A minicourse on random walks on groups

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Basics

Definition (Groups)

A group is a set G with a binary operation $\cdot : G \times G \to G$ satisfying three properties:

1. (associativity) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, $a, b, c \in G$
2. (identity) There exists $e \in G$ such that $a \cdot e = e \cdot a = a$ for all $a \in G$
3. (inverse) For every $g \in G$ there exists $g^{-1} \in G$ such that $g \cdot g^{-1} = g^{-1} \cdot g = e$.

A subset $H \subset G$ is a subgroup if it is closed under the group operation of G.

If $S \subset G$ is a subset, by $\langle S \rangle$ we denote the smallest subgroup containing S, this is the subgroup **generated** by S.

Some notation

- For every x, y in Γ we define $[x, y] = xyx^{-1}y^{-1}$. In particular, for any two subgroups Γ_0, Γ_1 we can define the commutator $[\Gamma_0, \Gamma_1]$ as the smallest subgroup of G that contains all $[x, y]$ for $x \in \Gamma_0, y \in \Gamma_1$.
- We will also denote $g^h = hgh^{-1}$ for $g, h \in G$.

Groups, basics

Let $H \leq G$ be a subgroup.

Definition

- A left coset is a subset $gH = \{gh : h \in H\}$. The set of all left cosets wrt H will be denoted by G/H.
- A subgroup $H \leq G$ is **normal** if for every $g \in G$ we have $gHg^{-1} = H$.
- The set of all left cosets wrt H will be denoted by G/H, with $|G/H| := [G : H]$ and called the **index** of H in G.

Exercise

Show that if H is a normal subgroup of G then the set G/H can be equipped with an operation $g_1H \cdot g_2H = (g_1g_2)H$, making it into a group, called the **quotient group**.

Group homomorphisms

Definition

A map $f : (G, \cdot_G) \to (H, \cdot_H)$ is a group homomorphism if

$$f(g_1 \cdot_G g_2) = f(g_1) \cdot_H f(g_2), \quad g_1, g_2$$

We say that

- $Kerf = \{g \in G : f(g) = e_H\}$, $Imf = \{h \in H : \exists g \in H : f(g) = h\}$.
- f is an endomorphism if $G = H$.
- f is an isomorphism if there exists a homomorphism $f^{-1} : H \to G$ such that $f \circ f^{-1} = f^{-1} \circ f = Id$.

Theorem (First isomorphism theorem)

If $f : G \to H$ is a homomorphism, then $Kerf \leq G$ is a normal subgroup with $G/Ker\, f$ isomorphic to H.

Group properties

Definition (Properites)

A group Γ is called

- abelian if $xy = yx$ for any $x, y \in \Gamma$
- finitely generated if there exists a finite collection $\{g_1, \dots g_n\} \subset \Gamma$ such that every $g \in \Gamma$ can be written as a finite product of g_i's.
- solvable if the sequence $\Gamma^{(0)} = \Gamma, \Gamma^{(n)} = [\Gamma^{(n-1)}, \Gamma^{(n-1)}]$ stabilizes
- nilpotent if the sequence $\Gamma^{(0)} = \Gamma, \Gamma^{(n)} = [\Gamma, \Gamma^{(n-1)}]$ stabilizes
- free if there exists a set $S \subset \Gamma$ so that Γ is generated by S and there are no relations between the generators (no non-trivial word can give identity)

In these notes I will restrict myself to finitely generated discrete groups (or compactly generated locally compact groups) unless stated otherwise. The strongest results can be restated for non f.g. groups but requires a bit more work.

Virtual notation

For a property P we say that a group Γ is **virtually** P if there exists a finite index subgroup $H \subset \Gamma$ such that P holds for H.

Group actions

Let G be a group. Observe that for a given object X with some structures, the set of all automorphisms $X \to X$ form a group $Aut(X)$.

Definition (Group actions)

Let X be a set which, possibly, has some additional structure. The correspondence $\rho : G \to Aut(X)$ defines a (**left**-)action if for every $g, h \in G$ and $h$we have

$$\rho(g)(\rho(h)(x)) = \rho(gh)(x).$$

In other words, ρ is a homomorphism. Notation: $G \curvearrowright X$.

Remark. We will mostly consider left group action, omitting the direction in the future.

Definition

Given a group Γ, we say that two actions X and Y are isomorphic, if there exists a morphism $\rho : X \to Y$ such that for every g the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow g & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

Examples of groups

- For $n \geq 1$ we will denote the free group by $F_n = \langle a_i^{\pm 1} \rangle_{1 \leq i \leq n}$, with the multiplication being concatenation with cancelling out inverses. For example, $a_1 \cdot a_2^{-1} \cdot a_3 \cdot a_3 = a_1 a_2^{-1} a_3^2$, but $a_1 \cdot a_1^{-1} \cdot a_2 = a_2$.
- In general, every finitely generated group can be represented in terms of its generators and relations between them:

$$\Gamma = \langle a_1, \dots a_n \mid e = w_1 = w_2 = w_3 = \dots \rangle,$$

  where w_i are words in a_k. If the set of relations can be made finite, Γ is called finitely presented.
- Let $\mathbb{F}$ be a field. Then we can define $GL_n(\mathbb{F})$ = invertible matrices with coefficients in $\mathbb{F}$. It is a group with respect to matrix multiplication.
- Similarly, $SL_n(\mathbb{F}) = \{A \in GL_n(\mathbb{F}) : \det(A) = 1\}$ is a subgroup of $GL_n(\mathbb{F})$.
- Consider $Aff(\mathbb{F}) = \{ax + b : a \in \mathbb{F}^\times, b \in \mathbb{F}\}$. It is a group with respect to composition.
- Consider $SL_n(\mathbb{Z})$. Show that this is a group with respect to matrix multiplication because the inverses are also integer-valued matrices.
- In general, any (finite-dimensional) Lie group G contains many interesting discrete subgroups, and we will discuss them much later!

Operations on groups

Definition

Let G, H be groups.

- The direct product $G \times H$ is a group with the operation

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2), \quad g_1, g_2 \in G, h_1, h_2 \in H.$$

- Suppose that H acts on G via $\varphi : H \to Aut(G)$, we define the **semi-direct** product $G \rtimes_\varphi H$ as follows:

$$(g_1, h_1)(g_2, h_2) = (g_1 \varphi_{h_1}(g_2), h_1 h_2), \quad g_1, g_2 \in G, h_1, h_2 \in H.$$

- The **free product** $G * H$ is a group with the underlying set being finite words $\{g_1 h_1 g_2 \dots h_n g_n\}$ with the operation being concatenation, with no additional relations imposed between elements of G, H.

More exotic groups

- Let A, B be countable groups. We define the **wreath product** $A \wr B$ as the semidirect product $\bigoplus_{b \in B} A \rtimes B$, where B acts on the direct sum $\bigoplus_B A$ as follows: for every finitely supported function $f : B \to A$ and $b, x \in B$ we have $(bf)(x) := f(b^{-1}x)$.
- In particular, we will consider the **lamplighter group** $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$.
- Fix $p, q \in \mathbb{N}$. The **Baumslag-Solitar** groups are a family of groups

$$BS(p, q) = \left\langle a, b \mid ba^m b^{-1} = a^n \right\rangle .$$

- The Basilica group $\mathcal{B}$ is a self-similar group acting on a rooted binary tree (with the vertex set represented by binary numbers), generated by two elements a, b and with multiplication defined by the following rule:

$$a(0x) = 0x, \qquad a(1x) = 1(bx),$$
$$b(0x) = 1x, \qquad b(1x) = 0(ax).$$

  It is an non-finitely presented group of exponential growth.

Examples of groups with some interesting actions

- Let $G = GL_n(\mathbb{R})$. Then G acts on $\mathbb{R}^n$ via matrix multiplication. It can be restricted to $SL_n(\mathbb{R})$ as well.
- Consider $G = PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\{\pm Id\}$. Then define the action of G on the upper half-plane model $\{z \in \mathbb{C} : Im(z) > 0\}$ as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

  Same can be done via taking $G = PSU(1,1)$, where $SU(1,1) = \left\{ \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} : |a|^2 - |b|^2 = 1 \right\}$, with $G = SU(1,1)/\{\pm Id\}$, and considering the action on $\mathbb{D} = \{|z| \le 1\}$

$$\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} z = \frac{az+b}{\overline{b}z+\overline{a}}.$$

  Both actions are isomorphic to each other.
- Consider $\Gamma = BS(p,q)$. Then we can consider its matrix representation via

$$a \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} q/p & 0 \\ 0 & 1 \end{pmatrix}.$$

  It is not always a faithful representation (the map does not need to be injective!)

Growth of groups

Let (G, S) is a finitely generated group equipped with a finite generating set S.

Definition

The word distance $|g|_S$ of an element $g \in G$ wrt S is the smallest $n \geq 0$ such that $g = s_1 \dots s_n$ for some $s_i \in S$.

Exercise

Show that $|g|$ is a **norm** on G: $|e| = 0$, $|g| = |g^{-1}|$ and $|gh| \leq |g| + |h|$.

We will denote $B(x, r) := \{g \in G : |x^{-1}g|\}$, with the growth function $b_r := |B(e, r)|$.

Definition

Let $f, g : (0, \infty) \to (0, \infty)$ be two increasing functions. We say that $f \prec g$ if there exist $C, \alpha > 0$ so that $g(n) \leq Cf(\alpha n)$, and we write $f \sim g$ if $f \prec g$ and $g \prec f$.

Gromov's theorem

Definition

Given a finitely generated G, we say

- G has **polynomial** growth if there exists $k \in \mathbb{N}$ and $C > 0$ such that $b_n \preceq Cn^k$.
- G has **intermediate/subexponential** growth if $\lim_{n\to\infty} b_n \succeq n^k$ for any $k \in \mathbb{N}$ and $b_n \prec \exp(\alpha n)$ for any $\alpha > 0$.
- G has **exponential** growth if there exists $\alpha > 0$ so that $b_n \succeq \exp(\alpha n)$.

Theorem

A finitely generated nilpotent group has polynomial growth.

