Homological dimensions of analytic Ore extensions

Petr Kosenko

April 2, 2022

Abstract

If A is an algebra with finite right global dimension, then for any automorphism α and α-derivation δ the right global dimension of $A[t; \alpha, \delta]$ satisfies

$$\operatorname{rgld} A \leq \operatorname{rgld} A[t; \alpha, \delta] \leq \operatorname{rgld} A + 1.$$

We extend this result to the case of holomorphic Ore extensions and smooth crossed products by $\mathbb{Z}$ of $\hat{\otimes}$-algebras.

Keywords: locally convex algebras, locally convex modules, homological dimensions, smooth crossed products
AMS Subject Classification: 46H30; 16E10

1 Introduction

Recall the following well-known theorem:

Theorem 1.1 ([13], Theorem 4.3.7)**.** If R is a ring then the following identity takes place for every $n \in \mathbb{N}$:

$$\operatorname{rgld} R[x_1, \dots x_n] = n + \operatorname{rgld} R.$$

The importance of this theorem lies in the fact that it immediately yields the Hilbert's syzygy theorem in the case when R is a field (see [[13], Corollary 4.3.8]).

This fact can be, indeed, generalized to Ore extensions $R[t; \alpha, \delta]$, as shown in [6]. It turns out that if the global dimension of R is finite, then the global dimension of $R[t; \alpha, \delta]$ either stays the same, or increases by one.

In this paper we adapt the arguments used in [6, ch. 7.5] to the topological setting in order to obtain the estimates for the right homological dimensions of analytic (also called holomorphic or topological) Ore extensions (see [8, ch. 4.1]) and smooth crossed products by $\mathbb{Z}$ (see [11] and [7]).

Below we state the result in the purely algebraic situation, which is provided in [6] and then we present its topological version.

Remark. There is a certain ambiguity in defining Ore extensions, which will be demonstrated below, so, to state the result in the algebraic setting, we need to fix an appropriate definition of an Ore extension:

Definition 1.1. Let A be an associative $\mathbb{C}$-algebra, $\alpha \in \operatorname{End}(A)$ and let $\delta : A \to A$ be a $\mathbb{C}$-linear map, such that the following relation holds for every $a, b \in A$:

$$\delta(ab) = \delta(a)b + \alpha(a)\delta(b).$$

Let us call such maps α-derivations.

Then the *Ore extension* of A w.r.t α and δ is the vector space

$$A[t; \alpha, \delta] = \left\{ \sum_{i=0}^{n} a_i t^i : a_i \in A \right\}$$

with the multiplication defined uniquely by the following conditions:

(1) The relation $ta = \alpha(a)t + \delta(a)$ holds for any $a \in A$.

(2) The natural inclusions $A \hookrightarrow A[t; \alpha, \delta]$ and $\mathbb{C}[t] \hookrightarrow A[t; \alpha, \delta]$ are algebra homomorphisms.

Also, if $\delta = 0$ and α is invertible, then one can define the *Laurent Ore extension* of A

$$A[t, t^{-1}; \alpha] = \left\{ \sum_{i=-n}^{n} a_i t^i : a_i \in A \right\}$$

with the multiplication defined the same way.

We would like to highlight a certain ambiguity: the authors of [6] define $A[t; \alpha, \delta]$ in a slightly different way:

Definition 1.2 ([6], pp. 1.2.1-1.2.6)**.** Let A be an algebra, $\tilde{\alpha} \in \mathrm{End}(A)$ and let $\tilde{\delta} : A \to A$ be a $\mathbb{C}$-linear map, such that the following relation holds for every $a, b \in A$:

$$\tilde{\delta}(ab) = \tilde{\delta}(a)\tilde{\alpha}(b) + a\tilde{\delta}(b).$$

Let us call such maps opposite α-derivations. Then the *(opposite) Ore extension* of A w.r.t $\tilde{\alpha}$ and $\tilde{\delta}$ is the vector space

$$A_{\mathrm{op}}[t; \tilde{\alpha}, \tilde{\delta}] = \left\{ \sum_{i=0}^{n} t^i a_i : a_i \in A \right\}$$

with the multiplication defined uniquely by the following conditions:

(1) The relation $at = t\tilde{\alpha}(a) + \tilde{\delta}(a)$ holds for any $a \in A$.

(2) The natural inclusions $A \hookrightarrow A_{\mathrm{op}}[t; \tilde{\alpha}, \tilde{\delta}]$ and $\mathbb{C}[t] \hookrightarrow A_{\mathrm{op}}[t; \tilde{\alpha}, \tilde{\delta}]$ are algebra homomorphisms.

Also, if $\tilde{\delta} = 0$ and $\tilde{\alpha}$ is invertible, then one can define the *(opposite) Laurent Ore extension* of A

$$A_{\mathrm{op}}[t, t^{-1}; \tilde{\alpha}] = \left\{ \sum_{i=-n}^{n} t^i a_i : a_i \in A \right\}$$

with the multiplication defined the same way.

It is easily seen that in the case of invertible α, the following algebra isomorphisms take place:

$$A[x; \alpha, \delta] \cong A_{\mathrm{op}}[x, \alpha^{-1}, -\delta\alpha^{-1}], \quad A[x, x^{-1}; \alpha] \cong A_{\mathrm{op}}[x, x^{-1}; \alpha^{-1}]. \tag{1}$$

Throughout the paper, we will work with Ore extensions in the sense of Definition 1.1 (if not stated otherwise).

Now we are ready to state the result in the purely algebraic case, which is contained in [6, Theorem 5.7.3]:

Theorem 1.2 ([6], Theorem 5.7.3)**.** Let A be an algebra, let σ be an automorphism and let δ be a σ-derivation, in the sense of Definition 1.2. Denote the right global dimension of a ring R by $\mathrm{dgr}(R)$. Then the following estimates hold:

(1) $\mathrm{dgr}\, A \leq \mathrm{dgr}\, A_{\mathrm{op}}[t; \sigma, \delta] \leq \mathrm{dgr}\, A + 1$ if $\mathrm{dgr}\, R < \infty$,

(2) $\mathrm{dgr}\, A \leq \mathrm{dgr}\, A_{\mathrm{op}}[t, t^{-1}; \sigma] \leq \mathrm{dgr}\, A + 1$,

(3) $\mathrm{dgr}\, A_{\mathrm{op}}[t, \sigma] = \mathrm{dgr}\, A + 1$,

(4) $\mathrm{dgr}\, A[t, t^{-1}] = \mathrm{dgr}\, A + 1$.

Remark. In fact, the above theorem still holds if we replace $A_{\mathrm{op}}[t; \sigma, \delta]$ with $A[t; \sigma, \delta]$ due to (1).

This paper is organized as follows: in the Section 2 we recall the important notions related to homological properties of topological modules, in particular, we provide definitions of homological dimensions for topological algebras and modules. In the Section 3 we obtain the estimates for the homological dimensions of holomorphic Ore extensions; we use the bimodules of relative differentials to construct the required projective resolutions. In the Section 4 we obtain the estimates for the smooth crossed products by $\mathbb{Z}$.

In the Appendix A we provide the computations of algebraic and topological bimodules of relative differentials for different types of Ore extensions.

Acknowledgments

We want to thank Alexei Pirkovskii for his assistance in writing this paper, Alexei's comments and ideas were very helpful and greatly improved this paper. Moreover, we thank the reviewer as well, whose remarks tremendously contributed to the quality and readability of the paper.

2 Homological dimensions

Remark. All algebras in this paper are considered to be complex, unital and associative. Also, we will be working only with unital modules.

2.1 Notation

Let us introduce some notation and important definitions. (see [2] and [10] for more details).

For locally convex Hausdorff spaces E, F we denote the completed projective tensor product of E, F by $E\hat{\otimes}F$.

Recall that a **Fréchet** space is a complete metrizable locally convex space. In other words, a complete locally convex space X is a Fréchet space if and only if the topology on X can be generated by a countable family of seminorms.

Denote by **LCS**, **Fr** the categories of complete locally convex spaces and Fréchet spaces, respectively. Also we will denote the category of vector spaces by **Lin**.

For a detailed introduction to the theory of locally convex spaces and algebras, and relevant examples, the reader can see [12], [4], [5], or [3].

Let A be a locally convex space with a multiplication $\mu : A \times A \to A$, such that (A, μ) is an algebra.

(1) If μ is separately continuous, then A is called **a locally convex algebra**.

(2) If A is a complete locally convex space, and μ is jointly continuous, then A is called a **$\hat{\otimes}$-algebra**.

If A, B are $\hat{\otimes}$-algebras and $\eta : A \to B$ is a continuous unital algebra homomorphism, then the pair (B, η) is called a **A-$\hat{\otimes}$-algebra**.

A $\hat{\otimes}$-algebra with the underlying locally convex space which is a Fréchet space is called a **Fréchet algebra**.

Recall that a seminorm $\|\cdot\|$ on a locally convex algebra A is called **submultiplicative** if $\|ab\| \leq \|a\| \, \|b\|$ for all $a, b \in A$. Recall that a locally convex algebra A is called m-convex if the topology on it can be defined by a family of submultiplicative seminorms. A complete locally m-convex algebra is called an **Arens-Michael algebra**.

Let A be a $\hat{\otimes}$-algebra and let M be a complete locally convex space with a structure of a left A-module. Also suppose that the natural map $A \times M \to M$ is jointly continuous. Then we will call M a left **A-$\hat{\otimes}$-module**. In a similar fashion we define right **A-$\hat{\otimes}$-modules** and **A-B-$\hat{\otimes}$-bimodules**.

A $\hat{\otimes}$-module with its underlying LCS being a Fréchet space over a Fréchet algebra is called a Fréchet A-$\hat{\otimes}$-module.

For $\hat{\otimes}$-algebras A, B we denote

$$\begin{aligned} A\text{-}\mathbf{mod} &= \text{the category of left } A\text{-}\hat{\otimes}\text{-modules}, \\ \mathbf{mod}\text{-}A &= \text{the category of right } A\text{-}\hat{\otimes}\text{-modules}, \\ A\text{-}\mathbf{mod}\text{-}B &= \text{the category of } A\text{-}B\text{-}\hat{\otimes}\text{-bimodules}. \end{aligned}$$

More generally, for a fixed category $\mathcal{C} \subseteq$ **LCS** we denote the following **full** subcategories:

$$\begin{aligned} A\text{-}\mathbf{mod}(\mathcal{C}) &= \text{the category of left } A\text{-}\hat{\otimes}\text{-modules whose underyling LCS belong to } \mathcal{C}, \\ \mathbf{mod}\text{-}A(\mathcal{C}) &= \text{the category of right } A\text{-}\hat{\otimes}\text{-modules whose underyling LCS belong to } \mathcal{C}, \\ A\text{-}\mathbf{mod}\text{-}B(\mathcal{C}) &= \text{the category of } A\text{-}B\text{-}\hat{\otimes}\text{-bimodules whose underyling LCS belong to } \mathcal{C}. \end{aligned}$$

Complexes of A-$\hat{\otimes}$-modules for a $\hat{\otimes}$-algebra A

$$\ldots \xrightarrow{d_{n+1}} M_{n+1} \xrightarrow{d_n} M_n \xrightarrow{d_{n-1}} M_{n-1} \xrightarrow{d_{n-2}} \ldots$$

are usually denoted by $\{M, d\}$.

