

The Arens-Michael envelopes of Laurent Ore extensions

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Abstract: For an Arens-Michael algebra A we consider a class of A - $\hat{\otimes}$ -bimodules which are invertible with respect to the projective bimodule tensor product. We call such bimodules topologically invertible over A . Given a Fréchet-Arens-Michael algebra A and an topologically invertible Fréchet A - $\hat{\otimes}$ -bimodule M , we construct an Arens-Michael algebra $\widehat{L}_A(M)$ which serves as a topological version of the Laurent tensor algebra $L_A(M)$.

Also, for a fixed algebra B we provide a condition on an invertible B -bimodule N which allows us to explicitly describe the Arens-Michael envelope of $L_B(N)$ as a topological Laurent tensor algebra. In particular, we provide an explicit description of the Arens-Michael envelope of an invertible Ore extension $A[x, x^{-1}; \alpha]$ for a metrizable algebra A .

Key words: Arens-Michael envelopes, topological bimodules, locally convex algebras, Ore extensions

Introduction

We would like to begin the paper by demonstrating a connection between Arens-Michael envelopes and non-commutative geometry.

Noncommutative geometry is a branch of mathematics which, in particular, arose from such fundamental results as the first Gelfand-Naimark theorem or the Nullstellensatz, or, more precisely, their categorical interpretations:

Theorem 0.1 (the first Gelfand-Naimark theorem) *Denote the category of commutative unital C^* -algebras by \mathbf{CUC}^* and the category of compact Hausdorff topological spaces by \mathbf{Comp} . Then the pair of functors $\mathcal{F} : \mathbf{Comp} \rightarrow \mathbf{CUC}^*$ and $\mathcal{G} : \mathbf{CUC}^* \rightarrow \mathbf{Comp}$, where $\mathcal{F}(X) = C(X)$ and $\mathcal{G}(A) = \text{Spec}_m(A)$, is an anti-equivalence of categories.*

Theorem 0.2 (Nullstellensatz) *Let \mathbb{K} be an algebraically closed field. Denote the category of affine algebraic \mathbb{K} -varieties by \mathbf{Aff} and the category of commutative finitely generated reduced unital \mathbb{K} -algebras by \mathbf{Alg} . Then a pair of functors $\mathcal{F} : \mathbf{Aff} \rightarrow \mathbf{Alg}$ and $\mathcal{G} : \mathbf{Alg} \rightarrow \mathbf{Aff}$, where $\mathcal{F}(X) = \mathbb{K}[X]$ and $\mathcal{G}(A) = \text{Spec}_m(A)$, is an anti-equivalence of categories.*

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1 As the reader can see, these theorems state that some category of geometrical objects is anti-equivalent
 2 to the category of algebras of functions on them. This observation, for example, serves as a motivation to think
 3 of noncommutative C^* -algebras as the function spaces on “noncommutative topological spaces”.

4 The notion of an Arens-Michael envelope was discovered by J. Taylor in [12] due to the problem of a multi-
 5 operator functional calculus existence. It is worth noting that the terminology the author used was different
 6 from that we use nowadays: Taylor defined the Arens-Michael envelopes as “completed locally multiplicative
 7 convex envelopes”. The current terminology is due to A. Helemskii, see [5].

8 The following theorem stands as a reason to study Arens-Michael envelopes in the context of noncom-
 9 mutative geometry.

10 **Theorem 0.3** *The Arens-Michael envelope of $\mathbb{C}[t_1, \dots, t_n]$ is topologically isomorphic to the algebra of holo-*
 11 *morphic functions $\mathcal{O}(\mathbb{C}^n)$ endowed with the compact-open topology.*

12 This fact, discovered by J. Taylor, can be formulated as follows: the Arens-Michael envelope of the algebra
 13 of regular functions on \mathbb{C}^n is isomorphic to the algebra of holomorphic functions on \mathbb{C}^n . In fact, the same
 14 holds true for an arbitrary affine complex algebraic variety, see [9, Example 3.6]. Therefore, it makes sense to
 15 define the algebra of “holomorphic functions” on a noncommutative affine algebraic variety as the Arens-Michael
 16 envelope of the algebra of “regular functions” on it. In other words, the notion of Arens-Michael envelope serves
 17 as a “bridge” between algebra and functional analysis.

18 In this paper we are concerned with “computing” the Arens-Michael envelopes for some interesting non-
 19 commutative associative finitely generated algebras over \mathbb{C} . By this we mean that, for an algebra A , we aim
 20 to explicitly construct the Arens-Michael algebra B which turns out to be isomorphic to the Arens-Michael
 21 envelope \widehat{A} . In most (non-degenerate) cases such algebra B is constructed as a power series algebra. In other
 22 words, the underlying locally convex space of B turns out to be a Köthe space.

23 Suppose that A is an algebra with an endomorphism $\alpha \in \text{End}(A)$ and an α -derivation $\delta : A \rightarrow A$.
 24 Then, under some reasonable conditions on α and δ , the Arens-Michael envelope of its Ore extension $A[t; \alpha, \delta]$
 25 admits a description in terms of the Arens-Michael envelope \widehat{A} .

26 A lot of naturally occurring noncommutative algebras can be represented as iterated Ore extensions, for
 27 example, q -deformations of classical algebras, such as $\text{Mat}_q(2)$ or $U_q(\mathfrak{g})$.

28 Let A be a unital associative complex algebra, $\alpha \in \text{End}(A)$ and $\delta : A \rightarrow A$ be an α -derivation. Consider
 29 the Ore extension $A[t; \alpha, \delta]$. Then there are several cases:

- 30 1. Suppose that the pair (α, δ) is “nice enough” in the sense that their extensions to \widehat{A} behave well enough
 31 with respect to the topology on \widehat{A} (m -localizable families of morphisms). Then the Arens-Michael
 32 envelope of $A[t; \alpha, \delta]$ admits a relatively simple description, see [9, Proposition 4.5] and [9, Theorem
 33 5.17].

34 Now suppose that $\delta = 0$ for simplicity.

- 35 2. Consider now the case of general $\alpha \in \text{End}(A)$. Then $A[t; \alpha]$ still admits an explicit description of the
 36 Arens-Michael envelope, which utilizes an analytic version of the notion of tensor algebra associated with
 37 a bimodule. However, the seminorms which were used to describe the topology on $A\{t; \alpha\} \simeq \widehat{A[t; \alpha]}$ are
 38 difficult to compute in the general case, see [9, Proposition 4.9], [9, Corollary 5.6] and [9, Example 4.3].

3. Now suppose that α is invertible. Then one can define the Laurent Ore extension $A[t, t^{-1}; \alpha]$. If the pair (α, α^{-1}) is "nice enough", then the Arens-Michael envelope admits a description similar to the one in Case 1, see [9, Proposition 4.15] and [9, Theorem 5.21].
4. In this paper we treat the case of arbitrary $\alpha \in \text{Aut}(A)$, in other words, we don't assume that the pair (α, α^{-1}) is (m)-localizable. It is worth mentioning that the approach is inspired by methods used in [9]. In particular, we introduce an analytic version of the Laurent tensor algebra associated with a topologically invertible bimodule.
5. The most general case, $A[t; \alpha, \delta]$, is still out of reach, unfortunately.

Let us present the main results of this paper. For a $\hat{\otimes}$ -algebra A and an A - $\hat{\otimes}$ -module M let us denote their Arens-Michael envelopes by \widehat{A} and \widehat{M} , respectively (see Definition 1.5 and Definition 1.9).

- (1) First of all, we define the notion of topologically invertible $\hat{\otimes}$ -bimodules, and we utilize several ideas from [10] in order to understand the following problem: if M and M^{-1} are invertible A -modules, is it true that \widehat{M} and $\widehat{M^{-1}}$ are topologically invertible over \widehat{A} ? We prove that this is true in a particular case (see Theorem 2.11), but the general case remains open.
- (2) Given a pair of topologically inverse $\hat{\otimes}$ -bimodules $\tilde{M}, \tilde{M}^{-1}$ over a $\hat{\otimes}$ -algebra \tilde{A} , we define the topological version of the Laurent tensor algebra via universal property, similarly to the algebraic case. We manage to prove that it exists (see Theorem 2.14) and it is unique in the metrizable setting. Moreover, we show that if A is an associative algebra, and M, M^{-1} are inverse bimodules, then, under reasonable conditions, the Arens-Michael envelope of the algebraic Laurent algebra $L_A(M)$ is the topological Laurent algebra corresponding to the Arens-Michael envelopes $\widehat{A}, \widehat{M}, \widehat{M^{-1}}$ (see Proposition 2.15).
- (3) If $\alpha : A \rightarrow A$ is an automorphism, we can define the A -bimodule A_α as follows: it coincides with A as a left A -module and for any $x \in A_\alpha$ and $a \in A$ we have $x \cdot a = x\alpha(a)$, where \cdot denotes the action of A on A_α via right multiplication. In our paper we prove that the Arens-Michael envelope of $A[t, t^{-1}; \alpha]$ is the topological Laurent algebra of A_α over A (see Corollary 2.16).
- (4) Finally, if A is a Fréchet-Arens-Michael algebra and α is an arbitrary continuous automorphism of A , we provide an explicit power series representation for the topological Laurent algebra of A_α over A (see Corollary 3.7).

The structure of this paper is as follows. In Section 1 we give the definitions of different types of topological algebras and their Arens-Michael envelopes. Subsections 2.1 and 2.2 are devoted to defining several algebraic constructions, in particular, the Laurent tensor algebra $L_A(M)$ of an invertible bimodule. Throughout the next subsections we introduce the analytic analogue of $L_A(M)$, formulate and prove one of the main results of the paper.

In Section 3 we tackle the special case $M = A_\alpha$ and describe the topological Laurent tensor algebra $\widehat{L}_A(A_\alpha)$ as explicitly as possible for any Fréchet-Arens-Michael algebra A and any continuous automorphism $\alpha : A \rightarrow A$. In Section 4 we state some open problems related to Arens-Michael envelopes. We also provide some examples in Appendix A.

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1. Definitions

1.1. Basic notions

All algebras are considered over \mathbb{C} , and assumed to be unital, associative.

Definition 1.1 *Let A be a locally convex space with a multiplication $\mu : A \times A \rightarrow A$, such that (A, μ) is an algebra.*

(1) *If μ is separately continuous then A is called a locally convex algebra.*

(2) *If A is a Fréchet space and μ is separately continuous then we call A a Fréchet algebra.*

(3) *If A is a complete locally convex space, and μ is jointly continuous then A is called a $\hat{\otimes}$ -algebra.*

Definition 1.2 *A locally convex algebra A is called m -convex if the topology on it can be defined by a family of submultiplicative seminorms.*

Definition 1.3 *A complete locally m -convex algebra is called an Arens-Michael algebra.*

For us it will be important to keep in mind the following examples of Arens-Michael algebras:

1. Any Banach algebra is an Arens-Michael algebra.
2. For any $n \in \mathbb{N}$ the algebra $\mathcal{O}(\mathbb{C}^n)$ of holomorphic functions on \mathbb{C}^n , endowed with the compact-open topology, is an Arens-Michael algebra.
3. For any locally compact space X the algebra of continuous functions $C(X)$, endowed with the compact-open topology, is an Arens-Michael algebra.

