KAM approach to outer billiards

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1 Introduction

Let us start with the definition of an outer (dual) billiard.

Definition 1.1. Let $K \subset \mathbb{R}^2$ be a bounded convex set. Then we define the *outer (dual) billiard map* $T : \mathbb{R}^2 \setminus K \to \mathbb{R}^2 \setminus K$ as follows: we require that the line segment $[x, Tx]$ is tangent to ∂K at the midpoint. Moreover, K should be on the right side of said segment.

Remark. In this definition we allow K to be a convex polygon. In this case T will be defined on $\mathbb{R}^2 \setminus L$, where

$$L = \bigcup_{k \in \mathbb{Z}} \bigcup_{i=1}^{n} T^k(L_i),$$

which go through the sides $L_1, \ldots, L_n$ of K. Nevertheless, it still follows that T is defined almost everywhere, which is enough for our purposes.

Thus, a bounded convex set $K \subset \mathbb{R}$ defines a discrete dynamical system called an *outer (dual) billiard*. The discovery of these systems is often attributed to B. H. Neumann, in the paper with a rather unusual title [Neu59].

However, even to this day, there are still quite a few unanswered questions about outer billiards. In this project we are going to discuss the following problem, attributed to B. H. Neumann and J. Moser.

Question: Given K, is there a point $x_0 \in \mathbb{R}$ such that its orbit under the outer billiard map $\{T^n x_0\}_{n \geq 0}$ is unbounded?

Below we provide several important examples which illustrate that the answer is affirmative for billiards with a relatively simple structure.

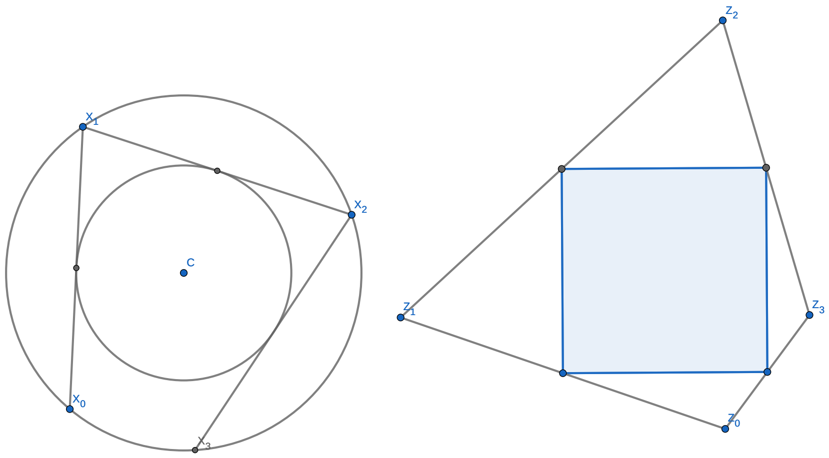

Figure 1: Circle and square dual billiards.

Example 1.1. Let K be a disk in $\mathbb{R}^2$. Then the Figure 1 shows that for any point X_0 its orbit lies on a circle with the same center. In other words, all circles with the same center are the invariant curves for this billiard.

Elliptic dual billiards behave in a very similar manner, as well: the invariant curves are homothetic ellipses. As a matter of fact, an elliptic billiard can be considered as a particular case of Poncelet Porism.

