Research plan

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My interests lie in geometric group theory with a probability theory flavor, and for the last couple of years I have been studying how random walks behave on cocompact Fuchsian groups acting geometrically on $\mathbb{H}^2$. At the moment I am working on the following conjecture, which goes back to Furstenberg's and Guivarc'h's works:

Conjecture 0.1 ([KL11], page 259)**.** For any finitely supported measure μ on $SL_d(\mathbb{R})$, whose support generates a discrete subgroup, the hitting measure for the random walk driven by μ is singular with respect to Lebesgue measure.

In other words, we are tackling the following natural question: **how the hitting measure is related to the Patterson-Sullivan (quasi-conformal) measures?**

For any finitely supported measure μ on $SL_2(\mathbb{Z})$, it is known since Guivarc'h-LeJan [GL90] that the hitting measure is singular. Kaimanovich-LePrince [KL11] produced on any countable Zariski dense subgroup of $SL_d(\mathbb{R})$ examples of finitely supported measures with singular hitting measure. Finally, it is known that the hitting measure is singular for any group acting non-cocompactly on $\mathbb{H}^n$ due to [RT21]. Their approach uses the method of cusp excursion.

However, as we can see, none of these results apply to **cocompact** groups. Simply put, we cannot exploit the behaviour of the random walk near cusps, as there are none. I have developed different techniques to prove the singularity of the harmonic measure for various families of cocompact Fuchsian groups, proving the singularity of the harmonic measure for Fuchsian groups with regular fundamental polygons (see [Kos20]) and for centrally symmetric fundamental polygons in [KT20] (joint with G. Tiozzo).

1 The current progress

Here are the main results that I have **already** obtained during my PhD program (which will serve as a basis for my PhD thesis).

Theorem 1.1 ([Kos20])**.** Let P be a regular hyperbolic polygon in the Poincaré disk $\mathbb{D}$, with $2m$ sides, satisfying the cycle condition, and let $S := \{t_1, t_2, \ldots, t_{2m}\}$ be the hyperbolic translations which identify opposite sides of P. Then, for any measure μ supported on the set S, the hitting measure ν on $S^1 = \partial\mathbb{D}$ is singular with respect to Lebesgue measure. Moreover, the Hausdorff dimension of ν is strictly less than 1.

Remark. The above theorem works for regular polygons with sum of angles being a strictly rational multiple of π as well, but there are some exceptions, described in the paper. Also, keep in mind that the first results only covers a countable family of fundamental polygons.

Theorem 1.2 ([KT20], Theorem 2)**.** Let P be a centrally symmetric hyperbolic polygon in the Poincaré disk $\mathbb{D}$, with $2m$ sides, satisfying the cycle condition, and let $S := \{t_1, t_2, \ldots, t_{2m}\}$ be the hyperbolic translations which identify opposite sides of P. Then, for any measure μ supported on the set S, the hitting measure ν on $S^1 = \partial\mathbb{D}$ is singular with respect to Lebesgue measure. Moreover, the Hausdorff dimension of ν is strictly less than 1.

The second result works not only for Fuchsian groups with regular fundamental polygons, but for any group with a centrally symmetric fundamental polygons. Moreover, it covers an uncountable family of fundamental polygons, see the provided figure for an example of such a polygon to which our theorem applies.

Let us briefly describe the approach used, which relies on the following theorem:

Theorem 1.3 ([BHM11, Corollary 1.4, Theorem 1.5], [Tan19])**.** Let Γ be a non-elementary hyperbolic group acting geometrically on $\mathbb{H}^2$, endowed with the geometric distance $d = d_{\mathbb{H}^2}$ induced from the action of Γ. Consider a generating probability measure μ on Γ with finite support. Let us also assume that μ is symmetric. Then the following conditions are equivalent:

