Homological dimensions of smooth crossed products

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Abstract. In this paper we provide upper estimates for the global projective dimensions of smooth crossed products $\mathscr{S}(G, A; \alpha)$ for $G = \mathbb{R}$ and $G = \mathbb{T}$ and a self-induced Fréchet-Arens-Michael algebra A. In order to do this, we provide a powerful generalization of methods which are used in the works of Ogneva and Helemskii.

Introduction

There are numerous papers dedicated to homological properties of smooth crossed products of Fréchet algebras and C*-algebras, see [3], [8], [9], [11], or [14], for example.

However, it seems that nothing is known about homological dimensions of smooth crossed products. In [6] we provided the estimates for homological dimensions of holomorphic Ore extensions and smooth crossed products by $\mathbb{Z}$ of unital $\hat{\otimes}$-algebras, and in this paper we show that the methods of the author's previous works and the paper [10] can be adapted to smooth crossed products by $\mathbb{R}$ and $\mathbb{T}$.

The idea behind the estimates lies in the construction of admissible $\widehat{\Omega}^1$-like sequences for the required *non-unital* algebras. What do we mean by that? Recall the definition of a bimodule of relative 1-forms:

Definition 0.1 ([1]). Suppose that R is an algebra, and A is an R-algebra. For an A-bimodule X we define a *R-derivation* as an additive map $d : A \to X$ such that:

1. $d(ab) = d(a) \cdot b + a \cdot d(b)$ for every $a, b \in A$,
2. $d(r) = 0$ for every $r \in R$,

where $\cdot$ denotes the outer multiplication in X as an A-bimodule.

Definition 0.2. Let R be a unital $\hat{\otimes}$-algebra, and let A denote a unital R-$\hat{\otimes}$-algebra (see Definition 3.1). A pair $\left(\widehat{\Omega}^1_R(A), d_A\right)$, which consists of an A-$\hat{\otimes}$-bimodule $\widehat{\Omega}^1_R(A)$ and a continuous R-derivation $d_A : A \longrightarrow \widehat{\Omega}^1_R(A)$, is called the *bimodule of relative 1-forms* of A, if this pair is universal in the following sense:

for every A-$\hat{\otimes}$-bimodule M and a continuous R-derivation $D : A \longrightarrow M$ there exists a unique continuous A-$\hat{\otimes}$-bimodule homomorphism $\widetilde{D} : \widehat{\Omega}^1_R(A) \longrightarrow M$ such that $D = \widetilde{D} \circ d_A$.

$$\begin{array}{ccc} \widehat{\Omega}^1_R(A) & \xrightarrow{\widetilde{D}} & M \\ {\scriptstyle d_A}\uparrow & \nearrow_{D} & \\ A & & \end{array}$$

This construction is a topological version of a construction presented in [1]. It is not hard to prove that $\widehat{\Omega}^1_R(A)$ is a well-defined object, moreover, this bimodule is a part of an extremely useful admissible sequence. The following theorem is the topological version of [1, Proposition 2.5].

Theorem 0.3 ([12], Proposition 7.2). Let R be a unital $\hat{\otimes}$-algebra and let A denote a unital R-$\hat{\otimes}$-algebra. Then there exists a sequence which splits in the categories A-**mod**-R and R-**mod**-A:

$$0 \longrightarrow \widehat{\Omega}^1_R(A) \xrightarrow{\;j\;} A \hat{\otimes}_R A \xrightarrow{\;m\;} A \longrightarrow 0, \tag{0.1}$$

where $m(a \otimes b) = ab$. In particular, this sequence is admissible.

In the paper [6] we utilized the sequence (0.1) in order to obtain the upper estimates for the homological dimensions of different types of non-commutative Ore-like extensions. In the case when all algebras are unital, this sequence proves to be quite useful because it turns out that $\widehat{\Omega}^1_R(A) \cong A \hat{\otimes}_R A$ as an R-$\hat{\otimes}$-module.

However, when $G = \mathbb{R}$ or $G = \mathbb{T}$, then, given a Fréchet-Arens-Michael algebra A, the algebras $\mathscr{S}(G, A; \alpha)$ are, in general, not unital. Nevertheless, we managed to obtain the exact sequences for these algebras, which look similar to (0.1), and which allowed us to derive the upper estimates for the global projective dimensions of $\mathscr{S}(\mathbb{R}, A; \alpha)$ and $\mathscr{S}(\mathbb{T}, A; \alpha)$.