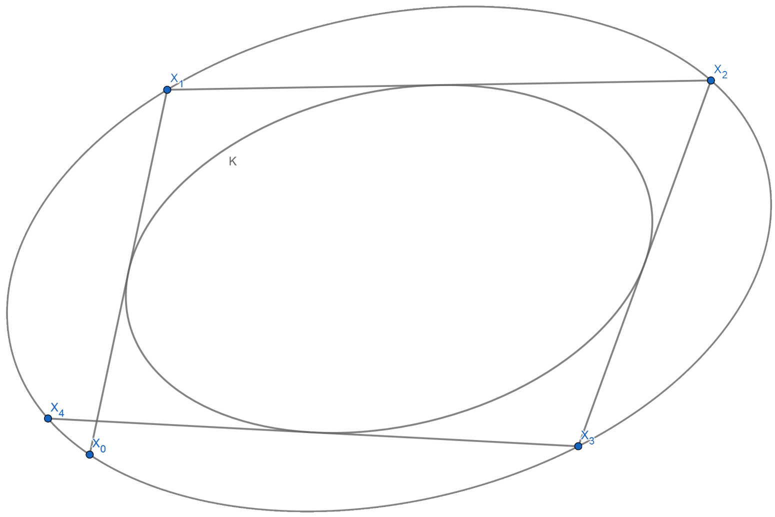

Figure 2: An elliptic billiard

Example 1.2. Now let K be a square. Taking the previous remark into account, we need to take care that X_0 does not lie on the sides of K. It is also easy to see that T will be a 4-periodic map.

These examples are shown in the Figure 1.

The problem of existence of unbounded orbits was open for a long time, and it was settled relatively recently(see Section 3). The first major breakthrough in this problem was achieved by J. Moser and R. Douady in [Mös73, Theorem 2.11] and [Dou82], respectively. Moser observed that outer billiards with sufficiently smooth boundary admit an integrable approximation, which allows us to use the KAM theory. However, this was just a sketch of the argument, and a few years later, R. Douady completed the proof in his paper [Dou82].

Theorem 1.1 (Moser, Douady)**.** Let K be a bounded convex set with a C^6-boundary. Then all orbits of the outer billiard map are bounded.

2 Moser's approach

First of all, recall that outer billiards preserve areas. For the proof we refer to [Tab05, Theorem 9.4]. This allows one to apply KAM methods in this problem, see [Mös73].

Before we state the result and how it can be applied to outer billiards, we need to the corresponding setting. We start by considering the polar coordinate system (r, θ) on $\mathbb{R}^2 \setminus \{0\}$.

We want to develop a theory for area-preserving bijective mappings $\phi : A \to \mathbb{R}^2$, where

$$A = \{(r, \theta) : a \leq r \leq b, 0 \leq \theta \leq 2\pi\},$$

and

$$\phi(r, \theta) = (f(r, \theta), \theta + g(r, \theta)),$$

where $f(r, \theta), g(r, \theta)$ have period 2π in θ. Also, for a function $f \in C^l(A)$ we denote

$$\|f\|_l = \sup_{m+n \leq l, x \in A} \left| \frac{\partial^{m+n} f}{\partial^m r \partial^n \theta}(x) \right|.$$

Finally, let us denote $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$.

Theorem 2.1 ([Mös62]). There exists $l > 1$ such that given the following data:

- a function $\alpha \in C^l([a,b])$,
- a positive real number $\nu > 0$ such that $|\alpha'(r)| \geq \nu > 0$,
- a positive real number $\varepsilon > 0$,

there exists a positive $\delta = \delta(\varepsilon, l, \alpha(r)) > 0$ such that for any area-preserving $\phi : A \to \mathbb{R}^2$ which sufficiently approximates the rotation map

$$\|f - r\|_l + \|g - \alpha(r)\| < \nu\delta$$

the following holds.

1. There exists an an invariant curve of the form

$$r(\xi) = c + u(\xi), \quad \theta(\xi) = \xi + v(\xi) \pmod{2\pi},$$

where $u, v \in C^1(\mathbb{T})$, in other words, the functions u and v have periods 2π, and

$$\|u\|_1 + \|v\|_1 < \varepsilon, \tag{1}$$

and c is a constant with $a < c < b$.

2. Moreover, the restriction of ϕ on such an invariant curve can be described by a rotation $\xi \to \xi + \omega$, where ω satisfies infinitely many conditions

$$\left| \frac{\omega}{2\pi} - \frac{p}{q} \right| > \gamma q^{-\tau}$$

for all $p, q \in \mathbb{Z}_{\geq 0}$ and some $\gamma, \tau > 0$.