Also, keep in mind that every Arens-Michael algebra is a $\hat{\otimes}$ -algebra.

Definition 1.4 *Let A be a $\hat{\otimes}$ -algebra and let M be a complete locally convex space equipped with an A -bimodule structure such that the natural maps $A \times M \rightarrow M$ and $M \times A \rightarrow M$ are jointly continuous. Then M is called a A - $\hat{\otimes}$ -bimodule.*

For a detailed introduction to the theory of locally convex spaces and algebras the reader can see [13], [7], [8] or [6].

1 **1.2. Arens-Michael envelopes**

Definition 1.5 ([5]) *Let A be an algebra. An Arens-Michael envelope of A is a pair (\widehat{A}, i_A) , where \widehat{A} is an Arens-Michael algebra and $i_A : A \rightarrow \widehat{A}$ is an algebra homomorphism, satisfying the following universal property: for any Arens-Michael algebra B and algebra homomorphism $\varphi : A \rightarrow B$ there exists a unique continuous algebra homomorphism $\widehat{\varphi} : \widehat{A} \rightarrow B$ extending φ , i.e. $\varphi = \widehat{\varphi} \circ i_A$:*

$$\begin{array}{ccc} \widehat{A} & \overset{\widehat{\varphi}}{\dashrightarrow} & B \\ i_A \uparrow & \nearrow \varphi & \\ A & & \end{array}$$

2 The Arens-Michael envelope of an algebra always exists and is unique up to a topological isomorphism,
 3 it is isomorphic to the completion of A with respect to the family of all submultiplicative seminorms on A .

4 We have already mentioned Theorem 0.3, which serves as a fundamental example of a computation of
 5 the Arens-Michael envelope. Here are some other important examples, which we borrow from [12] and [9]:

6 **Example 1.6** *Denote the free algebra with generators ξ_1, \dots, ξ_n over \mathbb{C} by F_n . Then its Arens-Michael*
 7 *envelope is a locally convex algebra, looks as follows:*

$$\mathcal{F}_n := \left\{ a = \sum_{w \in W_n} a_w \xi^w : \|a\|_\rho = \sum_{w \in W_n} |a_w| \rho^{|w|} < \infty \forall 0 < \rho < \infty \right\}.$$

8 *In particular, \mathcal{F}_n is a nuclear Fréchet algebra.*

9 **Example 1.7** *Let \mathfrak{g} be a finite-dimensional semisimple Lie algebra. The Arens-Michael envelope of $U(\mathfrak{g})$ is*
 10 *isomorphic to the direct product $\prod_{V \in \widehat{\mathfrak{g}}} \text{Mat}(V)$, where $\widehat{\mathfrak{g}}$ is the set of the equivalence classes of finite-dimensional*
 11 *irreducible representations of \mathfrak{g} .*

12 Sometimes the Arens-Michael envelope of an algebra is isomorphic to the zero algebra:

13 **Example 1.8** *Suppose that A is an algebra generated by x and y with the single relation $xy - yx = 1$. Then*
 14 *$\widehat{A} = 0$, because an arbitrary non-zero Banach algebra B cannot contain elements $x, y \in B$ such that $[x, y] = 1$.*

15 The definition of Arens-Michael envelopes can be given in case of bimodules, too.

16 **Definition 1.9** ([9]) *Let A be a algebra and suppose that M is an A -bimodule. Then an Arens-Michael*
 17 *envelope of M is a pair (\widehat{M}, i_M) , where \widehat{M} is a \widehat{A} - $\widehat{\otimes}$ -bimodule and $i_M : M \rightarrow \widehat{M}$ is a A -bimodule*
 18 *homomorphism, which satisfies the following universal property: for any \widehat{A} - $\widehat{\otimes}$ -bimodule N and A -bimodule*
 19 *homomorphism $f : M \rightarrow N$ there exists a unique continuous \widehat{A} - $\widehat{\otimes}$ -bimodule homomorphism $\widehat{f} : \widehat{M} \rightarrow N$ which*
 20 *extends f :*

$$\begin{array}{ccc} \widehat{M} & \overset{\widehat{f}}{\dashrightarrow} & N \\ i_M \uparrow & \nearrow f & \\ M & & \end{array}$$

1 In this paper we will use [11, Proposition 6.1], which, basically, states that the Arens-Michael functor
 2 commutes with quotients:

3 **Theorem 1.10** *Suppose that A is an algebra and $I \subset A$ is a two-sided ideal. Denote by J the closure of*
 4 *$i_A(I)$ in \widehat{A} . Then J is a closed two-sided ideal in \widehat{A} and the induced homomorphism $A/I \rightarrow \widehat{A}/J$ extends to*
 5 *a topological algebra isomorphism*

$$\widehat{(A/I)} \simeq (\widehat{A}/J)^\sim,$$

6 where \widetilde{A} denotes the completion of A as a locally convex space.

7 Moreover, if \widehat{A} is a Fréchet algebra, then we do not need to complete the quotient, so we have

$$\widehat{(A/I)} \simeq \widehat{A}/J.$$

8 **Remark.** As a corollary from this theorem and Example 1.6 we have that the Arens-Michael algebra of any
 9 finitely generated algebra over \mathbb{C} is a Fréchet algebra.

10 2. Topological analogues of invertible bimodules and their Laurent tensor algebras

11 2.1. Some algebraic constructions

12 Firstly, let us recall the definitions of some crucial algebraic constructions, which we will use throughout this
 13 paper:

14 **Definition 2.1** *Let A be an algebra and consider an endomorphism α of A . Then we define A_α as a A -*
 15 *bimodule which coincides with A as a left A -module and $x \circ a = x\alpha(a)$ for $x \in A_\alpha$, $a \in A$. Similarly, one*
 16 *defines an A -bimodule ${}_\alpha A$.*

17 **Definition 2.2** *Let A be an algebra, $\alpha \in \text{End}(A)$ and $\delta \in \text{Der}(A, {}_\alpha A)$, or, equivalently,*

$$\delta(ab) = \alpha(a)\delta(b) + \delta(a)b \quad \forall a, b \in A.$$

18 Then the Ore extension of A with respect to α and δ is the vector space

$$A[t; \alpha, \delta] = \left\{ \sum_{i=0}^n a_i t^i : a_i \in A \right\}$$

19 with the multiplication, which is uniquely defined by the following conditions:

- 20 1. The relation $ta = \alpha(a)t + \delta(a)$ holds for any $a \in A$
- 21 2. The natural inclusions $A \hookrightarrow A[t; \alpha, \delta]$ and $\mathbb{C}[t] \hookrightarrow A[t; \alpha, \delta]$ are algebra homomorphisms.

22 Also, if $\delta = 0$ and α is invertible, then one can define the Laurent Ore extension

$$A[t, t^{-1}; \alpha] = \left\{ \sum_{i=-n}^n a_i t^i : a_i \in A \right\}$$

23 with the multiplication defined in a similar way.

1 **2.2. Invertible bimodules and the Laurent tensor algebra**

2 **Remark.** We were not able to find any published references related to Laurent tensor algebras of invertible
 3 modules in the literature, so we decided to provide a basic but **completely self-contained** exposition in this
 4 section, inspired by the unpublished notes of R.C.Cannings and M.P.Holland, “Tensor Algebras of Invertible
 5 Bimodules”.

6 **Definition 2.3** Let a A be an algebra and consider an A -bimodule M . Then M is called an invertible A -
 7 bimodule if there exist an A -bimodule M^{-1} together with A -bimodule isomorphisms $i_1 : M \otimes_A M^{-1} \simeq A$ and
 8 $i_2 : M^{-1} \otimes_A M \simeq A$ (which we shall call convolutions) such that the following diagrams commute:

$$\begin{array}{ccc}
 M \otimes_A M^{-1} \otimes_A M & \xrightarrow{Id_M \otimes i_2} & M \otimes_A A & & M^{-1} \otimes_A M \otimes_A M^{-1} & \xrightarrow{Id_M \otimes i_1} & M^{-1} \otimes_A A \\
 \downarrow i_1 \otimes Id_M & & \downarrow m \otimes a \rightarrow ma & & \downarrow i_2 \otimes Id_M & & \downarrow n \otimes a \rightarrow na \\
 A \otimes_A M & \xrightarrow{a \otimes m \rightarrow am} & M & & A \otimes_A M^{-1} & \xrightarrow{a \otimes n \rightarrow an} & M^{-1}
 \end{array} \tag{2.1}$$

9 With any A -bimodule M one associates the tensor algebra $T_A(M)$:

$$T_A(M) := A \oplus \bigoplus_{n \in \mathbb{N}} M^{\otimes n},$$

10 where $M^{\otimes n} := \underbrace{M \otimes_A \dots \otimes_A M}_{n \text{ times}}$. In turn, for every invertible A -bimodule we can define a complex vector space
 11 which will be denoted by $L_A(M)$:

$$L_A(M) := \bigoplus_{n \in \mathbb{Z}} M^{\otimes n}, \tag{2.2}$$

12 where $M^{\otimes -n} := (M^{-1})^{\otimes n}$ and $M^{\otimes 0} := A$.

13 The elements belonging to $M^{\otimes n}$ for some $n \in \mathbb{Z}$ will be called *homogeneous* of degree n . The following
 14 proposition states that $L_A(M)$ admits a natural algebra structure:
 15

16 **Proposition 2.4** Suppose that A is an algebra and M is an invertible A -bimodule. Then $L_A(M)$ admits
 17 a unique multiplication μ such that $(L_A(M), \mu)$ becomes an associative algebra and μ satisfies the following
 18 conditions:

19 (1) the natural inclusions $j_M : T_A(M) \rightarrow L_A(M)$ and $j_{M^{-1}} : T_A(M^{-1}) \rightarrow L_A(M)$ are algebra homomor-
 20 phisms.

21 (2) for any $m \in M$ and $n \in M^{-1}$ we have $m \cdot n = i_1(m \otimes n)$ and $n \cdot m = i_2(n \otimes m)$.

22 **Proof** It suffices to define the multiplication on the homogeneous elements of $L_A(M)$. Fix
 23 $m_1 \otimes \dots \otimes m_k \in M \otimes \dots \otimes M$ and $n_1 \otimes \dots \otimes n_l \in M^{-1} \otimes \dots \otimes M^{-1}$. Then define

$$(m_1 \otimes \dots \otimes m_k) \cdot (n_1 \otimes \dots \otimes n_l) = (m_1 \otimes \dots \otimes m_{k-1} i_1(m_k \otimes n_1)) \cdot (n_2 \otimes \dots \otimes n_l)$$

24 and

$$(n_1 \otimes \dots \otimes n_l) \cdot (m_1 \otimes \dots \otimes m_k) = (n_1 \otimes \dots \otimes n_{l-1} i_2(n_l \otimes m_1)) \cdot (m_2 \otimes \dots \otimes m_k),$$

1 then we repeat the process until we get a homogeneous element of $L_A(M)$. The associativity of the resulting
 2 algebra is a straightforward corollary from the commutativity of (2.1) in the Definition 2.3. \square

3 Let us call $L_A(M)$ the *Laurent tensor algebra* of an invertible bimodule M . The following proposition
 4 immediately follows from the constructions of $T_A(M)$ and $L_A(M)$.