Theorem (Gromov)

Every group of polynomial growth is virtually nilpotent. Moreover, the growth is **exactly** *polynomial, so there exists $C > 0$ and $k \in \mathbb{N}$ such that $\frac{n^k}{C} \leq b_n \leq Cn^k$.*

Intermediate growth

Theorem (Grigorchuk)

There exists a group of intermediate growth.

Theorem (Shalom, Tao)

Let G be a finitely generated group with $b_n \leq n^{\log(\log n)^c}$ for some $c > 0$. Then G is virtually nilpotent.

Grigorchuk's gap conjecture

Let G be a finitely generated group with $b_n \prec \exp(\sqrt{n})$. Then G is virtually nilpotent.

The gap between $n^{\log(\log n)^c}$ is $\exp(\sqrt{n})$ is very big, but so far there are no known groups with growth in this regime.

Random walks on groups

Definition

A **random walk** on an (infinite) group Γ generated/induced by a probability measure μ on Γ and the initial distribution μ_0 is a sequence of random variables

$$X_n = g_0 g_1 \dots g_n,$$

where $\{g_i\}_{i \geq 1}$ are i.i.d μ-distributed Γ-valued random variables, with $g_0 \sim \mu_0$ and also independent.

Unless stated otherwise, we will usually assume $\mu_0 = \delta_e$.

If the **support** $\operatorname{supp} \mu := \{\gamma \in \Gamma : \mu(\gamma) > 0\}$ generates Γ as a semigroup, we call μ (and the respective random walk) **admissible/non-degenerate**.

Our goal

We aim to understand the long-term behaviour of random walks by classifying the stochastically significant and distinct events that can happen if we let the random walk run for a long amount of time!

Random walks on groups

Notation

We will refer to the pairs (Γ, μ), where Γ is a group equipped with a probability measure μ as **measured groups**.

Definition

Let (Γ, μ) be a measured group equppied with an measurable action on a probability space $(X, \mathcal{A}, \nu)$.

- We say that ν is a μ-**stationary** measure if for every $A \in \mathcal{A}$ we have

$$\nu(A) = \int_\Gamma \nu(g^{-1}(A)) d\mu(g) = \int_\Gamma g_*(\nu)(A) d\mu(g).$$

- In this case we will say that (X, ν) is a (Γ, ν)-space.

Self-similarity

This definition should be familiar to harmonic analysis folks who study self-similar measures!

Examples: $\mathbb{Z}^d$

Consider $\Gamma = \mathbb{Z}^d$. Let μ be a probability measure which is uniformly distributed on $\{\pm e_i\}_{1 \leq i \leq d}$. Then, due to Polya's theorem, the respective random walk on Γ will return to 0 infinitely often for $d = 1, 2$ (recurrence) and the opposite will hold for $d \geq 3$ (transience).

Definition

Let (Γ, μ) be a group equipped with a probability measure μ. A function $f : \Gamma \to \mathbb{C}$ is called μ-harmonic if $f(g) = \sum_h \mu(h) f(gh)$.

We will sometimes denote the space of bounded harmonic functions by $\mathrm{Har}^\infty(\Gamma, \mu)$. However, as we will see later, from the asymptotic standpoint, there is nothing interesting going on, due to the Choquet-Deny theorem:

Theorem (Choquet-Deny theorem)

Let Γ be a countable abelian group. Then, for every non-degenerate measure μ on Γ we have $\mathrm{Har}^\infty(\Gamma, \mu) = \mathbb{C}$.

(proof by D. Blackwell for $\mathbb{Z}^d$.)

Free group

If we consider μ to be a measure with weights $1/2n$ on all a_i and a_i^{-1}, then the word length will decrease with probability $1/2n$ (if you have a word $X_k = a_{i_1} \dots a_{i_k}$, then only $X_k a_{i_k}^{-1}$ has smaller length), so from the drift we see in the respective Markov chain, the X_k will stabilize to an infinite word. Let us denote the space of all infinite words by ∂F_n. The above argument shows that the pushforward $(S^{\mathbb{N}}, \mu^{\otimes \infty}) \to \partial F_n$ is well defined up to measure zero, so we can denote the resulting pushforward measure by ν.

Definition

The measure ν on ∂F_n will be referred to as the **hitting measure** on the random walk.

Exercise

Let $\Gamma = F_n$, and μ is a uniform measure on the generators.

- Express ν as an infinite product measure!
- Show that ν is a μ-stationary measure with respect to the standard action of F_n on ∂F_n, so $(\partial F_n, \nu)$ is, indeed, a (F_n, μ)-space.
- Find an example of a non-constant $f \in \mathrm{Har}^{\infty}(F_n, \mu)$.
- (*) Do the same if μ is an arbitrary non-uniform measure with the same support.

Poisson boundary: formal construction

Let Γ be a countable discrete group equipped with a measure μ and initial distribution μ_0.

Definition

Denote the space of sample paths by $\Gamma^{\mathbb{Z}_{\geq 0}}$, with the coordinate-wise action of Γ via multiplication from the left.

Let $f : \Gamma^{\mathbb{Z}_{\geq 0}} \to \Gamma^{\mathbb{Z}_{\geq 0}}$ be the map which is defined as follows,

$$f(g_0, g_1, g_2, \ldots) = (g_0, g_0 g_1, g_0 g_1 g_2 \ldots).$$

This will allow us to define the probability measure $\mathbb{P}_{\mu_0}$ as the pushforward of the product measure $\mu_0 \bigotimes \mu^{\otimes \mathbb{N}}$. We will denote the codomain and the measure by $(\Omega, \mathbb{P}_{\mu_0})$, with $\mathbb{P} = \mathbb{P}_{\delta_e}$.

Definition

For two sequences (g_i), (h_i), define the **orbit equivalence** relation as follows: $(g_i) \sim (h_i)$ iff there exists $m > 0$, $n > 0$ such that $g_{m+k} = h_{n+k}$ for $k > 0$.

Poisson boundary: formal definition

The naive idea is to take the space $(\Omega, \mathbb{P})$ and "quotient out" the equivalent sequences to make sure different asymptotic events are actually different. This is the idea behind **measurable quotients and partitions**, and this turns out to be quite difficult to do rigorously, done by Rokhlin a long time ago. We will just present the definition via the universal property. Define the time shift map

$$T : \Omega \to \Omega, \quad (Tg)_n = g_{n+1}.$$

Definition

Let (Γ, μ) be a measured group, and consider the time shift map $T : \Omega \to \Omega$, $(Tg)_n = g_{n+1}$. Then there exists a unique up to a measurable isomorphism (Γ, μ)-space (X, ν) equipped with a Γ-equivariant map $bnd : \Omega \to X$, such that for every Γ-equivariant map between (Γ, μ)-spaces $F : (\Omega, \mathbb{P}) \to (Y, \lambda)$ with $F \circ T = F$ we have the following commutative diagram:

$$\begin{array}{ccc} (\Omega, \mathbb{P}) & \xrightarrow{F} & (Y, \lambda) \\ \downarrow{\scriptstyle bnd} & \nearrow{\scriptstyle \exists! G} & \\ (X, \nu) & & \end{array}$$

Notation: $(X, \nu) = (\Omega, \mathbb{P}) // T = Pois(\Gamma, \mu)$.

Poisson boundary: formal definition

We can modify the equivalence relation with a weaker one:

Definition

Two sequences (x_n) and y_n in Ω are **tail equivalent** if there exists $N \geq 0$ such that $x_n = y_n$ for all $n \geq N$.

We can repeat the same procedure to define the **tail boundary** of (Γ, μ).

Theorem (Derriennic, Kaimanovich)

The tail and Poisson boundaries coincide $\mathbb{P}_{\mu_0}$ mod 0 for a given μ_0 iff for any integers $k, d \geq 0$ and any $\lambda \prec \mu_0\mu_k \wedge \mu_0\mu_{k+d}$ we have

$$\lim_{n\to\infty} ||\lambda\mu_n - \lambda\mu_{n+d}||_{TV} = 0.$$

Otherwise the above limit is > 2.
In particular, the Poisson and tail boundaries coincide for $\mu_0 = \delta_e$.

Poisson boundary: geometry

Ironically, contrary to the previous point, it is evident that the definition is just too hard to work with, and we cannot compute Poisson boundaries by explicitly computing measurable quotients. This is why one seeks to find **geometric realizations** of the Poisson boundary of (Γ, μ).

Definition (Topological realization)

- A (Γ, μ)-space (X, ν) is a μ-**boundary** if we have $(g_n)_* \nu \xrightarrow{wk^*} \delta_x$ $\mathbb{P}$-a.s.
- A μ-boundary is called **maximal** if it is not a non-trivial quotient of another μ-boundary.
- A **topological realization** of the Poisson boundary is a maximal μ-boundary equipped with the map $bnd : \Omega \to X$, where $(g_n)_* \nu \xrightarrow{wk^*} \delta_{bnd(g_n)}$.

Finding μ-boundaries is as easy as showing the stabilization of a feature related to the group or its action on a nice space in a long run of a random walk. Yet, it is still a bit difficult to show that a given μ-boundary is maximal using quotients.

Poisson boundary: harmonic functions

Definition

A function $f : (\Gamma, \mu) \to \mathbb{C}$ is μ-harmonic if $f(g) = \sum_h \mu(h) f(gh)$.

Observe that, given a (Γ, μ)-space (X, ν) plus a function $F \in L^\infty(X, \nu)$, the following integral always delivers you a bounded μ-harmonic function:

$$\chi_F(g) = \int_X F(x) dg_*(\nu)(x), \quad g \in \Gamma.$$

Thus we define a correspondence $L^\infty(X, \mu) \to \mathrm{Har}^\infty(\Gamma, \mu)$.

Definition

We say that a (Γ, μ)-space (X, ν) is an **analytic realization** of the Poisson boundary if the above map is an isometric isomorphism; in other words, integration along (X, ν) realizes all bounded harmonic functions.

This definition is nice because sometimes one can study bounded harmonic functions directly, and one can show the triviality of the Poisson boundary by showing non-existence of non-constant harmonic functions.

