Research statement

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1 Introduction

My interests lie in geometric group theory with a probability theory flavor, and, to a lesser extent, functional analysis.

1. Probabilistic methods have proven to be incredibly useful in the study of negatively curved surfaces and groups acting on them (in the sense of geometric group theory), see various works of Furstenberg, Guivar'ch, Gromov, Ledrappier and so on. As it turns out, studying random walks and Brownian motion gives insight into the underlying geometry of the objects we want to study. Furstenberg and Kesten (see [FK60], for example) were, arguably, the first to introduce methods to study random walks on matrix groups, and their papers are considered to be the starting point of this field of study. Since then, there have been many results concerning random walks on (finitely generated) groups by Furstenberg, Guivarc'h, Lalley, Kaimanovich, and many other authors.

   The notion of the (Gromov/Poisson/Martin) **boundary** of the hyperbolic spaces/groups we study is a key notion, as it turns out, random walks on groups of isometries of hyperbolic spaces tend to **converge** to the boundary almost surely, yielding a measure called the **hitting measure**.

   For the last couple of years we have been studying how random walks behave on cocompact Fuchsian groups acting geometrically on $\mathbb{H}^2$. Right now I am studying the following conjecture, which goes back to Furstenberg's and Guivarc'h's works:

   Conjecture 1.1 ([KL11], page 259)**.** For any finitely supported measure μ on $SL_d(\mathbb{R})$, whose support generates a discrete subgroup, the hitting measure for the random walk driven by μ is singular with respect to Lebesgue measure.

   In other words, we are tackling the following natural question: **how the hitting measure is related to the Patterson-Sullivan (quasi-conformal) measures?**

   For any finitely supported measure μ on $SL_2(\mathbb{Z})$, it is known since Guivarc'h-LeJan [GL90] that the hitting measure is singular. Kaimanovich-LePrince [KL11] produced on any countable Zariski dense subgroup of $SL_d(\mathbb{R})$ examples of finitely supported measures with singular hitting measure. Finally, it is known that the hitting measure is singular for any group acting non-cocompactly on $\mathbb{H}^n$ due to [RT21]. Their approach uses the method of cusp excursion.

   However, as we can see, none of these results apply to **cocompact** groups. Simply put, we cannot exploit the behaviour of the random walk near cusps, as there are none. I have developed different techniques to prove the singularity of the harmonic measure for various families of cocompact Fuchsian groups, proving the singularity of the harmonic measure for Fuchsian groups with regular fundamental polygons (see [Kos20]) and for centrally symmetric fundamental polygons in [KT20] (joint with G. Tiozzo).

2. My research in functional analysis is related to the following natural question: **how to do homological algebra for topological modules and topological algebras?** The most intuitive way to do this would be as follows: for topological vector (Fréchet, Banach) modules A, B, C we consider the sequences

   $$0 \to A \to B \to C \to 0$$

   which are exact in the usual sense and with all arrows being continuous, "exact" in the respective topologcal category. But it becomes quickly evident that we lose a lot of topological information by considering this exact structure.

Taylor and Helemskii (see [Tay72] and [Hel86] for details) are considered to be the first to realize that we need an additional condition: we want all morphisms in such sequences to split in the respective topological category. Then we get a theory which roughly resembles the purely algebraic one, where we can define projectivity, flatness, injectivity, derived functors, and so on.

I have been working on Arens-Michael envelopes and homological dimensions of various types of analytic extensions of Fréchet algebras. We were able to compute the Arens-Michael envelopes of the Laurent tensor algebra $L_A(M)$, generalizing the methods introduced in [Pir08]. Moreover, we derived lower and upper bounds for projective dimensions of holomorphic and smooth crossed products of Arens-Michael algebras.

In the next two sections I will provide a more detailed outline my current research in geometric group theory, and the past research in functional analysis.

2 Geometric group theory

2.1 Random walks on cocompact Fuchsian groups

First of all, recall, that a **Fuchsian group** is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$. It is well-known that any finite covolume Fuchsian group admits a polygonal fundamental domain, called the **fundamental (Dirichlet) polygon**. Moreover, a finite covolume Fuchsian group is uniquely defined by its fundamental polygon due to the **Poincare's theorem**: let $g \geq 0, r \geq 0, m_i \geq 2$, for $1 \leq i \leq r$ be integers such that

$$(2g-2) + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) > 0.$$

Then there is a Fuchsian group generated by the side-pairings of a hyperbolic n-gon with r proper cycles with the sums of angles along the cycles being equal to $\frac{2\pi}{m_i}$ for all $i = 1 \ldots r$.

Let us also recall the definition of random walks on groups:

Definition 2.1. Let (G, S) be a finitely generated group. A **random walk** on G is an infinite sequence of G-valued random variables of form

$$X_n = X_0 g_1 \ldots g_n,$$

where g_i are i.i.d. G-valued random variables, and X_0 (initial distribution) is independent from g_i. If g_i take values in S then we say that (X_n) is a **nearest-neighbor random walk**. If, in addition, g_1 is uniformly distributed then we will call (X_n) a **simple** random walk.

Denote the distribution of X_0 and g_1 by μ_0 and μ, respectively. Then the distribution of X_n is denoted by μ_n. Also, define the first-entrance function $F_\mu(x, y)$ as follows:

$$F_\mu(x, y) := \mathbb{P}^x(\exists n : X_n = y) = \mathbb{P}^e(\exists n : X_n = x^{-1}y).$$

This also allows us to define the **Green metric** as follows:

$$d_\mu(x, y) := -\log(F_\mu(x, y)) \quad \text{for all } x, y \in G.$$

It is well-known that symmetric RW's on hyperbolic groups tend to converge to the Gromov boundary, see [Kai00] or [MT18] for details.