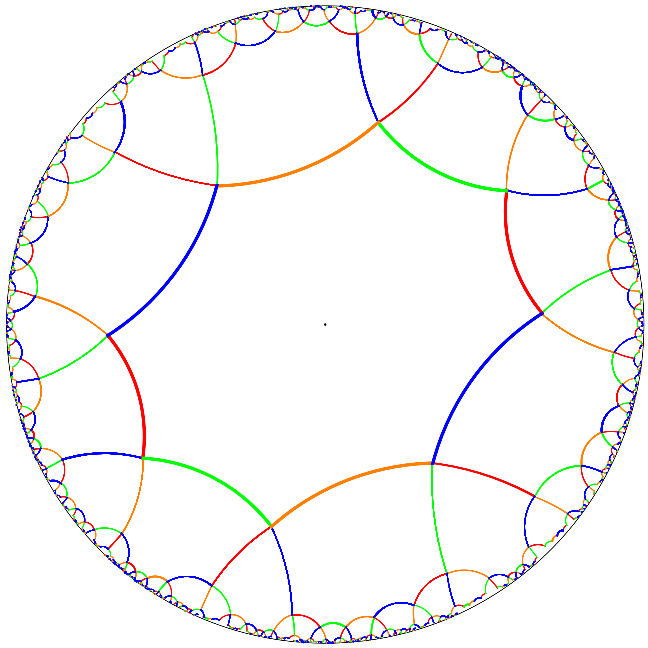

Figure 1: An example of a centrally symmetric hyperbolic octagon equipped with a pairing

(1) The equality $h_\mu = l_{d,\mu} v_d$ holds.

(2) The Hausdorff dimension of the exit measure μ_∞ on S^1 is equal to 1.

(3) The measure μ_∞ is equivalent to the Lebesgue measure on S^1.

(4) There exists a constant $C > 0$ such that for any $g \in \Gamma$ we have

$$|v_d d(e,g) - d_\mu(e,g)| \leq C.$$

The main idea is to find a hyperbolic element $g \in \Gamma$ and a point $x_0 \in \mathbb{H}^2$ such that

- $d_{\mathbb{H}^2}(e, g^k) := d(x_0, g^k.x_0) = k d_{\mathbb{H}^2}(e,g)$,
- $k d_{\mathbb{H}^2}(e,g) > k|g| \log(|\Sigma|) \geq d_\mu(e, g^k)$.

Then the implication (4) $\Rightarrow$ (1) in Theorem 1.3 implies that $h < lv$. However, finding such elements is not trivial at all. In the regular case we were able to establish that g can always be picked to be a generator, the proof boils down to concrete computations.

In [KT20] we were considering the case of arbitrary centrally symmetric polygon. Here we want to use the same idea but there is an issue: we **hope** that for some i the translation length has to be greater than the Green metric. But as the polygon is now arbitrary, there are no closed formulas for $l(t_i)$ nor $d_\mu(e, t_i)$. At the same time, we don't need to **explicitly** present such an element, we just need to prove that some generator works. So, we prove the following two inequalities:

Theorem 1.4. Consider a random walk on the free group

$$F_m = \left\langle s_1^{\pm 1}, \ldots, s_m^{\pm 1} \right\rangle,$$

defined by a **symmetric** probability measure μ' on the generators. If we denote $x_i := F_{\mu'}(1, s_i)$, then

$$\sum_{i=1}^{m} \frac{x_i}{1+x_i} = 1. \tag{1}$$

In particular, if we consider the induced measure μ on the generators of Γ, then

$$\frac{1}{1+e^{d_\mu(e,t_1)}} + \cdots + \frac{1}{1+e^{d_\mu(e,t_{2m})}} \geq 1 \tag{2}$$

Theorem 1.5. Let P be a centrally symmetric, hyperbolic polygon satisfying the cycle condition, with $2m$ sides, and let $S := \{t_1, \ldots, t_{2m}\}$ be the set of hyperbolic translations identifying opposite sides of P. Then we have

$$\sum_{t \in S} \frac{1}{1 + e^{\ell(t)}} < 1. \tag{3}$$

The inequality 2 still follows from estimates on the free group, but 3 is more challenging to prove. It could also be considered as yet another generalization of the **McShane's (in)equality**, similar to formulas obtained in [CS92], [And+96], and most recently, [He17]. It is easily seen that (2) and (3) imply that there is a suitable generator t_i.