Let A be a $\hat{\otimes}$-algebra and consider a left A-$\hat{\otimes}$-module Y and a right A-$\hat{\otimes}$-module X.

(1) A bilinear map $f : X \times Y \longrightarrow Z$, where $Z \in \mathbf{LCS}$, is called A-balanced if $f(x \circ a, y) = f(x, a \circ y)$ for every $x \in X, y \in Y, a \in A$.

(2) A pair $(X\hat{\otimes}_A Y, i)$, where $X\hat{\otimes}_A Y \in \mathbf{LCS}$, and $i : X \times Y \longrightarrow X\hat{\otimes}_A Y$ is a jointly continuous A-balanced map, is called the **completed projective tensor product of** X and Y, if for every $Z \in \mathbf{LCS}$ and jointly continuous A-balanced map $f : X \times Y \longrightarrow Z$ there exists a unique continuous linear map $\tilde{f} : X\hat{\otimes}_A Y \longrightarrow Z$ such that $f = \tilde{f} \circ i$.

For the proof of the existence and uniqueness of completed projective tensor products of $\hat{\otimes}$-modules the reader can see [2, ch. 2.3-2.4]. In this paper we would like to keep in mind a trivial, but nonetheless useful example:

Example 2.1. Let A be a $\hat{\otimes}$-algebra, and consider a left A-$\hat{\otimes}$-module M. Then

$$A\hat{\otimes}_A M \xrightarrow{\sim} M, \quad a \otimes m \mapsto a \cdot m, \tag{2}$$

is a topological isomorphism of left A-$\hat{\otimes}$-modules. Similar isomorphisms can be constructed for right A-$\hat{\otimes}$-modules and A-$\hat{\otimes}$-bimodules.

2.2 Projectivity and flatness

The following definitions will be given in the case of left modules; the definitions in the cases of right modules and bimodules are similar, just use the following category isomorphisms:

$$\mathbf{mod}\text{-}A \simeq A^{\mathrm{op}}\text{-}\mathbf{mod};\ A\text{-}\mathbf{mod}\text{-}B \simeq (A\hat{\otimes}B^{\mathrm{op}})\text{-}\mathbf{mod}.$$

Let us fix a $\hat{\otimes}$-algebra A. A complex of A-$\hat{\otimes}$-modules $\{M, d\}$ is called **admissible** if it is split exact in the category **LCS**. A morphism of A-$\hat{\otimes}$-modules $f : X \to Y$ is called **admissible** if it is one of the morphisms in an admissible complex.

An additive functor $F : A\text{-}\mathbf{mod} \to \mathbf{Lin}$ is called **exact** if for every admissible complex $\{M, d\}$ the corresponding complex $\{F(M), F(d)\}$ in **Lin** is exact.

(1) A module $P \in A$-**mod** is called **projective** if the functor $\mathrm{Hom}_A(P, -)$ is exact.

(2) A module $Y \in A$-**mod** is called **flat** if the functor $(-)\hat{\otimes}_A Y : \mathbf{mod}\text{-}A \to \mathbf{Lin}$ is exact.

(3) A module $X \in A$-**mod** is called **free** if X is isomorphic to $A\hat{\otimes}E$ for some locally convex space E.

The following result is well-known, and we will use it in our paper.

Proposition 2.1. Let X be a free left A-$\hat{\otimes}$-module. Then for every admissible sequence $\{M, d\}$ of right A-$\hat{\otimes}$-modules the complex $\{M\hat{\otimes}_A X, d\hat{\otimes}\mathrm{Id}_X\}$, where $(M\hat{\otimes}_A X)_i = M_i\hat{\otimes}_A X$ splits in **LCS**.

Let A be an algebra and let M be a right A-module (or a A-bimodule, resp.). For any endomorphism $\alpha : A \to A$ denote by M_α a right A-module (or an A-bimodule, resp.), which coincides with M as an abelian group (left A-module, resp.), and whose structure of right A-module is defined by $m \circ a = m\alpha(a)$. In a similar fashion one defines ${}_\alpha M$ for left modules.

The proof of the following lemma is very similar to the proof of [2, Proposition 4.1.5].

Lemma 2.1. Let R be a $\hat{\otimes}$-algebra and let A be a R-$\hat{\otimes}$-algebra.

(1) For every (projective/flat) left module $X \in R$-**mod** the left A-$\hat{\otimes}$-module $A\hat{\otimes}_R X \in A$-**mod** is (projective/flat). Moreover, for every projective (flat) right module $X \in \mathbf{mod}$-R the right A-module $X\hat{\otimes}_R A \in \mathbf{mod}$-$A$ is (projective/flat).

(2) For every projective bimodule $P \in R$-**mod**-R and $\alpha \in \mathrm{Aut}(R)$ the bimodule $A_\alpha \hat{\otimes}_R P \hat{\otimes}_R A \in A$-**mod**-$A$ is projective.

Proof.

(1) This part of the lemma is well-known to follow from [2, Theorem 3.1.27].

(2) First of all, notice that the following isomorphism of A-R-$\hat{\otimes}$-bimodules takes place:

$$A_\alpha \hat{\otimes}_R R \xrightarrow{\sim} A_\alpha.$$

In particular, $A_\alpha \cong A$ as a left A-$\hat{\otimes}$-module. Any projective bimodule is a retract of a free bimodule, in other words, there exist a locally convex space E and a retraction $\sigma : R \hat{\otimes} E \hat{\otimes} R \to P$. Notice that the map

$$\mathrm{Id}_{A_\alpha} \otimes \sigma \otimes \mathrm{Id}_A : A \hat{\otimes} E \hat{\otimes} A \cong A_\alpha \hat{\otimes}_R R \hat{\otimes} E \hat{\otimes} R \hat{\otimes}_R A \to A_\alpha \hat{\otimes}_R P \hat{\otimes}_R A$$

is a retraction of A-$\hat{\otimes}$-bimodules, and $A \hat{\otimes} E \hat{\otimes} A$ is a free A-$\hat{\otimes}$-bimodule.

□

2.3 Homological dimensions

Let $X \in A$-**mod**. Suppose that X can be included in a following admissible complex:

$$0 \leftarrow X \xleftarrow{\varepsilon} P_0 \xleftarrow{d_0} P_1 \xleftarrow{d_1} \dots \xleftarrow{d_{n-1}} P_n \leftarrow 0,$$

where every P_i is a projective module. Then we will call such complex a **projective resolution** of X of **length** n. Furthermore, we call resolutions of form

$$0 \leftarrow X \xleftarrow{\varepsilon} P_0 \xleftarrow{d_0} P_1 \xleftarrow{d_1} \dots \xleftarrow{d_{n-1}} P_n \leftarrow P_{n+1} \leftarrow \dots$$

where $P_n \neq 0$ for all $n \geq 0$ unbounded, and we define the length of an unbounded resolution as ∞. **Flat resolutions** are defined similarly.

It is known that A-**mod** has enough projectives, therefore, one is able define the notion of a derived functor in the topological case, for example, see [2, ch 3.3]. In particular, $\mathrm{Ext}^k_A(M, N)$ and $\mathrm{Tor}^A_k(M, N)$ are defined similarly to the purely algebraic situation.

Consider an arbitrary module $M \in A$-**mod**$(\mathcal{C})$ for a category $\mathcal{C} \subseteq$ **LCS** such that A-mod$(\mathcal{C})$ has enough projectives. For example, we can consider an admissible category $\mathcal{C}$ in the sense of [10, Definition 2.4]. Then due to [2, Theorem 3.5.4] following number is well-defined and we have the following identities:

$$\begin{aligned}\mathrm{dh}^{\mathcal{C}}_A(M) &:= \min\{n \in \mathbb{Z}_{\geq 0} : \mathrm{Ext}^{n+1}_A(M, N) = 0 \text{ for every } N \in A\text{-}\mathbf{mod}(\mathcal{C})\} = \\ &= \{\text{the length of a shortest projective resolution of } M \text{ in } A\text{-}\mathbf{mod}(\mathcal{C})\} \in \{-\infty\} \cup [0, \infty].\end{aligned}$$

We define $\mathrm{dh}^{\mathcal{C}}_A(0) = -\infty$ and if every projective resolution of M is unbounded, we set $\mathrm{dh}^{\mathcal{C}}_A(M) = \infty$.

As we can see, this number doesn't depend on the choice of the category $\mathcal{C}$:

$$\begin{aligned}\mathrm{dh}^{\mathcal{C}}_A(M) &= \min\{n \in \mathbb{Z}_{\geq 0} : \mathrm{Ext}^{n+1}_A(M, N) = 0 \text{ for every } N \in A\text{-}\mathbf{mod}(\mathcal{C})\} \leq \\ &\leq \min\{n \in \mathbb{Z}_{\geq 0} : \mathrm{Ext}^{n+1}_A(M, N) = 0 \text{ for every } N \in A\text{-}\mathbf{mod}\} = \mathrm{dh}^{\mathbf{LCS}}_A(M),\end{aligned}$$

$$\begin{aligned}\mathrm{dh}^{\mathbf{LCS}}_A(M) &= \{\text{the length of a shortest projective resolution of } M \text{ in } A\text{-}\mathbf{mod}\} \leq \\ &\leq \{\text{the length of a shortest projective resolution of } M \text{ in } A\text{-}\mathbf{mod}(\mathcal{C})\} = \mathrm{dh}^{\mathcal{C}}_A(M).\end{aligned}$$

So we will denote this invariant by $\mathrm{dh}_A(M)$, and we will call it the **projective homological dimension** of M.

If A is a Fréchet algebra, and M is a left Fréchet A-module, then we can define the **weak homological dimension** of M:

$$\begin{aligned}\mathrm{w.dh}_A(M) = \min\{n\in\mathbb{Z}_{\geq 0} : \mathrm{Tor}^A_{n+1}(N,M)=0 \text{ and } \mathrm{Tor}^A_n(N,M) \text{ is Hausdorff for every } N\in \textbf{mod-}A(\textbf{Fr})\} = \\ = \{\text{the length of the shortest flat resolution of } M\}\in\{-\infty\}\cup[0,\infty].\end{aligned}$$

We define $\mathrm{w.dh}^{\mathcal{C}}_A(0)=-\infty$ and if every flat resolution of M is unbounded we set $\mathrm{dh}^{\mathcal{C}}_A(M)=\infty$.

