will discuss measure-preserving group actions of discrete and locally compact abelian groups.

Definition 5.1. Let $(G, +)$ be an abelian group.

An action of G on a system $(X, \mathcal{F}, \mu)$ is called measure-preserving if for every $A \in \mathcal{F}$ and $g \in G$ we have

$$\mu(g^{-1} \cdot A) = \mu(A) \text{ for every } g \in G. \tag{4}$$

A system $(X, \mathcal{F}, \mu)$ with a measure-preserving group action is called a G-measure-preserving system (measure-preserving G-system).

Some authors denote the action of G by defining the operators $T_g : X \to X$, where $T_g(x) := g \cdot x$, and measure-preserving G-systems are denoted by $(X, \mathcal{F}, \mu, (T_g)_{g \in G})$.

For example, for $G = (\mathbb{Z}, +)$ we get a standard measure-preserving system $(X, \mathcal{F}, \mu, T)$, where T denotes the action of 1. However, we also require T to have a well-defined measurable inverse, so not every measure-preserving system is a $\mathbb{Z}$-system.

Definition 5.2. [BM00, Definition 4.1] Let $(G, +)$ be a locally compact abelian group. Then a measure preserving G-system $(X, \mathcal{F}, \mu, (T_g)_{g\in G})$ is called strongly k-mixing if for every family of subsets $(A_i)_{1\le i\le k}$ and all sequences $g_i(n) \in G$, with

$$g_i^{-1}(n)g_j(n) \to \infty \text{ for } 1 \le i < j \le k \text{ and as } n \to \infty,$$

we have

$$\lim_{n\to\infty} \mu\left(\bigcap_{i=1}^{k} g_i(n) \cdot A_i\right) = \prod_{i=1}^{k} \mu(A_i).$$

For example, any countable discrete group is locally compact, so the above definition can be applied to $G = (\mathbb{Z}^k, +)$ for $k \ge 1$. Also let us state a useful lemma for checking whether a group G-system is strongly mixing.

Lemma 5.1. [BM00, Lemma 4.2] Let G be an infinite discrete group acting by automorphisms on a compact abelian group X. Denote the dual group of X by $\hat{X}$. If the dual action of G on $\hat{X} \setminus \{1\}$ is free, then the G-system $(X, \mathrm{Bor}(X), \mu, (T_g)_{g\in G})$ is strongly mixing, where μ denotes the Haar measure on X.

5.1 Two approaches to the Ledrappier's counterexample

F. Ledrappier in the paper [Led78] presented a counterexample to the Rokhlin's problem for $G = (\mathbb{Z}^2, +)$ (the paper is written in French). There are at least two very similar ways to explain the idea behind the counterexample, one way is described in the book [BM00], and the other way is presented in the paper [De 06].

(a) Define

$$Y = \{0,1\}^{\mathbb{Z}^2} = \{f : \mathbb{Z}^2 \to \mathbb{F}_2\}$$

which is a compact abelian group with respect to the Tychonoff(product) topology. Consider the natural action of $\mathbb{Z}^2$ on this space:

$$((k,l) \cdot f)(m,n) = f(m-k, n-l).$$

Now we let us define an invariant subgroup $X \subset Y$ as follows:

$$X = \{f \in Y : f(m,n) = f(m-1,n) + f(m+1,n) + f(m,n-1) + f(m,n+1)\}.$$

Being a compact abelian group, Y admits the Haar measure μ. Moreover, it is a $\mathbb{Z}^2$-invariant subgroup, so we consider the natural $\mathbb{Z}^2$-system (X, μ).

Proposition 5.1. [BM00, Proposition 4.4, Corollary 4.10]. This $\mathbb{Z}^2$-system is strongly 2-mixing, but not 5-mixing.

Proof. The proof of the fact that this system is strongly 2-mixing is quite lengthy and technical, but let us present the key points of it. The main idea is to apply the Lemma 5.1 to the action of $\mathbb{Z}^2$ on $\hat{X}$.