Now we are ready to present the main results obtained during my PhD program:

Theorem 2.1 ([Kos20]). Let P be a regular hyperbolic polygon in the Poincaré disk $\mathbb{D}$, with $2m$ sides, satisfying the cycle condition, and let $S := \{t_1, t_2, \ldots, t_{2m}\}$ be the hyperbolic translations which identify opposite sides of P. Then, for any measure μ supported on the set S, the hitting measure ν on $S^1 = \partial\mathbb{D}$ is singular with respect to Lebesgue measure. Moreover, the Hausdorff dimension of ν is strictly less than 1.

Remark. The above theorem works for regular polygons with sum of angles being a strictly rational multiple of π as well, but there are some exceptions, described in the paper. Also, keep in mind that the first results only covers a countable family of fundamental polygons.

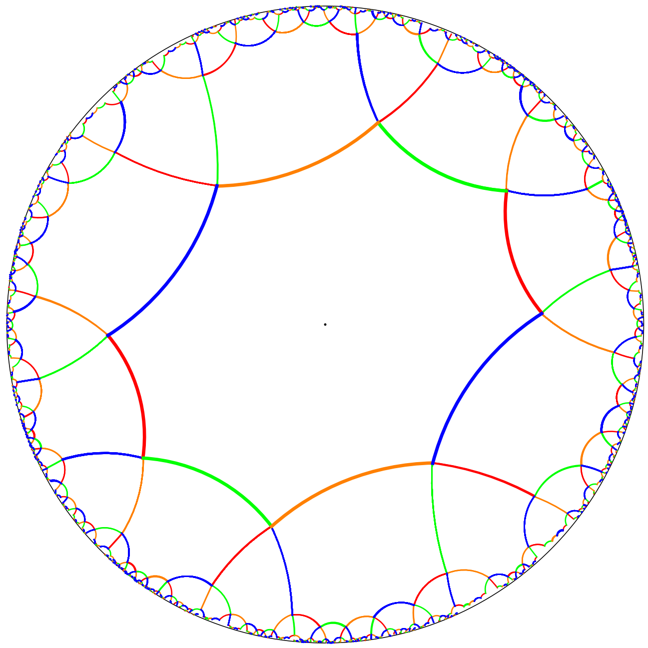

Figure 1: An example of a centrally symmetric hyperbolic octagon equipped with a pairing

Theorem 2.2 ([KT20], Theorem 2)**.** *Let P be a centrally symmetric hyperbolic polygon in the Poincaré disk $\mathbb{D}$, with $2m$ sides, satisfying the cycle condition, and let $S := \{t_1, t_2, \ldots, t_{2m}\}$ be the hyperbolic translations which identify opposite sides of P. Then, for any measure μ supported on the set S, the hitting measure ν on $S^1 = \partial\mathbb{D}$ is singular with respect to Lebesgue measure. Moreover, the Hausdorff dimension of ν is strictly less than 1.*

The second result that we have obtained works not only for Fuchsian groups with regular fundamental polygons, but for any group with a centrally symmetric fundamental polygons. Moreover, it covers an uncountable family of fundamental polygons.

2.2 Connection to the fundamental inequality

Before outlining our approach to proving these results, let us talk about a seemingly different problem.

Let (G, d) be a finitely generated metric group with a left-invariant distance d. Consider a nearest-neighbour random walk (X_i) defined by a probability measure μ with support in the generating set of G. Then we can define the following invariants: **Avez entropy** h, **drift** l and **logarithmic volume** v:

$$v_d := \lim_{n\to\infty} \frac{\log |B_n|}{n} \qquad \text{(logarithmic volume)}$$
$$h_\mu := \lim_{n\to\infty} \frac{-\mathbb{E}[\log \mu_n]}{n} \qquad \text{(Avez entropy)}$$
$$l_{d,\mu} := \lim_{n\to\infty} \frac{\mathbb{E}[d(e, X_n)]}{n} \qquad \text{(drift),}$$

where $B_n = \{g \in G : d(e, g) \le n\}$.

If these invariants are well-defined, they alone can provide a lot of information about a random walk on a group. In particular, $h = 0$ if and only if the Poisson boundary of the random walk is trivial (see [KV83], [Kai00]). Moreover, they are related via the **fundamental inequality** (for proofs see [Gui80], [Ver01], [BHM08]):

$$h_\mu \le l_{d,\mu} v_d. \tag{1}$$

There is a well-known problem, which was considered by Y. Guivarc'h, V. Kaimanovich, S. Lalley, A. Vershik, S. Gouëzel, and many others (see [Gui80], [GL90], [Gou14], [Ver01], [Le 08], [KL11], [BHM11], [GMM18] for example). The following theorem shows that establishing the strictness of the fundamental inequality is, essentially, equivalent to Conjecture:

Theorem 2.3 ([BHM11, Corollary 1.4, Theorem 1.5], [Tan19])**.** Let Γ be a non-elementary hyperbolic group acting geometrically on $\mathbb{H}^2$, endowed with the geometric distance $d = d_{\mathbb{H}^2}$ induced from the action of Γ. Consider a generating probability measure μ on Γ with finite support. Let us also assume that μ is symmetric. Then the following conditions are equivalent:

(1) The equality $h_\mu = l_{d,\mu} v_d$ holds.