Let us recall that for $A = \mathbb{C}$ and $\alpha = \mathrm{Id}_{\mathbb{C}}$ we have $\mathscr{S}(\mathbb{R}, A; \alpha) \simeq \mathscr{S}(\mathbb{R})$. In [10] it is shown that the projective homological dimension of $\mathscr{S}(\mathbb{R}^n)$ equals n for $n \geq 1$.

As for the general case, we conjecture that the estimates for homological dimensions should look as follows:

Conjecture 0.4. Let A be a Fréchet-Arens-Michael algebra (not necessarily unital) with a smooth m-tempered action α of $\mathbb{R}$ or $\mathbb{T}$ on A. Denote the left (projective) global dimension by dgl. Then for $G = \mathbb{R}$ or $G = \mathbb{T}$ we have

$$\mathrm{dgl}(A) \leq \mathrm{dgl}(\mathscr{S}(G, A; \alpha)) \leq \mathrm{dgl}(A) + 1.$$

The main results of this paper are Theorems 2.19, 2.20, 3.12, 3.13. In particular, we have proven a weak form of the above conjecture.

Theorem 0.5. Let A be a projective Fréchet-Arens-Michael algebra, which satisfies the following condition: the multiplication map $m : A\hat{\otimes}_A A \longrightarrow A$ is an A-$\hat{\otimes}$-bimodule isomorphism. Also let α denote a smooth m-tempered action of $\mathbb{R}$ or $\mathbb{T}$ on A. Denote the left (projective) global dimension by dgl. Then for $G = \mathbb{R}$ or $G = \mathbb{T}$ we have

$$\mathrm{dgl}(\mathscr{S}(G, A; \alpha)) \leq \max\{\mathrm{dgl}(A), 1\} + 1$$

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1. Preliminaries

1.1. Notation

Remark. All algebras in this paper are defined over the field of complex numbers and assumed to be associative. Moreover, we allow the algebras to be *non-unital*, in contrast to [6].

Definition 1.1. A *Fréchet* space is a complete metrizable locally convex space.

Let us introduce some notation (see [5] and [13] for more details). Denote by **LCS**, **Fr** the categories of complete locally convex spaces, Fréchet spaces, respectively. Also we will denote the category of vector spaces by **Lin**.

For a locally convex Hausdorff space E we will denote its completion by $\tilde{E}$. Also for locally convex Hausdorff spaces E, F the notation $E\hat{\otimes}F$ denotes the completed projective tensor product of E, F.

By A_+ we will denote the unitization of an algebra A. By A^{op} we denote the opposite algebra.

Definition 1.2. A complete locally convex algebra with jointly continuous multiplication is called a $\hat{\otimes}$*-algebra.*

A $\hat{\otimes}$-algebra with the underlying locally convex space which is a Fréchet space is called a *Fréchet algebra.*

Definition 1.3. A locally convex algebra A is called m-convex if the topology on it can be defined by a family of submultiplicative seminorms.

Definition 1.4. A complete locally m-convex algebra is called an *Arens-Michael algebra.*

Definition 1.5. Let A be a $\hat{\otimes}$-algebra and let M be a complete locally convex space which is also a left A-module. Also suppose that the natural map $A \times M \to M$ is jointly continuous. Then we will call M a left *A-$\hat{\otimes}$-module.* In a similar fashion we define right *A-$\hat{\otimes}$-modules* and *A-B-$\hat{\otimes}$-bimodules.*

A $\hat{\otimes}$-module over a Fréchet algebra which is itself a Fréchet space is called a Fréchet A-$\hat{\otimes}$-module.