First of all, we need to compute the dual group of X. It turns out that there is a natural bijection between $\mathrm{Fin}(\mathbb{Z}) \times \mathrm{Fin}(\mathbb{Z})$ and $\hat{X}$, where $\mathrm{Fin}(\mathbb{Z})$ denotes the set of all finite subsets of $\mathbb{Z}$. For any finite subsets F, G of $\mathbb{Z}$ define an element $\sigma_{F,G} \in \hat{X}$ as follows:

$$\sigma_{F,G}(f) = \prod_{n\in F} (-1)^{f(n,0)} \prod_{m\in G} (-1)^{f(m,1)}$$

Now we claim that the correspondence

$$(F, G) \mapsto \sigma_{F,G}$$

is a bijection.

We use this bijection to identify $\hat{X}$ with the sum of $\mathbb{F}_2$-group algebras $\mathbb{F}_2[\mathbb{Z}] \oplus \mathbb{F}_2[\mathbb{Z}]$. It turns out that this action is generated by two elements $T, S \in \mathrm{Aut}(\mathbb{F}_2[\mathbb{Z}] \oplus \mathbb{F}_2[\mathbb{Z}])$, where

$$T = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ 1 & \delta_{-1} + \delta_0 + \delta_1 \end{pmatrix}$$

Moreover, $T^p S^q(f, g) \neq (f, g)$ for $(f, g) \neq (0, 0)$ and any $p, q \in \mathbb{Z}$, where $q \geq 0$. This almost immediately yields that the action of G on $\hat{X} \setminus \{1\} = \mathbb{F}_2[\mathbb{Z}] \oplus \mathbb{F}_2[\mathbb{Z}] \setminus \{(0, 0)\}$ is free, and then we can apply the Lemma 5.1.

The second part of this proposition is much easier, so we will provide a complete proof of that.

Define $E = \{y \in X : y(0, 0) = 1\}$ and consider the five sets

$$\begin{aligned} A_1 &= E = \{y \in X : y(0, 0) = 1\}, \\ A_2 &= (2^n, 0) \cdot E = \{y \in X : y(-2^n, 0) = 1\}, \\ A_3 &= (-2^n, 0) \cdot E = \{y \in X : y(2^n, 0) = 1\}, \\ A_4 &= (0, 2^n) \cdot E = \{y \in X : y(0, -2^n) = 1\}, \\ A_5 &= (0, -2^n) \cdot E = \{y \in X : y(0, 2^n) = 1\}. \end{aligned}$$

We claim that 5-mixing breaks on these sets. To prove this, we notice that $\mu(A_i) \neq 0$ for $i = 1, \ldots, 5$, but their intersection is empty.

From the definition of A_i we have

$$A_1 \cap \cdots \cap A_5 = \{y \in X : y(0, 0) = y(2^n, 0) = y(-2^n, 0) = y(0, 2^n) = y(0, -2^n) = 1\},$$

but for every harmonic function $f \in X$ and $m, n, k \in \mathbb{Z}$ we have

$$f(m, n) = f(m - 2^k, n) + f(m + 2^k, n) + f(m, n - 2^k) + f(m, n + 2^k) \pmod 2,$$

this can be proven via induction. Suppose that there is $y \in A_1 \cap \cdots \cap A_5$, then we set $f = y$, $(m, n) = (0, 0)$, $k = n$:

$$1 = y(0, 0) = y(2^n, 0) + y(-2^n, 0) + y(0, 2^n) + y(0, -2^n) = 0 \pmod 2,$$

which yields a contradiction. □

The authors of the book do not prove that this counterexample is not 3-mixing, but that is precisely why we present another approach.