Let A be a $\hat{\otimes}$-algebra. Then we can define the following invariants of A:

$$\mathrm{dgl}_{\mathcal{C}}(A)=\sup\{\mathrm{dh}_A(M): M\in A\textbf{-mod}(\mathcal{C})\} - \textbf{the left global dimension of } A.$$

$$\mathrm{dgr}_{\mathcal{C}}(A)=\sup\{\mathrm{dh}_{A^{\mathrm{op}}}(M): M\in \textbf{mod-}A(\mathcal{C})\} - \textbf{the right global dimension of } A.$$

$$\mathrm{db}(A)=\mathrm{dh}_{A\hat{\otimes}A^{\mathrm{op}}}(A) - \textbf{the bidimension of } A.$$

For a Fréchet algebra A we can consider **weak dimensions**.

$$\begin{aligned}\mathrm{w.dg}(A) &= \sup\{\mathrm{w.dh}_A(M): M\in A\textbf{-mod}(\textbf{Fr})\} = \\ &= \sup\{\mathrm{w.dh}_A(M): M\in \textbf{mod-}A(\textbf{Fr})\} - \textbf{the weak global dimension of } A.\end{aligned}$$

$$\mathrm{w.db}(A)=\mathrm{w.dh}_{A\hat{\otimes}A^{\mathrm{op}}}(A) - \textbf{the weak bidimension of } A.$$

Unfortunately, we are not aware whether global dimensions depend on the choice of $\mathcal{C}$. We will denote

$$\mathrm{dgl}(A):=\mathrm{dgl}_{\textbf{LCS}}(A),\quad \mathrm{dgr}(A):=\mathrm{dgr}_{\textbf{LCS}}(A).$$

For more details the reader can consult [2].

The following theorem demonstrates one of the most important properties of homological dimensions.

Theorem 2.1. [2, Proposition 3.5.5] Let A be a $\hat{\otimes}$-algebra. If $0\to X'\to X\to X''\to 0$ is an admissible sequence of left A-$\hat{\otimes}$-modules, then

$$\begin{aligned}\mathrm{dh}_A(X)&\leq \max\{\mathrm{dh}_A(X'),\mathrm{dh}_A(X'')\}\\ \mathrm{dh}_A(X')&\leq \max\{\mathrm{dh}_A(X),\mathrm{dh}_A(X'')-1\}\\ \mathrm{dh}_A(X'')&\leq \max\{\mathrm{dh}_A(X),\mathrm{dh}_A(X')+1\}.\end{aligned}$$

In particular, $\mathrm{dh}_A(X)=\max\{\mathrm{dh}_A(X'),\mathrm{dh}_A(X'')\}$ except when $\mathrm{dh}_A(X)<\mathrm{dh}_A(X'')=\mathrm{dh}_A(X')+1$.

Moreover, the same estimates hold for weak homological dimensions of Fréchet modules over Fréchet algebras, for the proof see [9].

Proposition 2.2. [9, Proposition 4.1, Corollary 4.4] Let A be a Fréchet algebra, then for every $M\in A\textbf{-mod}(\textbf{Fr})$ we have

$$\mathrm{w.dh}_A(M)=\min\{n\colon \mathrm{Ext}^{n+1}_A(M,N^*)=0 \text{ for every } N\in\textbf{mod-}A(\textbf{Fr})\},$$

where Y^* denotes the strong dual of Y.

As a corollary, for every admissible sequence $0\to X'\to X\to X''\to 0$ of left Fréchet A-$\hat{\otimes}$-modules we have the following estimates:

$$\begin{aligned}\mathrm{w.dh}_A(X)&\leq \max\{\mathrm{w.dh}_A(X'),\mathrm{w.dh}_A(X'')\}\\ \mathrm{w.dh}_A(X')&\leq \max\{\mathrm{w.dh}_A(X),\mathrm{w.dh}_A(X'')-1\}\\ \mathrm{w.dh}_A(X'')&\leq \max\{\mathrm{w.dh}_A(X),\mathrm{w.dh}_A(X')+1\}.\end{aligned}\tag{3}$$

In particular, $\mathrm{w.dh}_A(X)=\max\{\mathrm{w.dh}_A(X'),\mathrm{w.dh}_A(X'')\}$ except when $\mathrm{w.dh}_A(X)<\mathrm{w.dh}_A(X'')=\mathrm{w.dh}_A(X')+1$.

Proposition 2.3. Let R be a $\hat{\otimes}$-algebra, let A be a R-$\hat{\otimes}$-algebra, which is free as a left R-module, and let M be a right R-$\hat{\otimes}$-module.

(1) For every projective resolution of M in **mod**-R

$$0 \leftarrow M \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \dots$$

the complex

$$0 \leftarrow M\hat{\otimes}_R A \leftarrow P_0\hat{\otimes}_R A \leftarrow P_1\hat{\otimes}_R A \leftarrow P_2\hat{\otimes}_R A \leftarrow \dots \tag{4}$$

is a projective resolution of $M\hat{\otimes}_R A$ in the category of right A-$\hat{\otimes}$-modules. In particular,

$$\mathrm{dh}_{A^{\mathrm{op}}}(M\hat{\otimes}_R A) \leq \mathrm{dh}_{R^{\mathrm{op}}}(M).$$

(2) Moreover, assume that M, A and R are all metrizable as locally convex spaces. Then for every flat resolution of M in **mod**-R

$$0 \leftarrow M \leftarrow F_0 \leftarrow F_1 \leftarrow F_2 \leftarrow \dots$$

the complex

$$0 \leftarrow M\hat{\otimes}_R A \leftarrow F_0\hat{\otimes}_R A \leftarrow F_1\hat{\otimes}_R A \leftarrow F_2\hat{\otimes}_R A \leftarrow \dots \tag{5}$$

is a flat resolution of $M\hat{\otimes}_R A$ in the category of right A-$\hat{\otimes}$-modules. In particular,

$$\mathrm{w.dh}_{A^{\mathrm{op}}}(M\hat{\otimes}_R A) \leq \mathrm{w.dh}_{R^{\mathrm{op}}}(M).$$

Proof.

(1) Lemma 2.1 implies that $P_i\hat{\otimes}_R A$ is a projective right A-$\hat{\otimes}$-module for all i. The complex (4) is admissible, because the functor $(-)\hat{\otimes}_R A$ for a free A preserves admissibility (due to Proposition 2.1), so it defines a projective resolution of $M\hat{\otimes}_R A$ in the category **mod**-A.

(2) Due to Lemma 2.1 the modules $F_i\hat{\otimes}_R A$ are flat right A-$\hat{\otimes}$-modules for all i. Then the rest of the proof is the same as in (1).

□

Lemma 2.2. Let R be a $\hat{\otimes}$-algebra, and consider an R-$\hat{\otimes}$-algebra A.

1. If A is projective as a right R-$\hat{\otimes}$-module, and M is projective as a right A-$\hat{\otimes}$-module, then M is projective as a right R-$\hat{\otimes}$-module. In this case for every $X \in$ **mod**-A we have

$$\mathrm{dh}_{R^{\mathrm{op}}}(X) \leq \mathrm{dh}_{A^{\mathrm{op}}}(X). \tag{6}$$

2. If A is projective as a right R-$\hat{\otimes}$-module, and M is flat as a right A-$\hat{\otimes}$-module, then M is flat as a right R-$\hat{\otimes}$-module. If R, A are Fréchet algebras, then for every $X \in$ **mod**-A(**Fr**) we have

$$\mathrm{w.dh}_{A^{\mathrm{op}}}(X) \leq \mathrm{w.dh}_{R^{\mathrm{op}}}(X). \tag{7}$$

Proof. (1) Due to [2, Theorem 3.1.27] we can fix an isomorphism of right R-$\hat{\otimes}$-modules $A \oplus S \cong X\hat{\otimes}R$ for some right R-$\hat{\otimes}$-module S and for some complete locally convex space X.

Suppose that M is a projective right A-$\hat{\otimes}$-module. Then due to [2, Theorem 3.1.27] there exists a right A-$\hat{\otimes}$-module N such that, for some complete locally convex space E, we have $M \oplus N \cong E\hat{\otimes}A$ as right A-modules, hence there exist isomorphisms in **mod**-R

$$M \oplus N \oplus (E\hat{\otimes}S) \cong E\hat{\otimes}A \oplus (E\hat{\otimes}S) \cong E\hat{\otimes}(X\hat{\otimes}R) = (E\hat{\otimes}X)\hat{\otimes}R.$$

(2) As in the proof of [6, Lemma 7.2.2], we notice that for any right R-$\hat{\otimes}$-module N we have

$$M\hat{\otimes}_R N \cong M\hat{\otimes}_A(A\hat{\otimes}_R N),$$

and this immediately implies our statement.

□

Lemma 2.3. If R is a $\hat{\otimes}$-algebra, $\alpha : R \to R$ is an automorphism and M is a right R-$\hat{\otimes}$-module, then $\mathrm{dh}_{R^{\mathrm{op}}}(M_\alpha) = \mathrm{dh}_{R^{\mathrm{op}}}(M)$ and if R is a Fréchet algebra, and M is a Fréchet R-$\hat{\otimes}$-module, then $\mathrm{w.dh}_{R^{\mathrm{op}}}(M_\alpha) = \mathrm{w.dh}_{R^{\mathrm{op}}}(M)$.

Proof. The proof relies on the fact that $(\cdot)_\alpha : \mathbf{mod}\text{-}R \to \mathbf{mod}\text{-}R$ and ${}_\alpha(\cdot) : R\text{-}\mathbf{mod} \to R\text{-}\mathbf{mod}$ can be viewed as functors between $\mathbf{mod}\text{-}R$ and $R\text{-}\mathbf{mod}$, which preserve admissibility of morphisms, projectivity and flatness of modules.

Indeed, if $f : M \to N$ is an admissible module homomorphism, then $f_\alpha : M_\alpha \to N_\alpha$ is admissible, because $\Box f_\alpha = \Box f$, where $\Box : \mathbf{mod}\text{-}R \to \mathbf{LCS}$ denotes the forgetful functor. The same goes for ${}_\alpha(\cdot)$.

Let $P \in \mathbf{mod}\text{-}R$ be projective. Let us show that the functor $\mathrm{Hom}(P_\alpha, -)$ is exact. However, we know that for every admissible epimorphism $\varphi : X \to Y$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}(P_\alpha, X) & \xrightarrow{\varphi_*} & \mathrm{Hom}(P_\alpha, Y) \\ \downarrow{\scriptstyle (\cdot)_{\alpha^{-1}}} & & \downarrow{\scriptstyle (\cdot)_{\alpha^{-1}}} \\ \mathrm{Hom}(P, X_{\alpha^{-1}}) & \longrightarrow & \mathrm{Hom}(P, Y_{\alpha^{-1}}) \end{array},$$

where the left and right arrows are isomorphisms. However, projectivity of P implies that the bottom arrow is a surjection, therefore, the top arrow is a surjection as well.