For arbitrary $\hat{\otimes}$-algebras A, B we denote

$$A\textbf{-mod} = \text{the category of left } A\text{-}\hat{\otimes}\text{-modules},$$

$$\textbf{mod-}A = \text{the category of right } A\text{-}\hat{\otimes}\text{-modules},$$

$$A\textbf{-mod-}B = \text{the category of } A\text{-}B\text{-}\hat{\otimes}\text{-bimodules}.$$

For unital $\hat{\otimes}$-algebras A, B we denote

$$A\textbf{-unmod} = \text{the category of unital left } A\text{-}\hat{\otimes}\text{-modules},$$

$$\textbf{unmod-}A = \text{the category of unital right } A\text{-}\hat{\otimes}\text{-modules},$$

$$A\textbf{-unmod-}B = \text{the category of unital } A\text{-}B\text{-}\hat{\otimes}\text{-bimodules}.$$

Let A be a $\hat{\otimes}$-algebra, and consider a complex of A-$\hat{\otimes}$-modules:

$$\dots \xrightarrow{d_{n+1}} M_{n+1} \xrightarrow{d_n} M_n \xrightarrow{d_{n-1}} M_{n-1} \xrightarrow{d_{n-2}} \dots,$$

then we will denote this complex by $\{M, d\}$.

Definition 1.6. Let A be a $\hat{\otimes}$-algebra and consider a left A-$\hat{\otimes}$-module Y and a right A-$\hat{\otimes}$-module X.

(1) A bilinear map $f : X \times Y \longrightarrow Z$, where $Z \in \mathbf{LCS}$, is called A-balanced if $f(x \circ a, y) = f(x, a \circ y)$ for every $x \in X, y \in Y, a \in A$.
(2) A pair $(X\hat{\otimes}_A Y, i)$, where $X\hat{\otimes}_A Y \in \mathbf{LCS}$, and $i : X \times Y \longrightarrow X\hat{\otimes}_A Y$ is a continuous A-balanced map, is called the *completed projective tensor product of* X and Y, if for every $Z \in \mathbf{LCS}$ and continuous A-balanced map $f : X \times Y \longrightarrow Z$ there exists a unique continuous linear map $\tilde{f} : X\hat{\otimes}_A Y \longrightarrow Z$ such that $f = \tilde{f} \circ i$.

1.2. Projectivity and homological dimensions

The following definitions will be given in the case of left modules; the definitions in the cases of right modules and bimodules are similar, just use the following category isomorphisms: for unital A, B we have

$$\textbf{unmod-}A \simeq A^{\mathrm{op}}\textbf{-unmod} \quad A\textbf{-unmod-}B \simeq (A\hat{\otimes}B^{\mathrm{op}})\textbf{-unmod}$$

Let A be a unital $\hat{\otimes}$-algebra.

Definition 1.7. A complex of A-$\hat{\otimes}$-modules $\{M, d\}$ is called *admissible* $\iff$ it splits in the category $\mathbf{LCS}$. A morphism of A-$\hat{\otimes}$-modules $f : X \to Y$ is called *admissible* if it is one of the morphisms in an admissible complex.

Definition 1.8. An additive functor $F : A\textbf{-unmod} \to \mathbf{Lin}$ is called *exact* $\iff$ for every admissible complex $\{M, d\}$ the corresponding complex $\{F(M), F(d)\}$ in $\mathbf{Lin}$ is exact.

Definition 1.9. Suppose that A and B are unital $\hat{\otimes}$-algebras.

(1) A module $P \in A$-**unmod** is called *projective* $\iff$ the functor $\mathrm{Hom}_A(P, -)$ is exact.

(2) A module $X \in A$-**unmod** is called *free* $\iff$ X is isomorphic to $A \hat{\otimes} E$ for some $E \in \mathbf{LCS}$.

Now we consider the general, non-unital case. Let A be a $\hat{\otimes}$-algebra. Any left $\hat{\otimes}$-module over an algebra A can be viewed as a unital $\hat{\otimes}$-module over A_+, in other words, the following isomorphism of categories takes place:

$$A\text{-}\mathbf{mod} \cong A_+\text{-}\mathbf{unmod}, \quad A\text{-}\mathbf{mod}\text{-}B \cong A_+ \hat{\otimes} B_+^{\mathrm{op}}\text{-}\mathbf{unmod}.$$

By using this isomorphism we can define projective and free modules in the non-unital case.

Definition 1.10. Suppose that A and B are $\hat{\otimes}$-algebras.

(1) A module $P \in A$-**mod** is called *projective* $\iff$ the module P is projective in the category A_+-**unmod**

(2) A module $X \in A$-**mod** is called *free* $\iff$ X is isomorphic to $A_+ \hat{\otimes} E$ for some $E \in \mathbf{LCS}$.