Poisson boundary: harmonic functions

Theorem

Let (X, ν) be the Poisson boundary of a measured group (Γ, μ).

- *(X, ν) realizes all bounded μ-harmonic functions on Γ.*
- *(X, ν) is a maximal μ-boundary.*

Proof.

- Let f be a bounded μ-harmonic function. The Martingale Convergence Theorem ensures that $f(\omega_n)$ converges for $\mathbb{P}$-a.e. $\omega = (\omega_n)$, so f can be extended to a T-invariant function on Ω ($\mathbb{P}$-mod 0). By the universal property there has to exist a function g on (X, ν) such that $f(\omega) = g(bnd(\omega))$. However, it is not difficult to check

$$\int_X g(\gamma x) d\nu(x) = \int_\Omega f(\gamma\omega) d\mathbb{P}(\omega) = \int_\Omega f(\omega) d\mathbb{P}_\gamma(\omega) = f(\gamma).$$

- This is just the same argument, where we start with a continuous $f \in C(X)$ and consider the μ-harmonic function $\xi : \gamma \mapsto \int_X f(\gamma\xi) d\nu(\xi)$.

□

Entropy criterion

Definition

Let (G, μ) be a measured group. Then we define the **entropy** of μ as $H(\mu) = -\sum_g \mu(g) \log(\mu(g))$.

The following is a corollary of the Fekete's lemma.

Theorem

If $H(\mu) < \infty$, then $H(\mu^{*k}) < \infty$ and the limit

$$h(\mu) := \lim_{k \to \infty} \frac{H(\mu^{*k})}{k}$$

*exists. This limit is called the **asymptotic entropy** of the random walk.*

Theorem (Liuoville property + Entropy criterion)

Let μ be a non-degenerate measure on a discrete countable group Γ. TFAE:

- *The Poisson boundary of (G, μ) is trivial*
- *There are no non-constant bounded μ-harmonic functions on G (Liuoville)*

If $H(\mu) < \infty$ then above are equivalent to $h = 0$.

Wait, but do we actually determine Poisson boundaries?

How do we show that a candidate μ-boundary is maximal?

Definition (Conditional RW)

Let (X, ν) be a μ-boundary of (Γ, μ), and fix a point $\xi \in X$. Define the **conditional random walk** on Γ as a Markov process defined via

$$\mu_n^\xi(g) = \mu^{*n}(g) \frac{dg_* \nu}{d\nu}(\xi).$$

Definition (Relative entropy)

Let (X, ν) be a μ-boundary of (Γ, μ). The relative entropy at $\xi \in X$ is the limit

$$h_\xi = \lim_{k \to \infty} \frac{H(\mu_n^\xi)}{k}.$$

Theorem (Kaimanovich, Sobieczky '12)

Let μ be a non-degenerate measure on a countable discrete G with $H(\mu) < \infty$. TFAE:

- *A μ-boundary (X, ν) is maximal.*
- *$h_\xi = 0$ for ν-a.e $\xi \in X$.*

Geometric criteria for Poisson boundary

The above criterion serves as a foundation for practical criteria for maximality of μ-boundaries.

Definition

Let (X, x_0, d, μ) be a metric μ-boundary of (G, μ), with the action preserving distances. In particular, we can identify G with a subgroup of $\mathrm{Isom}(X, d)$.

- $G < \mathrm{Isom}(X, d)$ has **exponentially bounded growth** if there exists $C > 0$ such that

$$|\{g \in G : d(x_0, g.x_0) \leq R\}| \leq Ce^{CR}.$$

- We say that μ has finite first moment if

$$\int_\Gamma d(x_0, g.x_0) d\mu(g) < \infty.$$

It is not difficult to see that finite support will imply finite first moment. We can also replace $d(x_0, g.x_0)$ with $f(d(x_0, g.x_0))$ to modify the moment condition for our needs (common candidates are $f = \log, \exp$).

Geometric criteria for Poisson boundary

Theorem (Ray approximation criterion)

Let:

- Γ *be a countable group*
- $G < Isom(X, d)$ *is an action of exponentially bounded growth*
- μ *has finite first moment wrt* d
- (B, ν) *is a* μ*-boundary.*

Assume that there exist maps $\pi_n : B \to G$ *such that*

$$\lim_{n\to\infty} \frac{d(X_n, \pi_n(bnd(X_\infty)))}{n} = 0.$$

Then (B, ν) *is maximal.*

This criterion works well if we can somehow show that a random walks tracks paths which diverge to infinity – then we can just take π_n to be the n-th point in our path. Moreover, there are more approximation criteria, which we are not going to talk about in much detail.

Geometric criteria for Poisson boundary

To formulate the strip criterion we need to consider the backward random walk, with $\tilde{\mu}(g) = \mu(g^{-1})$.

Theorem (Strip approximation criterion)

Let:

- Γ *be a countable group*
- $G < Isom(X, d)$ *is an action of exponentially bounded growth*
- μ *has finite first moment wrt* d
- $(B_,\nu_-)$ *and* (B_+, ν_+) *are a* $\tilde{\mu}$*-boundary and* μ*-boundary respectively.*

Assume that there exists a measurable Γ*-equivariant map* S *assigning to pairs of points* $(b_-, b_+) \in B_- \times B_+$ *non-empty "strips"* $S(b_-, b_+) \subset \Gamma$ *such that for all* $g \in \Gamma$ *and* $\nu_- \otimes \nu_+$*-a.e* $(b_-, b_+) \in B_- \times B_+$ *we have*

$$\frac{1}{n} \log |S(b_-, b_+)g \cap B(e, d(e, x_n))| \xrightarrow{\mathbb{P}} 0,$$

with respect to $x = (x_n)_{n \geq 0}$*, then* (B_+, ν_+) *is maximal.*

Triviality of the Poisson boundary

The problem of classification of measured groups with trivial Poisson boundary is extremely difficult, but here is the state of the art.

1. (Blackwell, '55) There are no non-constant bounded μ-harmonic functions on $\Gamma = \mathbb{Z}^d$ for any measure μ.
2. (Choquet, Deny '60) Same but for any abelian Γ.
3. (Dynkin, Maljutov, '61) Same but for any virtually nilpotent Γ.
4. (Furstenberg '60s) The Poisson boundary of a non-amenable group Γ is always non-trivial for a non-degenerate μ.
5. (Kaimanovich-Vershik, '83) There exists an amenable group of exponential growth (lamplighter $\mathbb{Z}_2 \wr \mathbb{Z}$) and a finitely supported symmetric μ with trivial Poisson boundary.
6. (Kaimanovich-Vershik, Rosenblatt, '83) Every amenable group admits a measure μ with trivial Poisson boundary.
7. (Frisch, Hartman, Tamuz, Ferdowsi, '19) The reverse is true – if Γ is a finitely generated group so that its Poisson boundary is trivial for any non-degenerate μ, then Γ is virtually nilpotent.

Triviality of the Poisson boundary

Exercise

- Prove that the Poisson boundary of a recurrent random walk is trivial. Hint: martingale convergence theorem.
- Use the entropy criterion to show that any group Γ of subexponential growth has trivial Poisson boundary with respect to any finitely supported non-degenerate μ.

Exercise (Proof of the Choquet-Deny theorem)

Let Γ be an abelian group and let μ be a non-degenerate measure.

1. Consider $H = Har^{[0,1]}(\Gamma, \mu)$ to be a collection of bounded μ-harmonic functions with range in $[0, 1]$. Show that this is a Γ-invariant compact convex subset of $\mathrm{Har}^{\infty}(\Gamma, \mu)$.
2. Suppose that $f \in H$ is **extremal**, so it cannot be a non-trivial convex combination of harmonic functions. Show that f is constant.
3. Conclude via Krein-Milman theorem and prove the Choquet-Deny's theorem.

Warning!

Warning

Due to [Ershler] and [Frisch-Chawla] we know that there are measures on groups of subexponential growth with non-trivial Poisson boundary.

Theorem (Erschler, '04)

There exists a measured group (Γ, μ), such that

- Γ *is a group of subexponential growth with growth faster than* $\exp(n^\alpha)$ *for any* $\alpha \in (0, 1)$
- *The measure* μ *is a symmetric non-degenerate measure with finite entropy*
- *There is a non-constant bounded* μ*-harmonic function on* Γ*, so the Poisson boundary of* (Γ, μ) *is non-trivial.*

Free group

The free group is, arguably, one of the simplest groups one could study where the Poisson boundary is non-trivial. Before we have shown that if we take $\Gamma = F_n$ and μ to be uniform on $S = \{a_i^{\pm 1}\}$, then ∂F_n is a μ-boundary.

- (Furstenberg, '60s) For any non-degenerate μ and $n > 1$ the respective random walk on F_n converges to ∂F_n, thus obtaining a μ-boundary $(\partial F_n, \nu)$
- (Dynkin-Maljutov '61) For any μ supported on the standard generators $(\partial F_n, \nu)$ is the Poisson boundary.
- (Derrienic '63) Same but for any finitely supported μ.
- (Kaimanovich '00) Same but for any non-degenerate μ with finite entropy + finite logarithmic moment, using ray or strip approx. criteria.
- (Forghani-Tiozzo '19) Same but for finite entropy **or** finite logarithmic moment
- (Chawla-Frisch '25+) An example of μ_τ for which $(\partial F_n, \nu)$ is not the Poisson boundary.

Exercise

- Follow up on the previous exercise and establish using an approximation criterion that $(\partial F_n, \nu)$ is the Poisson boundary for uniform μ.

Hyperbolic spaces and groups

Definition

- A metric space (X, d) is **geodesic** if for every $x, y \in X$ there exists an isometry $i : [a, b] \to X$ with $i(a) = x, i(b) = y$.
- (thin triangles) A geodesic metric space (X, d) is called (Gromov) δ-hyperbolic if for every triple $x, y, z \in X$ we have

$$[x, z] \subseteq B_\delta([x, y] \cup [y, z])$$

  for every choice of the respective geodesics.

There are many nice definitions for hyperbolicity which rely on some "negative curvature" feature.