Let $F \in \mathbf{mod}\text{-}R$ be flat. Then $F_\alpha\hat{\otimes}_R X \simeq F\hat{\otimes}_R({}_{\alpha^{-1}}X)$ and we already know that ${}_{\alpha^{-1}}(\cdot)$ preserves admissibility. □

3 Estimates for the bidimension and projective global dimensions of holomorphic Ore extensions

3.1 Bimodules of relative differentials

Firstly, let us give several necessary algebraic definitions.

Definition 3.1. Let S be an algebra, A be an S-algebra and M be an A-bimodule. Then an additive map $\delta : A \to M$ is called an **S-derivation** if the following relations hold for every $a, b \in A$ and $s \in S$:

1. $\delta(ab) = \delta(a)b + a\delta(b)$,
2. $\delta(s) = 0$.

Remark. In particular, any S-derivation is an S-linear map: for every $a \in A$ and $s \in S$ we have $\delta(sa) = s\delta(a)$ and $\delta(as) = \delta(a)s$.

Example 3.1. Let α be an endomorphism of an algebra A. Then an α-derivation is precisely an $\mathbb{C}$-derivation $\delta : A \to {}_\alpha A$.

The following definition is due to J. Cuntz and D. Quillen, see [1]:

Definition 3.2. Suppose that S is an algebra and (A, η) is an S-algebra, where $\eta : S \to A$ is an algebra homomorphism. Denote by $\overline{A} = A/\mathrm{Im}(\eta(S))$ the S-bimodule quotient. Then we can define the bimodule of **relative differential 1-forms** $\Omega^1_S(A) = A \otimes_S \overline{A}$. The elementary tensors in $\Omega^1_S(A)$ are usually denoted by $a_0 \otimes \overline{a_1} = a_0 da_1$. The A-bimodule structure on $\Omega^1_S(A)$ is uniquely defined by the following relations:

$$b \circ (a_0 da_1) = ba_0 da_1, \quad (a_0 da_1) \circ b = a_0 d(a_1 b) - a_0 a_1 db.$$

The bimodule of relative differential 1-forms together with the canonical S-derivation

$$d_A : A \to \Omega^1_S(A), \quad d_A(a) = 1 \otimes \overline{a} = da$$

has the following universal property:

Proposition 3.1 ([1], Proposition 2.4)**.** For every A-bimodule M and an S-derivation $D : A \to M$ there is a unique A-bimodule morphism $\varphi : \Omega^1_S(A) \to M$ such that the following diagram is commutative:

$$\begin{array}{ccc} \Omega^1_S(A) & \overset{\exists!\,\varphi}{\dashrightarrow} & M \\ {\scriptstyle d_A}\uparrow & \nearrow_{D} & \\ A & & \end{array} \tag{8}$$

Proposition 3.2 ([1], Proposition 2.5)**.** The following sequence of A-bimodules is exact:

$$0 \longrightarrow \Omega^1_S A \xrightarrow{\ j\ } A \otimes_S A \xrightarrow{\ m\ } A \longrightarrow 0, \tag{9}$$

where $j(a_0 \otimes \overline{a_1}) = j(a_0 da_1) = a_0 a_1 \otimes 1 - a_0 \otimes a_1$ and m denotes the multiplication.

In the appendix of this paper we compute the bimodule of relative differential 1-forms of Ore extensions, see Proposition A.1.

3.2 A topological version of the bimodule of relative differentials

The following definition serves as a topological version of Definition 3.1.

Definition 3.3. Let R be a $\hat{\otimes}$-algebra, and suppose that (A, η) is a R-$\hat{\otimes}$-algebra. Denote by $(A/\overline{\mathrm{Im}(\eta(R))})^{\sim} = \overline{A}$ the R-$\hat{\otimes}$-bimodule quotient, where $(\cdot)^{\sim}$ stands for the completion. Then we can define the **(topological) bimodule of relative differential 1-forms** $\widehat{\Omega}^1_R(A) := A \otimes_R \overline{A}$. The elementary tensors are usually denoted by $a_0 \hat{\otimes} \overline{a_1} = a_0 \mathrm{d} a_1$.

The structure of A-$\hat{\otimes}$-bimodule on $\widehat{\Omega}^1_R(A)$ is uniquely defined by the following relations:

$$b \circ (a_0 \mathrm{d}a_1) = b a_0 \mathrm{d}a_1, (a_0 \mathrm{d}a_1) \circ b = a_0 \mathrm{d}(a_1 b) - a_0 a_1 \mathrm{d}b \quad \text{for every } a_0, a_1, b \in A.$$

Remark. To avoid confusion with the algebraic bimodules of differential 1-forms, here we use the notation $\widehat{\Omega}^1_R(A)$, unlike in [8].

We will need to recall the following propositions related to topological bimodule of relative differential 1-forms.

Theorem 3.1. [8, p. 99] For every A-$\hat{\otimes}$-bimodule M and a continuous R-derivation $D : A \to M$ there exists a unique A-$\hat{\otimes}$-bimodule morphism $\varphi : \widehat{\Omega}^1_R(A) \to M$ such that the following diagram is commutative:

$$\begin{array}{ccc} \widehat{\Omega}^1_R(A) & \overset{\exists!\,\varphi}{\dashrightarrow} & M \\ {\scriptstyle d_A}\uparrow & \nearrow_{D} & \\ A & & \end{array}, \tag{10}$$

where $d_A(a) = da$.

Proposition 3.3. [8, Proposition 7.2] The short exact sequence

$$0 \longrightarrow \widehat{\Omega}^1_R(A) \xrightarrow{\ j\ } A \hat{\otimes}_R A \xrightarrow{\ m\ } A \longrightarrow 0,$$

where $j(a_0 da_1) = a_0 \otimes a_1 - a_0 a_1 \otimes 1$ and $m(a_0 \otimes a_1) = a_0 a_1$, splits in the categories A-**mod**-R and R-**mod**-A.

3.3 Holomorphic Ore extensions

To give the definition of the holomorphic Ore extension of a $\hat{\otimes}$-algebra, associated with an endomorphism and derivation, we need to recall the definition of localizable morphisms.

For a locally convex space X let us denote the space of all continuous linear maps $X \to X$ by $L(X)$.

Definition 3.4. [8, Definition 4.1] Let X be a LCS and consider a family $\mathcal{F} \subset L(X)$ of continuous linear maps $X \to X$.

Then a seminorm $\|\cdot\|$ on X is called **$\mathcal{F}$-stable** if for every $T \in \mathcal{F}$ there exists a constant $C_T > 0$, such that

$$\|Tx\| \le C_T \|x\| \quad \text{for every } x \in A.$$

Definition 3.5.

(1) Let X be a LCS.

A family of continuous linear operators $\mathcal{F} \subset L(X)$ is called **localizable**, if the topology on X can be defined by a family of $\mathcal{F}$-stable seminorms.

(2) Let A be an Arens-Michael algebra.

A family of continuous linear operators $\mathcal{F} \subset L(A)$ is called m-**localizable**, if the topology on A can be defined by a family of submultiplicative $\mathcal{F}$-stable seminorms.

Now we will state the theorem which proves the existence of certain $\hat{\otimes}$-algebras which would be reasonable to call the "holomorphic Ore extensions". By $\mathcal{O}(\mathbb{C})$ we denote the space of entire holomorphic functions equipped with the topology of uniform convergence on compact subsets of $\mathbb{C}$.

Theorem 3.2. [8, Section 4.1] Let A be a $\hat{\otimes}$-algebra and suppose that $\alpha : A \to A$ is a localizable endomorphism of A, $\delta : A \to A$ is a localizable α-derivation of A.

Then there exists a unique multiplication on the tensor product $A \hat{\otimes} \mathcal{O}(\mathbb{C})$, such that the following conditions are satisfied:

(1) The resulting algebra, which is denoted by $\mathcal{O}(\mathbb{C}, A; \alpha, \delta)$, is an A-$\hat{\otimes}$-algebra.

(2) The natural inclusion

$$A[z; \alpha, \delta] \hookrightarrow \mathcal{O}(\mathbb{C}, A; \alpha, \delta)$$

induced by the inclusion $\mathbb{C}[z] \to \mathcal{O}(\mathbb{C})$, where z stands for the identity map $\mathbb{C} \to \mathbb{C}$, is an algebra homomorphism.

(3) Moreover, if the pair (α, δ) is m-localizable, then for every Arens-Michael A-$\hat{\otimes}$-algebra B the following natural isomorphism takes place:

$$\mathrm{Hom}(A[z; \alpha, \delta], B) \cong \mathrm{Hom}(\mathcal{O}(\mathbb{C}, A; \alpha, \delta), B).$$

Moreover, let α be invertible, and suppose that the pair (α, α^{-1}) is localizable. Then there exists a unique multiplication on the tensor product $A \hat{\otimes} \mathcal{O}(\mathbb{C}^\times)$, such that the following conditions are satisfied:

(1) The resulting algebra, which is denoted by $\mathcal{O}(\mathbb{C}^\times, A; \alpha)$, is a $\hat{\otimes}$-algebra.

(2) The natural inclusion

$$A[z; \alpha, \alpha^{-1}] \hookrightarrow \mathcal{O}(\mathbb{C}^\times, A; \alpha)$$

induced by the inclusion $\mathbb{C}[z, z^{-1}] \to \mathcal{O}(\mathbb{C}^\times)$, where z stands for the identity map $\mathbb{C} \to \mathbb{C}$, is an algebra homomorphism.

(3) Moreover, if the pair (α, α^{-1}) is m-localizable, then for every Arens-Michael A-$\hat{\otimes}$-algebra B the following natural isomorphism takes place:

$$\mathrm{Hom}(A[z; \alpha, \alpha^{-1}], B) \cong \mathrm{Hom}(\mathcal{O}(\mathbb{C}^\times, A; \alpha), B).$$

And if we replace the word "localizable" with "m-localizable" in this theorem, then the resulting algebras will become Arens-Michael algebras.

By considering $A_{\mathrm{op}}[z;\alpha,\tilde{\delta}]$ for an endomorphism α and an (opposite) derivation $\tilde{\delta}$, we can formulate a version of Theorem 3.2 for opposite Ore extensions, the proof is basically the same in this case. As a result, we can construct $\hat{\otimes}$-algebras $\mathcal{O}_{op}(\mathbb{C},A;\alpha,\delta)$ and $\mathcal{O}_{op}(\mathbb{C}^\times,A;\alpha)$ with underlying LCS isomorphic to $\mathcal{O}\hat{\otimes}A$, and multiplication extended from $A_{\mathrm{op}}[z;\alpha,\tilde{\delta}]$. In fact, we need this to prove the following corollary:

Corollary 3.1. Let R be a $\hat{\otimes}$-algebra equipped with an endomorphism $\alpha : R \to R$ and a α-derivation δ such that the pair (α,δ) is localizable.

(1) If $A = \mathcal{O}(\mathbb{C},R;\alpha,\delta)$, then A is free as a left R-$\hat{\otimes}$-module.