As it turns out, there is no ambiguity, a unital module is projective in the sense of Definition 1.9 if and only if it is projective in the sense of Definition 1.10.

Definition 1.11. Let $X \in A$-**mod**. Suppose that X can be included in the following admissible complex:

$$0 \leftarrow X \xleftarrow{\varepsilon} P_0 \xleftarrow{d_0} P_1 \xleftarrow{d_1} \dots \xleftarrow{d_{n-1}} P_n \leftarrow 0 \leftarrow 0 \leftarrow \dots,$$

for a chain where every P_i is a non-zero projective module. Then we call this admissible complex a *projective resolution* of X of *length* n. If X is included in an admissible complex of a form

$$0 \leftarrow X \xleftarrow{\varepsilon} P_0 \xleftarrow{d_0} P_1 \xleftarrow{d_1} \dots \xleftarrow{d_{n-1}} P_n \xleftarrow{d_n} P_{n+1} \xleftarrow{d_{n+1}} \dots,$$

where $P_i \neq 0$ for all $i \geq 0$, then it is called an *unbounded projective resolution*, with its length being equal to ∞.

This allows us to define the notion of a derived functor in the topological case, for example, see [5, ch 3.3]. In particular, $\mathrm{Ext}_A^k(M, N)$ and $\mathrm{Tor}_k^A(M, N)$ are defined similarly to the purely algebraic situation.

Definition 1.12. Consider an arbitrary non-zero module $M \in A$-**mod**. Then the following number is well-defined:

$$\begin{aligned} \mathrm{dh}_A(M) &= \min\{n \in \mathbb{Z}_{\geq 0} : \mathrm{Ext}_A^{n+1}(M, N) = 0 \text{ for every } N \in A\text{-}\mathbf{mod}\} = \\ &= \{\text{the length of a shortest projective resolution of } M\} \in [0, \infty]. \end{aligned}$$

It is called the *projective (homological) dimension* of M.

Remark. Following a standard convention, we define $\mathrm{dh}_A(0) = -\infty$.

Definition 1.13. Let A be a $\hat{\otimes}$-algebra. Then we can define the following invariants of A:

$$\mathrm{dgl}(A) = \sup\{\mathrm{dh}_A(M) : M \in A\text{-}\mathbf{mod}\} - \textit{the left global dimension of } A.$$

$$\mathrm{dgr}(A) = \sup\{\mathrm{dh}_A(M) : M \in \mathbf{mod}\text{-}A\} - \textit{the right global dimension of } A.$$

1.3. Algebra of rapidly decreasing functions

Recall the definition of the space of rapidly decreasing functions on $\mathbb{R}^n$.

Definition 1.14. For $n > 0$ define the Fréchet space

$$\mathscr{S}(\mathbb{R}^n) := \{f : \mathbb{R}^n \to \mathbb{C} : \|f\|_{k,l} = \sup_{x \in \mathbb{R}^n} |x^k D^l(f)| < \infty \text{ for all } k, l \in \mathbb{Z}^n_{\geq 0}\},$$

where $x^k = x_1^{k_1} \dots x_n^{k_n}$ and $D^l(f) = \frac{\partial^{l_1}}{\partial x_1^{l_1}} \dots \frac{\partial^{l_n}}{\partial x_n^{l_n}} f$. The topology on $\mathscr{S}(\mathbb{R}^n)$ is defined by the system $\{\|f\|_{k,l} : k, l \in \mathbb{Z}^n_{\geq 0}\}$.

There are two natural ways to define the multiplication on $\mathscr{S}(\mathbb{R}^n)$:

$$(f \cdot g)(x) = f(x)g(x) \quad \text{(pointwise product)}$$

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\mathrm{d}y. \quad \text{(convolution product)}$$

The following theorem is well-known.

Theorem 1.15. Fix $n \in \mathbb{N}$.

(1) $(\mathscr{S}(\mathbb{R}^n), \cdot)$ is a Fréchet-Arens-Michael algebra.
(2) The Fourier transform induces an isomorphism of Arens-Michael algebras

$$\mathcal{F}_n : (\mathscr{S}(\mathbb{R}^n), \cdot) \longrightarrow (\mathscr{S}(\mathbb{R}^n), *),$$

$$\mathcal{F}_n(f)(x) = \int_{\mathbb{R}^n} f(y)e^{-2\pi i \langle x, y \rangle} dy_1 \dots dy_n.$$

Proof. (1) The proof is very similar to the proof that $C^\infty(\mathbb{R}^n)$ is a Fréchet-Arens-Michael algebra, which can be found in [7, Section 4.4.(2)].
(2) See [2, Theorem 8.22, Corollary 8.28] for the proof.