5 **Proposition 2.5** *Suppose that A is an algebra and α is an automorphism of A .*

6 (1) A_α and $A_{\alpha^{-1}}$ are inverse A -bimodules with respect to the maps

$$i_1(a \otimes b) = a\alpha(b), \quad i_2(b \otimes a) = b\alpha^{-1}(a).$$

7 (2) Moreover, $T(A_\alpha) \simeq A[t; \alpha]$, $L(A_\alpha) \simeq A[t, t^{-1}; \alpha]$.

8 The algebra $L_A(M)$ satisfies the following universal property:

9 **Definition 2.6** *Let A be an algebra and consider an A -algebra B with respect to a homomorphism $\theta : A \rightarrow B$
 10 together with A -bimodule homomorphisms $\alpha : M \rightarrow B$, $\beta : M^{-1} \rightarrow B$. Then we will call the triple of
 11 morphisms $(\theta, \alpha, \beta, B)$ compatible if and only if the following diagram is commutative:*

$$\begin{array}{ccccc} M \otimes_A M^{-1} & \xrightarrow{i_1} & A & \xleftarrow{i_2} & M^{-1} \otimes_A M \\ \downarrow \alpha \otimes \beta & & \downarrow \theta & & \downarrow \beta \otimes \alpha \\ B \otimes_A B & \xrightarrow{m} & B & \xleftarrow{m} & B \otimes_A B \end{array} \quad (2.3)$$

12 **Proposition 2.7** *The triple of morphisms $(i_A, i_M, i_{M^{-1}}, L_A(M))$, where all morphisms are tautological inclu-
 13 sions into $L_A(M)$, is a universal compatible triple, i.e. for any other algebra B and any compatible triple of
 14 morphisms $(\theta, \alpha, \beta, B)$ there exists a unique A -algebra homomorphism $f : L_A(M) \rightarrow B$ such that the following
 15 diagrams commute:*

$$\begin{array}{ccc} \begin{array}{ccc} L_A(M) & \xrightarrow{f} & B \\ i_A \uparrow & \nearrow \theta & \\ A & & \end{array} & \begin{array}{ccc} L_A(M) & \xrightarrow{f} & B \\ i_M \uparrow & \nearrow \alpha & \\ M & & \end{array} & \begin{array}{ccc} L_A(M) & \xrightarrow{f} & B \\ i_{M^{-1}} \uparrow & \nearrow \beta & \\ M^{-1} & & \end{array} \end{array} \quad (2.4)$$

16 **Proof** It suffices to check the existence, as the uniqueness will follow as a standard category-theoretic argument.

17 And the existence is straightforward: for every $m_1 \otimes \cdots \otimes m_k \in M^{\otimes k}$ define

$$f(m_1 \otimes \cdots \otimes m_k) = \alpha(m_1) \dots \alpha(m_k),$$

18 and for $n_1 \otimes \cdots \otimes n_l \in M^{\otimes -l}$ we define

$$f(n_1 \otimes \cdots \otimes n_l) = \beta(n_1) \dots \beta(n_l),$$

19 and $f(a) = \theta(a)$ for every $a \in A$.

20 The commutativity of (2.3) ensures that f is a well-defined homomorphism of A -algebras. And the
 21 diagrams (2.4) commute due to the construction of f . \square

2.3. Topologically invertible bimodules

Now, using the language of locally convex vector spaces, we will construct topological versions of the notions we described above.

Definition 2.8 Let A be a $\hat{\otimes}$ -algebra and M be an A - $\hat{\otimes}$ -bimodule. Then we will call M a topologically invertible A - $\hat{\otimes}$ -bimodule if there exist an A - $\hat{\otimes}$ -bimodule M^{-1} and two A - $\hat{\otimes}$ -bimodule topological isomorphisms $i_1 : M \hat{\otimes}_A M^{-1} \simeq A$ and $i_2 : M^{-1} \hat{\otimes}_A M \simeq A$, such that the following diagrams commute:

$$\begin{array}{ccc}
 M \hat{\otimes}_A M^{-1} \hat{\otimes}_A M & \xrightarrow{Id_M \hat{\otimes} i_2} & M \hat{\otimes}_A A & & M^{-1} \hat{\otimes}_A M \hat{\otimes}_A M^{-1} & \xrightarrow{Id_M \hat{\otimes} i_1} & M^{-1} \hat{\otimes}_A A \\
 \downarrow i_1 \hat{\otimes} Id_M & & \downarrow m \otimes r \rightarrow mr & & \downarrow i_2 \hat{\otimes} Id_M & & \downarrow n \otimes r \rightarrow nr \\
 A \hat{\otimes}_A M & \xrightarrow{r \otimes m \rightarrow rm} & M & & A \hat{\otimes}_A M^{-1} & \xrightarrow{r \otimes n \rightarrow rn} & M^{-1}
 \end{array} \tag{2.5}$$

The following proposition is the topological version of the Proposition 2.5.

Proposition 2.9 Let A be a $\hat{\otimes}$ -algebra and suppose that α is an automorphism of A . Then A_α and $A_{\alpha^{-1}}$ are topologically inverse A - $\hat{\otimes}$ -bimodule with respect to the maps

$$i_1(a \otimes b) = a\alpha(b), \quad i_2(b \otimes a) = b\alpha^{-1}(a).$$

More information on topologically invertible bimodules can be found in [10].

There is a natural question related to Arens-Michael envelopes: is it true that the Arens-Michael envelope of an invertible bimodule is topologically invertible? At the moment we can state a conjecture:

Conjecture 2.10 Suppose that A is an algebra and M is an invertible A -bimodule. Then there exist topological A - $\hat{\otimes}$ -bimodule isomorphisms $\hat{i}_1 : \widehat{M \hat{\otimes}_A M^{-1}} \rightarrow \widehat{A}$ and $\hat{i}_2 : \widehat{M^{-1} \hat{\otimes}_A M} \rightarrow \widehat{A}$, satisfying the following conditions:

- (1) \widehat{M} is a topologically invertible \widehat{A} - $\hat{\otimes}$ -bimodule w.r.t. \hat{i}_1 and \hat{i}_2 .
- (2) The following diagram is commutative:

$$\begin{array}{ccccc}
 M \otimes_A M^{-1} & \xrightarrow{i_1} & A & \xleftarrow{i_2} & M^{-1} \otimes_A M \\
 \downarrow i_M \otimes i_{M^{-1}} & & \downarrow i_A & & \downarrow i_{M^{-1}} \otimes i_M \\
 \widehat{M} \hat{\otimes}_A \widehat{M^{-1}} & \xrightarrow{\hat{i}_1} & \widehat{A} & \xleftarrow{\hat{i}_2} & \widehat{M^{-1}} \hat{\otimes}_A \widehat{M}
 \end{array} \tag{2.6}$$

where the left arrow maps $a \otimes b$ to $i_M(a) \otimes i_{M^{-1}}(b)$, and the right arrow maps $b \otimes a$ to $i_{M^{-1}}(b) \otimes i_M(a)$.

It turns out that there is a particular case in which, at least, the first statement of the above conjecture holds.

Remark. Here we consider the Arens-Michael envelopes of $\hat{\otimes}$ -algebras and $\hat{\otimes}$ -bimodules, see [9, Section 3] for the details.

Theorem 2.11 Consider a $\hat{\otimes}$ -algebra A and a pair of topologically invertible A - $\hat{\otimes}$ -bimodules M, M^{-1} . Suppose that the following condition is satisfied:

$$M \hat{\otimes}_A \widehat{A} \simeq \widehat{A} \hat{\otimes}_A M, M^{-1} \hat{\otimes}_A \widehat{A} \simeq \widehat{A} \hat{\otimes}_A M^{-1} \text{ as } A\text{-}\hat{\otimes}\text{-bimodules.}$$

1 Then \widehat{M} and \widehat{M}^{-1} are topologically invertible \widehat{A} -bimodules.

2 **Proof** This is an immediate corollary of the fact that $\widehat{M} \simeq \widehat{A} \widehat{\otimes}_A M \widehat{\otimes}_A \widehat{A}$, which is the statement of [9, Remark
3 3.8], and [10, Proposition 10.4]. The idea is to write the following chain of \widehat{A} - $\widehat{\otimes}$ -bimodule isomorphisms:

$$\begin{aligned} \widehat{M} \widehat{\otimes}_{\widehat{A}} \widehat{M}^{-1} &\stackrel{R3.8}{\simeq} \widehat{A} \widehat{\otimes}_A M \widehat{\otimes}_A \widehat{A} \widehat{\otimes}_{\widehat{A}} \widehat{A} \widehat{\otimes}_A M^{-1} \widehat{\otimes}_A \widehat{A} \simeq \widehat{A} \widehat{\otimes}_A M \widehat{\otimes}_A \widehat{A} \widehat{\otimes}_A M^{-1} \widehat{\otimes}_A \widehat{A} \stackrel{P10.4(ii)}{\simeq} \\ &\simeq \widehat{A} \widehat{\otimes}_A M \widehat{\otimes}_A M^{-1} \widehat{\otimes}_A \widehat{A} \widehat{\otimes}_A \widehat{A} \stackrel{(1)}{\simeq} \widehat{A} \widehat{\otimes}_A M \widehat{\otimes}_A M^{-1} \widehat{\otimes}_A \widehat{A} \xrightarrow{i_1} \widehat{A} \widehat{\otimes}_A A \widehat{\otimes}_A \widehat{A} \stackrel{R3.8}{\simeq} \widehat{A}. \end{aligned} \quad (2.7)$$

4 In a similar fashion we can show that the associativity diagrams commute. \square

5 As a corollary, consider an algebra A and a pair of invertible bimodules M, M^{-1} of at most countable
6 dimension. Then [9, Proposition 2.3] implies the following statements:

7 (1) A_s is a $\widehat{\otimes}$ -algebra, $M_s, (M^{-1})_s$ are A_s - $\widehat{\otimes}$ -bimodules.

8 (2) These bimodules are topologically invertible as A_s -bimodules.

9 Suppose that A_s, M_s and $(M^{-1})_s$ satisfy the conditions of the Theorem 2.11. Then \widehat{M} and \widehat{M}^{-1} are
10 topologically invertible \widehat{A} -bimodules.