(2) If α is invertible, and (α,α^{-1}) is a localizable pair, then $A = \mathcal{O}(\mathbb{C}^\times,R;\alpha)$ is free as a left and right R-$\hat{\otimes}$-module.

(3) If α is invertible, and the pairs (α,δ) and $(\alpha^{-1},\delta\alpha^{-1})$ are m-localizable, then $A = \mathcal{O}(\mathbb{C},R;\alpha,\delta)$ is free as a left and right R-$\hat{\otimes}$-module.

Proof.

1. This already follows from the fact that $\mathcal{O}(\mathbb{C},R;\alpha,\delta)$ is isomorphic to $R\hat{\otimes}\mathcal{O}(\mathbb{C})$ in R-**mod**.

2. This is the immediate corollary of [8, Lemma 4.12]. It shows that the map

$$\tau : \mathcal{O}(\mathbb{C}^\times)\hat{\otimes}R \to R\mathcal{O}(\mathbb{C}^\times), \quad \tau(z^n \otimes r) = \alpha^n(r) \otimes z^n,$$

where z stands for the identity map $\mathbb{C} \to \mathbb{C}$, is a well-defined continuous left R-$\hat{\otimes}$-module map. However, we can define its inverse as follows:

$$\gamma : R\hat{\otimes}\mathcal{O}(\mathbb{C}) \to \mathcal{O}(\mathbb{C})\hat{\otimes}R, \quad \gamma(r \otimes z^n) = z^n \otimes \alpha^{-n}(r).$$

Thus γ will be a continuous inverse of τ in **LCS**. It remains to construct an isomorphism of right R-$\hat{\otimes}$-modules

$$\mathcal{O}(\mathbb{C}^\times,R;\alpha) \to \mathcal{O}(\mathbb{C}^\times)\hat{\otimes}R, \quad rz^n \mapsto r \otimes z^n \mapsto \gamma(r \otimes z^n).$$

It is, indeed, an R-$\hat{\otimes}$-module homomorphism because for any $r, r' \in R$ we have

$$(rz^n)r' = r\alpha^n(r')z^n \mapsto (r\alpha^n(r')) \otimes z^n \mapsto \gamma((r\alpha^n(r')) \otimes z^n) = (z^n \otimes \alpha^{-n}(r))r' = \gamma(r \otimes z^n)r'.$$

3. Here the idea is similar. Observe that the isomorpisms (1) can be extended via the universal properties to the topological isomorphisms of R-$\hat{\otimes}$-algebras as follows: consider the algebra homomorphisms

$$R[t;\alpha,\delta] \xrightarrow{\sim} R_{op}[t;\alpha^{-1},-\delta\alpha^{-1}] \hookrightarrow \mathcal{O}_{\mathrm{op}}(\mathbb{C},R;\alpha^{-1},-\delta\alpha^{-1}),$$
$$R_{op}[t;\alpha^{-1},-\delta\alpha^{-1}] \xrightarrow{\sim} R[t;\alpha,\delta] \hookrightarrow \mathcal{O}(\mathbb{C},R;\alpha,\delta).$$

Universal property allows us to extended these maps to continuous algebra homomorphisms

$$\mathcal{O}(\mathbb{C},R;\alpha,\delta) \to \mathcal{O}_{\mathrm{op}}(\mathbb{C},R;\alpha^{-1},-\delta\alpha^{-1}),$$
$$\mathcal{O}_{\mathrm{op}}(\mathbb{C},R;\alpha^{-1},-\delta\alpha^{-1}) \to \mathcal{O}(\mathbb{C},R;\alpha,\delta).$$

Now notice that the extensions are continuous and inverse on dense subsets of the holomorphic Ore extensions, therefore, they are actually inverse to each other. However, the opposite Ore extensions are free as right R-$\hat{\otimes}$-modules by definition.

Notice that to apply the universal properties we need the the morphisms to be m-localizable, so that the holomorphic Ore extensions are Arens-Michael algebras.

□

3.4 Upper estimates for the bidimension

Theorem 3.3. Suppose that R is a $\hat{\otimes}$-algebra, and A is one of the two $\hat{\otimes}$-algebras:

(1) $A = \mathcal{O}(\mathbb{C}, R; \alpha, \delta)$, where the pair $\{\alpha, \delta\}$ is localizable.

(2) $A = \mathcal{O}(\mathbb{C}^\times, R; \alpha)$, where the pair $\{\alpha, \alpha^{-1}\}$ is localizable.

Then we have

$$\mathrm{db}(A) \leq \mathrm{db}(R) + 1.$$

Proof. Due to Proposition 3.3 and Proposition A.2, we have the following sequence of A-$\hat{\otimes}$-bimodules, which splits in the categories R-**mod**-A and A-**mod**-R:

$$0 \longrightarrow A_\alpha \hat{\otimes}_R A \xrightarrow{j} A \hat{\otimes}_R A \xrightarrow{m} A \longrightarrow 0, \tag{11}$$

where m is the multiplication operator. Let

$$0 \leftarrow R \leftarrow P_0 \leftarrow \cdots \leftarrow P_n \leftarrow 0 \tag{12}$$

be a projective resolution of R in R-**mod**-R. Notice that (12) splits in R-**mod** and **mod**-R, because all objects in the resolution are projective as left and right R-$\hat{\otimes}$-modules ([2, Corollary 3.1.18]). Therefore, we can apply the functors $A_\alpha \hat{\otimes}_R (-)$ and $A \hat{\otimes}_R (-)$ to (12) and the resulting complexes of A-R-$\hat{\otimes}$-bimodules are still admissible:

$$0 \leftarrow A \leftarrow A \hat{\otimes}_R P_0 \leftarrow \cdots \leftarrow A \hat{\otimes}_R P_n \leftarrow 0 \tag{13}$$

$$0 \leftarrow A_\alpha \leftarrow A_\alpha \hat{\otimes}_R P_0 \leftarrow \cdots \leftarrow A_\alpha \hat{\otimes}_R P_n \leftarrow 0. \tag{14}$$

Recall that A is a free left R-$\hat{\otimes}$-module due to Corollary 3.1, so the functor $(-)\hat{\otimes}_R A$ preserves admissibility, due to Proposition 2.1, therefore the following complexes of A-$\hat{\otimes}$-bimodules are admissible:

$$0 \leftarrow A \hat{\otimes}_R A \leftarrow A \hat{\otimes}_R P_0 \hat{\otimes}_R A \leftarrow \cdots \leftarrow A \hat{\otimes}_R P_n \hat{\otimes}_R A \leftarrow 0 \tag{15}$$

$$0 \leftarrow A_\alpha \hat{\otimes}_R A \leftarrow A_\alpha \hat{\otimes}_R P_0 \hat{\otimes}_R A \leftarrow \cdots \leftarrow A_\alpha \hat{\otimes}_R P_n \hat{\otimes}_R A \leftarrow 0 \tag{16}$$

Lemma 2.1 implies that (15) and (16) define projective resolutions for $A \hat{\otimes}_R A$ and $A_\alpha \hat{\otimes}_R A$. Now we can apply Theorem 2.1 to (11), so we get

$$\mathrm{db}(\mathrm{A}) = \mathrm{dh}_{A^e}(A) \leq \max\{\mathrm{dh}_{A^e}(A \hat{\otimes}_R A), \mathrm{dh}_{A^e}(A_\alpha \hat{\otimes}_R A) + 1\} \leq n + 1.$$

In other words, we have obtained the desired estimate

$$\mathrm{db}(A) \leq \mathrm{db}(R) + 1.$$

□

3.5 Upper estimates for the right global and weak global dimensions

Theorem 3.4. Let R be a $\hat{\otimes}$-algebra. Suppose that A is one of the two $\hat{\otimes}$-algebras:

(1) $A = \mathcal{O}(\mathbb{C}, R; \alpha, \delta)$, where α is invertible, and the pair $\{\alpha, \delta\}$ is localizable.

(2) $A = \mathcal{O}(\mathbb{C}^\times, R; \alpha)$, where the pair $\{\alpha, \alpha^{-1}\}$ is localizable.

Then the right global dimension of A can be estimated as follows:

$$\mathrm{dgr}(A) \leq \mathrm{dgr}(R) + 1,$$

and a similar estimate holds for the weak dimensions if R is a Fréchet algebra:

$$\mathrm{w.dg}(A) \leq \mathrm{w.dg}(R) + 1.$$

Proof. Suppose that M is a right A-$\hat{\otimes}$-module. Then we can apply the functor $M\hat{\otimes}_A(-)$ to the sequence (11). Notice that the resulting sequence of right A-$\hat{\otimes}$-modules

$$0 \longrightarrow M\hat{\otimes}_A A_\alpha \hat{\otimes}_R A \xrightarrow{\mathrm{Id}_M\otimes j} M\hat{\otimes}_A A\hat{\otimes}_R A \xrightarrow{\mathrm{Id}_M\otimes m} M\hat{\otimes}_A A \longrightarrow 0$$

is isomorphic to the sequence

$$0 \longrightarrow M_\alpha \hat{\otimes}_R A \xrightarrow{j'} M\hat{\otimes}_R A \xrightarrow{m} M \longrightarrow 0. \tag{17}$$

Since (11) splits in A-**mod**-R, (17) splits in **mod**-R, in particular, this is an admissible short exact sequence.

Now notice that we can apply Theorem 2.1 to (17), so we get

$$\mathrm{dh}_{A^{\mathrm{op}}}(M) \le \max\{\mathrm{dh}_{A^{\mathrm{op}}}(M\hat{\otimes}_R A), \mathrm{dh}_{A^{\mathrm{op}}}(M_\alpha\hat{\otimes}_R A)+1\} \le \mathrm{dh}_{R^{\mathrm{op}}}(M)+1$$

due to Corollary 3.1, Proposition 2.3 and Lemma 2.3. Hence, the following estimate holds:

$$\mathrm{dgr}(A) \le \mathrm{dgr}(R)+1.$$

For the weak dimensions we apply the second part of Proposition 2.2 to the sequence (17):

$$\mathrm{w.dh}_{A^{\mathrm{op}}}(M) \le \max\{\mathrm{w.dh}_{A^{\mathrm{op}}}(M\hat{\otimes}_R A), \mathrm{w.dh}_{A^{\mathrm{op}}}(M_\alpha\hat{\otimes}_R A)\}+1 \le \mathrm{w.dh}_{R^{\mathrm{op}}}(M)+1.$$

□

3.6 Lower estimates

In order to obtain lower estimates, we need to formulate the following lemma:

Proposition 3.4. Suppose that R is a $\hat{\otimes}$-algebra, and A is a R-$\hat{\otimes}$-algebra. Assume that there exists a morphism of left R-$\hat{\otimes}$-modules $\varphi : A \to R\hat{\otimes}E$ such that $\varphi(1) = 1\otimes x$ for some $x \in E$.