□

From now on we will write $\mathscr{S}(\mathbb{R}^n)$ instead of $(\mathscr{S}(\mathbb{R}^n), \cdot)$ and $\mathscr{S}(\mathbb{R}^n)_{\mathrm{conv}}$ instead of $(\mathscr{S}(\mathbb{R}^n), *)$.

1.4. $\widehat{\Omega}^1$-like admissible sequences for $\mathscr{S}(\mathbb{R})$

In order to determine the homological dimensions of $\mathscr{S}(\mathbb{R}^n)$ in [10], Helemskii and Ogneva used a simple and natural $\widehat{\Omega}^1$-like admissible sequence for $\mathscr{S}(\mathbb{R})$. It was constructed using Hadamard's lemma.

Lemma 1.16 (Hadamard's lemma). Let $f \in \mathscr{S}(\mathbb{R}^n)$, such that $f(0, x_2, \dots, x_n) = 0$ for all $(x_2, \dots, x_n) \in \mathbb{R}^{n-1}$. Then there exists a function $g \in \mathscr{S}(\mathbb{R}^n)$ such that

$$f(x_1, \dots, x_n) = x_1 g(x_1, \dots, x_n).$$

More generally, suppose that $f(x) = 0$ on a hyperplane in $\mathbb{R}^n$ defined by the equation $a_1x_1 + \cdots + a_nx_n = 0$. Then there exists $g \in \mathscr{S}(\mathbb{R}^n)$ such that

$$f(x_1, \ldots, x_n) = (a_1x_1 + \cdots + a_nx_n)g(x_1, \ldots, x_n).$$

Recall that $\mathscr{S}(\mathbb{R}^2)$ admits the following structure of an $\mathscr{S}(\mathbb{R})$-$\hat{\otimes}$-bimodule:

$$(\varphi \cdot f)(x, y) = \varphi(x)f(x, y), \quad (f \cdot \varphi)(x, y) = f(x, y)\varphi(y)$$

for any $\varphi \in \mathscr{S}(\mathbb{R})$, $f \in \mathscr{S}(\mathbb{R}^2)$, $x, y \in \mathbb{R}$.

The Theorem 1.1 gives a similar $\mathscr{S}(\mathbb{R})_{\text{conv}}$-$\hat{\otimes}$-bimodule structure on $\mathscr{S}(\mathbb{R}^2)_{\text{conv}}$.

Proposition 1.17 ([10], Proposition 3). The following diagram is commutative, moreover, the rows of the diagram are short exact sequences of $\mathscr{S}(\mathbb{R})$-$\hat{\otimes}$-bimodules which split in the categories $\mathscr{S}(\mathbb{R})$-**mod** and **mod**-$\mathscr{S}(\mathbb{R})$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{S}(\mathbb{R}^2) & \xrightarrow{\;j\;} & \mathscr{S}(\mathbb{R}^2) & \xrightarrow{\;\pi\;} & \mathscr{S}(\mathbb{R}) & \longrightarrow & 0 \\ & & \wr\uparrow & & \wr\uparrow & & \uparrow\mathrm{Id} & & \\ 0 & \longrightarrow & \mathscr{S}(\mathbb{R})\hat{\otimes}\mathscr{S}(\mathbb{R}) & \xrightarrow{\;k\;} & \mathscr{S}(\mathbb{R})\hat{\otimes}\mathscr{S}(\mathbb{R}) & \xrightarrow{\;m\;} & \mathscr{S}(\mathbb{R}) & \longrightarrow & 0 \end{array} \tag{1.1}$$

where

$$\begin{aligned} j(f)(x, y) &= (x - y)f(x, y) && \text{for all } f \in \mathscr{S}(\mathbb{R}^2), \\ \pi(f)(x) &= f(x, x) && \text{for all } f \in \mathscr{S}(\mathbb{R}^2) \\ k(f \otimes g) &= fx \otimes g - f \otimes gx && \text{for all } f \in \mathscr{S}(\mathbb{R}^2), \\ m(f \otimes g) &= fg && \text{for all } f \in \mathscr{S}(\mathbb{R}^2). \end{aligned}$$

Let us restate the above proposition for $\mathscr{S}(\mathbb{R})_{\text{conv}}$. First of all, we will formulate a lemma which can be considered as the “Fourier dual” to Hadamard’s lemma.