11 **Proposition 2.12** Conjecture 2.10 holds in the case of $M = A_\alpha$, where A is an arbitrary associative algebra
12 and $\alpha \in \text{Aut}(A)$.

13 **Proof** We refer to [9, Corollary 5.6] which states that $\widehat{A}_\alpha \simeq (\widehat{A})_\alpha$. Taking the necessary isomorphisms from
14 the Proposition 2.9, we get (1), and the following computation proves the commutativity of the left side of (2.6):

$$\widehat{i}_1(i_A(a) \otimes i_A(b)) = i_A(a\alpha(b)) = i_A \circ i_1(a \otimes b) \quad (a \in M, b \in M^{-1})$$

15 A similar argument also shows that the right quadrant of the diagram (2.6) is commutative too. \square

16 2.4. Topological Laurent tensor algebras

17 Fix an Arens-Michael algebra A and a pair of topologically inverse A - $\widehat{\otimes}$ -bimodules M and M^{-1} .

18 **Definition 2.13** Let B be an Arens-Michael algebra, which is an A -algebra with respect to a continuous homo-
19 morphism $\theta : A \rightarrow B$, also let $\alpha : M \rightarrow B, \beta : M^{-1} \rightarrow B$ be continuous A - $\widehat{\otimes}$ -bimodule homomorphisms. Then
20 we will call the triple $(\theta, \alpha, \beta, B)$ topologically compatible if and only if the following diagram is commutative:

$$\begin{array}{ccccc} M \widehat{\otimes}_A M^{-1} & \xrightarrow{i_1} & A & \xleftarrow{i_2} & M^{-1} \widehat{\otimes}_A M \\ \downarrow \alpha \widehat{\otimes} \beta & & \downarrow \theta & & \downarrow \beta \widehat{\otimes} \alpha \\ B \widehat{\otimes}_A B & \xrightarrow{m} & B & \xleftarrow{m} & B \widehat{\otimes}_A B \end{array} \quad (2.8)$$

21 Now we will formulate and prove one of the main theorems of the paper:

1 **Theorem 2.14** Let A be a Fréchet-Arens-Michael algebra and consider topologically inverse Fréchet A - $\hat{\otimes}$ -
 2 bimodules M, M^{-1} . Then there exist an Arens-Michael algebra $\widehat{L}_A(M)$ and a topologically compatible triple
 3 of morphisms $(\theta, \alpha, \beta, \widehat{L}_A(M))$ that satisfies the following universal property: for every Arens-Michael algebra
 4 B and a topologically compatible triple of morphisms $(\theta', \alpha', \beta', B)$ there exists a unique continuous A -algebra
 5 homomorphism $f : \widehat{L}_A(M) \rightarrow B$ such that the following diagrams commute:

$$\begin{array}{ccc}
 \widehat{L}_A(M) & \xrightarrow{f} & B \\
 \theta \uparrow & \nearrow & \\
 A & &
 \end{array}
 \quad
 \begin{array}{ccc}
 \widehat{L}_A(M) & \xrightarrow{f} & B \\
 \alpha \uparrow & \nearrow & \\
 M & &
 \end{array}
 \quad
 \begin{array}{ccc}
 \widehat{L}_A(M) & \xrightarrow{f} & B \\
 \beta \uparrow & \nearrow & \\
 M^{-1} & &
 \end{array}
 \quad (2.9)$$

6 We will call it the topological(or analytic) Laurent tensor algebra of the A - $\hat{\otimes}$ -bimodule M .

7 The proof of the existence of the universal object will be given in the next subsection. What we want to
 8 do now is to establish the connection between analytic Laurent tensor algebras and Arens-Michael envelopes.

9 **Proposition 2.15** Now suppose that A is an algebra and M, M^{-1} is a pair of (algebraically) inverse A -
 10 bimodules. Suppose that the following condition holds for \widehat{A}, \widehat{M} and \widehat{M}^{-1} :

11 (1) The underlying LCS of \widehat{A}, \widehat{M} and \widehat{M}^{-1} are Fréchet spaces.

12 (2) \widehat{M} and \widehat{M}^{-1} are topologically inverse as \widehat{A} - $\hat{\otimes}$ -bimodules which satisfy Conjecture 2.10.

13 Then, if $(\theta, \alpha, \beta, \widehat{L}_{\widehat{A}}(\widehat{M}))$ is the resulting topologically compatible triple in Theorem 2.14, then $\widehat{L}_A(\widehat{M}) \simeq \widehat{L}_{\widehat{A}}(\widehat{M})$.

14 **Proof** Firstly we need to construct an algebra homomorphism $i : L_A(M) \rightarrow \widehat{L}_{\widehat{A}}(\widehat{M})$. Consider the following
 15 morphisms: $\theta i_A : A \rightarrow \widehat{A} \rightarrow \widehat{L}_{\widehat{A}}(\widehat{M})$, $\alpha i_M : M \rightarrow \widehat{M} \rightarrow \widehat{L}_{\widehat{A}}(\widehat{M})$ and $\beta i_{M^{-1}} : M^{-1} \rightarrow \widehat{M}^{-1} \rightarrow \widehat{L}_{\widehat{A}}(\widehat{M})$. It turns
 16 out that this triple of morphisms is (algebraically) compatible, however, this statement is not as obvious as one
 17 might think: look at the diagram, commutativity of which we aim to prove:

$$\begin{array}{ccccc}
 M \otimes_A M^{-1} & \xrightarrow{i_1} & A & \xleftarrow{i_2} & M^{-1} \otimes_A M \\
 \downarrow \alpha i_M \otimes \beta i_{M^{-1}} & & \downarrow \theta i_A & & \downarrow \beta i_{M^{-1}} \otimes \alpha i_M \\
 \widehat{L}_{\widehat{A}}(\widehat{M}) \otimes_A \widehat{L}_{\widehat{A}}(\widehat{M}) & \xrightarrow{m} & \widehat{L}_{\widehat{A}}(\widehat{M}) & \xleftarrow{m} & \widehat{L}_{\widehat{A}}(\widehat{M}) \otimes_A \widehat{L}_{\widehat{A}}(\widehat{M})
 \end{array} \quad (2.10)$$

18 Notice that we deal with the algebraic tensor product of $\widehat{L}_{\widehat{A}}(\widehat{M})$, not with completed projective tensor product.

19 However, we can write

$$\begin{aligned}
 \theta i_A \circ i_1(x \otimes y) &\stackrel{2.6}{=} \theta \circ \hat{i}_1(i_M(x) \otimes i_{M^{-1}}(y)) \stackrel{2.8}{=} m \circ (\alpha \hat{\otimes} \beta)(i_M(x) \otimes i_{M^{-1}}(y)) = \\
 &= m \circ \varphi \circ (\alpha i_M \otimes i_{M^{-1}} \beta)(x \otimes y) = m(\alpha i_M \otimes i_{M^{-1}} \beta)(x \otimes y),
 \end{aligned}$$

20 where

$$\varphi : \widehat{L}_{\widehat{A}}(\widehat{M}) \otimes_A \widehat{L}_{\widehat{A}}(\widehat{M}) \rightarrow \widehat{L}_{\widehat{A}}(\widehat{M}) \hat{\otimes}_{\widehat{A}} \widehat{L}_{\widehat{A}}(\widehat{M}), \quad \varphi(b_1 \otimes b_2) = b_1 \otimes b_2.$$

1 If we denote the algebra $\widehat{L}_{\widehat{A}}(\widehat{M})$ by B , then the argument can be illustrated by the following three-dimensional
 2 diagram:

$$\begin{array}{ccccc}
 & M \otimes_A M^{-1} & \xrightarrow{i_1} & A & \xleftarrow{i_2} & M^{-1} \otimes_A M \\
 & \swarrow & & \swarrow i_A & & \swarrow \\
 \widehat{M} \widehat{\otimes}_{\widehat{A}} \widehat{M}^{-1} & \xrightarrow{\widehat{i}_1} & \widehat{A} & \xleftarrow{\widehat{i}_2} & \widehat{M}^{-1} \widehat{\otimes}_{\widehat{A}} \widehat{M} & \\
 \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
 B \otimes_A B & \xrightarrow{m} & B & \xleftarrow{m} & B \otimes_A B & \\
 \downarrow \varphi & \downarrow & \downarrow Id_B & \downarrow & \downarrow \varphi & \\
 B \widehat{\otimes}_{\widehat{A}} B & \xrightarrow{m} & B & \xleftarrow{m} & B \widehat{\otimes}_{\widehat{A}} B &
 \end{array} \tag{2.11}$$

3 And if the triple $(\theta i_A, \alpha i_M, \beta i_{M^{-1}}, \widehat{L}_{\widehat{A}}(\widehat{M}))$ is compatible, we get the morphism $i : L_A(M) \rightarrow \widehat{L}_{\widehat{A}}(\widehat{M})$.

4 Secondly, we need to prove that the pair $(\widehat{L}_{\widehat{A}}(\widehat{M}), i)$ satisfies the universal property. Let us consider
 5 the following continuous morphisms: $\widehat{\varphi}|_{\widehat{A}} : \widehat{A} \rightarrow X$, $\widehat{\varphi}|_{\widehat{M}} : \widehat{M} \rightarrow X$ and $\widehat{\varphi}|_{\widehat{M}^{-1}} : \widehat{M}^{-1} \rightarrow X$, which come
 6 from the respective universal properties. The first map is an algebra homomorphism, and the latter are \widehat{A} -
 7 bimodule morphisms. The resulting triple is topologically compatible, and the argument is basically the same
 8 as the one we gave in the first step of the proof, we only need to keep in mind that elementary tensors span a
 9 dense subspace in a completed projective tensor products of locally convex spaces From that we get a unique
 10 \widehat{A} -algebra morphism $\widehat{\varphi} : \widehat{L}_{\widehat{A}}(\widehat{M}) \rightarrow X$. The last thing that is left is to show that it really extends φ . However,
 11 if we restrict φ on A , M or M^{-1} , the statement holds, so it is true for $L_A(M)$. \square

12 The following is a corollary of Propositions 2.12 and 2.15.

13 **Corollary 2.16** *Suppose that A is an associative algebra with the Arens-Michael envelope which is a Fréchet
 14 algebra, and let $\alpha \in \text{Aut}(A)$ be an arbitrary algebra automorphism. Then the following isomorphism takes place:*

$$(A[t, t^{-1}; \alpha])^{\widehat{}} = L_A(\widehat{A}_\alpha) \cong \widehat{L}_{\widehat{A}}(\widehat{A}_\alpha).$$

15 2.5. Constructing the universal object

16 To prove Theorem 2.14 we need to utilize the construction of the analytic tensor algebra, described in [9].
 17 This approach is natural because the resulting topological Laurent tensor algebra is expected to contain an
 18 appropriate version of the topological tensor algebra, and this object is already defined in [9].