Examples/exercises

- Compact spaces are δ-hyperbolic.
- Trees are δ-hyperbolic.
- Free products of cyclic groups are δ-hyperbolic.
- The hyperbolic space $\mathbb{H}^2$ is δ-hyperbolic, with $\delta = \log(3)$.
- $\mathbb{R}^n$ is hyperbolic only for $n = 1$.

Hyperbolic groups

Theorem

A finitely generated group Γ is δ-hyperbolic if its Cayley graph with respect to a choice of a generating set S is δ-hyperbolic wrt word metric.

Exercise

- Recall that $f : (X, d_X) \to (Y, d_Y)$ is a **quasi-isometry** if we have

$$\frac{d_X(x, y)}{C} - A \leq d_Y(f(x), f(y)) \leq C d_X(x, y) + A.$$

  Show that hyperbolicity is preserved under quasi-isometries.
- Use this to show that δ-hyperbolicity does not depend on the choice of a generating set, so it is a true group property.
- Show that the free groups are δ-hyperbolic (what is δ?), and any group which contains $\mathbb{Z}^2$ cannot be δ-hyperbolic for any $\delta > 0$.

Milnor-Svarc lemma

Definition

An action of a group Γ on a proper metric space (X, d) via isometries is

- **cocompact** if there is a compact set $K \subset X$ so that $X = \bigcup_{g \in \Gamma} gK$.
- **totally discontinuous** if for every $x \in X$ and open nbhd $x \in U$ the set of $g \in \Gamma$ such that $U \cap gU \neq \emptyset$ is finite.
- **geometric** if it is cocompact and totally discontinuous.

Theorem (Milnor-Svarc lemma)

Let Γ be a group which admits a geometric action on a proper metric space (X, d). Then Γ is finitely generated, with the map $g \mapsto gx_0$ being a quasi-isometry with respect to the word distance defined by said generating set. In particular, if (X, d) is hyperbolic, then so is Γ.

This theorem allows us to effortlessly show hyperbolicity of groups which act on hyperbolic spaces.

More examples of hyperbolic groups

Definition

Let G be a locally compact group. We will say that a subgroup $\Gamma \leq G$ is a **lattice**, if the quotient space $\Gamma \setminus G$ admits a finite G-invariant Borel measure.
We will say that Γ is a cocompact lattice if $\Gamma \setminus G$ is a compact space.

On the notation + exercise

Consider $G = PSL_2(\mathbb{R})$. Consider the subgroup $K = \begin{pmatrix} \cos(\lambda) & -\sin(\lambda) \\ \sin(\lambda) & \cos(\lambda) \end{pmatrix}$.

- Show that the Riemannian structure on the quotient G/K defines a symmetric space isomorphic to $\mathbb{H}^2$.
- Show that $PSL_2(\mathbb{Z}) \leq PSL_2(\mathbb{R})$ is a lattice
- Show that a lattice $\Gamma \leq G$ is cocompact if and only if the quotient $\Gamma \backslash G/K$ is compact if and only if there is a compact fundamental domain with respect to the action of Γ on G/K.
- Show that any cocompact lattice in $PSL_2(\mathbb{R})$ is a hyperbolic group.
- Show that $SL_2(\mathbb{Z})$ is lattice but not cocompact, yet it is a hyperbolic group.
- (*) Show that $SL_3(\mathbb{Z})$ is not a hyperbolic group.

Gromov boundary

Let (X, d) be a hyperbolic space.

Definition (Gromov product)

For $x, y, z \in X$ define

$$(x, y)_z = \frac{d(x, z) + d(y, z) - d(x, y)}{2}.$$

Definition (Gromov/geodesic boundary)

Fix $x_0 \in X$. Define the **Gromov boundary** $\partial_{x_0} X$ as a set of asymptotic equivalence classes of geodesic rays $\gamma : [0, \infty) \to X$ with $\gamma(0) = x_0$, with $\gamma \sim \gamma'$ if $\sup_t d(\gamma(t), \gamma'(t)) < \infty$.
The topology on ∂X is defined using the basis of neighbourhoods

$$V(p, r) = \{q \in \partial_{x_0} X \mid \exists \gamma_1 \sim p, \gamma_2 \sim q, \gamma_1(0) = \gamma_2(0) = x, \liminf_{t \to \infty} (\gamma_1(t), \gamma_2(t))_x \geq r\}$$

around each point $p \in \partial X$.

The Gromov boundary does not depend on the choice of x_0, moreover, both ∂X and $X \cup \partial X$ are compact.

Gromov boundary: examples

- The Gromov boundary of $\mathbb{R}$ consists of just two points, which correspond to $\pm\infty$.
- The Gromov boundary of a tree T can be identified with its space of ends ∂T.
- Any quasi-isometry induces a homeomorphism between the Gromov boundaries. So, as long as a hyperbolic group Γ acts geometrically on a hyperbolic space X, the Gromov boundaries of Γ and X are the same.

In general, we have the following theorem.

Theorem

Let Γ be a hyperbolic group. Exactly one of the following occurs.

- Γ *is finite and* $\partial\Gamma$ *is empty*
- Γ *is virtually cyclic and* $|\partial\Gamma| = 2$
- Γ *contains a subgroup isomorphic to* F_2 *and* $\partial\Gamma$ *is an infinite perfect compact metrizable space.*

Remark. This theorem shows that any non-virtually cyclic hyperbolic group is non-amenable!

Random walks on hyperbolic groups

The following theorem is a powerful result which establishes the Poisson boundary for hyperbolic groups.

Theorem (Kaimanovich '01)

Let (Γ, μ) be a hyperbolic group equipped with a non-degenerate μ with the finite entropy and finite first logarithmic moment.

- *Then the respective random walk almost surely converges to the Gromov boundary $\partial\Gamma$, so $(\partial\Gamma, \nu)$ is a μ-boundary with respect to the hitting distribution ν.*
- *Moreover, if μ has finite entropy and finite logarithmic moment, $(\partial\Gamma, \nu)$ is maximal, so it is a model for the Poisson boundary.*

The proof is not super difficult once you have ray/strip approximation criterion at your disposal. The recent paper by Chawla, Frisch, Forghani, Tiozzo removes the logarithmic moment condition using a new approximation criterion but finite entropy turns out to be strictly necessary.

Strengthening of the previous theorem

We would like to also understand random walks on groups which might act on nice hyperbolic spaces but not necessarily in a geometric fashion.

Example

Consider $\Gamma = PSL_2(\mathbb{Z})$ acting on $\mathbb{H}^2 = \{Im(z) > 0\}$. Then it is well-known that the fundamental domain for this action has a cusp, so the action cannot be cocompact.

Thankfully, we have a strengthening of the above theorem by Maher-Tiozzo.

Theorem (Maher-Tiozzo '18)

Let (X, d) be a separable proper hyperbolic space, and consider $\Gamma \leq Isom(X)$.

- *If μ is a non-degenerate measure on Γ, then the random walk almost surely converges to ∂X.*
- *If, in addition, μ has a finite entropy + finite logarithmic moment wrt the distance on Γ induced by d + Γ is itself a hyperbolic group, then $(\partial X, \nu)$ is the maximal μ-boundary.*

Example/exercise: random walks on $PSL_2(\mathbb{Z})$

Let us study random walks on $\Gamma = PSL_2(\mathbb{Z})$.

- Let $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Show that $[S]$ and $[T]$ generate Γ.
- (*) Show that S and ST **freely** generate Γ, and use this to establish an isomorphism $\Gamma \simeq \mathbb{Z}_2 * \mathbb{Z}_3$. Hint: can be done directly by studying the relations in Γ, cleverly apply the ping-pong lemma or use the Bass-Serre theory.
- Make sure that both [Kai] and [MT] apply to (Γ, μ) for any finitely supported μ. In particular, the previous discussion implies that there are two distinct geometric realizations for the Poisson boundary of Γ:
  1. If we consider the isomorphism $\Gamma = \mathbb{Z}_2 * \mathbb{Z}_3$, then we can identify the Gromov boundary $\partial\Gamma$ of Γ with the Cantor-like space of infinite words $\{a, b, b^{-1}\}^{\mathbb{N}}$
  2. A more interesting realization is obtained by using the action on $\mathbb{H}^2$, so that we can identify the Poisson boundary with $\mathbb{RP}^1 = \mathbb{R} \cup \{\infty\} \cong S^1$.

Higher rank Lie groups and lattices

Remark. The following assumes intricate knowledge of Lie groups, I will provide a concrete example in the following slides.

Definition

Let $G \leq GL_n(\mathbb{R})$ be a closed subgroup.

- G is **connected** if G is connected as a topological space.
- G is **semi-simple** if G has no non-trivial closed connected abelian normal subgroups.
- The **(real) rank** of G is the dimension of the maximal abelian subgroup (Cartan subgroup) it contains.

Consider a connected semi-simple real Lie group of real rank ≥ 2 with finite center. Consider its maximal compact subgroup K, and take $S = G/K$ to be the corresponding Riemannian symmetric space with the origin $o \simeq K$.

Theorem (Iwasawa decomposition)

Any connected semi-simple Lie semigroup G can be decomposed as a product $G = KAN$, where K is the maximal compact subgroup, A is the Cartan subgroup, and N is a nilpotent subgroup.

Higher rank Lie groups and lattices, cont.

- Fix the dominant Weyl chamber $\mathcal{A}^+$ in the Cartan Lie algebra $\mathcal{A}$ of the Cartan subgroup A, and denote by $\mathcal{A}_1^+$ the intersection of $\mathcal{A}^+$ with the unit sphere wrt the Killing form $\langle \cdot, \cdot \rangle$.
- The exponential map is just the matrix exponential $\exp(A) = \sum_n \frac{A^n}{n!}$.
- Any point $x \in S$ can be represented as $x = k(\exp a)o$, where $k \in K$, $a := r(x) \in \overline{\mathcal{A}^+}$ is the uniquely determined **radial part** of x. Then the Riemannian distance satisfies $dist(o, x) := ||r(x)||$.
- In a similar fashion to the Gromov boundary, define the **visual compactification** ∂S to be a set of equivalence classes of geodesic rays from o equipped with the cone topology. For any $a \in \overline{\mathcal{A}_1^+}$ denote $\partial_a S = \{t \mapsto g \exp(ta)o : g \in G\}$. These are exactly the orbits of the G-action on ∂S, and we can identify each orbit corresponding to the interior vectors $\alpha \in \mathcal{A}_1^+$ with the **Furstenberg boundary** $B = G/P$, where P is the minimal parabolic subgroup of G.