Lemma 1.18. Let $f \in \mathscr{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}} f(t, x_2, \ldots, x_n)dt = 0$ for any $(x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$. Then there exists a function $g \in \mathscr{S}(\mathbb{R}^n)$ satisfying

$$f(x_1, \ldots, x_n) = \frac{\partial}{\partial x_1} g(x_1, \ldots, x_n).$$

More generally, if there is a non-zero vector $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ such that the integral $\int_{\mathbb{R}} f(x + tv)dt = 0$ for any $x \in \mathbb{R}^n$, then there exists a function $g \in \mathscr{S}(\mathbb{R}^n)$ satisfying

$$f(x_1, \ldots, x_n) = \left(\sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i} \right) g(x_1, \ldots, x_n).$$

Proof. WLOG we can assume that $v = e_1 = (1, 0, \ldots, 0)$, otherwise we apply a suitable linear change of variables.

Let us consider $\tilde{f} := \mathcal{F}_n(f) \in \mathscr{S}(\mathbb{R}^n)$, this is a well-defined function due to Theorem 1.15. Then

$$\begin{aligned}\tilde{f}(0, x_2, \ldots, x_n) &= \int_{\mathbb{R}^n} f(y) e^{-2\pi i (x_2 y_2 + \cdots + x_n y_n)} dy = \\ &= \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} f(t, y_2, \ldots, y_n) dt \right) e^{-2\pi i (x_2 y_2 + \cdots + x_n y_n)} dy_2 \ldots dy_n\end{aligned}$$

due to Fubini's theorem. However, $\int_{\mathbb{R}} f(t, y_2, \ldots, y_n) dt = 0$ due to our assumption. Therefore, $\tilde{f}$ satisfies the conditions of Lemma 1.16, so there exists a function $\tilde{g}$ such that $\tilde{f}(x) = x_1 \tilde{g}$. We finish the argument by applying the inverse Fourier transform to obtain

$$f = \frac{\partial}{\partial x_1} \left(\frac{1}{2\pi i} \mathcal{F}_n^{-1}(\tilde{g}) \right).$$

□

In the same fashion, we can formulate the "Fourier dual" to Proposition 1.17.

Proposition 1.19. The following diagram is commutative, moreover, the rows of the diagram are short exact sequences of $\mathscr{S}(\mathbb{R})_{\mathrm{conv}}$-$\hat{\otimes}$-bimodules which split in the categories $\mathscr{S}(\mathbb{R})_{\mathrm{conv}}$-**mod** and **mod**-$\mathscr{S}(\mathbb{R})_{\mathrm{conv}}$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{S}(\mathbb{R}^2)_{\mathrm{conv}} & \xrightarrow{j} & \mathscr{S}(\mathbb{R}^2)_{\mathrm{conv}} & \xrightarrow{\pi} & \mathscr{S}(\mathbb{R})_{\mathrm{conv}} & \longrightarrow & 0 \\ & & \wr\uparrow & & \wr\uparrow & & \mathrm{Id}\uparrow & & \\ 0 & \longrightarrow & \mathscr{S}(\mathbb{R})_{\mathrm{conv}} \hat{\otimes} \mathscr{S}(\mathbb{R})_{\mathrm{conv}} & \xrightarrow{k} & \mathscr{S}(\mathbb{R})_{\mathrm{conv}} \hat{\otimes} \mathscr{S}(\mathbb{R})_{\mathrm{conv}} & \xrightarrow{m} & \mathscr{S}(\mathbb{R})_{\mathrm{conv}} & \longrightarrow & 0 \end{array} \tag{1.2}$$

where

$$\begin{aligned} j(f)(x,y) &= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) f(x,y) && \text{for all } f \in \mathscr{S}(\mathbb{R}^2), \\ \pi(f)(x) &= \int_{\mathbb{R}} f(y, x-y) dy && \text{for all } f \in \mathscr{S}(\mathbb{R}^2). \\ k(f \otimes g) &= f' \otimes g - f \otimes g' && \text{for all } f \in \mathscr{S}(\mathbb{R}^2), \\ m(f \otimes g) &= f * g && \text{for all } f \in \mathscr{S}(\mathbb{R}^2). \end{aligned}$$

In the next section we will show that the diagram (1.2) can be generalized if we replace $\mathscr{S}(\mathbb{R})$ with smooth crossed products of Fréchet-Arens-Michael algebras by $\mathbb{R}$ and $\mathbb{T}$.