19 Suppose that A is an Arens-Michael algebra and M is an A - $\widehat{\otimes}$ -bimodule. Fix a directed generating
 20 family of seminorms $\{\|\cdot\|_\nu : \nu \in \Lambda\}$ on M . Consider the locally convex space

$$\widehat{T}_A(M)^+ = \left\{ (x_n) \in \prod_{i=1}^{\infty} M^{\widehat{\otimes} n} : \|(x_n)\|_{\nu, \rho} := \sum_{n=1}^{\infty} \|x_n\|_\nu^{\widehat{\otimes} n} \rho^n < \infty, \nu \in \Lambda, 0 < \rho < \infty \right\}, \tag{2.12}$$

21 where $M^{\widehat{\otimes} n} := \underbrace{M \widehat{\otimes}_A \dots \widehat{\otimes}_A M}_{n \text{ times}}$. By definition, the seminorms $\|\cdot\|_{\nu, \rho}$ generate the topology on $\widehat{T}_A(M)^+$.

1 **Definition 2.17** *The topological (analytic) tensor algebra of M is a locally convex space*

$$\widehat{T}_A(M) := A \oplus \widehat{T}_A(M)^+.$$

2 In [9] it is proven that $\widehat{T}_A(M)$ admits a multiplication which makes a natural inclusion $f : T_A(M) \rightarrow \widehat{T}_A(M)$
 3 into an algebra homomorphism and turns $\widehat{T}_A(M)$ into an Arens-Michael algebra. We also will use [9, Proposition
 4 4.8], which states that $\widehat{T}_A(M)$ is defined by its universal property.

5 Fix an Fréchet-Arens-Michael algebra A , a pair of topologically inverse Fréchet A - $\widehat{\otimes}$ -bimodules M and
 6 M^{-1} with respect to the topological A -bimodule isomorphisms $i_1 : M \widehat{\otimes}_A M^{-1} \rightarrow A$ and $i_2 : M^{-1} \widehat{\otimes}_A M \rightarrow A$.

7 Now, for any $x \in M$ and $y \in M^{-1}$ consider the elements

$$(0, 0, (x, 0) \otimes (0, y), 0, \dots) - (i_1(x \otimes y), 0, 0, \dots) \in \widehat{T}_A(M) \quad (2.13)$$

8 and

$$(0, 0, (0, y) \otimes (x, 0), 0, \dots) - (i_2(y \otimes x), 0, 0, \dots) \in \widehat{T}_A(M). \quad (2.14)$$

9 It would be reasonable to assume that these elements are equal to zero in $\widehat{T}_A(M)$. This idea serves as a
 10 motivation for the following definition:

11 **Definition 2.18** *Let $\widehat{L}_A(M)' := \widehat{T}_A(M \oplus M^{-1})/I$, where I is the closure of the two-sided ideal generated by
 12 elements of form (2.13) and (2.14) for any $x \in M$, $y \in M^{-1}$.*

13 **Remark.** Actually, this is the only place where we use the Fréchet assumption. If $\widehat{T}_A(M \oplus M^{-1})$ is not
 14 Fréchet, the quotient might not be complete, we would have to complete the resulting algebra and the following
 15 proof, in fact, will still work, however this assumption makes everything easier.

16 Let us also denote some morphisms associated with this object:

$$j_0 : A \hookrightarrow \widehat{T}_A(M \oplus M^{-1})$$

$$j_M : M \hookrightarrow M \oplus M^{-1} \rightarrow \widehat{T}_A(M \oplus M^{-1})$$

$$j_{M^{-1}} : M^{-1} \hookrightarrow M \oplus M^{-1} \rightarrow \widehat{T}_A(M \oplus M^{-1})$$

$$\pi : \widehat{T}_A(M \oplus M^{-1}) \rightarrow \widehat{L}_A(M)'$$

20 .

21 If A is an Fréchet-Arens-Michael algebra, so are $\widehat{T}_A(M \oplus M^{-1})$ and $\widehat{L}_A(M)'$. Consider the triple of
 22 morphisms $i_A = \pi \circ j_0$, $i_M = \pi \circ j_M$, $i_{M^{-1}} = \pi \circ j_{M^{-1}}$.

23 **Lemma 2.19** *The triple $(i_A, i_M, i_{M^{-1}}, \widehat{L}_A(M)')$ is topologically compatible.*

24 **Proof** We need to prove the commutativity of the following diagram:

$$\begin{array}{ccccc} M \widehat{\otimes}_A M^{-1} & \xrightarrow{i_1} & A & \xleftarrow{i_2} & M^{-1} \widehat{\otimes}_A M \\ \downarrow i_M \otimes i_{M^{-1}} & & \downarrow i_A & & \downarrow i_{M^{-1}} \otimes i_M \\ \widehat{L}_A(M)' \widehat{\otimes}_A \widehat{L}_A(M)' & \xrightarrow{m} & \widehat{L}_A(M)' & \xleftarrow{m} & \widehat{L}_A(M)' \widehat{\otimes}_A \widehat{L}_A(M)' \end{array} \quad (2.15)$$

1 For every $x \in M$ and $y \in M^{-1}$ we can consider an elementary tensor $x \otimes y \in M \hat{\otimes}_A M^{-1}$.

$$\begin{aligned} m \circ (i_M \otimes i_{M^{-1}})(x \otimes y) &= m((0, (x, 0), 0, 0, \dots) \cdot (0, (0, y), 0, 0, \dots)) = (0, 0, (x, 0) \otimes (0, y), 0, \dots) \\ &= (i_1(x \otimes y), 0, 0, \dots) = i_A \circ i_1(x \otimes y). \end{aligned}$$

2 We finish the proof by using the fact that elementary tensors span a dense subspace in $M \hat{\otimes}_A M^{-1}$. □

3 **Proposition 2.20** $\widehat{L}_A(M)' \simeq \widehat{L}_A(M)$.

4 **Proof** We must check that the triple $(i_A, i_M, i_{M^{-1}}, \widehat{L}_A(M)')$ satisfies the universal property. Suppose that
 5 B is a Banach algebra and that $(\theta, \gamma, \delta, B)$ is an topologically compatible triple. Consider the direct sum of
 6 γ and δ : $\gamma \oplus \delta : M \oplus M^{-1} \rightarrow B$. It is a continuous A -bimodule morphism which, by [9, Proposition 4.8],
 7 can be uniquely extended to a continuous A -algebra morphism $\varphi : \widehat{T}_A(M \oplus M^{-1}) \rightarrow B$. From the fact that
 8 $(\theta, \gamma, \delta, B)$ is topologically compatible it easily follows that $\varphi(I) = 0$, so, in fact, we obtain a unique continuous
 9 A -algebra homomorphism $\tilde{\varphi} : \widehat{L}_A(M)' \rightarrow B$. Due to the construction it extends θ , γ and δ . □

10 This concludes the proof of the Theorem 2.14.

11 3. The case of $M = A_\alpha$

12 3.1. Localizable linear maps between locally convex spaces

13 **Definition 3.1** Let A be an Arens-Michael algebra and let $\mathcal{F} \subset \mathcal{L}(A)$ be a family of continuous linear maps
 14 $A \rightarrow A$.

15 Then \mathcal{F} is called an m -localizable family (see [9]) if the topology on A can be defined by a family
 16 of submultiplicative seminorms $\{\|\cdot\|_\lambda\}_{\lambda \in \Lambda}$, satisfying the following property: for every $T \in \mathcal{F}$ there exists a
 17 constant $C_T > 0$, such that

$$\|Ta\|_\lambda \leq C_T \|a\|_\lambda \quad \text{for every } a \in A.$$

18 An operator $T \in \mathcal{L}(E)$ is called m -localizable $\iff \{T\}$ is a m -localizable family.

19 Suppose now that A is an Arens-Michael algebra, α is a continuous automorphism of A , such that
 20 $\{\alpha, \alpha^{-1}\}$ is a m -localizable family. Fix a generating family of seminorms $\{\|\cdot\|_\lambda : \lambda \in \Lambda\}$, then we can define
 21 the following vector space:

$$\mathcal{O}(\mathbb{C}^\times, A) := \left\{ f = \sum_{i=-\infty}^{\infty} a_i t^i : \|f\|_{\lambda, \rho} := \sum_{i=-\infty}^{\infty} \|a_i\|_\lambda \rho^i < \infty \forall \lambda \in \Lambda, 0 < \rho < \infty \right\}. \quad (3.1)$$

22 This vector space with topology, generated by $\|\cdot\|_{\lambda, \rho}$ becomes a complete locally convex space. Moreover, [9,
 23 Lemma 4.12] and [9, Proposition 4.15] state that in our case $\mathcal{O}(\mathbb{C}^\times, A)$ admits a unique multiplication, which
 24 is compatible with α (i.e. $ta = \alpha(a)t, t^{-1}a = \alpha^{-1}(a)t^{-1}$ for every $a \in A$) and makes $\mathcal{O}(\mathbb{C}^\times, A)$ into an
 25 Arens-Michael algebra, which is denoted by $\mathcal{O}(\mathbb{C}^\times, A; \alpha)$.

26 **Proposition 3.2** Under assumptions made above, $\widehat{L}_A(A_\alpha) \simeq \mathcal{O}(\mathbb{C}^\times, A; \alpha)$.

1 **Proof** Firstly, we must consider natural morphisms

$$\begin{aligned} i_A &: A \hookrightarrow \mathcal{O}(\mathbb{C}^\times, A; \alpha), \\ i_{A_\alpha} &: A_\alpha \rightarrow \mathcal{O}(\mathbb{C}^\times, A; \alpha), i_{A_\alpha}(1) = t \\ i_{A_{\alpha^{-1}}} &: A_{\alpha^{-1}} \rightarrow \mathcal{O}(\mathbb{C}^\times, A; \alpha), i_{A_{\alpha^{-1}}}(1) = t^{-1}. \end{aligned}$$

2 We aim to prove that the triple of morphisms $(i_A, i_{A_\alpha}, i_{A_{\alpha^{-1}}}, \mathcal{O}(\mathbb{C}^\times, A; \alpha))$ is a topologically compatible triple,
3 which satisfies the universal property. The first part is obviously true due to the construction of $\mathcal{O}(\mathbb{C}^\times, A; \alpha)$.

4 Suppose that $(\theta, \alpha, \beta, B)$ is another topologically compatible triple. Notice that

$$\alpha(1)\beta(1) = \beta(1)\alpha(1) = 1,$$

5 so $\alpha(1) \in B$ is an invertible element. Then, due to [9, Proposition 4.14], there exists a unique continuous algebra
6 homomorphism $f : \mathcal{O}(\mathbb{C}^\times, A; \alpha) \rightarrow B$, $f(t) = \alpha(1)$. It easily seen that $fi_A = \theta$, $fi_{A_\alpha} = \alpha$, $fi_{A_{\alpha^{-1}}} = \beta$. \square

7 3.2. The general case

8 In this section A is an Arens-Michael algebra and α is an automorphism of A . We aim to obtain a description
9 of $\widehat{T}_A(A_\alpha \oplus A_{\alpha^{-1}})$, similar to the description of $\widehat{T}_A(A_\alpha)$, obtained in [9, Proposition 4.9], without putting any
10 localizability assumptions on the pair (α, α^{-1}) .