Then $i : M \to M\hat{\otimes}_R A, i(m) = m\otimes 1$ is an admissible monomorphism for every $M \in$ **mod**-R. In particular, it is a coretraction between the underlying locally convex spaces.

Proof. Look at the following diagram:

$$M \xrightarrow{i} M\hat{\otimes}_R A \xrightarrow{\mathrm{Id}_M\otimes\varphi} M\hat{\otimes}_R R\hat{\otimes}E \xrightarrow{\pi\otimes\mathrm{Id}_E} M\hat{\otimes}E, \qquad M \xrightarrow{m\to m\otimes x} M\hat{\otimes}E$$

where $\pi : M\hat{\otimes}_R R \to M, \pi(m\otimes r) = mr$.

Due to the Hahn-Banach theorem there exists a functional $f \in E^*$ such that $f(x) = 1$, so the map $m \to m\otimes x$ admits a right inverse, which is uniquely defined by $n\otimes y \to f(y)n$, therefore $m \mapsto m\otimes x$ is a coretraction therefore i is a coretraction as well. □

Proposition 3.5. Let R be a Fréchet algebra, and assume that $\mathrm{dgr}_{\mathbf{Fr}}(R) < \infty$. Suppose that the following conditions hold:

(1) Let A be a Fréchet R-$\hat{\otimes}$-algebra which is a free left R-$\hat{\otimes}$-module, moreover, we can choose an isomorphism of left R-$\hat{\otimes}$-modules $\varphi : A \to R\hat{\otimes}E$ in such a way that $\varphi(1) = 1\otimes x$ for some $x \in E$.

(2) A is projective as a right R-$\hat{\otimes}$-module.

Then $\mathrm{dgr}_{\mathbf{Fr}}(R) \le \mathrm{dgr}_{\mathbf{Fr}}(A)$ and $\mathrm{w.dg}(R) \le \mathrm{w.dg}(A)$.

Proof. Fix a Fréchet module $M \in \mathbf{mod}\text{-}R(\mathbf{Fr})$ such that $\mathrm{dh}_{R^{\mathrm{op}}}(M) = \mathrm{dgr}_{\mathbf{Fr}}(R) = n$. Proposition 3.4 states that the map $M \to M \hat{\otimes}_R A$, $m \to m \otimes 1$ is an admissible monomorphism, so there exists a short admissible sequence

$$0 \to M \overset{i}{\hookrightarrow} M \hat{\otimes}_R A \to N \to 0.$$

Notice that $\mathrm{dh}_{R^{\mathrm{op}}}(M) = \mathrm{dgr}_{\mathbf{Fr}}(R)$ and $\mathrm{dh}_{R^{\mathrm{op}}}(N) \leq \mathrm{dgr}_{\mathbf{Fr}}(R)$, therefore $\mathrm{dh}_{R^{\mathrm{op}}}(M \hat{\otimes}_R A) = \mathrm{dgr}_{\mathbf{Fr}}(R)$ due to Theorem 2.1.

Now recall that A is projective as a right R-$\hat{\otimes}$-module, therefore, due to Lemma 2.2, we have

$$\mathrm{dgr}_{\mathbf{Fr}}(R) = \mathrm{dh}_{R^{\mathrm{op}}}(M \hat{\otimes}_R A) \overset{L2.2}{\leq} \mathrm{dh}_{A^{\mathrm{op}}}(M \hat{\otimes}_R A),$$

which implies $\mathrm{dgr}_{\mathbf{Fr}}(R) \leq \mathrm{dgr}_{\mathbf{Fr}}(A)$.

A similar argument works for weak dimensions as well: we find a module $M' \in \mathbf{mod}\text{-}R(\mathbf{Fr})$ of maximal dimension: $\mathrm{w.dh}_{R^{op}}(M) = \mathrm{w.dg}(R)$. Then we prove

$$\mathrm{w.dg}(R) = \mathrm{w.dh}_{R^{\mathrm{op}}}(M \hat{\otimes}_R A) \overset{L2.2}{\leq} \mathrm{w.dh}_{A^{\mathrm{op}}}(M \hat{\otimes}_R A) \leq \mathrm{w.dg}(A)$$

in a similar fashion by using Proposition 2.2 and Lemma 2.2.

□

As a quick corollary from Proposition 3.5 and Corollary 3.1 we obtain lower estimates for the homological dimensions.

Theorem 3.5. Let R be a Fréchet algebra, and suppose that $\mathrm{dgr}_{\mathbf{Fr}}(R) < \infty$ and A is one of the two $\hat{\otimes}$-algebras:

(1) $A = \mathcal{O}(\mathbb{C}, R; \alpha, \delta)$, where α is invertible, and the pairs (α, δ) and $(\alpha^{-1}, \delta\alpha^{-1})$ are m-localizable.

(2) $A = \mathcal{O}(\mathbb{C}^{\times}, R; \alpha)$, where the pair (α, α^{-1}) is m-localizable.

Then the conditions of Proposition 3.5 are satisfied. As a corollary, we have the following estimates:

$$\mathrm{dgr}_{\mathbf{Fr}}(R) \leq \mathrm{dgr}_{\mathbf{Fr}}(A), \quad \mathrm{w.dg}(R) \leq \mathrm{w.dg}(A).$$

Proof. We have nothing to prove, because Corollary 3.1 ensures that A is free as a left and right R-$\hat{\otimes}$-module, both conditions follow from this and the construction of holomorphic Ore extensions. □

4 Homological dimensions of smooth crossed products by $\mathbb{Z}$

First of all, let us recall the definition of the space of rapidly decreasing sequences:

$$\begin{aligned} s = &\left\{ (a_n) \in \mathbb{C}^{\mathbb{Z}} : \|a\|_k = \sup_{n \in \mathbb{Z}} |a_n|(|n|+1)^k < \infty \ \forall k \in \mathbb{N} \right\} = \\ \cong &\left\{ (a_n) \in \mathbb{C}^{\mathbb{Z}} : \|a\|_k^2 = \sum_{n \in \mathbb{Z}} |a_n|^2(|n|+1)^{2k} < \infty \ \forall k \in \mathbb{N} \right\} = \\ \cong &\left\{ (a_n) \in \mathbb{C}^{\mathbb{Z}} : \|a\|_k = \sum_{n \in \mathbb{Z}} |a_n|(|n|+1)^k < \infty \ \forall k \in \mathbb{N} \right\}. \end{aligned}$$

It is well-known that s is a nuclear Fréchet space.

The following definitions and theorems are due to L. Schweitzer, see [11] or [7] for more detail.

Definition 4.1. Suppose that R is a Fréchet algebra equipped with an automorphism $\alpha \in \mathrm{Aut}(R)$. Then α defines an *m-tempered* $\mathbb{Z}$-action if the topology on R can be defined by a sequence of submultiplicative seminorms $\{\|\cdot\|_m : m \in \mathbb{N}\}$ such that for every $m, t \in \mathbb{N}$ there exists a polynomial $p \in \mathbb{R}[t]$ satisfying

$$\left\|\alpha^t(r)\right\|_m \leq |p(t)| \, \|r\|_m$$

for any $t \in \mathbb{Z}$ and $r \in R$.

Definition 4.2. Let A be a Fréchet algebra with a fixed generating system of seminorms $\{\|\cdot\|_\lambda, \lambda \in \Lambda\}$ Then we can define the following locally convex space:

$$\mathscr{S}(\mathbb{Z}, A) = \left\{ f = (f_m)_{m\in\mathbb{Z}} \in A^{\mathbb{Z}} : \|f\|_{\lambda,k} := \sum_{n\in\mathbb{Z}} \|f_n\|_\lambda (|n|+1)^k < \infty \text{ for all } \lambda \in \Lambda, k \in \mathbb{N} \right\}.$$

Theorem 4.1 ([11], Theorem 3.1.7)**.** Let R be a Fréchet-Arens-Michael algebra with an m-tempered $\mathbb{Z}$-action. Then the space $\mathscr{S}(\mathbb{Z}, R)$ endowed with the multiplication

$$(f * g)_k = \sum_{n\in\mathbb{Z}} f_n \alpha^n(g_{k-n}), \quad f, g \in \mathscr{S}(\mathbb{Z}, R).$$

becomes a Fréchet-Arens-Michael algebra. This algebra is denoted by $\mathscr{S}(G, \mathbb{Z}; \alpha)$ and called the **smooth crossed product** by $\mathbb{Z}$.

Proposition 4.1. Consider the following multiplication on $\mathscr{S}(\mathbb{Z}, R)$:

$$(f *' g)_k = \sum_{m\in\mathbb{Z}} \alpha^{-m}(f_{k-m}) g_m.$$

Then the following locally convex algebra isomorphism takes place:

$$i : \mathscr{S}(\mathbb{Z}, R; \alpha) \to (\mathscr{S}(\mathbb{Z}, R), *'), \quad i(f)_k = \alpha^{-k}(f_k).$$

Moreover, this is an isomorphism of unital algebras.

Proof. First of all, let us prove that i is a topological isomorphism of locally convex spaces. First of all, let us show that i is well-defined and continuous. Fix a countable non-decreasing generating family $\{\|\cdot\|_t\}_{t\in\mathbb{N}}$ of seminorms on R given by Definition 4.2. Then

$$||i(f)||_{t,k} = \sum_{n\in\mathbb{Z}} ||\alpha^{-n}(f_n)||_t (|n|+1)^k.$$

Because the action is m-tempered, we know that for all $n \in \mathbb{Z}$ we have $||\alpha^n(f_n)||_t \leq |p(n)|||f_n||_t$ for some $t' \in \mathbb{N}$ and $C > 1$. Therefore,

$$||i(f)||_{t,k} \leq \sum_{n\in\mathbb{Z}} ||f_n||_t |p(-n)|(|n|+1)^k.$$

Considering the fact that p is a polynomial, we can always find a constant $C' > 1$ and $k' > k$ such that

$$|p(-n)|(|n|+1)^k \leq C'(|n|+1)^{k'},$$

therefore,

$$||i(f)||_{t,k} \leq C'||f||_{t,k'}.$$

Observe that the map $(f_k) \mapsto (\alpha^k(f_k))$ is a continuous inverse of i. Therefore, i is an isomorphism of LCS. Now notice that

$$(i(f) *' i(g))(k) = \sum_{m\in\mathbb{Z}} \alpha^{-m}(i(f)_{k-m}) i(g)_m = \sum_{m\in\mathbb{Z}} \alpha^{-k}(f_{k-m}) \alpha^{-m}(g_m) =$$

$$= \sum_{m\in\mathbb{Z}} \alpha^{-k}(f_{-m}) \alpha^{-m-k}(g_{m+k}) = \alpha^k \left(\sum_{m\in\mathbb{Z}} f_{-m} \alpha^{-m}(g_{m+k}) \right) =$$

$$= \alpha^k \left(\sum_{m\in\mathbb{Z}} f_m \alpha^m(g_{k-m}) \right) = (i(f * g))_k.$$

therefore, i is an algebra homomorphism. □

For example, let us consider a Fréchet-Arens-Michael algebra R with a m-tempered action of $\mathbb{Z}$ and fix a generating family of submultiplicative seminorms $\{\|\cdot\|_\lambda \mid \lambda \in \Lambda\}$ on R, such that

$$\|\alpha_1^n(r)\|_\lambda \leq p(n)\,\|r\|_\lambda\,, \quad (r \in R, n \in \mathbb{Z}),$$

where p is a polynomial depending on $\lambda \in \Lambda$. In this case the algebra R is contained in $\mathscr{S}(\mathbb{Z}, R; \alpha)$:

$$R \hookrightarrow \mathscr{S}(\mathbb{Z}, R; \alpha), \quad r \mapsto re_0,$$

where $(re_i)_j := \delta_{ij} r$.