(2) The Hausdorff dimension of the exit measure μ_∞ on S^1 is equal to 1.

(3) The measure μ_∞ is equivalent to the Lebesgue measure on S^1.

(4) There exists a constant $C > 0$ such that for any $g \in \Gamma$ we have

$$|v_d d(e,g) - d_\mu(e,g)| \leq C.$$

2.3 Our approach

Now we are ready describe our approach. The idea is to find a hyperbolic element $g \in \Gamma$ and a point $x_0 \in \mathbb{H}^2$ such that

- $d_{\mathbb{H}^2}(e, g^k) = k d_{\mathbb{H}^2}(e,g)$,
- $k d_{\mathbb{H}^2}(e,g) > k|g| \log(|\Sigma|) \geq d_\mu(e, g^k)$.

Then the implication (4) $\Rightarrow$ (1) in Theorem 2.3 implies that $h < lv$. However, finding such elements is not trivial at all. In the regular case we were able to establish that g can always be picked to be a generator, the proof boils down to concrete computations.

In [KT20] we were considering the case of arbitrary centrally symmetric polygon. Here we want to use the same idea but there is an issue: we **hope** that for some i the translation length has to be greater than the Green metric. But as the polygon is now arbitrary, there are no closed formulas for $l(t_i)$ nor $d_\mu(e, t_i)$. At the same time, we don't need to **explicitly** present such an element, we just need to prove that some generator works. So, we prove the following two inequalities:

Theorem 2.4. Consider a random walk on the free group

$$F_m = \left\langle s_1^{\pm 1}, \ldots, s_m^{\pm 1} \right\rangle,$$

defined by a **symmetric** probability measure μ' on the generators. If we denote $x_i := F_{\mu'}(1, s_i)$, then

$$\sum_{i=1}^{m} \frac{x_i}{1 + x_i} = 1. \tag{2}$$

In particular, if we consider the induced measure μ on the generators of Γ, then

$$\frac{1}{1 + e^{d_\mu(e,t_1)}} + \cdots + \frac{1}{1 + e^{d_\mu(e,t_{2m})}} \geq 1 \tag{3}$$

Theorem 2.5. Let P be a centrally symmetric, hyperbolic polygon satisfying the cycle condition, with $2m$ sides, and let $S := \{t_1, \ldots, t_{2m}\}$ be the set of hyperbolic translations identifying opposite sides of P. Then we have

$$\sum_{t \in S} \frac{1}{1 + e^{\ell(t)}} < 1. \tag{4}$$

The inequality 3 still follows from estimates on the free group, but 4 is more challenging to prove. It could also be considered as yet another generalization of the **McShane's (in)equality**, similar to formulas obtained in [CS92], [And+96], and most recently, [He17].

2.4 Asymptotics of the Green function on trees

What happens if we try to generalize the same result for arbitrary supports? As it turns out, the inequality

$$\frac{1}{1+e^{d_\mu(e,t_1)}} + \cdots + \frac{1}{1+e^{d_\mu(e,t_{2m})}} \geq 1$$

might fail, so we need to come up with different methods. We decided to look at a particular family of supports: $S_k = \{t^i_1, \ldots t^i_{2m}\}_{i=1,\ldots,k}$ for $k \geq 1$, in other words, we suppose that the probablitity measure μ is symmetric and supported on all powers of the generators up to k.

As it turns out, we can still lift this random walk on Γ to a random walk on the free group $F_m = \left\langle s_1^{\pm 1}, \ldots, s_m^{\pm 1} \right\rangle$ generated by μ', and we were able to derive a powerful generalization of (2):

Theorem 2.6. Let us define a matrix $(P_k)_{\substack{1\leq i\leq k\\ 1\leq j\leq k}} \in \mathrm{Mat}_k(\mathbb{R})$ as follows:

$$(P_k)_{ij} = F_{\mu'}(s_1^{-k+i}, s_1^j; \{s_1, \ldots, s_1^k\} \setminus s_1^j).$$

If we also denote by S a permutation matrix where $S_{ij} = \delta_{i,k-j}$, then we have the following formula for the boundary measure of a cylinder set:

$$SP_k(E+SP_k)^{-1}\begin{pmatrix}1\\1\\ \vdots \\ 1\end{pmatrix} = \begin{pmatrix}\nu(C(s_1))\\ \vdots \\ \nu(C(s_1^k)).\end{pmatrix}. \tag{5}$$

The resemblance of $\dfrac{SP_k}{E+SP_K}$ to $\dfrac{x_i}{1+x_i}$ (2) should be evident. Using this theorem, we aim to prove the following conjecture:

Conjecture 2.1. Let $\mathrm{supp}\mu = \{s_i^j\}_{\substack{1\leq i\leq m\\ 1\leq |j|\leq k}}$ for some $k \geq 1$. Then for every $1 \leq i \leq m$ we have

$$\lim_{l\to\infty} F_\mu(e, s_i^l)^{\frac{1}{l}} \geq \frac{\nu(C(s_i))}{1-\nu(C(s_i))}.$$

In particular, for $k = 1$ we have an equality:

$$F_\mu(e, s_i^l) = F_\mu(e, s_i)^l = \left(\frac{\nu(C(s_i))}{1-\nu(C(s_i))}\right)^l.$$

As a relatively quick corollary using the already standard technique of comparing the hyperbolic and Green distance, we get the following corollary:

Corollary 2.1. Let P be a centrally symmetric hyperbolic polygon in the Poincaré disk $\mathbb{D}$, with $2m$ sides, satisfying the cycle condition, and let $S := \{t_1, t_2, \ldots, t_{2m}\}$ be the hyperbolic translations which identify opposite sides of P. Then, for any measure μ supported on the set $\{t_i^j\}$ for $j = 1, \ldots, k$, where $k \geq 1$, the hitting measure ν on $S^1 = \partial\mathbb{D}$ is singular with respect to Lebesgue measure. Moreover the Hausdorff dimension of ν is strictly less than 1.