11 For every tuple $w \in W_2$ we denote the k -th symbol of w by $w(k)$. Also consider the functions
12 $c_1 : W_2 \rightarrow \mathbb{Z}_{\geq 0}$ and $c_2 : W_2 \rightarrow \mathbb{Z}_{\geq 0}$ which count the number of instances of 1 and 2 in a tuple, respectively.
13 Also denote $c(w) = c_1(w) - c_2(w)$. For every element in W_2 define an A - $\widehat{\otimes}$ -bimodule as follows:

- 14 (1) $A_\emptyset := A$
- 15 (2) $A_{(1)} := A_\alpha$, $A_{(2)} := A_{\alpha^{-1}}$
- 16 (3) for every $w_1, w_2 \in W_2$ we have $A_{w_1 w_2} := A_{w_1} \widehat{\otimes}_A A_{w_2}$

17 Let $w \in W_2$ be a non-empty element and let $1 \leq k \leq |w|$. Replace all numbers 2 in w with -1 and
18 denote the new tuple by w' . Let us define a function $p(w, k)$ as follows:

$$p(w, k) = \sum_{i=1}^k w'(j) = \sum_{i=1}^k 3 - 2w(j). \tag{3.2}$$

19 **Proposition 3.3** For every $w \in W_2$ consider a mapping

$$i_w : \prod_{i=1}^{|w|} A_{\alpha^{w'(i)}} \rightarrow A_{\alpha^n}, \quad i_w(x_1, \dots, x_{|w|}) := x_1 \prod_{i=2}^{|w|} \alpha^{p(w, i-1)}(x_i),$$

20 where $n = c(w)$.

21 Then i_w is a continuous A -balanced map which induces a A - $\widehat{\otimes}$ -bimodule isomorphism

22 $i_w : A_w \simeq A_{\alpha^n}$.

1 **Proof** First of all, let us prove that i_w is an A -balanced map:

$$\begin{aligned} i_w(x_1, \dots, x_i \circ r, x_{i+1}, \dots, x_{|w|}) &= \\ &= x_1 \alpha^{p(w,1)}(x_2) \dots \alpha^{p(w,i-1)}(x_i \alpha^{w'(i)}(r)) \alpha^{p(w,i)}(x_{i+1}) \dots \alpha^{p(w,|w|-1)}(x_{|w|}) = \\ &= x_1 \alpha^{p(w,1)}(x_2) \dots \alpha^{p(w,i-1)}(x_i) \alpha^{w'(i)+p(w,i-1)}(r) \alpha^{p(w,i)}(x_{i+1}) \dots \alpha^{p(w,|w|-1)}(x_{|w|}). \end{aligned}$$

2 However, by definition, $w'(i) + p(w, i - 1) = p(w, i)$, so we get

$$\begin{aligned} x_1 \alpha^{p(w,1)}(x_2) \dots \alpha^{p(w,i-1)}(x_i) \alpha^{w'(i)+p(w,i-1)}(r) \alpha^{p(w,i)}(x_{i+1}) \dots \alpha^{p(w,|w|-1)}(x_{|w|}) &= \\ = x_1 \alpha^{p(w_1)}(x_2) \dots \alpha^{p(w,i-1)}(x_i) \alpha^{p(w,i)}(r) \alpha^{p(w,i)}(x_{i+1}) \dots \alpha^{p(w,|w|-1)}(x_{|w|}) &= \\ = i_w(x_1, \dots, x_i, r \circ x_{i+1}, \dots, x_{|w|}). \end{aligned}$$

3 Therefore, i_w is balanced.

4 Now suppose that $f : \prod_{i=1}^{|w|} A_{\alpha^{w'(i)}} \rightarrow M$ is a continuous A -balanced A - $\hat{\otimes}$ -bimodule homomorphism.

5 Then we define

$$\tilde{f} : A_{\alpha^n} \rightarrow M, \quad \tilde{f}(a) = f(a, 1, \dots, 1).$$

6 This map is a well-defined homomorphism of A - $\hat{\otimes}$ -bimodules: for any $b \in A$ we have

$$\tilde{f}(ba) = f(ba, 1, \dots, 1) = bf(a, 1, \dots, 1) = b\tilde{f}(a),$$

7

$$\begin{aligned} \tilde{f}(a)b &= f(a, 1, \dots, 1) \circ b = f(a, 1, \dots, \alpha^{p(w,|w|)}(b)) = \\ &= f(a, 1, \dots, 1, \alpha^{p(w,|w|-1)+p(w,|w|)}(r), 1) = \dots = f(a\alpha^n(b), 1, \dots, 1). \end{aligned}$$

8 We prove that $f = \tilde{f} \circ i_w$ by using a similar argument, which we will omit here. □

9 **Lemma 3.4** *The following diagram is commutative:*

$$\begin{array}{ccc} A_{w_1} \hat{\otimes}_A A_{w_2} & \xlongequal{\quad} & A_{w_1 w_2} \\ \downarrow i_{w_1} \otimes i_{w_2} & & \downarrow i_{w_1 w_2} \\ A_{\alpha^{k_1}} \hat{\otimes}_A A_{\alpha^{k_2}} & \xrightarrow{\varphi} & A_{\alpha^{k_1+k_2}} \end{array} \quad (3.3)$$

10 where $\varphi(a \otimes b) = a\alpha^{k_1}(b)$.

11 **Proof** Again, it suffices to look at elementary tensors. Let $x = x_1 \otimes \dots \otimes x_{|w_1|} \in A_{w_1}$ and

12 $y = y_1 \otimes \dots \otimes y_{|w_2|} \in A_{w_2}$. Then we have

$$\begin{aligned} \varphi \circ (i_{w_1} \otimes i_{w_2})(x \otimes y) &= \varphi \left(\prod_{i=1}^{|w_1|} \alpha^{p(w_1,i-1)}(x_i) \otimes \prod_{i=1}^{|w_2|} \alpha^{p(w_2,i-1)}(y_i) \right) = \\ &= \prod_{i=1}^{|w_1|} \alpha^{p(w_1,i-1)}(x_i) \cdot \prod_{i=1}^{|w_2|} \alpha^{p(w_2,i-1)+k_1}(y_i). \end{aligned}$$

1 Notice that $k_1 = p(w_1, |w_1|)$, therefore,

$$\prod_{i=1}^{|w_1|} \alpha^{p(w_1, i-1)}(x_i) \cdot \prod_{i=1}^{|w_2|} \alpha^{p(w_2, i-1)+k_1}(y_i) = \prod_{i=1}^{|w_1|} \alpha^{p(w_1 w_2, i-1)}(x_i) \cdot \prod_{i=|w_1|+1}^{|w_2|} \alpha^{p(w_1 w_2, i-1)}(y_i) = i_{w_1 w_2}(x \otimes y).$$

2 □

3 Fix a generating family of seminorms $\{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ on A .

4 **Definition 3.5** Define the following locally convex space:

$$A\{x_1, x_2; \alpha\} = \left\{ a = \sum_{w \in W_2} a_w x^w : \|a\|_{\lambda, \rho} = \sum_{w \in W_2} \|a_w\|_\lambda^{(w)} \rho^{|w|} < \infty \forall \lambda \in \Lambda, 0 < \rho < \infty \right\}, \quad (3.4)$$

5 where $\|r\|_\lambda^{(w)}$ are seminorms on A , induced by the seminorm on A_{α_n} (for $n = c(w)$)

$$\|r\|_\lambda^{(w)} = \inf_{r = \sum_{j=1}^k i_w(r_{1,j} \otimes \dots \otimes r_{|w|,j})} \sum_{j=1}^k \|r_{1,j}\|_\lambda \dots \|r_{|w|,j}\|_\lambda,$$

6 identifying A_{α_n} with A , as their underlying LCS are the same. Finally, we define $\|\cdot\|_\lambda^{(\emptyset)} = \|\cdot\|_\lambda$.

7 **Remark.** Actually, $\|\cdot\|_\lambda^{(1)} = \|\cdot\|_\lambda^{(2)} = \|\cdot\|_\lambda$ due to the definition of i_w .

8 The space $A\{x_1, x_2; \alpha\}$ with the topology, generated by $\|\cdot\|_{\lambda, \rho}$, is a complete locally convex space.

9 **Theorem 3.6** The space $A\{x_1, x_2; \alpha\}$ admits a unique multiplication which satisfies the following conditions:

10 (1) the natural inclusions $A[t; \alpha] \hookrightarrow A\{x_1, x_2; \alpha\}$ and $A[s; \alpha^{-1}] \hookrightarrow A\{x_1, x_2, \alpha\}$, where

11 $\sum a_n t^n \rightarrow \sum a_n x_1^n$ and $\sum a_n s^n \rightarrow \sum a_n x_2^n$, are algebra homomorphisms.

12 (2) there exists a canonical topological A -algebra isomorphism $\psi : A\{x_1, x_2; \alpha\} \rightarrow \widehat{T}_A(A_\alpha \oplus A_{\alpha^{-1}})$.

13 As a corollary, $A\{x_1, x_2; \alpha\}$ becomes an Arens-Michael algebra.

14 **Proof** Fix a generating directed family of seminorms $\{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ on A . For every $k > 0$ we identify

15 $(A_\alpha \oplus A_{\alpha^{-1}})^{\widehat{\otimes} k}$ with $\bigoplus_{|w|=k} A_w$. If we denote the projective tensor product of k copies of $\|\cdot\|_\lambda + \|\cdot\|_\lambda$ by $\|\cdot\|_{\lambda, \lambda}^{\widehat{\otimes} n}$,

16 we can rewrite the definition of $\widehat{T}_A(A_\alpha \oplus A_{\alpha^{-1}})$ as follows:

$$\widehat{T}_A(A_\alpha \oplus A_{\alpha^{-1}}) = \left\{ (x_w) \in \prod_{w \in W_2} A_w : \|(x_w)\|_{\lambda, \rho} = \sum_{n \geq 0} \|(x_w)_{|w|=n}\|_{\lambda, \lambda}^{\widehat{\otimes} n} \rho^n < \infty, \lambda \in \Lambda, 0 < \rho < \infty \right\}.$$

17 Moreover, notice that for every $x_w \in A_w$, $\lambda \in \Lambda$ we have

$$\|(x_w)_{|w|=n}\|_{\lambda, \lambda}^{\widehat{\otimes} n} = \sum_{|w|=n} \|i_w(x_w)\|_\lambda^{(w)}$$

1 by the definition of $\|\cdot\|_\lambda^{(w)}$.