A concrete and most useful example

Let $G = SL_n(\mathbb{R})$ for $n \geq 3$.

- G is a connected semi-simple Lie group with finite center of real rank $n-1$.
- In the Iwasawa decomposition we can choose $K = SO_n$, $A =$ diagonal matrices, $N =$ upper triangular matrices with 1-s on the main diagonal, $P =$ all upper triangular matrices.
- Thus we can identify

$$\mathcal{A} = \{\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^d : \sum \alpha_i = 0\} \tag{1}$$

$$\mathcal{A}^+ = \{\alpha \in \mathcal{A} : \alpha_1 > \alpha_2 > \cdots > \alpha_n\} \tag{2}$$

$$\mathcal{A}_1^+ = \{\alpha \in \mathcal{A}^+ : ||\alpha||_2 = 1\}. \tag{3}$$

  In particular, if x is a diagonalizable matrix, then $r(x)$ is just x written wrt the eigenbasis of x.
- If $\alpha \in \mathcal{A}_1^+$, then we can identify the visual boundary with the space of **full flags**

$$\mathcal{V} = \{V_i\}, \quad \{0\} = V_0 \subset V_1 \subset \cdots \subset V_{n-1} = V_n = \mathbb{R}^d.$$

  For all non-interior roots α the flag space will partially degenerate.

Oseledets' multiplicative theorem

Theorem (Oseledets, Kaimanovich)

Let (Γ, μ) be a discrete measured subgroup of a connected semi-simple Lie group G with finite center. Then there exists a vector $\lambda(\mu) \in \overline{\mathcal{A}^+}$ (the Lyapunov vector) such that

$$\frac{\log(r(x_n))}{n} \to \lambda$$

for $\mathbb{P}$-a.e sample path $x = (x_n) \in \Omega$. Moreover, for $\mathbb{P}$-a.e sample path $x = (x_n) \in \Omega$ there exists a uniquely determined positive definite symmetric matrix $g = g(x) = k(\exp \lambda)k^{-1}$, $k \in K$ such that

$$\log ||g^{-n} x_n|| = o(n).$$

Random walks on higher rank lattices

As a corollary from the above theorem, we get the following identification result.

Theorem

Let $\Gamma \subset G$ be a discrete subgroup of a connected semisimple Lie group G with finite center so that μ has the finite first moment.

- *If the Lyapunov vector $\lambda(\mu) = 0$, then the Poisson boundary of (Γ, μ) is trivial.*
- *If the Lyapunov vector $\lambda(\mu) \neq 0$, then the respective random walk converges to the visual boundary G/P of the symmetric space G/K, and $(G/P, \nu)$ is the maximal μ-boundary wrt the hitting measure.*

As discrete subgroups of higher-rank Lie groups tend to be non-hyperbolic, only this result can guarantee the identification of the Poisson boundary.

Exercise

Verify that for $G = SL_2(\mathbb{R})$ and Γ hyperbolic the above theorem is consistent with the previous results.

For Zariski-dense Γ the above theorem was strengthened by [Chwala-Forghani-Frisch-Tiozzo] a few weeks ago, requiring only finite entropy condition.

Random walks on rational affine group: preliminaries

Definition

Let $a \in \mathbb{Q}$, and let $\mathcal{P} \subset \mathbb{N}$ denote the set of prime numbers. For any $p \in \mathcal{P}$ define $|a|_p = p^{-k}$ to be the p-adic norm of a – the maximal $k \in \mathbb{Z}$ so that $a = p^k r$ where p divides neither numerator or denominator of r.

Definition

Fix a prime number p. Define the p-adic integers $\mathbb{Z}_p$ as $\mathbb{Z}$ equipped with the ultrametric defined by the $d_p(x, y) = |x - y|_p$. In a similar way, we define $\mathbb{Q}_p$ as the $\mathbb{Q}$ equipped with the same ultrametric.

Exercise

Show that every p-adic rational $x \in \mathbb{Q}_p$ can be uniquely written as a converging series

$$x = \sum_{k}^{\infty} a_j p^j$$

for some $k \leq 0$, with x being a p-adic integer iff $k \geq 0$.

Random walks on rational affine group: result

The following result belongs to Sara Broffeiro, and is quite interesting from the standpoint of arithmetic dynamics.

Theorem (Broffeiro, '06)

Let $\Gamma = Aff(\mathbb{Q})$, and consider a probability measure μ which is not supported on an abelian subgroup, satisfying the moment condition

$$\int_{Aff(\mathbb{Q})} \left(\sum_{p \in \mathcal{P}} |\ln |a|_p| + \sum_{p \in \mathcal{P} \cup \infty} \ln^+ |b|_p \right) d\mu(a, b) < \infty.$$

Denoting the p-drift

$$\phi_p = \int_{Aff(\mathbb{Q})} \ln |a|_p d\mu(a, b),$$

the random walk $\mathbb{P}$-a.s. converges to the associated adele space $B^* = \prod_{p \in \mathcal{P} \cup \infty, \phi_p < 0} \mathbb{Q}_p$ with the hitting measure ν. Moreover, ν is the only μ-stationary measure in B^*, and (B^*, ν) is the Poisson boundary of (Γ, μ).

Examples: affine + p-adic self-similar measures

- Fix a prime p. Consider $\mu = (\underbrace{x \mapsto px}_{1/p}, \dots, \underbrace{x \mapsto px + (p-1)}_{1/p})$. Show that $Pois(\Gamma, \mu) = (\mathbb{Q}_p, Haar)$
- Consider $\mu = \frac{1}{2}\delta_{x \mapsto 3x} + \frac{1}{2}\delta_{x \mapsto 3x+2}$. Find the hitting measure on $\mathbb{Q}_3$ and compute its Hausdorff dimension.

Random walks on the lamplighter group

Theorem (Kaimanovich)

Let $\Gamma = \mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$. Consider μ to be a symmetric measure supported on the set of generators $\{L, R, S\}$, where L, R move the lamplighter left/right, and S switches the lamp. Then the Poisson boundary of (Γ, μ) is trivial.

There are at least two ways to approach this nice theorem:

Exercise

- Use strong large deviation estimates to show that the lamplighter's distance from the origin at time n cannot exceed $n^{3/4}$ with high probability, then use this to show that the entropy grows sublinearly.
- A more elegant way is to find coupled random walks starting from any two distinct lamp configurations: show that one can first match the positions of the lamplighter, then use recurrence to eventually match the lamp states as well.

Random walks on wreath products

It is very important that the projection of the random walk to the base group is recurrent for triviality of the Poisson boundary: as one suspects, transience guarantees the lamp stabilization, yielding at the very least a μ-boundary!

Theorem (Frisch-Silva, '23+)

Consider two countable groups A, B, and a probability measure on $A \wr B$ with finite entropy, such that the lamp configurations stabilize almost surely. Denote the Poisson boundary of the induced random walk on B by $(\partial B, \nu_B)$. Then $A^B \times \partial B$ with the corresponding hitting measure ν is a model for the Poisson boundary.
Moreover, if μ is non-degenerate, and has finite first moment wrt a word distance, then the Poisson boundary can be modelled by lamp configurations A^B alone.

- We provide the strongest theorem available, for complete history I will refer to the introduction of this paper.
- The first half does not require non-degeneracy!
- It is still open if we can recover the trajectory of the lamplighter without the finite first moment condition.

Kesten's amenability criterion

Definition (Amenability)

Let Γ be a discrete group. Γ is **amenable** if it admits a finite left-invariant mean.

This notion admits many equivalent characterizations, see https://arxiv.org/abs/1705.04091. We are interested in a nice random walk-related criterion for amenability due to Kesten.

Definition

Define the **spectral radius** of a random walk as the limit

$$\rho_\mu = \limsup_{n\to\infty} (\mu^{*n}(x))^{1/n},$$

which exists and is the same for all $x \in \Gamma$.

Theorem (Kesten '59)

Let Γ be a discrete group. TFAE:

- *Γ is amenable*
- *The return probability $\mu^{*n}(e, e)$ has subexponential decay.*
- *For any symmetric non-degenerate μ on Γ we have $\rho_\mu = 1$.*

Corollaries of Kesten's theorem

Usually this theorem is used to groups that we already know to be non-amenable (free, hyperbolic...) to show that the spectral radius is strictly less than 1, and then use this fact to get immediate estimates on the return probabilities.

Exercise

Compute the spectral radius of the simple random walk on $\mathbb{F}_n$.

In a certain sense, the free group is the only group for which we really can compute the spectral radius numerically. The most studied groups in this regard seem to be surface groups $\Gamma_g = \pi_1(S_g) = \langle a_i, b_i \mid [a_1, b_1] \dots [a_g, b_g] = e \rangle$ for $S_g =$ orientable compact Riemann surface of genus ≥ 2:

Known results

Let $g = 2$, and consider the simple RW on $\pi_1(S_g)$.

- (Nagnibeda '97) $\rho \leq 0.662816$
- (Bartholdi '04) $\rho \geq 0.662421$
- (Gouezel '15) $\rho \geq 0.662772$

Liouville property and amenability

Lemma

Let (Γ, μ) be a measured group. Then the Poisson boundary of (Γ, μ) is trivial iff for every $\gamma \in \operatorname{supp}\mu$ and for $\mathbb{P}$-a.e. $\omega = (\omega_n)$ we have $\lim_n \frac{\mu^{*(n-1)}(\gamma^{-1}\omega_n)}{\mu_n(\omega_n)} = 1$.