We are still working on upgrading this method to arbitrary supports to resolve Conjecture 1.1 in the most general case.

3 Functional analysis

3.1 Arens-Michael envelopes

Let us define the notion of the Arens-Michael envelope.

Definition 3.1 ([Hel93])**.** Let A be an algebra. An *Arens-Michael envelope* of A is a pair $(\widehat{A}, i_A)$, where $\widehat{A}$ is an Arens-Michael algebra and $i_A : A \to \widehat{A}$ is an algebra homomorphism, satisfying the following universal property:

for any Arens-Michael algebra B and algebra homomorphism $\varphi : A \to B$ there exists a unique continuous algebra homomorphism $\hat{\varphi} : \widehat{A} \to B$ extending φ, i.e. $\varphi = \hat{\varphi} \circ i_A$:

$$\begin{array}{ccc} \widehat{A} & \overset{\hat{\varphi}}{\dashrightarrow} & B \\ {\scriptstyle i_A}\uparrow & \nearrow_{\varphi} & \\ A & & \end{array}$$

Here are some important examples.

Example 3.1. Denote the free algebra with generators $\xi_1, \ldots, \xi_n$ over $\mathbb{C}$ by F_n. Then its Arens-Michael envelope is a locally convex algebra, looks as follows:

$$\mathcal{F}_n := \left\{ a = \sum_{w \in W_n} a_w \xi^w : \|a\|_\rho = \sum_{w \in W_n} |a_w| \rho^{|w|} < \infty \; \forall \, 0 < \rho < \infty \right\}.$$

In particular, $\mathcal{F}_n$ is a nuclear Fréchet algebra.

Example 3.2. Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra. The Arens-Michael envelope of $U(\mathfrak{g})$ is isomorphic to the direct product $\prod_{V \in \hat{\mathfrak{g}}} \mathrm{Mat}(V)$, where $\hat{\mathfrak{g}}$ is the set of the equivalence classes of finite-dimensional irreducible representations of $\mathfrak{g}$.

Sometimes the Arens-Michael envelope of an algebra is isomorphic to the zero algebra:

Example 3.3. Suppose that A is an algebra generated by x and y with the single relation $xy - yx = 1$. Then $\widehat{A} = 0$, because an arbitrary non-zero Banach algebra B cannot contain elements $x, y \in B$ such that $[x, y] = 1$.

As we can see, Arens-Michael envelopes of finitely generated algebras can be trivial, and determining which algebras admit the trivial Arens-Michael envelope is an important and difficult problem. Moreover, in the non-trivial case it is highly desirable to express the Arens-Michael envelope as a concrete space of power series. In general, it can be expressed as a quotient of $\mathcal{F}_n$. Both problems are, open, and there are partial results in [Kos22] and [Ari21].

3.1.1 Describing the Arens-Michael envelope of Laurent tensor algebras

First of all, let us recall the notion of Laurent tensor algebras.

Definition 3.2. Let a A be an algebra and consider an A-bimodule M. Then M is called an *invertible A-bimodule* if there exist an A-bimodule M^{-1} together with A-bimodule isomorphisms $i_1 : M \otimes_A M^{-1} \simeq A$ and $i_2 : M^{-1} \otimes_A M \simeq A$ (which we shall call *convolutions*) such that the following diagrams commute:

$$\begin{array}{ccc} M \otimes_A M^{-1} \otimes_A M & \xrightarrow{Id_M \otimes i_2} & M \otimes_A A \\ \downarrow{\scriptstyle i_1 \otimes Id_M} & & \downarrow{\scriptstyle m \otimes a \to ma} \\ A \otimes_A M & \xrightarrow{a \otimes m \to am} & M \end{array} \qquad \begin{array}{ccc} M^{-1} \otimes_A M \otimes_A M^{-1} & \xrightarrow{Id_M \otimes i_1} & M^{-1} \otimes_A A \\ \downarrow{\scriptstyle i_2 \otimes Id_M} & & \downarrow{\scriptstyle n \otimes a \to na} \\ A \otimes_A M^{-1} & \xrightarrow{a \otimes n \to an} & M^{-1} \end{array} \tag{6}$$

With any A-bimodule M one associates the tensor algebra $T_A(M)$:

$$T_A(M) := A \oplus \bigoplus_{n \in \mathbb{N}} M^{\otimes n},$$

where $M^{\otimes n} := \underbrace{M \otimes_A \cdots \otimes_A M}_{n \text{ times}}$. In turn, for every invertible A-bimodule we can define a complex vector space which will be denoted by $L_A(M)$:

$$L_A(M) := \bigoplus_{n \in \mathbb{Z}} M^{\otimes n}, \tag{7}$$

where $M^{\otimes -n} := (M^{-1})^{\otimes n}$ and $M^{\otimes 0} := A$.