2 For any element $a \in A\{x_1, x_2; \alpha\}$ we define ψ as follows:

$$(\psi(a))_w = a_w \otimes 1 \cdots \otimes 1.$$

3 Therefore, for any $0 < \rho < \infty$ and $\lambda \in \Lambda$ we have

$$\begin{aligned} \|\psi(a)\|_{\lambda, \rho} &= \sum_{n=0}^{\infty} \left(\|(a_w \otimes 1 \cdots \otimes 1)_{|w|=n}\|_{\lambda}^{\hat{\otimes} n} \right) \rho^n = \sum_{n=0}^{\infty} \left(\sum_{|w|=n} \|i_w(a_w \otimes 1 \cdots \otimes 1)\|_{\lambda}^{(w)} \right) \rho^n = \\ &= \sum_{n=0}^{\infty} \left(\sum_{|w|=n} \|a_w\|_{\lambda}^{(w)} \right) \rho^n = \|a\|_{\lambda, \rho}. \end{aligned}$$

4 Therefore, we have proven that ψ is a topological isomorphism of locally convex spaces, and Lemma 3.4 ensures
5 that ψ is an algebra homomorphism, and the existence of natural inclusions

$$T_A(A_\alpha) \hookrightarrow \widehat{T}_A(A_\alpha \oplus A_{\alpha^{-1}}), \quad T_A(A_{\alpha^{-1}}) \hookrightarrow \widehat{T}_A(A_\alpha \oplus A_{\alpha^{-1}})$$

6 implies (1). □

7 **Corollary 3.7** *Suppose that A is a Fréchet-Arens-Michael algebra and α is an automorphism of A . Then*

$$\widehat{L}_A(A_\alpha) \simeq A\{x_1, x_2; \alpha\} / \overline{(x_1x_2 - 1, x_2x_1 - 1)}.$$

8 We also provide some examples of explicit computations of $A\{x, y; \alpha\}$ in Appendix A.

9 4. Open questions

10 1. How can one characterize the Arens-Michael envelopes among all Arens-Michael algebras? In particular,
11 is every Arens-Michael algebra isomorphic to the Arens-Michael envelope of an associative algebra?

12 **Remark.** This question is non-trivial because in general the Arens-Michael envelope of an Arens-Michael
13 algebra A is not, in general, isomorphic to A itself, even when A is a Banach algebra, as follows from
14 [4, Theorems 5.1.17, 5.1.18]. The basic idea is that there might be multiple non-equivalent norms on an
15 algebra which turn it into a Banach algebra.

16 2. Consider an element $f \in F_2 = \mathbb{C}\langle x, y \rangle$. Is there a way to determine whether the Arens-Michael envelope
17 of $\widehat{F_2/(f)}$ is isomorphic to the zero algebra? This question is equivalent to finding a non-zero Banach
18 algebra B and a non-trivial pair $(x, y) \in B^2$ satisfying $f(x, y) = 0$. In particular, is it a decidable
19 problem?

20 A slightly different case, featuring the relations of form $f(x, y) = [x, y] - h(y)$, where h is a holomorphic
21 function, is considered in a recent preprint [3] of O. Aristov (in Russian).

22 3. Does Conjecture 2.10 hold for every algebra A and an invertible R -bimodule M ?

23 4. There are a lot of interesting algebras for which the Arens-Michael envelopes are yet to be explicitly
24 described. For example, consider the quantum universal enveloping algebra $U_q(\mathfrak{sl}_2)$.

Definition 4.1 The quantum universal enveloping algebra $U_q(\mathfrak{sl}_2)$ is the associative unital algebra generated by E, F, K, K^{-1} with the following relations:

$$KE = q^2EK, KF = q^{-2}FK, [E, F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

This algebra can be represented as an iterated Ore extension:

$$U_q(\mathfrak{sl}_2) \simeq \mathbb{C}[K, K^{-1}][F; \alpha_0][E; \alpha_1, \delta],$$

and we have the following result due to D. Pedchenko ^{*}:

Theorem 4.2 Consider $|q| = 1, q \neq -1, 1$. Then

$$\widehat{U_q(\mathfrak{sl}_2)} \simeq \left\{ f = \sum_{i,j \in \mathbb{Z}_{\geq 0}, k \in \mathbb{Z}} c_{ijk} K^i F^j E^k : \|f\|_\rho := \sum_{i,j,k} |c_{ijk}| \rho^{i+j+k} < \infty \forall \rho > 0 \right\},$$

with multiplication, uniquely defined by the relations in Definition 4.1.

When $|q| \neq 1$ this representation becomes useless to us, because the morphisms cease to be m -localizable.

In fact, this problem was what motivated us to tackle the description of the Arens-Michael envelope of Laurent Ore extensions in the general case. Consider the following isomorphism:

$$U_q(\mathfrak{sl}_2) \simeq \frac{\mathbb{C}\langle E, F \rangle [K, K^{-1}; \alpha]}{([E, F] - \frac{K - K^{-1}}{q - q^{-1}})},$$

where $\alpha(E) = q^2E$ and $\alpha(F) = q^{-2}F$. Then we use the main result:

$$(\mathbb{C}\langle E, F \rangle [K, K^{-1}; \alpha])^\wedge \simeq \widehat{L_{\mathcal{F}_2}((\mathcal{F}_2)_\alpha)} \simeq \mathcal{F}_2\{x_1, x_2; \alpha\} / \overline{(x_1x_2 - 1, x_2x_1 - 1)}. \quad (4.1)$$

Unfortunately, the algebra $\mathcal{F}_2\{x, y; \alpha\}$ itself turned out to be too difficult to describe explicitly. As the readers can see, Example A.5 demonstrates how computationally difficult this approach is.

However, there is some progress made by O. Aristov [†], where he provides a description of the Arens-Michael envelope of $U_q(\mathfrak{g})$ for a semisimple Lie algebra \mathfrak{g} and $|q| \neq 1$ in terms of the equivalence classes of irreducible finite-dimensional representations of $U_q(\mathfrak{g})$, in the spirit of J. Taylor's result. Unfortunately, this approach does not provide a power-series representation of $\widehat{U_q(\mathfrak{sl}_2)}$ for $|q| \neq 1$. So, for now, (4.1) is the closest thing to a power series representation of the Arens-Michael envelope of $U_q(\mathfrak{g})$ that is available to us.

^{*}D. Pedchenko (2020). The Arens-Michael envelopes of the Jordanian Plane and $U_q(\mathfrak{sl}(2))$ [online]. Website <https://arxiv.org/abs/2009.06477> [accessed 17 February 2022].

[†]O. Aristov (2020). Banach space representations of Drinfeld-Jimbo algebras and their complex-analytic forms [online]. Website <https://arxiv.org/abs/2012.12565> [accessed 17 February 2022].

1 **A. Several examples of explicit computations of $\widehat{T}_A(A_\alpha)$ and $\widehat{L}_A(A_\alpha)$**

2 Here we will provide several examples, which illustrate the complexity of "extensions" $\widehat{T}_A(A_\alpha) = A\{x; \alpha\}$, $\widehat{T}_A(A_\alpha \oplus$
 3 $A_{\alpha^{-1}}) = A\{x, y; \alpha\}$ (and $\widehat{L}_A(A_\alpha)$ as a corollary) even for the simplest and most natural cases.

4 We want to consider the case of non- m -localizable pairs $\{\alpha, \alpha^{-1}\}$, because the m -localizable case has
 5 been already treated in Section 3.1.

6 **Lemma A.1** Consider an Arens-Michael algebra A with topology, generated by a family of seminorms $\{\|\cdot\|_\lambda : \lambda \in \Lambda\}$
 7 and $\alpha \in \text{Aut}(A)$. Denote

$$w_n = (\underbrace{1, 1, \dots, 1, 1}_{n \text{ times}}, \underbrace{2, 2, \dots, 2, 2}_{n \text{ times}}), \quad w'_n = (\underbrace{2, 2, \dots, 2, 2}_{n \text{ times}}, \underbrace{1, 1, \dots, 1, 1}_{n \text{ times}}).$$

8 Suppose that there is an element $r \in A$ such that

$$\lim_{n \rightarrow \infty} (\|r\|_\lambda^{(w_n)} \rho^{2n}) = 0 \quad (\text{A.1})$$

9 for every $\lambda \in \Lambda$ and $\rho > 0$ (in other words, the sequence $(\|r\|_\lambda^{(w_n)})$ is rapidly decaying), or

$$\lim_{n \rightarrow \infty} (\|r\|_\lambda^{(w'_n)} \rho^{2n}) = 0. \quad (\text{A.2})$$

10 Then $r \in I \subset A\{x_1, x_2; \alpha\}$, where I is the smallest closed two-sided ideal, which contains $x_1 x_2 - 1$ and $x_2 x_1 - 1$.

11 In particular, if there exists an invertible element $r \in A$, which satisfies (A.1) or (A.2), then $\widehat{L}_A(A_\alpha) = 0$.

12 **Proof** Notice that $x_1^k x_2^k - 1 \in I$ for any $k > 0$, therefore, $r - r x_1^k x_2^k \in I$, but

$$\|(r - r x_1^k x_2^k) - r\|_{\lambda, \rho} = \|r\|_\lambda^{(w_k)} \rho^{2k} \xrightarrow{k \rightarrow \infty} 0.$$

13 As we can see, the sequence $r - r x_1^k x_2^k$ converges to r in the topology of $A\{x_1, x_2; \alpha\}$ due to the assumptions
 14 in our Lemma, therefore, $r \in \bar{I}$. □

15 **Example A.2** Consider $A = C(\mathbb{R})$ and $\alpha(f)(x) = f(x - 1)$ for $f \in C(\mathbb{R})$, $x \in \mathbb{R}$. Recall that the topology on
 16 A is generated by the family $\|f\|_K := \sup_{x \in K} |f(x)|$, where $K \subset \mathbb{R}$ is a compact subset. Notice that instead of all
 17 K we could take all the intervals $[-x, x]$ for $x > 0$ or even $[-a_n, a_n]$, where (a_n) is an arbitrary increasing
 18 unbounded sequence.

19 Let $|w| > 1$. Then we can write down a lower estimate for $\|\cdot\|_{[-n, n]}^{(w)}$ as follows:

$$\begin{aligned} \|f\|_{[-n, n]}^{(w)} &= \inf_{f = \sum_{j=1}^k i_w(f_{1,j} \otimes \dots \otimes f_{|w|,j})} \sum_{j=1}^k \|f_{1,j}\|_{[-n, n]} \dots \|f_{|w|,j}\|_{[-n, n]} = \\ &= \inf_{f = \sum_{j=1}^k i_w(f_{1,j} \otimes \dots \otimes f_{|w|,j})} \sum_{j=1}^k \|f_{1,j}\|_{[-n, n]} \dots \left\| \alpha^{p(w, |w|-1)}(f_{|w|,j}) \right\|_{[-n+p(w, |w|-1), n+p(w, |w|-1)]} \geq \\ &\geq \inf_{f = \sum_{j=1}^k i_w(f_{1,j} \otimes \dots \otimes f_{|w|,j})} \sum_{j=1}^k \|f_{1,j}\|_{I_n^{(w)}} \dots \left\| \alpha^{p(w, |w|-1)}(f_{|w|,j}) \right\|_{I_n^{(w)}} \geq \|f\|_{I_n^{(w)}}, \end{aligned}$$

1 where

$$I_n^{(w)} = \bigcap_{i=1}^{|w|-1} [-n + p(w, i), n + p(w, i)]$$

2 for $|w| > 1$, and $I_n^{(w)} = [-n, n]$ for $|w| \leq 1$.