Hence, $\mathscr{S}(\mathbb{Z}, R; \alpha)$ becomes a unital R-$\hat{\otimes}$-algebra, and in the appendix we prove Proposition A.3, which states that the structure of $\widehat{\Omega}^1_R(\mathscr{S}(\mathbb{Z}, R; \alpha))$ is similar to the algebraic and holomorphic cases. This gives us an opportunity to formulate the following theorem:

Theorem 4.2. Let R be a Fréchet-Arens-Michael algebra with an m-tempered $\mathbb{Z}$-action α. If we denote $A = \mathscr{S}(\mathbb{Z}, R; \alpha)$, then we have

$$\mathrm{db}(A) \leq \mathrm{db}(R) + 1, \quad \mathrm{dgr}_{\mathbf{Fr}}(A) \leq \mathrm{dgr}_{\mathbf{Fr}}(R) + 1, \quad \mathrm{w.dg}(A) \leq \mathrm{w.dg}(R) + 1.$$

The proof of the theorem is very similar to the proofs of Theorems 3.3 and 3.4. We just use Proposition A.3 to establish the sequence (11) for smooth products, and then we proceed with the same argument.

Lemma 4.1. The module $\mathscr{S}(\mathbb{Z}, R; \alpha)$ is free as a left and right R-$\hat{\otimes}$-module.

Proof. We observe that $\mathscr{S}(\mathbb{Z}, R; \alpha)$ is a left R-$\hat{\otimes}$-module because its underlying locally convex space $\mathscr{S}(\mathbb{Z}, R)$ is isomorphic to $s \otimes R$, and the map

$$(f_n) \otimes r \mapsto (f_n r)$$

is an isomorphism of left R-$\hat{\otimes}$-modules. We finish the argument by proving that $(\mathscr{S}(G, \mathbb{Z}), *')$ is isomorphic to $R \otimes s$ as a right R-$\hat{\otimes}$-module in a similar fashion, then invoking Proposition 4.1. ☐

Lemma 4.1 together with Proposition 3.5 allows us to obtain the lower estimates.

Theorem 4.3. Let R be Fréchet-Arens-Michael algebra with $\mathrm{dgr}_{\mathbf{Fr}}(R) < \infty$ and with a m-tempered $\mathbb{Z}$-action α. Denote $A = \mathscr{S}(\mathbb{Z}, R; \alpha)$. Then the conditions of Proposition 3.5 are satisfied. In particular, we have

$$\mathrm{dgr}_{\mathbf{Fr}}(R) \leq \mathrm{dgr}_{\mathbf{Fr}}(A), \quad \mathrm{w.dg}(R) \leq \mathrm{w.dg}(A).$$

A Relative bimodules of differential 1-forms of Ore extensions

We will provide a sketch of a proof for the following proposition, whereas for the full proof we refer to [6, Proposition 7.5.2].

Proposition A.1. Let R be a $\mathbb{C}$-algebra. Suppose that

(1) $A = R[t; \alpha, \delta]$, where $\alpha : R \to R$ is an endomorphism and $\delta : R \to R$ is an α-derivation.

(2) $A = R[t, t^{-1}; \alpha]$, where $\alpha : R \to R$ is an automorphism.

Then $\Omega^1_R(A)$ is isomorphic as an A-bimodule to $A_\alpha \otimes_R A$.

Sketch of the proof. The first part of the proof works for the both cases. Define the map $\varphi : A_\alpha \times A \to \Omega^1_R(A)$ as follows:

$$\varphi(f, g) = f\mathrm{d}(tg) - ft\mathrm{d}g = f(\mathrm{d}t)g, \quad (f, g \in A).$$

First of all, we prove that this map is balanced.

Therefore, φ induces a well-defined homomorphism of A-bimodules $\varphi : A_\alpha \otimes_R A \to \Omega^1_R(A)$.

We will use the universal property of $\Omega^1_R(A)$ to construct the inverse morphism.

(1) Suppose that $A = R[t;\alpha,\delta]$. Consider the following linear mapping:

$$D : A \to A_\alpha \otimes_R A, \quad D(rt^n) = \sum_{k=0}^{n-1} rt^k \otimes t^{n-k-1}.$$

As it turns out, D is an R-derivation.

(2) Suppose that $A = R[t, t^{-1};\alpha]$. Consider the following linear mapping:

$$D : A \to A_\alpha \otimes_R A, \quad D(rt^n) = \begin{cases} \sum_{k=0}^{n-1} rt^k \otimes t^{n-k-1}, & \text{if } n \geq 0, \\ -\sum_{k=1}^{|n|} rt^{-k} \otimes t^{n+k-1}, & \text{if } n < 0. \end{cases}$$

As in the first case, this map turns out to be an R-derivation.

The rest of this proof works in the both cases. Notice that $\varphi \circ D = d_A$, because $(\varphi \circ D)(t) = d_A(t)$, and both $\varphi \circ D$ and d_A are R-derivations. Therefore, these derivations have to agree on the subalgebra generated by R, t (and t^{-1} in the invertible case), which is A itself.

□

The following proposition was already proven by A. Yu. Pirkovskii, see [8, Proposition 7.8], but we present another proof, which utilizes the derivations introduced in the sketch of the proof of Proposition A.1; it even works in the case of *localizable* morphisms. Moreover, the proof can be carried over to the case of smooth crossed products by $\mathbb{Z}$, as we will see later.

Proposition A.2. Let R be an Arens-Michael algebra. Suppose that A is one of the following $\hat{\otimes}$-algebras:

(1) $A = \mathcal{O}(\mathbb{C}, R;\alpha,\delta)$, where $\alpha : R \to R$ is an endomorphism and $\delta : R \to R$ is a α-derivation, such that the pair (α,δ) is a localizable pair of morphisms.

(2) $A = \mathcal{O}(\mathbb{C}^\times, R;\alpha)$, where $\alpha : R \to R$ is an automorphism, such that the pair (α,α^{-1}) is a localizable pair of morphisms.

Then $\widehat{\Omega}^1_R(A)$ is isomorphic to $A_\alpha \hat{\otimes}_R A$.

Proof. Fix a generating family of seminorms $\{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ on R such that for every $\lambda \in \Lambda$ there exists $C > 0$ such that for all $x \in R$ we have $\|\alpha(x)\|_\lambda \leq C\|x\|_\lambda$ and $\|\delta(x)\|_\lambda \leq C\|x\|_\lambda$. Define the map $A_\alpha \times A \to \widehat{\Omega}^1_R(A)$ as in the proof of Proposition 0.1:

$$\varphi(f,g) = f\mathrm{d}(zg) - fz\mathrm{d}g = f(\mathrm{d}z)g, \quad (f,g \in A).$$

This is a R-balanced map (the proof is literally the same as in Proposition A.1), also it is easily seen from the continuity of the bimodule action of A on $\Omega^1_R(A)$ that this map is continuous. Therefore, this map induces a continuous A-$\hat{\otimes}$-bimodule homomorphism $A_\alpha \hat{\otimes}_R A \to \widehat{\Omega}^1_R(A)$.

(1) Suppose that $A = \mathcal{O}(\mathbb{C}, R;\alpha,\delta)$. Recall that the topology on A is generated by the family $\{\|\cdot\|_{\lambda,\rho}\}_{\lambda\in\Lambda,\rho>1}$, where

$$\left\| \sum_{n\geq 0} f_n z^n \right\|_{\lambda,\rho} = \sum \|f_n\| \rho^n.$$

Consider the following linear map:

$$D : R[z;\alpha,\delta] \to A_\alpha \hat{\otimes}_R A, \quad D(rz^n) = \sum_{k=0}^{n-1} rz^k \otimes z^{n-k-1}.$$

For now it is defined on the dense subset of A; let us prove that this map is continuous. Fix $\lambda_1, \lambda_2 \in \Lambda$ and $\rho_1, \rho_2 \in \mathbb{R}_{\geq 0}$. Denote the projective tensor product of $\|\cdot\|_{\lambda_1,\rho_1}$ and $\|\cdot\|_{\lambda_2,\rho_2}$ by γ. Then for every $f = \sum_{k=0}^{m} f_k z^k \in R[z;\alpha,\delta] \subset A$ we have

$$\gamma(D(f)) \leq \sum_{k=1}^{m} \gamma(D(f_k z^k)) = \sum_{k=1}^{m} \gamma\left(\sum_{l=0}^{k-1} f_k z^l \otimes z^{k-l-1}\right) \leq \sum_{k=1}^{m} \|f_k\|_{\lambda_1} \sum_{l=0}^{k-1} \rho_1^l \rho_2^{k-l-1} \leq$$

$$\leq \sum_{k=1}^{m} \|f_k\|_{\lambda_1} (2\max\{\rho_1,\rho_2,1\})^k = \|f\|_{\lambda_1, 2\max\{\rho_1,\rho_2,1\}}.$$

(2) Suppose that $A = \mathcal{O}(\mathbb{C}^\times, R; \alpha)$. Recall that the topology on A is generated by the family $\{\|\cdot\|_{\lambda,\rho}\}_{\lambda\in\Lambda,\rho>1}$, where

$$\left\| \sum_{n\in\mathbb{Z}} f_n z^n \right\|_{\lambda,\rho} = \sum \|f_n\| \rho^{|n|}.$$

Consider the following linear map:

$$D : R[t,t^{-1};\alpha] \to A_\alpha \hat{\otimes}_R A, \quad D(rz^n) = \begin{cases} \sum_{k=0}^{n-1} rz^k \otimes z^{n-k-1}, & \text{if } n \geq 0, \\ -\sum_{k=1}^{|n|} rz^{-k} \otimes z^{n+k-1}, & \text{if } n < 0. \end{cases}$$