The elements belonging to $M^{\otimes n}$ for some $n \in \mathbb{Z}$ will be called *homogeneous* of degree n. The following proposition states that $L_A(M)$ admits a natural algebra structure:

Proposition 3.1. Suppose that A is an algebra and M is an invertible A-bimodule. Then $L_A(M)$ admits a unique multiplication μ such that $(L_A(M), \mu)$ becomes an associative algebra and μ satisfies the following conditions:

(1) the natural inclusions $j_M : T_A(M) \to L_A(M)$ and $j_{M^{-1}} : T_A(M^{-1}) \to L_A(M)$ are algebra homomorphisms.

(2) for any $m \in M$ and $n \in M^{-1}$ we have $m \cdot n = i_1(m \otimes n)$ and $n \cdot m = i_2(n \otimes m)$.

In [Pir08] the author computed the Arens-Michael envelope of $T_A(A_\alpha)$, where A is an Arens-Michael algebra, α is an automorphism, and A_α is a module which is isomorphic to A as a vector space, with left action induced by the multiplication in A, and the right action is deformed by α: $m \circ a = m\alpha(a)$. In [Kos22] we have generalized the construction to $L_A(M)$. This required us to introduce a topological version of Laurent tensor algebras (denoted $\widehat{L}_A(M)$) via universal property, proving the existence and uniqueness.

These are the results that I have obtained in [Kos22].

Theorem 3.1. Let A be a Fréchet-Arens-Michael algebra and consider topologically inverse Fréchet A-$\hat{\otimes}$-bimodules M, M^{-1}. Then there exist an Arens-Michael algebra $\widehat{L}_A(M)$ and a topologically compatible triple of morphisms $(\theta, \alpha, \beta, \widehat{L}_A(M))$ that satisfies the following universal property: for every Arens-Michael algebra B and a topologically compatible triple of morphisms $(\theta', \alpha', \beta', B)$ there exists a unique continuous A-algebra homomorphism $f : \widehat{L}_A(M) \to B$ such that the following diagrams commute:

$$\begin{array}{ccc} \widehat{L}_A(M) & \xrightarrow{f} & B \\ \uparrow{\scriptstyle\theta} & \nearrow{\scriptstyle\theta'} & \\ A & & \end{array} \qquad \begin{array}{ccc} \widehat{L}_A(M) & \xrightarrow{f} & B \\ \uparrow{\scriptstyle\alpha} & \nearrow{\scriptstyle\alpha'} & \\ M & & \end{array} \qquad \begin{array}{ccc} \widehat{L}_A(M) & \xrightarrow{f} & B \\ \uparrow{\scriptstyle\beta} & \nearrow{\scriptstyle\beta'} & \\ M^{-1} & & \end{array} \tag{8}$$

We will call it the *topological(or analytic) Laurent tensor algebra* of the A-$\hat{\otimes}$-bimodule M.

Conjecture 3.1. Suppose that A is an algebra and M is an invertible A-bimodule. Then there exist topological A-$\hat{\otimes}$-bimodule isomorphisms $\hat{i}_1 : \widehat{M} \hat{\otimes}_{\widehat{A}} \widehat{M^{-1}} \to \widehat{A}$ and $\hat{i}_2 : \widehat{M^{-1}} \hat{\otimes}_{\widehat{A}} \widehat{M} \to \widehat{A}$, satisfying the following conditions:

(1) $\widehat{M}$ is a topologically invertible $\widehat{A}$-$\hat{\otimes}$-bimodule w.r.t. i_1 and i_2.

(2) The following diagram is commutative:

$$\begin{array}{ccccc} M \otimes_A M^{-1} & \xrightarrow{i_1} & A & \xleftarrow{i_2} & M^{-1} \otimes_A M \\ \downarrow{\scriptstyle i_M \otimes i_{M^{-1}}} & & \downarrow{\scriptstyle i_A} & & \downarrow{\scriptstyle i_{M^{-1}} \otimes i_M} \\ \widehat{M} \hat{\otimes}_{\widehat{A}} \widehat{M^{-1}} & \xrightarrow{\hat{i}_1} & \widehat{A} & \xleftarrow{\hat{i}_2} & \widehat{M^{-1}} \hat{\otimes}_{\widehat{A}} \widehat{M} \end{array} \tag{9}$$

where the left arrow maps $a \otimes b$ to $i_M(a) \otimes i_{M^{-1}}(b)$, and the right arrow maps $b \otimes a$ to $i_{M^{-1}}(b) \otimes i_M(a)$.

Proposition 3.2. Now suppose that A is an algebra and M, M^{-1} is a pair of (algebraically) inverse A-bimodules. Suppose that the following condition holds for $\widehat{A}$, $\widehat{M}$ and $\widehat{M^{-1}}$:

(1) The underlying LCS of $\widehat{A}$, $\widehat{M}$ and $\widehat{M^{-1}}$ are Fréchet spaces.

(2) $\widehat{M}$ and $\widehat{M^{-1}}$ are topologically inverse as $\widehat{A}$-$\hat{\otimes}$-bimodules which satisfy Conjecture 3.1.

Then, if $(\theta, \alpha, \beta, \widehat{L}_{\widehat{A}}(\widehat{M}))$ is the resulting topologically compatible triple, then $\widehat{L_A(M)} \simeq \widehat{L}_{\widehat{A}}(\widehat{M})$.