(b) de la Rue uses probabilistic notation to explain the counterexample, so an unprepared reader might get a little confused about what is going on, but the idea here is almost the same. Let us define

$$\tilde{X} = \{f \in Y : f(m, n) + f(m + 1, n) + f(m, n + 1) = 0\}.$$

It is not hard to show that $\tilde{X}$ is a $\mathbb{Z}^2$-invariant compact abelian subgroup of Y, and it admits a unique Haar measure $\tilde{\mu}$. The author of the paper presents an explicit construction of this measure using Bernoulli trials. If we translate this argument to the measure-theoretic language, we define $\tilde{\mu}$ in such a way that

$$\tilde{\mu}(\{f \in X : f(0,j) = 0\}) = \tilde{\mu}(\{f \in X : f(0,j) = 1\}) = \frac{1}{2},$$
$$\tilde{\mu}(\{f \in X : f(i,0) = 0\}) = \tilde{\mu}(\{f \in X : f(i,0) = 1\}) = \frac{1}{2}$$

for every $j \in \mathbb{Z}$ and $i \in \mathbb{N}$. Intuitively, this uniquely defines $\tilde{\mu}$, because every function $f \in X$ is uniquely determined by values on $(0,j)$ and $(i,0)$ for all $j \in \mathbb{Z}$, $i \in \mathbb{N}$.

For example, we know that for any $f \in X$ we have $f(1,j) = f(0,j) + f(0,j+1)$ for every $j \in \mathbb{Z}$, so

$$\tilde{\mu}(\{f \in X : f(1,j) = 0\}) = \tilde{\mu}(f \in X : f(0,j) = f(0,j+1)) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Proposition 5.2. [De 06] The $\mathbb{Z}^2$-system $(\tilde{X}, \tilde{\mu})$ is strongly 2-mixing, but not 3-mixing.

The idea is to consider the behavior of the system on the following three sets:

$$R_1 = \{(i,j) \in \mathbb{Z}^2 : 0 < j < -i, i < 0\},$$
$$R_2 = \{(i,j) \in \mathbb{Z}^2 : i > 0, j > 0\},$$
$$R_3 = \{(i,j) \in \mathbb{Z}^2 : 0 < i < -j, j < 0\}.$$

Now consider two cylinder sets

$$A = \{f \in X : f(i_1, j_1) = 0\},$$
$$B = \{f \in X : f(i_2, j_2) = 0\},$$

where (i_1, j_1) and (i_2, j_2) belong to different R_k. Again, notice that the values of any function $f \in X$ on R_1 depend only on the values at $(0,j)$ for $j < 0$, on R_2 – only on the values at $(0,j)$ for $j \geq 0$, and R_3 – only on the values at $(i,0)$ for $i < 0$. This, basically, implies that A and B are independent.

But if we let A, B to be arbitrary cylinder sets, then we can separate the supports of A, B using the action of $\mathbb{Z}^2$, and then move them in such a way that they will belong to different R_k, and we already know the independence there. Then we approximate any measurable set by a cylinder set to obtain strong mixing.

That is the author's idea for how to show strong mixing, and showing that this system is not strongly 3-mixing requires ideas, which are roughly similar to ones used in the second part of the Proposition 5.1.

What we mean by that that for every $i, j \in \mathbb{Z}, n \in \mathbb{Z}$, and $f \in X$ we have

$$f(i,j) + f(i + 2^n, j) + f(i, j + 2^n) = 0. \tag{5}$$

However, this equality implies that the sets

$$A = \{f \in X : f(i,j) = 0\},$$
$$B = \{f \in X : f(i + 2^n, j) = 0\},$$
$$C = \{f \in X : f(i, j + 2^n) = 0\}.$$

are "far from independent", because

$$\mu(A)\mu(B)\mu(C) = \frac{1}{8},$$

and

$$\mu(A \cap B \cap C) \overset{(5)}{=} \mu(A \cap B) = \frac{1}{4}.$$

But n can be made arbitrarily large.

In another lecture notes [Tao11] Terrence Tao, inspired by the Ledrappier's counterexample, presents an example of a strongly 2-mixing but not 3-mixing $\mathbb{F}_3[t]$-system.

Both Tao and de la Rue admit that there are "strong obstacles" to transferring these counterexamples to the case $G = \mathbb{Z}$. However, de la Rue constructed a stationary process ($\mathbb{Z}$-system) which is strongly 2-mixing, but not 3-mixing **with respect to a conditional sigma-algebra**, so there are conditional counterexamples to the Rokhlin's problem.

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