3. Finally, any ω which satisfies the conditions above will generate this curve.

Essentially, this result is considered the Moser's contribution to KAM theory, and it guarantees the existence of invariant curves for finitely differentiable mappings. Moser proved this theorem for $l \geq 333$, later l was brought down to 5 by Russman in [Rüß70]. In [Mös73] Moser claims that the smoothness of the boundary can be brought down to $C^{3+\beta}$, but, as we will see in Section 3, the parameter l has to be strictly greater than 1.

However, from this statement it is not entirely clear how this theorem can be applied to outer billiards. Let us provide a sketch of the argument. The idea is that a simpler system

$$\varphi(r, \theta) = (r, \theta + \alpha(r)) \tag{2}$$

is integrable by definition – the system is written in angle-action variables, and the invariant tori are just the circles $|x| = r$. We are trying to approximate the outer billiard system by a rotation system.

Remark. The condition $|\alpha'(r)| \neq 0$ is equivalent to **non-degeneracy** of (2), and this is precisely what allows one to apply KAM theory in this setting.

Even then, if we take a point close to the boundary curve of K, then the map isn't really close to a rotation mapping, so what we should do is to consider a point **far away** from K. From this perspective K is really close to small circle, and circles are roughly mapped to circles. But this exactly means that the assumptions of the theorem should be satisfied for annuli $A = \{a \leq r \leq b\}$ for a very big $a > 0$. To make this argument rigorous, one, for example, needs to ascertain that

- we can find an approximating function α such that $|\alpha'(r)|$ is bounded away from 0 with a nice $\nu > 0$,
- and that the condition (1) also holds.

In particular, we get the following result:

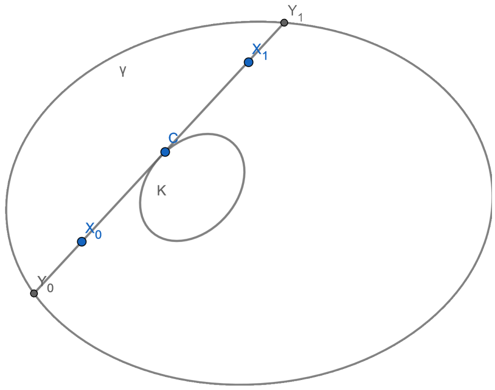

Figure 3: The argument for a convex invariant curve γ

Lemma 2.1. Let K be a bounded convex domain with C^l boundary curve for big enough l. Then the outer billiard admits invariant curves which are arbitrarily far away from K.

As an easy corollary of this lemma one can show that each point has a bounded orbit, because every point will lie inside a certain invariant curve, therefore, the domain bounded by an invariant curve will also be invariant. Indeed, the Figure 3 illustrates the argument.

If γ is an invariant curve, and X_0 is a point inside the domain, bounded by γ, then the segment, connecting X_0 and its image X_1, intersects γ in some points Y_0 and Y_1. However, X_i lie between Y_i, so X_1 will still be inside the domain.

3 Other results

Here we list some other known results about dual billiards.

1. It turns out that one cannot relax the differentiability condition to C^1, because in this case we have the following counterexamples:

   - Floris Takens proved in [Tak71] that in C^1 case there exists a domain K and a point x_0 such that Theorem 2.1 does not hold. In other words, the closure of the orbit of x_0 can contain points on the boundary of A.
   - Philip Boyland proved in [Boy96] that there exists a domain K such that the closure of the ω-orbit of a point can contain points in the boundary of K itself.

2. Now it is known that the Neumann-Moser question has an affirmative answer: using a computational approach, R.E. Schwartz was able to provide an example of a polygonal billiard with an unbounded orbit. The example itself arises from the "Penrose Kite" (however, the author studies an equivalent kite given by Figure 4), a nice illustration is provided in the paper itself, [Sch07, Figure 1.2]. The proof is computer-assisted, for further discussion reader can see [Sch07] and [Sch09].