Proof via entropy criterion + Kolmogorov 0-1 law (essentially proving the trviality of the tail boundary)

Theorem (Kaimanovich-Vershik)

Let Γ be a group equipped with an aperiodic non-degenerate measure μ. TFAE:

- *The Poisson boundary of (Γ, μ) is trivial.*
- *The measures μ_n converge weakly to a left-invariant mean on Γ.*

Corollary

Let (Γ, μ) be a non-amenable group with a non-degenerate μ. Then the Poisson boundary of (Γ, μ) is non-trivial.

Proof of the theorem

- ($\Rightarrow$) By aperiodicity WLOG we assume $\mu(e) > 0$. Therefore,

$$\lim_n \frac{\mu^{*(n-1)}(\gamma^{-1}\omega_n)}{\mu^{*n}(\omega_n)} = \lim_n \frac{\mu^{*(n-1)}(\gamma^{-1}\omega_n)}{\mu^{*(n-1)}(\omega_n)} = \lim_n \frac{\mu^{*n}(\gamma^{-1}\omega_n)}{\mu^{*n}(\omega_n)} = 1,$$

so

$$\mu^{*n}\{x \in \Gamma : \left|1 - \frac{\mu^{*n}(\gamma^{-1}\omega_n)}{\mu^{*n}(\omega_n)}\right| > \varepsilon\} \to 0$$

for all $\gamma \in \mathrm{supp}\mu$ and $\varepsilon > 0$. Non-degeneracy of μ implies the claim.

- ($\Leftarrow$) Assume that f is a bounded μ-harmonic function. Then for every $x \in \Gamma$ and $n > 0$ we have by definition

$$f(x) = \sum_h f(h)\mu^{*n}(x^{-1}h),$$

and

$$f(e) = \sum_h f(h)\mu^{*n}(h).$$

Subtracting the two formulas, we get

$$f(x) - f(e) = \sum_h f(h)(\mu^{*n}(x^{-1}h) - \mu^{*n}(h)).$$

By weak convergence of the convolutions, the RHS converges to 0, so $f(x) = f(e)$ for every $x \in \Gamma$ and f is a constant function.

Self-similar groups

Definition

Let X be a finite set. Denote the rooted tree with vertices encoded by words in the alphabet X by X^*, denoting the root by the empty word $\emptyset$. So, w_1 is connected to w_2 if there exists $x \in X$ so that $w_1 = w_2 x$ or $w_2 = w_1 x$.

Denote the group of graph automorphisms of X^* by $Aut(X^*)$.

Definition

We say that $\Gamma \leq Aut(X^*)$ is a self-similar group if for every $g \in \Gamma$ and $v \in X^*$ the restriction of g to the subtree rooted at v still belongs to Γ.

Sometimes it is useful to consider the **wreath recursion** map:

$$\psi : \Gamma \to Sym(X) \wr \Gamma, \quad \psi(g) = (\sigma_g, g),$$

where $\sigma_g(x) := g(x)$. Keep in mind that defining the set of rewriting rules is the same as embedding Γ as a subgroup of $Sym(X) \wr Aut(X^*)$.

Examples of self-similar groups

- Take $X = \{0, 1\}$ and define Γ via the rule

$$a(0w) = 1w, \quad a(1w) = 0(aw).$$

  Check that a has infinite order, so $\Gamma = \mathbb{Z}$.

- Take $X = \{0, 1\}$ and define Γ via rules

$$\begin{aligned} a(0w) &= 1w, & a(1w) &= 0w, \\ b(0w) &= 0(aw), & b(1w) &= 1(cw), \\ c(0w) &= 0(aw), & c(1w) &= 1(dw), \\ d(0w) &= 0w, & d(1w) &= 1(bw). \end{aligned}$$

Not an exercise

The resulting group, called **the Grigorchuk group**, is the first known example of an infinite group (*) of intermediate growth (**).

More examples/exercises

Take $X = \{0, 1\}$ and define Γ via the rules

$$\begin{aligned} a(0w) = 1(bw), \quad a(1w) &= 0(aw), \\ b(0w) = 0(bw), \quad b(1w) &= 1(aw). \end{aligned}$$

Exercise

- Identify $X = \mathbb{Z}/2\mathbb{Z} = F_2$. Show that

$$\begin{aligned} (b^{-1}a)(w_1 w_2 w_3 \cdots) &= (w_1 + 1) w_2 w_3 \ldots, \\ b(w_1 w_2 w_3 \ldots) &= w_1 (w_2 + w_1)(w_3 + w_2) \ldots \end{aligned}$$

- Now identify X^* with the formal power series $x = \sum x_k t^k \in F_2[[t]]$. Show that Γ is isomorphic to the group generated by transformations

$$\varphi_\sigma(F(t)) = F(t) + 1, \quad \varphi_b(F(t)) = (1 + t)F(t).$$

- Finally, use the above identification to show that $\Gamma \simeq (\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$.

Bartholdi-Virag's treatment of Basilica group

Basilica group $\mathcal{B}$

$$a(0x) = 0x, \qquad a(1x) = 1(bx),$$
$$b(0x) = 1x, \qquad b(1x) = 0(ax).$$

It is outright baffling that it took so long to discover the reverse application of Kesten's theorem!

Theorem (Bartholdi-Virag, '05)

There exists a symmetric non-degenerate random walk on the Basilica group with subexponential decay of the return probabilities. In particular, the Basilica group is amenable.

To this day it is essentially the only way to prove its amenability.

Exercise

- (**) Construct an explicit Folner sequence on the Basilica group.

Their method turned out to be extremely flexible, allowing us to show amenability for a large class of self-similar groups, but the complete classification of amenable self-similar groups is still far away.

Münchhausen's trick

Consider the matrix "representation" of the Basilica group:

$$a = \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}.$$

Let $\mu = \alpha(\delta_a + \delta_{a^{-1}}) + \beta(\delta_b + \delta_{b^{-1}})$ with $2\alpha + 2\beta = 1$. Define

$$M^\mu = \alpha(a + a^{-1}) + \beta(b + b^{-1}) = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} = \begin{pmatrix} \alpha(b + b^{-1}) & \beta(1+\alpha) \\ \beta(1 + a^{-1}) & 2\alpha \end{pmatrix}.$$

Think of μ_{xy} as the term that fully accounts for first digit switch from x to y. It is not difficult to see that if we start from $(e, 0)$, at some point the random walk will return to $(g, 0)$. The induced random walk will be given by the measure

$$\mu^0 = \mu_{00} + \underbrace{\mu_{01}\mu_{10}}_{0\mapsto 1\mapsto 0} + \underbrace{\mu_{01}\mu_{11}\mu_{10}}_{0\mapsto 1\mapsto 1\mapsto 0} + \cdots = \mu_{00} + \mu_{01}(1 - \mu_{11})^{-1}\mu_{10}.$$

The above argument by Kaimanovich serves as an elegant restatement of the proof of amenability of $\mathcal{B}$ by Bartholdi-Virag.

Theorem (Kaimanovich '05)

Let Γ be the Basilica group and let α, β define the probability measure $\mu = \alpha(\delta_a + \delta_{a^{-1}}) + \beta(\delta_b + \delta_{b^{-1}})$.

- *Check that $\mu^0 = \beta + \frac{\beta}{2}(a + a^{-1}) + \alpha(b + b^{-1})$.*
- *Prove that $h(\mu) \leq h(\mu^0)$ for any choice of weights α, β.*
- *Choose weights in such a way that $2\alpha^2 = \beta^2$, so that*

$$\mu^0 = \beta + (1 - \beta)\mu.$$

- *Show that $h(\mu) \leq h(\mu^0) = (1 - \beta)h(\mu)$. As $\beta \in (0, 1)$, this implies $h(\mu) = 0$.*

Thus Γ is amenable.

Contracting self-similar groups

Definition

A self-similar group Γ acting on a tree T is called **contracting** if there exists a finite set $\mathcal{N} \subset \Gamma$ so that for every $g \in \Gamma$ there exists $k \geq 1$ such that $g[v] \in \mathcal{N}$ for all vertices at a height $\geq k$.

A big objective is to understand the following conjecture:

Conjecture

All contracting self-similar groups are amenable.

The idea of constructing a self-similar walk to show subexponential decay of return probabilites/sublinear entropy turned out to be highly successful but there are contracting self-similar groups which are not Liouville for some finitely supported random walks.

Martin boundary

Definition

Given a measured group (Γ, μ),

- We define the **Green function** as

$$G_\mu(x, y) = \sum_{k \geq 0} \mu^{*k}(x^{-1}y).$$

- We define **the Martin kernel** as

$$K_\mu(x, y) := \frac{G(x, y)}{G(e, y)}.$$

Simple exercise

Show that transience of a random walk on a group is equivalent to $G_\mu(e, e) < \infty$.

Definition

The **Martin compactification** of a measured group (Γ, μ) is the closure of the functions $K_\mu(\cdot, y)$ in the continuous functions $C(\Gamma)$ wrt pointwise convergence. We define the Martin boundary as $\partial_M(\Gamma) = \overline{\{K_\mu(\cdot, y)\}} \setminus \{K_\mu(\cdot, y)\}$.

Unlike the Poisson boundary, this is a true topological invariant of (Γ, μ).

Martin representation theorem

Below we will assume that the random walk is transient. Martin boundary to the Poisson boundary is like positive harmonic functions are to bounded ones – but the representation theorem is a bit more tricky to formulate.

A subtle abuse of notation

If $\xi \in \partial_M \Gamma$, then, by definition, it is obtained as the pointwise limit of some sequence of functions $K(x, y_n)$, we will denote $\xi(x) := K_\mu(x, \xi)$, thus extending the Martin kernel to $\Gamma \times \partial_M \Gamma$.

Theorem (Choquet-Martin representation)

Let (Γ, μ) be a measured group with $G_\mu(e, e) < \infty$. Then for every positive μ-harmonic function f there exists a unique positive finite Borel measure ν_f on $\partial_\mu \Gamma$ so that

$$f(x) = \int K_\mu(x, \xi) d\nu_f(\xi).$$

The measure ν_1 which represents the constant function $f \equiv 1$ is called the **harmonic measure**.