We applied this proposition for $M = A_\alpha$ to compute the Arens-Michael envelope of $A[t, t^{-1}; \alpha]$ for any algebra A and automorphism α.

3.2 Homological dimensions of various topological extensions of Fréchet algebras

Let $X \in A$-**mod**. Suppose that X can be included in a following admissible complex:

$$0 \leftarrow X \xleftarrow{\varepsilon} P_0 \xleftarrow{d_0} P_1 \xleftarrow{d_1} \dots \xleftarrow{d_{n-1}} P_n \leftarrow 0,$$

where every P_i is a projective module. Then we will call such complex a **projective resolution** of X of **length** n. Furthermore, we call resolutions of form

$$0 \leftarrow X \overset{\varepsilon}{\leftarrow} P_0 \overset{d_0}{\longleftarrow} P_1 \overset{d_1}{\longleftarrow} \dots \overset{d_{n-1}}{\longleftarrow} P_n \leftarrow P_{n+1} \leftarrow \dots$$

where $P_n \neq 0$ for all $n \geq 0$ unbounded, and we define the length of an unbounded resolution as ∞. **Flat resolutions** are defined similarly.

It is known that A-**mod** has enough projectives, therefore, one is able define the notion of a derived functor in the topological case, for example, see [Hel86, ch 3.3]. In particular, $\operatorname{Ext}^k_A(M,N)$ and $\operatorname{Tor}^A_k(M,N)$ are defined similarly to the purely algebraic situation.

Consider an arbitrary module $M \in A$-$\mathbf{mod}(\mathcal{C})$ for a category $\mathcal{C} \subseteq \mathbf{LCS}$ such that A-mod$(\mathcal{C})$ has enough projectives. For example, we can consider an admissible category $\mathcal{C}$ in the sense of [Pir12, Definition 2.4]. Then due to [Hel86, Theorem 3.5.4] following number is well-defined and we have the following identities:

$$\begin{aligned}\operatorname{dh}^{\mathcal{C}}_A(M) &:= \min\{n \in \mathbb{Z}_{\geq 0} : \operatorname{Ext}^{n+1}_A(M,N) = 0 \text{ for every } N \in A\text{-}\mathbf{mod}(\mathcal{C})\} = \\ &= \{\text{the length of a shortest projective resolution of } M \text{ in } A\text{-}\mathbf{mod}(\mathcal{C})\} \in \{-\infty\} \cup [0,\infty].\end{aligned}$$

We define $\operatorname{dh}^{\mathcal{C}}_A(0) = -\infty$ and if every projective resolution of M is unbounded, we set $\operatorname{dh}^{\mathcal{C}}_A(M) = \infty$.

As we can see, this number doesn't depend on the choice of the category $\mathcal{C}$:

$$\begin{aligned}\operatorname{dh}^{\mathcal{C}}_A(M) &= \min\{n \in \mathbb{Z}_{\geq 0} : \operatorname{Ext}^{n+1}_A(M,N) = 0 \text{ for every } N \in A\text{-}\mathbf{mod}(\mathcal{C})\} \leq \\ &\leq \min\{n \in \mathbb{Z}_{\geq 0} : \operatorname{Ext}^{n+1}_A(M,N) = 0 \text{ for every } N \in A\text{-}\mathbf{mod}\} = \operatorname{dh}^{\mathbf{LCS}}_A(M),\end{aligned}$$

$$\begin{aligned}\operatorname{dh}^{\mathbf{LCS}}_A(M) &= \{\text{the length of a shortest projective resolution of } M \text{ in } A\text{-}\mathbf{mod}\} \leq \\ &\leq \{\text{the length of a shortest projective resolution of } M \text{ in } A\text{-}\mathbf{mod}(\mathcal{C})\} = \operatorname{dh}^{\mathcal{C}}_A(M).\end{aligned}$$

So we will denote this invariant by $\operatorname{dh}_A(M)$, and we will call it the **projective homological dimension** of M.

If A is a Fréchet algebra, and M is a left Fréchet A-module, then we can define the **weak homological dimension** of M:

$$\begin{aligned}\operatorname{w.dh}_A(M) &= \min\{n \in \mathbb{Z}_{\geq 0} : \operatorname{Tor}^A_{n+1}(N,M) = 0 \text{ and } \operatorname{Tor}^A_n(N,M) \text{ is Hausdorff for every } N \in \mathbf{mod}\text{-}A(\mathbf{Fr})\} = \\ &= \{\text{the length of the shortest flat resolution of } M\} \in \{-\infty\} \cup [0,\infty].\end{aligned}$$

We define $\operatorname{w.dh}^{\mathcal{C}}_A(0) = -\infty$ and if every flat resolution of M is unbounded we set $\operatorname{dh}^{\mathcal{C}}_A(M) = \infty$.