2 Future plans

- Right now I am working on generalizing Theorem 1.2 to random walks with **arbitrary** finite supports using linear-algebraic methods in order to study the asymptotic behavior of the Green function. The idea is still, essentially, the same: one aims to find a hyperbolic element $g \in \Gamma$, where Γ is our Fuchsian group, such that $\tau(g) > d_\mu(e, g)$ (τ stands for the translation length), but if the random walk isn't just supported on generators, we cannot guarantee that choosing a generator will suffice. However, we conjecture that the following estimate holds for the respective **covering** random walk on the free group.

  Conjecture 2.1. Consider the free group $F_m = \langle s_1^{\pm 1}, \ldots s_m^{\pm 1} \rangle$, and let $\text{supp}\mu = \{s_i^j\}_{\substack{1 \le i \le m \\ 1 \le |j| \le k}}$ for some $k \geq 1$. Then for every $1 \le i \le m$ we have

  $$\lim_{l \to \infty} F_\mu(e, s_i^l)^{\frac{1}{l}} \geq \frac{\nu(C(s_i))}{1 - \nu(C(s_i))}.$$

  In particular, for $k = 1$ we have an equality:

  $$F_\mu(e, s_i^l) = F_\mu(e, s_i)^l = \left(\frac{\nu(C(s_i))}{1 - \nu(C(s_i))} \right)^l.$$

  As a relatively quick corollary using the already standard technique of comparing the hyperbolic and Green distance, we get the following corollary:

  Corollary 2.1. Let P be a centrally symmetric hyperbolic polygon in the Poincaré disk $\mathbb{D}$, with $2m$ sides, satisfying the cycle condition, and let $S := \{t_1, t_2, \ldots, t_{2m}\}$ be the hyperbolic translations which identify opposite sides of P. Then, for any measure μ supported on the set $\{t_i^j\}$ for $j = 1, \ldots, k$, where $k \geq 1$, the hitting measure ν on $S^1 = \partial\mathbb{D}$ is singular with respect to Lebesgue measure. Moreover the Hausdorff dimension of ν is strictly less than 1.

  We are still working on upgrading this method to arbitrary supports to resolve Conjecture 0.1 in the most general case.

- The key tool that we use, Theorem 1.3, implies that the Hausdorff dimension of the hitting measure is strictly less than 1. It is natural to look for better upper bounds for the dimension. There is some progress done for the uniform nearest-neighbour random walks, but the problem is wide open for arbitrary nearest-neighbour random walks, let alone arbitrary RW's with finite support.

  Finally, it is reasonable to assume that an effective approach has to strengthen the methods used to prove [BHM11, Corollary 1.4, Theorem 1.5], it is quite likely that a stronger bound on the Hausdorff dimensions implies some stronger relations between the geometric distance and the Green distance on Γ.

- Finally, we aim to settle the conjecture for groups acting geometrically on $\mathbb{H}^n$ for $n > 2$. Some related results are available for $n = 3$ (Kleinian groups), see [And+96], but we are nowhere close to settling the cocompact case yet. The high-dimensional case also provides additional challenges, such as the non-existence of an elegant classification of higher-dimensional ($n > 3$) Kleinian groups, unlike in the two- and three-dimensional settings. However, not all is lost, as the Poincare's fundamental polygon theorem can be generalized to hyperbolic polyhedra in higher dimensions, and the first step should be doing the necessary computations for concrete hyperbolic polyhedra and respective groups.

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