Why you should not study Martin boundaries

- Martin boundaries are tricky to identify because we don't have the same handy entropy criteria.
- Somehow it does not carry the same asymptotic information about the random walk as the Poisson boundary does – the Martin representation formula is much less elegant and it does not represent positive functions in an equally constructive way compared to the Poisson formula. However, usually the points in $\partial_M \Gamma$ correspond 1-1 to **minimal** positive harmonic functions
- (Cartwright-Sawyer) Long before Chawla-Frisch result, it has been known that a stopping time modification of a simple RW on a free group can change the Martin boundary. In general, the way the Green function depends on the measure is very much not understood even for very nice groups and measure classes.
- Even describing the Martin boundary for nice random walks on $\mathbb{Z}^d$ requires some heavy machinery, see results of Ney-Spitzer.
- Finally, the Martin boundary lacks the "continuous-to-discrete" principle that works extremely well for Poisson boundaries – we cannot exploit geometric actions to compute the Martin boundary.

In case previous warning did not work!

Let us outline known results and methods one can use to identify the Martin boundary.

Definition

A positive μ-harmonic function h is minimal if $v \leq h$ on Γ implies $h = cv$.

- The Martin representation formula implies that if f is a minimal μ-harmonic function, then ν_f has to be a δ-measure, so $f = cK_\xi$ for some $c > 0$ and $\xi \in \partial_M \Gamma$.
- The key idea that for many nice groups we can show the reverse: every point in the Martin boundary corresponds to a minimal function! Combined with the Ancona-type inequality, we can actually show that the Martin boundary is homeomorphic to a geometric boundary.

Ancona's inequality

Let us show the original inequality, proved by Ancona and with the argument fixed and upgraded by Blachere-Haissinsky-Mathieu. Let us denote

$$F_\mu(x,y) = \frac{G\mu(x,y)}{G_\mu(e,e)}.$$

Theorem (Ancona)

Let (Γ, μ) be a non-elementary hyperbolic group equipped with a symmetric non-degererenate finitely supported (with finite exponential moment wrt word metric) measure μ. Then there exists a constant $C > 0$ such that

$$F_\mu(x,z)F_\mu(z,y) \leq F_\mu(x,y) \leq CF_\mu(x,z)F_\mu(z,y)$$

for any z on a geodesic segment $[x,y]$ in the Cayley graph.

This theorem essentially, shows convergence via geodesics to the Martin boundary should be the same as in the Gromov boundary.
Stronger versions of Ancona's inequality work with weaker "negative curvature" assumptions (relative hyperbolicity, acylindrically hyperbolic...)

Martin boundaries for measures with bad moments

Theorem (Gouezel, '15)

Let Γ be a non-amenable finitely presented Γ and a sequence y_n escaping to infinity in Γ. There exists a symmetric non-degenerate μ on Γ so that y_n does not converge in the Martin boundary.

The naive idea is to put weights with bad heavy tails on μ, so that is becomes "cheaper" for a random walk to make a big jump compared to a sequence of shorter jumps. This theorem prohibits a nice description of the Martin boundary for measures with bad moments, and every geometric argument which can be used to establish the Martin boundary has to operate in the exponential moment regime.

On a theorem of Chawla-Frisch

Question

Is it possible to find a group Γ and a space X so that for **every** probability measure μ on Γ there exists a probability measure ν on X so that the random walk $\mathbb{P}$-a.s. converges to X and (X, ν) is a maximal μ-boundary?

In other words, can Poisson boundary be a pure group invariant? Turns out that the answer is no!

Theorem (Chawla-Frisch, '25+)

There exists a non-degenerate probability measure μ on F_2, which satisfies the following. Denote the maximal μ-boundary of (F_2, μ) by (X, ν). Then there exists a randomized stopping time τ, so that $Har^\infty(F_2, \mu) \subsetneq Har^\infty(F_2, \mu_\tau)$. In particular, (X, ν) is a non-maximal μ_τ-boundary of (F_2, μ).

Remarks

- The argument can be generalized to any non-virtually nilpotent group and any non-degenerate μ by invoking the technique of **switching elements**.
- Even if we impose the finite entropy condition, the problem of realizing the Poisson boundary is open in general, with positive results for the hyperbolic groups.

Outline of the proof

Definition

A randomized stopping time τ is a collection of stopping times $(\tau)_{i\geq 0}$ weighted by a probability measure p on $\mathbb{Z}_{\geq 0}$.
Given a measured group (Γ, μ), we define $\mu_\tau = \sum_{i\geq 0} p(i)\mu_{\tau_i}$.

- We will start with the standard lamplighter group $\Gamma = \mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$.
- Consider the projection homomorphism $\pi : F_2 = \langle a, b\rangle \to \Gamma$, with

$$\pi(a) = (0, e_0), \pi(b) = (1, 0).$$

- Consider the simple RW on $F_2 = \langle a, b\rangle$, and take $\mu' = \pi_* \mu$.
- We already know that the Poisson boundary of (Γ, μ') is trivial invoking the argument of Kaimanovich.
- We will construct a randomized stopping time τ for (Γ, μ') so that the Poisson boundary of $(\Gamma, (\mu')_\tau)$ becomes non-trivial.
- This yields a non-trivial bounded μ'_τ-harmonic function f, which can be pulled back to a $\mu_{\pi\circ\tau}$-harmonic but not μ-harmonic function on F_2.

Sketch of the proof I: records

Definition

- Given a probability measure $p \in Prob(\mathbb{Z}_{\geq 0})$, consider $(X_i)_{i>0}$ to be i.i.d samples drawn according to p. Let T_k be the k-th record time, given by

$$T_0 = 1, \quad T_k = \inf\{i > T_{k-1}, X_i \geq X_j \ \forall j < i\}.$$

- Denote the k-th record value by $R_k = X_{T_k}$.
- A probability measure p has **eventually simple records** if a.s. for sufficiently large k we have $R_{k+1} > R_k$.

Lemma

Assume that $p(i) > 0$ *and* $\sum_i \left(\frac{p(i)}{\sum_{j \leq i} p(j)} \right)^2 < \infty$. *Then* p *has eventually simple records. Moreover, there exists a non-decreasing function* Φ *such that a.s.* $T_{k+1} < \Phi(R_k)$ *for* $k \gg 0$.

- We will use such p as our weight for the stopping time. There are many suitable measures, the only important thing tis to guarantee the eventual simplicity of records.

Sketch of the proof II: stopping time

Denote the respective random walk on Γ by (X_n, φ_n), where X_n denotes the position of the lamplighter and $\varphi : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ keeps track of lamp configurations.

- Given two sequences $(s_n)_{n\geq 0}, (r_n)_{n\geq 0}$, we define

$$\tau_n = \inf\{t \geq 1 : \varphi_t|_{[-s_n, s_n]} \equiv 0, |X_t| \geq r_n\}.$$

- Recurrence ensures that all stopping times are almost surely finite.
- Let us carefully choose the sequences so that $s_1 < r_1 < s_2 < r_2 < \dots$. Pick $s_1 = r_1 = 0$, and assume that r_i, s_i are picked up to $i = k - 1$.
- For any subset of indices $(i_1, \dots, i_{\Phi(k)}) \subset \{1, ..., k-1\}^{\Phi_k}$ consider the random sequence $(g_j)_{1\leq j\leq \Phi(k)}$, where g_j are independently drawn according to $\mu_{\tau_{i_j}}$. Denote $w_j = g_1 \dots g_j$ and denote the respective lamp configuration after applying w_j by ϕ_j.
- Pick s_k large enough so that for every choice of sequence $(i_1, \dots, i_{\Phi(k)})$ with probability $\geq 1 - 2^{-k}$ the support of φ_j is contained in $[-s_k/3, s_k/3]$ for all $1 \leq j \leq \Phi_k$. Then we set $r_k = 3s_k$.

Sketch of the proof III: non-triviality of Poisson boundary

- First of all, observe that $(\mu')_\tau$ is non-degenerate. We will establish the the sequence of lamp configurations $(\varphi_j)_{j\geq 0}$ almost surely converges.
- Consider the record time (T_k) wrt p. Given some $k \geq k_0$, consider the RW at time $T_{k+1}-1$,

$$w_{T_{k+1}-1} = \underbrace{g_1 \dots g_{T_k-1}}_{a} g_{T_k} \underbrace{g_{T_k+1} \dots g_{T_{k-1}-1}}_{b}.$$

- Observe that both a, b are products of at most $\Phi(R_k)$ increments.
- Then due to Borel-Cantelli and the way we chose s_k, r_k, a.s. for $k >> 0$ we have

$$\varphi_{T_k-1}\big|_{[-s_{R_k}/3, s_{R_k}/3]} = \varphi_j\big|_{[-s_{R_k}/3, s_{R_k}/3]}$$

  for all $T_k \leq j \leq T_{k+1}-1$. Since the records are increasing and a.s. go to infinity, the limit $\lim_{j\to\infty} \varphi_j$ exists a.s.
- As μ_τ is irreducible, the limiting lamp configuration is not a.s. constant, so it defines a non-trivial tail event in the Poisson boundary.

Sketch of the proof IV: reduction

- Consider the lazy simple RW on F_2 given by $\mu = \frac{1}{4}\delta_a + \frac{1}{4}\delta_b + \frac{1}{4}\delta_{a^{-1}} + \frac{1}{4}\delta_{b^{-1}}$. Pullback of a stopping time is a stopping time, so $(\partial F_2, \nu)$ is a μ-boundary and a $(\mu)_{\pi\circ\tau}$-boundary, as stopping times do not change the convergence.
- However, consider the non-trivial μ'-harmonic function f on Γ. Its pullback to F_2 will be $\mu_{\pi\circ\tau}$-harmonic but not μ'-harmonic.
- We show this by establishing that any bounded μ'-harmonic function which is constant on the fibers of π has to be constant.
- Therefore, $Har^{\infty}(F_2, \mu) \subsetneq Har^{\infty}(F_2, \mu_{\pi\circ\tau})$.

Properties of the harmonic measure

Suppose we have a measured group (Γ, μ) for which we understand a concrete geometric realization $(\partial\Gamma, \nu)$ of its Poisson boundary as a measure space.