3 Notice that if the intersection is empty, then we say that the respective seminorm is identically zero.

4 If we denote

$$k_{min}(w) = \begin{cases} \min_{1 \leq i \leq |w|-1} p(w, i) & , |w| > 1, \\ 0 & , |w| \leq 1, \end{cases} \quad k_{max}(w) = \begin{cases} \max_{1 \leq i \leq |w|-1} p(w, i) & , |w| > 1, \\ 0 & , |w| \leq 1, \end{cases}$$

5 then

$$I_n^{(w)} = [-n + k_{max}(w), n + k_{min}(w)].$$

6 Now we aim to prove that $\|f\|_{[-n, n]}^{(w)} = \|f\|_{I_n^{(w)}}$. Consider the representation

$$f = \alpha^{k_{max}(w)}(g) + \alpha^{k_{min}(w)}(h) + (f - \alpha^{k_{max}(w)}(g) - \alpha^{k_{min}(w)}(h)) \text{ for any } g, h \in C(\mathbb{R}).$$

7 Thus we get an upper estimate:

$$\|f\|_{[-n, n]}^{(w)} \leq \inf_{g, h \in C(\mathbb{R})} \|g\|_{[-n, n]} + \|h\|_{[-n, n]} + \|f - \alpha^{k_{max}(w)}(g) - \alpha^{k_{min}(w)}(h)\|_{[-n, n]} \quad (\text{A.3})$$

8 Denote the function $\tilde{f} = f - \alpha^{k_{max}(w)}(g) - \alpha^{k_{min}(w)}(h)$. Suppose that $-n + k_{max} \leq n + k_{min}$. Then consider the
9 following g_m and h_m :

$$g_m(x) = \begin{cases} f(x + k_{max}), & x < -n - 1/m \\ (-m(x + n))f(x + k_{max}), & x \in [-n - 1/m, -n] \\ 0, & x > -n \end{cases} \quad (\text{A.4})$$

$$h_m(x) = \begin{cases} 0, & x < n \\ (m(x - n))f(x + k_{min}), & x \in [n, n + 1/m] \\ f(x + k_{min}), & x > n + 1/m \end{cases} \quad (\text{A.5})$$

10 Then we have to look at the right hand side of (A.3):

$$\|g_m\|_{[-n, n]} = \|h_m\|_{[-n, n]} = 0,$$

$$\alpha^{k_{max}(w)}(g_m)(x) = \begin{cases} f(x), & x < -n - 1/m + k_{max}(w) \\ (-m(x - k_{max}(w) + n))f(x), & x \in [-n - 1/m + k_{max}(w), -n + k_{max}(w)] \\ 0, & x > -n + k_{max}(w) \end{cases} \quad (\text{A.6})$$

$$\alpha^{k_{min}(w)}(h_m)(x) = \begin{cases} 0, & x < n + k_{min}(w) \\ (m(x - k_{min}(w) - n))f(x), & x \in [n + k_{min}(w), n + k_{min}(w) + 1/m] \\ f(x), & x > n + 1/m + k_{min}(w) \end{cases} \quad (\text{A.7})$$

$$\tilde{f}(x) = \begin{cases} 0 & x < -n - 1/m + k_{max}(w) \\ (1 + m(x - k_{max}(w) + n))f(x) & x \in [-n - 1/m + k_{max}(w), -n + k_{max}(w)] \\ f(x) & x \in [-n + k_{max}(w), n + k_{min}(w)] = I_n^{(w)} \\ (1 - m(x - k_{min}(w) - n))f(x) & x \in [n + k_{min}(w), n + 1/m + k_{min}(w)] \\ 0 & x > n + 1/m + k_{min}(w), \end{cases} \quad (\text{A.8})$$

1 so

$$\|f\|_{[-n,n]}^{(w)} = \|\tilde{f}\|_{[-n,n]} \leq \|f\|_{[-n+k_{max}(w)-1/m, n+1/m+k_{min}(w)]}.$$

2 By taking $m \rightarrow 0$ we get the desired equality.

3 If $-n + k_{max}(w) > n + k_{min}(w)$, then we can look at $g_3(x)$ and $h_3(x)$. Notice that

$$-n + k_{max}(w) - 1/3 > n + k_{min}(w) + 1/3,$$

4 so the computations above show us that the supports of $\alpha^{k_{max}(w)}(g_3)$ and $\alpha^{k_{min}(w)}(h_3)$ have empty intersection,

5 so $\tilde{f} \equiv 0$ for g_3 and h_3 , therefore

$$\|f\|_{[-n,n]}^{(w)} = \|\tilde{f}\|_{[-n,n]} = 0.$$

6 This argument worked for $|w| > 1$, but we know that

$$\|\cdot\|_{[-n,n]}^{(w)} = \|\cdot\|_{[-n,n]}$$

7 when $|w| \leq 1$.

8 To sum everything up, we have deduced that the algebras $C(\mathbb{R})\{x; \alpha\}$ and $C(\mathbb{R})\{x_1, x_2; \alpha\}$ look as follows:

$$C(\mathbb{R})\{x; \alpha\} = \left\{ a = \sum_{k \geq 0} a_k x^k : \|a\|_{n,\rho} := \|a_0\|_{[-n,n]} + \sum_{k \geq 1} \|a_k\|_{[-n+k-1, n-k+1]} \rho^k < \infty, \forall n > 0, 0 < \rho < \infty \right\},$$

$$C(\mathbb{R})\{x_1, x_2; \alpha\} = \left\{ a = \sum_{w \in W_2} a_w x^w : \|a\|_{n,\rho} := \sum_{w \in W_2} \|a_w\|_{I_n^{(w)}} \rho^{|w|} < \infty, \forall w \in W_2, 0 < \rho < \infty \right\}.$$

9 It is easily seen that the isomorphism $C(\mathbb{R})\{x; \alpha\} \simeq C(\mathbb{R})[[x]]$ takes place, because

10 $\|\cdot\|_{[-n,n]}^{(k)} = 0$ for $k > 2n$. Therefore, Lemma A.1 implies that $\widehat{L}_A(A_\alpha) = 0$, because the $(\|1\|_{[-n,n]}^{(w_k)})_{k \in \mathbb{N}} \in c_{00}$.

11 **Remark.** As the readers can see, computing the topological tensor algebras using just the definition
12 requires a lot of effort. In the examples that will follow next, we are going to use several tricks which simplify
13 the computations.

14 **Example A.3** What happens if we consider the shift automorphism on the algebra of holomorphic functions
15 $\mathcal{O}(\mathbb{C})$ instead of $C(\mathbb{R})$? We get a result, which is similar to what we got in the Example A.2, as stated in [9,
16 Example 4.3]:

1 **Proposition A.4** Let $R = \mathcal{O}(\mathbb{C})$. Consider an automorphism $\alpha(f)(z) = f(z - 1)$. Then
 2 $R\{x; \alpha\} \cong R[[x]]$ as locally convex spaces, where the topology on $R[[x]]$ is generated by $\{\|\cdot\|_n\}_{n \in \mathbb{N}}$, where
 3 $\|\sum_k^\infty a_k x^k\|_n = \|a_n\|$.

4 The proof cleverly utilizes Mergelyan's approximation theorem. In particular, A. Yu. Pirkovskii proves that
 5 $\|\cdot\|_\rho^{(n+1)} = 0$ for $n > \lfloor 2\rho \rfloor + 1$, therefore, $\|\cdot\|_\rho^{(w_{n+1})} \leq \|\cdot\|_\rho^{(n+1)} = 0$, so $\widehat{L}_R(R_\alpha) = 0$, as well. Equivalently, we
 6 have

$$(\mathbb{C}[x][y, y^{-1}; \alpha])^\wedge = 0,$$

7 where $\alpha(f)(x) = f(x - 1)$.

8 **Example A.5** Let $A = \mathcal{O}(\mathbb{C})$ and consider the automorphism $\beta_q : A \rightarrow A$, $\beta_q(f)(z) = f(qz)$, where $|q| \neq 1$.
 9 Fix a generating family of seminorms $\{\|\cdot\|_\rho : 0 < \rho < \infty\}$ on A , where

$$\|f\|_\rho = \left\| \sum_k f^{(k)} z^k \right\|_\rho := \sum_k |f^{(k)}| \rho^k.$$

10 Let us try to describe $\widehat{L}_A(A_{\beta_q})$. To do this, we will need the following lemma:

11 **Lemma A.6** Let α be an automorphism of an associative algebra A . Suppose that for any Banach algebra B
 12 and any algebra homomorphism $\varphi : A[x; \alpha] \rightarrow B$ the element $\varphi(x)$ is nilpotent in B . Then

$$(A[x; \alpha])^\wedge \simeq \widehat{A}[[x]], \quad (A[x, x^{-1}; \alpha])^\wedge \simeq 0$$

13 as a locally convex space.

14 **Proof** Our argument follows the idea given in [9, Example 5.1]. If $\|\cdot\|_\nu$ is a submultiplicative seminorm on
 15 A , then for every $n > 0$ we can define the following submultiplicative seminorm $\|\cdot\|_{\nu, n}$ on $A[x; \alpha]$ as follows:

$$\left\| \sum_{i=0}^m a_i x^i \right\|_{\nu, n} = \sum_{i=0}^n \|a_i\|_\nu.$$

16 Now, suppose that $\|\cdot\|$ is a submultiplicative seminorm on $A[x; \alpha]$. As $\|x^N\| = 0$ for some $N > 0$, we get that

$$\left\| \sum_{i=0}^m a_i x^i \right\| = \left\| \sum_{i=0}^{N-1} a_i x^i \right\| \leq \sum_{i=0}^{N-1} \|x^i\| \|a_i\| \leq (\max_i \|x^i\|) \sum_{i=0}^{N-1} \|a_i\|.$$

17 However, $(\max_i \|x^i\|)$ is just a constant, and we have proven that $\|\cdot\|$ has to be dominated by a seminorm of
 18 form $\|\cdot\|_{\nu', N-1}$. Therefore, the Arens-Michael envelope of $(A[x; \alpha])^\wedge$ is the completion of $A[x; \alpha]$ with respect
 19 to $\|\cdot\|_{\nu, n}$ for all ν and $n > 0$, but this completion is, indeed, isomorphic to $\widehat{A}[[x]]$.

20 For the second isomorphism we just need to notice that there are no non-trivial Banach algebras
 21 with invertible nilpotents, so there are no non-trivial homomorphisms from $A[x, x^{-1}; \alpha]$ to Banach algebras.
 22 Therefore, the Arens-Michael envelope is trivial. □

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