Let A be a $\hat{\otimes}$-algebra. Then we can define the following invariants of A:

$$\operatorname{dgl}_{\mathcal{C}}(A) = \sup\{\operatorname{dh}_A(M) : M \in A\text{-}\mathbf{mod}(\mathcal{C})\} - \textbf{the left global dimension of } A.$$

$$\operatorname{dgr}_{\mathcal{C}}(A) = \sup\{\operatorname{dh}_{A^{\mathrm{op}}}(M) : M \in \mathbf{mod}\text{-}A(\mathcal{C})\} - \textbf{the right global dimension of } A.$$

$$\operatorname{db}(A) = \operatorname{dh}_{A \hat{\otimes} A^{\mathrm{op}}}(A) - \textbf{the bidimension of } A.$$

For a Fréchet algebra A we can consider **weak dimensions**.

$$\begin{aligned}\operatorname{w.dg}(A) &= \sup\{\operatorname{w.dh}_A(M) : M \in A\text{-}\mathbf{mod}(\mathbf{Fr})\} = \\ &= \sup\{\operatorname{w.dh}_A(M) : M \in \mathbf{mod}\text{-}A(\mathbf{Fr})\} - \textbf{the weak global dimension of } A.\end{aligned}$$

$$\operatorname{w.db}(A) = \operatorname{w.dh}_{A \hat{\otimes} A^{\mathrm{op}}}(A) - \textbf{the weak bidimension of } A.$$

Unfortunately, we are not aware whether global dimensions depend on the choice of $\mathcal{C}$. We will denote

$$\operatorname{dgl}(A) := \operatorname{dgl}_{\mathbf{LCS}}(A), \quad \operatorname{dgr}(A) := \operatorname{dgr}_{\mathbf{LCS}}(A).$$

For more details the reader can consult [Hel86].

Then we define holomorphic Ore extensions:

Theorem 3.2. [Pir08, Section 4.1] Let A be a $\hat{\otimes}$-algebra and suppose that $\alpha : A \to A$ is a localizable endomorphism of A, $\delta : A \to A$ is a localizable α-derivation of A.

Then there exists a unique multiplication on the tensor product $A \hat{\otimes} \mathcal{O}(\mathbb{C})$, such that the following conditions are satisfied:

(1) The resulting algebra, which is denoted by $\mathcal{O}(\mathbb{C}, A; \alpha, \delta)$, is an A-$\hat{\otimes}$-algebra.

(2) The natural inclusion

$$A[z; \alpha, \delta] \hookrightarrow \mathcal{O}(\mathbb{C}, A; \alpha, \delta)$$

induced by the inclusion $\mathbb{C}[z] \to \mathcal{O}(\mathbb{C})$, where z stands for the identity map $\mathbb{C} \to \mathbb{C}$, is an algebra homomorphism.

(3) Moreover, if the pair (α, δ) is m-localizable, then for every Arens-Michael A-$\hat{\otimes}$-algebra B the following natural isomorphism takes place:

$$\operatorname{Hom}(A[z; \alpha, \delta], B) \cong \operatorname{Hom}(\mathcal{O}(\mathbb{C}, A; \alpha, \delta), B).$$

Moreover, let α be invertible, and suppose that the pair (α, α^{-1}) is localizable. Then there exists a unique multiplication on the tensor product $A\hat{\otimes}\mathcal{O}(\mathbb{C}^{\times})$, such that the following conditions are satisfied:

(1) The resulting algebra, which is denoted by $\mathcal{O}(\mathbb{C}^{\times}, A; \alpha)$, is a $\hat{\otimes}$-algebra.

(2) The natural inclusion

$$A[z; \alpha, \alpha^{-1}] \hookrightarrow \mathcal{O}(\mathbb{C}^{\times}, A; \alpha)$$

induced by the inclusion $\mathbb{C}[z, z^{-1}] \to \mathcal{O}(\mathbb{C}^{\times})$, where z stands for the identity map $\mathbb{C} \to \mathbb{C}$, is an algebra homomorphism.

(3) Moreover, if the pair (α, α^{-1}) is m-localizable, then for every Arens-Michael A-$\hat{\otimes}$-algebra B the following natural isomorphism takes place:

$$\operatorname{Hom}(A[z; \alpha, \alpha^{-1}], B) \cong \operatorname{Hom}(\mathcal{O}(\mathbb{C}^{\times}, A; \alpha), B).$$

And if we replace the word “localizable” with “m-localizable” in this theorem, then the resulting algebras will become Arens-Michael algebras.

We are, finally, fully prepared to formulate the estimates proven in [Kos17].

Theorem 3.3. Suppose that R is a $\hat{\otimes}$-algebra, and A is one of the two $\hat{\otimes}$-algebras:

(1) $A = \mathcal{O}(\mathbb{C}, R; \alpha, \delta)$, where the pair $\{\alpha, \delta\}$ is localizable.

(2) $A = \mathcal{O}(\mathbb{C}^{\times}, R; \alpha)$, where the pair $\{\alpha, \alpha^{-1}\}$ is localizable.

Then we have

$$\operatorname{db}(A) \leq \operatorname{db}(R) + 1.$$

Theorem 3.4. Let R be a $\hat{\otimes}$-algebra. Suppose that A is one of the two $\hat{\otimes}$-algebras:

(1) $A = \mathcal{O}(\mathbb{C}, R; \alpha, \delta)$, where α is invertible, and the pair $\{\alpha, \delta\}$ is localizable.

(2) $A = \mathcal{O}(\mathbb{C}^{\times}, R; \alpha)$, where the pair $\{\alpha, \alpha^{-1}\}$ is localizable.

Then the right global dimension of A can be estimated as follows:

$$\operatorname{dgr}(A) \leq \operatorname{dgr}(R) + 1,$$

and a similar estimate holds for the weak dimensions if R is a Fréchet algebra:

$$\operatorname{w.dg}(A) \leq \operatorname{w.dg}(R) + 1.$$

Theorem 3.5. Let R be a Fréchet algebra, and suppose that $\operatorname{dgr}_{\mathbf{Fr}}(R) < \infty$ and A is one of the two $\hat{\otimes}$-algebras:

(1) $A = \mathcal{O}(\mathbb{C}, R; \alpha, \delta)$, where α is invertible, and the pairs (α, δ) and $(\alpha^{-1}, \delta\alpha^{-1})$ are m-localizable.