3. The most recent result was obtained in [DF09, Theorem 1.1], where it was proven that an outer billiard around a semicircle admits an open ball which "escapes to infinity". In other words, we can find an open ball such that for any point in this ball its orbit is unbounded. Finally, they prove that this behavior is stable in a following sense: the statement holds for domains which are sufficiently close to a semicircle, see [DF09, Lemma 2.2, 2.3].

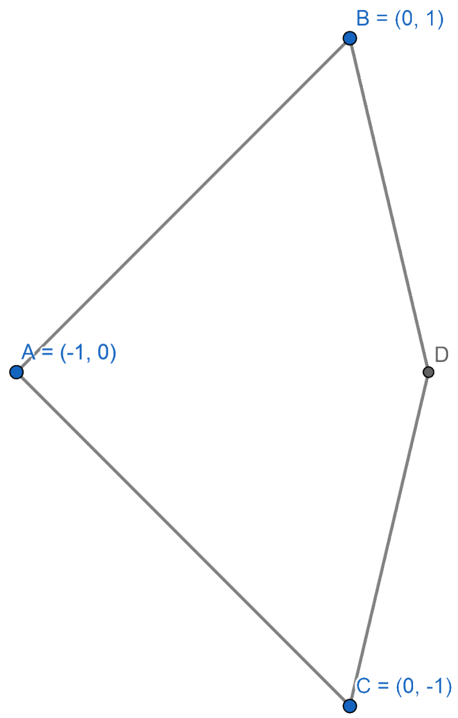

Figure 4: The coordinates of the point D are $(\phi^{-3}, 0)$, where ϕ stands for the golden ratio

References

[Boy96] Philip Boyland. "Dual billiards, twist maps and impact oscillators". In: *Nonlinearity* 9.6 (1996), p. 1411.

[DF09] Dmitry Dolgopyat and Bassam Fayad. "Unbounded Orbits for Semicircular Outer Billiard". In: *Annales Henri Poincaré*. Vol. 10. 2. Springer. 2009, pp. 357–375.

[Dou82] R Douady. "Applications du th eor eme des tores invariants, Th ese de 3 eme Cycle, Univ". In: *Paris VII* (1982).

[Mös62] J Möser. "On invariant curves of area-preserving mappings of an annulus". In: *Nachr. Akad. Wiss. Göttingen, II* (1962), pp. 1–20.

[Mös73] Jürgen Möser. *Stable and Random Motions in Dynamical Systems: With Special Emphasis on Celestial Mechanics (AM-77)*. REV - Revised. Princeton University Press, 1973. ISBN: 9780691089102. URL: `http://www.jstor.org/stable/j.ctt1bd6kg5`.

[Neu59] B Neumann. "Sharing ham and eggs". In: *Iota, Manchester University* (1959), pp. 14–18.

[Rüß70] Helmut Rüßmann. "Kleine Nenner I : über invariante Kurven differenzierbarer Abbildungen eines Kreisringes". In: (1970).

[Sch07] Richard Evan Schwartz. "Unbounded orbits for outer billiards I". In: *Journal of Modern Dynamics* 1.1930-5311 (2007), p. 371. ISSN: 1930-5311. DOI: `10.3934/jmd.2007.1.371`. URL: `http://aimsciences.org//article/id/e37ffe56-8e7c-472b-a52d-93e299b3b079`.

[Sch09] Richard Evan Schwartz. *Outer Billiards on Kites (AM-171)*. Princeton University Press, 2009.

[Tab05] Serge Tabachnikov. *Geometry and billiards*. Vol. 30. American Mathematical Soc., 2005.

[Tak71] Floris Takens. "A C1 counterexample to Moser's twist theorem". In: *Indagationes Mathematicae (proceedings)* 74 (Dec. 1971), pp. 379–386. DOI: `10.1016/S1385-7258(71)80045-8`.