Question

Assume that X carries a natural/invariant measure λ. Can we study the harmonic measure ν on it own and compare it to λ?

This question turns out to be quite subtle! It is even difficult when restricted to very narrow classes of measures μ.

Hausdorff dimension

Let (X, d, ν) be a metric measure space.

Definition

Hausdorff measure Let $A \subset X$ be a subset. For every $\alpha \geq 0, \Delta > 0$ we define

$$\mathcal{H}^{\alpha}_{\Delta}(E) := \inf\left\{ \sum_{i=1}^{\infty} diam_d(E_i)^{\alpha} : E \subset \cup E_i, diam(E_i) \leq \Delta \right\}.$$

The α-Hausdorff measure of E is

$$\mathcal{H}^{\alpha}(E) := \sup_{\Delta > 0} \mathcal{H}^{\alpha}_{\Delta}(E).$$

The Hausdorff dimension of E is

$$\dim_H(E) := \inf\{\alpha \geq 0 : \mathcal{H}^{\alpha}(E) = 0\} = \sup\{\alpha \geq 0 : \mathcal{H}^{\alpha}(E) > 0\}.$$

Definition

The Hausdorff dimension of ν is defined by

$$\dim_H \nu := \inf\{\dim_H E : \nu(E^c) = 0\}.$$

Asymptotic invariants

Let (Γ, μ) be a measured group equipped with a left-invariant distance d.

Definition

$$l = \lim_{n\to\infty} \frac{\sum_g \mu^{*n}(g) d(e,g)}{n} \qquad \text{(drift)},$$

$$h = \lim_{n\to\infty} \frac{-\sum_g \mu^{*n}(g) \log(\mu^{*n}(g))}{n} \qquad \text{(entropy)}$$

$$v = \lim_{n\to\infty} \frac{\log(|\{g \in \Gamma : d(e,g) \leq n\}|)}{n} \qquad \text{(logarithmic volume)}.$$

Theorem (Fundamental inequality)

Suppose that metric measured group (Γ, μ, d) satisfies the finite entropy + finite first moment condition + finite logarithmic volume condition. Then all three invariants are all finite and

$$h \leq lv.$$

Question:

When is this inequality strict?

Gromov boundaries, cont.

Theorem

Let (X, d) be a proper hyperbolic space. Then for any two non-equivalent geodesics $\gamma_1 \neq \gamma_2$ in ∂X there exists a bi-inifnite geodesic γ so that $\gamma_+ \sim \gamma_1$ and $\gamma_- \sim \gamma_2$.

Definition (Visual distance)

Let (X, x_0, d) be a proper hyperbolic space. Fix $\varepsilon > 0$. We say that a distance d_ε on ∂X is a **visual distance** if

- The metric d_ε generates the cone topology on ∂X
- There exists a constant $C > 0$ so that for any $a, b \in \partial X$, a biinfinite geodesic γ connecting a, b and $y \in X$ so $d(x_0, y) = d(x_0, \gamma)$

$$\frac{1}{C} e^{-\varepsilon d(x_0, y)} \leq d_\varepsilon(a, b) \leq C e^{-\varepsilon d(x_0, y)}.$$

Theorem

For any proper hyperbolic space (X, d) there exists $\varepsilon_0 > 0$ so that d_ε exists for any $\varepsilon \in (0, \varepsilon_0)$. Moreover, any visual distances are bi-Lipschitz equivalent for the same parameter.

Blachère-Haïssinsky-Mathieu criterion

The main tool which allows us to determine the inequality is a powerful result of S. Blachère, P. Haïssinsky and P. Mathieu.

Theorem (BHM, Theorem 1.5)

Let (Γ, d, μ) be a non-elementary hyperbolic measured group with μ satisfying a superexponential moment. Identify the Poisson boundary with the Gromov boundary $(\partial\Gamma, \nu)$, equipped with the **quasiconformal measure** ν_d*, which is the Hausdorff measure on $(\partial\Gamma, d_\varepsilon)$ for small enough ε. TFAE:*

1. $h = lv$,
2. *The hitting measure ν is equivalent to ν_d.*
3. $\sup_{g\in\Gamma} |d_\mu(e,g) - d_{\mathbb{H}^2}(e,g)| < \infty$, *where*

$$d_\mu(e,g) = -\log(F_\mu(e,g))$$

   is the **Green metric**, $F_\mu(e,g)$ *denotes the* **first-passage function**, *and* $d_{\mathbb{H}^2}(e,g) = d(o, go)$.
4. *The Hausdorff dimension of the hitting measure on $(\partial\Gamma, d_\varepsilon)$ equals $v = \frac{h}{l}$.*

Exercise

Let $\Gamma = F_n$, and μ be symmetric with supp$\mu = \{a_i^{\pm 1}\}$.

- Compute v with respect to the word distance.
- Assume the weights are uniform, compute h, l.
- Verify that the quasiconformal measure is exactly the product measure on ∂F_n.
- Assuming the weights are uniform, check that $h = lv$, confirming your previous computations.
- (Ledrappier '01, K.-Tiozzo '21) Consider the first-passage function $F_\mu(e, x) = \mathbb{P}(\exists n : X_n = x)$. Show that

$$F_\mu(e, a_i) = \frac{\nu(C(a_i))}{1 - \nu(C(a_i))},$$

  where $C(a_i) =$ infinite words which start with a_i.
- Use the above to show that $h = lv$ only for the uniform μ. Keep in mind that you are unable to compute h, l in general case.
- (**) (K. '25) Show that if the measure is allowed to sit on the powers $\{a_i^k\}$, then

$$\liminf_{n\to\infty} F_\mu(e, (a_i)^n)^{1/n} \geq \frac{\nu(C(a_i))}{1 - \nu(C(a_i))}.$$

- (*) Show that the above is not true for general μ.

Cool research problem

Let $\Gamma = PSL_2(\mathbb{Z})$ and consider $\mu = \frac{1}{3}\delta_S + \frac{1}{3}\delta_{S^{-1}} + \frac{1}{3}\delta_T$.

- Consider a sample path $T^{m_1}ST^{m_2}ST^{m_3}\ldots$, and define the correspondence

$$T^{m_1}ST^{m_2}ST^{m_3}S\cdots \mapsto m_1 - \cfrac{1}{m_2 - \cfrac{1}{m_3 - \cfrac{1}{\cdots}}} \in \mathbb{R}.$$

  Show that this correspondence is Γ-equivariant on $\mathbb{R}$.
- Consider the pushforward η of the hitting measure ν with respect to this correspondence. This is the distribution of a random continued fraction with i.i.d. geometric terms m_i.
- (*) Find asymptotics for the $\eta([n, n + \frac{1}{k}])$ as $n \to \infty$ and k is fixed.
- (**) Use them to show that the Hausdorff dimension of η is strictly less than 1.
- (***) Can you find the precise formula for the Hausdorff dimension? What if we change the weights?

Here be dragons: biggest open questions

There still a lot of open questions regarding random walks on groups.

Stability conjecture

Let Γ be a non-virtually nilpotent group. Then the triviality of the Poisson boundary of (Γ, μ_S) does not depend on the choice of generating set S, where μ_S is the uniform measure on S.

Still open for many amenable groups of exponential growth.

Realizing the Poisson boundary

Is it true that for every group Γ there exists a single space X which realizes the Poisson boundary of (G, μ) for all μ with finite entropy?

Remark: Due to recent result of Chawla-Frisch, this is not true if we drop the entropy condition.

Here be dragons: intermediate growth

Gap conjecture

Any group with growth $\prec e^{\sqrt{n}}$ is virtually nilpotent.

(most experts don't believe this conjecture)

Question

Is there a finitely presented group of intermediate growth?

Conjecture by Grigorchuk-Pak

If H is a group of intermediate group, then it contains two distinct infinite commuting subgroups H_1, H_2.

If true, it resolves the $p_c < 1$ conjecture positively.

Here be dragons: singularity

This slide is dedicated to the problems related to the subtle properties of the harmonic measure.

Singularity conjecture: rank 1, dimension 2

Let $\Gamma \leq PSU(1,1)$ be a discrete subgroup equipped with a finitely supported non-degenerate measure μ. Then, identifying the Poisson boundary with (S_1, ν), the hitting measure ν is singular with respect to the harmonic measure.

Singularity conjecture: rank 1, high dimensions

Let $\Gamma \leq \mathrm{Isom}(\mathbb{H}^n)$ be a discrete subgroup equipped with a finitely supported non-degenerate measure μ. Then, identifying the Poisson boundary with (S^{n-1}, ν), the hitting measure ν is singular with respect to the harmonic measure

Singularity conjecture: higher rank

Let $\Gamma \leq G$ be a discrete subgroup of a semisimple Lie group with trivial center and real rank ≥ 2 equipped with a finitely supported non-degenerate measure μ. Then, identifying the Poisson boundary with the associated visual boundary $(G/P, \nu)$ of the symmetric space G/K, the hitting measure ν is singular with respect to the obvious pushforward of the Haar measure on G.

Singularity: cont.

All of the above conjectures are still open as of fall of 2025, with the two-dimensional conjecture seeing the most progress.
These conjectures are, in spirit, similar to the conjectures related to affine and p-adic self-similar measures.

Bernoulli convolution problem

Fix $\lambda \in (1/2, 1)$. Consider the probability measure ν on $\mathbb{R}$ (uniquely) defined by

$$\int_{\mathbb{R}} f(x) d\nu(x) = \frac{1}{2} \int_{\mathbb{R}} f(\lambda x + 1) d\nu(x) + \frac{1}{2} \int_{\mathbb{R}} f(\lambda x - 1) d\nu(x).$$

Then ν is absolutely continuous wrt Lebesgue measure iff λ is not a Pisot number.

p-adic self-similar problem

Let $\Gamma \leq \mathrm{Aff}(\mathbb{Q})$ be a subgroup equipped with a non-degenerate probability measure μ. If we identify the Poisson boundary with the adele space $\prod_p Q_p$, then when is the hitting measure singular with respect to the product of the Haar measures?

Both problems are open, with the p-adic self-similarity related to difficult problems in arithmetic dynamics.