(2) $A = \mathcal{O}(\mathbb{C}^{\times}, R; \alpha)$, where the pair (α, α^{-1}) is m-localizable.

Then we have

$$\operatorname{dgr}_{\mathbf{Fr}}(R) \leq \operatorname{dgr}_{\mathbf{Fr}}(A), \quad \operatorname{w.dg}(R) \leq \operatorname{w.dg}(A).$$

Definition 3.3. Let A be a Fréchet algebra with a fixed generating system of seminorms $\{\|\cdot\|_\lambda, \lambda \in \Lambda\}$ Then we can define the following locally convex space:

$$\mathscr{S}(\mathbb{Z}, A) = \left\{ f = (f_m)_{m\in\mathbb{Z}} \in A^{\mathbb{Z}} : \|f\|_{\lambda,k} := \sum_{n\in\mathbb{Z}} \|f_n\|_\lambda (|n|+1)^k < \infty \text{ for all } \lambda \in \Lambda, k \in \mathbb{N} \right\}.$$

Theorem 3.6 ([Sch93], Theorem 3.1.7)**.** Let R be a Fréchet-Arens-Michael algebra with an m-tempered $\mathbb{Z}$-action. Then the space $\mathscr{S}(\mathbb{Z}, R)$ endowed with the multiplication

$$(f * g)_k = \sum_{n\in\mathbb{Z}} f_n \alpha^n(g_{k-n}), \quad f, g \in \mathscr{S}(\mathbb{Z}, R).$$

becomes a Fréchet-Arens-Michael algebra. This algebra is denoted by $\mathscr{S}(G, \mathbb{Z}; \alpha)$ and called the **smooth crossed product** by $\mathbb{Z}$.

Theorem 3.7. Let R be a Fréchet-Arens-Michael algebra with an m-tempered $\mathbb{Z}$-action α. If we denote $A = \mathscr{S}(\mathbb{Z}, R; \alpha)$, then we have

$$\operatorname{db}(A) \le \operatorname{db}(R) + 1, \quad \operatorname{dgr}_{\mathbf{Fr}}(A) \le \operatorname{dgr}_{\mathbf{Fr}}(R) + 1, \quad \operatorname{w.dg}(A) \le \operatorname{w.dg}(R) + 1.$$

Definition 3.4. Let A be a Fréchet-Arens-Michael algebra, and let $G = \mathbb{R}$ or $G = \mathbb{T}$. Then the action α of G on A via automorphisms is called

(a) *m-tempered* (as in [Sch93]), if there exists a generating family of submultiplicative seminorms $\{\|\cdot\|_m\}_{m\in\mathbb{N}}$ on A such that for every $m \in \mathbb{N}$ there is a polynomial $p_m(x) \in \mathbb{R}[x]$, satisfying

$$\|\alpha_x(a)\|_m \le |p_m(x)| \, \|a\|_m \quad (a \in A, x \in G).$$

(b) *C^∞-m-tempered* or *smooth m-tempered* , if the following conditions are satisfied:

(1) for every $a \in A$ the function

$$\alpha_x(a) : G \longrightarrow A, \quad x \mapsto \alpha_x(a),$$

is C^∞-differentiable,

(2) there exists a generating family of submultiplicative seminorms $\{\|\cdot\|_m\}_{m\in\mathbb{N}}$ on A such that for any $k \ge 0$ and $m > 0$ there exists a polynomial $p_{k,m} \in \mathbb{R}[x]$, satisfying

$$\left\|\alpha_x^{(k)}(a)\right\|_m \le |p_{k,m}(x)| \, \|a\|_m \quad (k \in \mathbb{N}, x \in G, a \in A).$$

The following theorem can be considered as a definition of smooth crossed products.

Theorem 3.8 ([Sch93], Theorem 3.1.7)**.** Let A be a Fréchet-Arens-Michael algebra with an m-tempered action of one of the groups $G = \mathbb{R}$ or $G = \mathbb{T}$. Then the space $\mathscr{S}(G, A)$ endowed with the following multiplication:

$$(f *_\alpha g)(x) = \int_G f(y)\alpha_y(g(x-y))dy$$

becomes a Fréchet-Arens-Michael algebra.

When $G = \mathbb{R}$, we will denote this algebra by $\mathscr{S}(\mathbb{R}, A; \alpha)$, and in the case $G = \mathbb{T}$ we will write $C^\infty(\mathbb{T}, A; \alpha)$.

Theorem 3.9 ([Kos21])**.** Let A be a projective Fréchet-Arens-Michael algebra, which satisfies the following condition: the multiplication map $m : A\hat{\otimes}_A A \longrightarrow A$ is an A-$\hat{\otimes}$-bimodule isomorphism. Also let α denote a smooth m-tempered action of $\mathbb{R}$ or $\mathbb{T}$ on A. Denote the left (projective) global dimension by dgl. Then for $G = \mathbb{R}$ or $G = \mathbb{T}$ we have

$$\operatorname{dgl}(\mathscr{S}(G, A; \alpha)) \le \max\{\operatorname{dgl}(A), 1\} + 1$$

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