For now it is defined on the dense subset of A; let us prove that this map is continuous. Fix $\lambda_1, \lambda_2 \in \Lambda$ and $\rho_1, \rho_2 \in \mathbb{R}_{\geq 0}$. Denote the projective tensor product of $\|\cdot\|_{\lambda_1,\rho_1}$ and $\|\cdot\|_{\lambda_2,\rho_2}$ by γ. Suppose that $n \geq 0$. Then we have

$$\gamma(D(rz^n)) = \gamma\left(\sum_{l=0}^{n-1} rz^l \otimes z^{n-l-1}\right) \leq \|r\|_{\lambda_1} \sum_{l=0}^{n-1} \rho_1^l \rho_2^{n-l-1} \leq$$

$$\leq n\|r\|_{\lambda_1} \max\{\rho_1,\rho_2\}^{n-1} \leq \|r\|_{\lambda_1} 2^n \max\{\rho_1,\rho_2,1\}^n \leq \|rz^n\|_{\lambda_1, 2\max\{\rho_1,\rho_2,1\}}.$$

If $n < 0$, then

$$\gamma(D(rz^n)) = \gamma\left(\sum_{l=1}^{|n|} rz^{-l} \otimes z^{n+l-1}\right) \leq \|r\|_{\lambda_1} \sum_{l=1}^{|n|} \rho_1^{|-l|} \rho_2^{|n+l-1|} = \|r\|_{\lambda_1} \sum_{l=1}^{|n|} \rho_1^l \rho_2^{1-l-n} \leq$$

$$\leq \|r\|_{\lambda_1} 2^{|n|} \max\{\rho_1,\rho_2,1\}^{|n|+1} \leq \|rz^n\|_{\lambda_1, 2\max\{\rho_1^2,\rho_2^2,1\}}.$$

Therefore, for every $f = \sum_{-m}^{m} f_k z^k \in R[z,z^{-1};\alpha] \subset A$ we have

$$\gamma(D(f)) = \sum_{k=-m}^{m} \gamma\left(D(f_k z^k)\right) \leq \sum_{k=-m}^{m} \left\|f_k z^k\right\|_{\lambda_1, 2\max\{\rho_1^2,\rho_2^2,1\}} = \|f\|_{\lambda_1, 2\max\{\rho_1^2,\rho_2^2,1\}}.$$

Therefore, this map can be uniquely extended to the whole algebra A; we will denote the extension by D, as well. Notice D is also an R-derivation: the equality $D(ab) - D(a)b - aD(b) = 0$ holds for $R[z;\alpha,\delta] \times R[z;\alpha,\delta] \subset A \times A$, which is a dense subset of $A \times A$. Therefore, $D(ab) = D(a)b + aD(b)$ for every $a, b \in A$.

Notice that $\varphi \circ D = d_A$. Denote the extension of $D : A \to A_\alpha \hat{\otimes}_R A$ by $\widetilde{D} : \widehat{\Omega}^1_R(A) \to A_\alpha \hat{\otimes}_R A$, so $D = \widetilde{D} \circ d_A$. Therefore we can derive from the universal property of $\widehat{\Omega}^1_R(A)$ that $\varphi \circ \widetilde{D} = \mathrm{Id}$. And

$$\widetilde{D} \circ \varphi(a \otimes b) = a(\widetilde{D} \circ \varphi(1 \otimes 1))b = a\widetilde{D}(dt)b = a \otimes b.$$

Therefore, the equality $\widetilde{D} \circ \varphi = \mathrm{Id}$ holds on a dense subset of $A_\alpha \hat{\otimes}_R A$, but $\widetilde{D} \circ \varphi$ is continuous, therefore, $\widetilde{D} \circ \varphi = \mathrm{Id}$ holds everywhere on $A_\alpha \hat{\otimes}_R A$. ☐

Proposition A.3. Let R be a Fréchet-Arens-Michael algebra and consider an m-tempered action α of $\mathbb{Z}$ on R. If we denote the algebra $\mathscr{S}(\mathbb{Z}, R; \alpha)$ by A, then $\widehat{\Omega}^1_R(A)$ is isomorphic to $A_{\alpha_1} \hat{\otimes}_R A$.

Proof. Fix a generating system of submultiplicative seminorms $\{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ on R, such that for every $\lambda \in \Lambda$ there exists a polynomial p such that for all $x \in R$ and $n \in \mathbb{N}$ we have

$$\|\alpha_n(x)\|_\lambda = \|\alpha_1^n(x)\|_\lambda \leq p(n)\,\|x\|_\lambda \quad (x \in R, \lambda \in \Lambda).$$

Define the map $\varphi : A_{\alpha_1} \times A \to \widehat{\Omega}^1_R(A)$ as follows:

$$\varphi(f,g) = f\mathrm{d}(e_1 * g) - (f * e_1)\mathrm{d}g = f(\mathrm{d}e_1)g.$$

It is a continuous R-balanced linear map (the proof is literally the same as in the previous propositions).

Now let us denote by $\mathcal{A}$ the dense subalgebra of A (algebraically) generated by R, z and z^{-1}, and let us consider the following linear map:

$$D : \mathcal{A} \to A_{\alpha_1}\hat{\otimes}_R A, \quad D(re_n) = \begin{cases} \sum\limits_{k=0}^{n-1} re_k \otimes e_{n-k-1}, & \text{if } n \geq 0 \\ -\sum\limits_{k=1}^{|n|} re_{-k} \otimes e_{n+k-1}, & \text{if } n < 0 \end{cases}.$$

Let us prove that it is a well-defined and continuous map from $\mathcal{A}$ to $A_{\alpha_1}\hat{\otimes}_R A$. Fix $\lambda_1, \lambda_2 \in \Lambda$ and $k_1, k_2 \in \mathbb{Z}_{\geq 0}$. Denote the projective tensor product of $\|\cdot\|_{\lambda_1,k_1}$ and $\|\cdot\|_{\lambda_2,k_2}$ by γ. Let $n \geq 1$, then we have

$$\begin{aligned}\gamma(D(re_n)) &= \gamma\left(\sum_{k=0}^{n-1} re_k \otimes e_{n-k-1}\right) \leq \|r\|_{\lambda_1}(n^{k_2} + 2^{k_1}(n-1)^{k_2} + \dots + n^{k_1}) \leq \\ &\leq \|r\|_{\lambda_1}(n^{\max\{k_1,k_2\}} + 2^{\max\{k_1,k_2\}}(n-1)^{\max\{k_1,k_2\}} + \dots + n^{\max\{k_1,k_2\}}) \leq \\ &\leq \|r\|_{\lambda_1} n^{2\max\{k_1,k_2\}+1} \leq \|re_n\|_{\lambda_1, 2\max\{k_1,k_2\}+1}.\end{aligned}$$

For $n < 0$ the argument is pretty much the same:

$$\begin{aligned}\gamma(D(re_n)) &= \gamma\left(\sum_{k=1}^{|n|} re_{-k} \otimes e_{n+k-1}\right) \leq \|r\|_{\lambda_1}(2^{k_1}(|n|+1)^{k_2} + 3^{k_1}|n|^{k_2} + \dots (|n|+1)^{k_1}2^{k_2}) \leq \\ &\leq \|r\|_{\lambda_1}(2^{\max\{k_1,k_2\}}(|n|+1)^{\max\{k_1,k_2\}} + \dots + (|n|+1)^{\max\{k_1,k_2\}}2^{\max\{k_1,k_2\}}) \leq \\ &\leq \|r\|_{\lambda_1}|n|(|n|+1)^{2\max\{k_1,k_2\}} \leq \|r\|_{\lambda_1}(|n|+1)^{2\max\{k_1,k_2\}+1} = \|re_n\|_{\lambda_1, 2\max\{k_1,k_2\}+1}.\end{aligned}$$

It is easily seen that for every $f \in \mathcal{A}$ we have

$$\gamma(Df) \leq \sum_{m\in\mathbb{Z}} \gamma(D(f^{(m)}e_m)) \leq 2\sum_{m\in\mathbb{Z}} \left\|f^{(m)}e_m\right\|_{\lambda_1, 2\max\{k_1,k_2\}+1} = 2\,\|f\|_{\lambda_1, 2\max\{k_1,k_2\}+1}.$$

Then D is a R-derivation which can be uniquely extended to the whole algebra A, the extension $\widetilde{D}$ is the inverse of φ, the proof is the same as in Proposition A.2. □

References

[1] Joachim Cuntz and Daniel Quillen, *Algebra extensions and nonsingularity*, J. Amer. Math. Soc. **8** (1995), no. 2, 251–289. MR1303029

[2] A. Ya. Helemskii, *The homology of Banach and topological algebras*, Mathematics and its Applications (Soviet Series), vol. 41, Kluwer Academic Publishers Group, Dordrecht, 1989. Translated from the Russian by Alan West. MR1093462

[3] ———, *Lectures and exercises on functional analysis*, Translations of Mathematical Monographs, vol. 233, American Mathematical Society, Providence, RI, 2006. Translated from the 2004 Russian original by S. Akbarov. MR2248303

[4] Hans Jarchow, *Locally convex spaces*, Mathematische Leitfäden. [Mathematical Textbooks], B. G. Teubner, Stuttgart, 1981. MR632257

[5] Anastasios Mallios, *Topological algebras. Selected topics*, North-Holland Mathematics Studies, vol. 124, North-Holland Publishing Co., Amsterdam, 1986. Notas de Matemática [Mathematical Notes], 109. MR857807

[6] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian rings*, Revised, Graduate Studies in Mathematics, vol. 30, American Mathematical Society, Providence, RI, 2001. With the cooperation of L. W. Small. MR1811901

[7] N. Christopher Phillips and Larry B. Schweitzer, *Representable K-theory of smooth crossed products by* $\mathbf{R}$ *and* $\mathbf{Z}$, Trans. Amer. Math. Soc. **344** (1994), no. 1, 173–201. MR1219733

[8] A. Yu. Pirkovskiĭ, *Arens-Michael envelopes, homological epimorphisms, and relatively quasifree algebras*, Tr. Mosk. Mat. Obs. **69** (2008), 34–125. MR2549445

[9] ______, *Weak homological dimensions and biflat Köthe algebras*, Mat. Sb. **199** (2008), no. 5, 45–80. MR2421805

[10] ______, *Homological dimensions and Van den Bergh isomorphisms for nuclear Fréchet algebras*, Izv. Ross. Akad. Nauk Ser. Mat. **76** (2012), no. 4, 65–124. MR3013272

[11] Larry B. Schweitzer, *Dense m-convex Fréchet subalgebras of operator algebra crossed products by Lie groups*, Internat. J. Math. **4** (1993), no. 4, 601–673. MR1232986

[12] François Treves, *Topological vector spaces, distributions and kernels*, Academic Press, New York-London, 1967. MR0225131

[13] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR1269324

P. Kosenko, Department of Mathematics, Higher School of Economics, Moscow, Usacheva str. 6, 119048

E-mail address, P. Kosenko: pkosenko@hse.ru

Department of Mathematics, University of Toronto, 40 St. George St., Toronto, ON, Canada, M5S 2E4

E-mail address: petr.kosenko@mail.utoronto.ca

Personal webpage: http://www.math.utoronto.ca/pkosenko/

ORCID: https://orcid.org/0000-0002-4150-0613