2. $\widehat{\Omega}^1$-like admissible sequences for smooth crossed products

2.1. Smooth m-tempered actions and smooth crossed products

Definition 2.1. Let E be a Hausdorff topological vector space. For a function $f:\mathbb{R}^n\to E$ and $x\in\mathbb{R}^n$ we denote

$$\frac{\partial f}{\partial x_i}(x):=\lim_{h\to 0}\frac{f(x_1,\ldots,x_i+h,\ldots,x_n)-f(x_1,\ldots,x_i,\ldots,x_n)}{h}.$$

Let us define $\mathbb{T}=\{z\in\mathbb{C}:|z|=1\}$. From now on, we will canonilcally identify $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z}$.

Definition 2.2. Let X be a Fréchet space with topology, generated by a sequence of seminorms $\{\|\cdot\|_m : m\in\mathbb{N}\}$.
(1) The space $\mathscr{S}(\mathbb{T}^n,X):=C^\infty(\mathbb{T}^n,X)$ is a Fréchet space with respect to the system

$$\left\{\|f\|_{k,m}=\sup_{x\in\mathbb{T}^n}\left\|D^k(f)(x)\right\|_m : k\in\mathbb{Z}^n_{\geq 0}, m\in\mathbb{N}\right\}.$$

(2) Define the following space:

$$\mathscr{S}(\mathbb{R}^n,X)=\left\{f:\mathbb{R}^n\to X : \|f\|_{k,l,m}:=\sup_{x\in\mathbb{R}^n}\left\|x^lD^k(f)(x)\right\|_m<\infty, k,l\in\mathbb{Z}^n_{\geq 0}, m\in\mathbb{N}\right\},$$

where $D^k(f)=\frac{\partial^{k_1}}{\partial x_1^{k_1}}\cdots\frac{\partial^{k_n}}{\partial x_n^{k_n}}f$. The topology on $\mathscr{S}(\mathbb{R}^n,X)$ is defined by the system
$\{\|f\|_{k,l,m} : k,l\in\mathbb{Z}^n_{\geq 0}, m\in\mathbb{N}\}$.

The following proposition was proven in [4], also see [7, Chapter 11.2] for some related results.

Proposition 2.3. Let A be a Fréchet space. Then the natural maps

$$\mathscr{S}(\mathbb{R}^n)\hat{\otimes}A\to\mathscr{S}(\mathbb{R}^n,A),\quad f\otimes a\mapsto(x\mapsto f(x)a),$$
$$\mathscr{S}(\mathbb{T}^n)\hat{\otimes}A\to\mathscr{S}(\mathbb{T}^n,A),\quad f\otimes a\mapsto(x\mapsto f(x)a).$$

are topological isomorphisms for $n\in\mathbb{N}$. As a corollary, we have

$$\mathscr{S}(\mathbb{R}^m)\hat{\otimes}\mathscr{S}(\mathbb{R}^n)\cong\mathscr{S}(\mathbb{R}^m,\mathscr{S}(\mathbb{R}^n))\cong\mathscr{S}(\mathbb{R}^{n+m}),$$
$$\mathscr{S}(\mathbb{T}^m)\hat{\otimes}\mathscr{S}(\mathbb{T}^n)\cong\mathscr{S}(\mathbb{T}^m,\mathscr{S}(\mathbb{T}^n))\cong\mathscr{S}(\mathbb{T}^{n+m}),$$

This proposition gives us another way to differentiate and integrate vector-valued Schwartz functions.

Definition 2.4. Let A be a Fréchet algebra. Then for $G=\mathbb{T},\mathbb{R}$ we define the derivative
$\frac{d}{dx}:\mathscr{S}(G,A)\to\mathscr{S}(G,A)$ and the integral $\int_G:\mathscr{S}(G,A)\to A$ using the universal property of the completed projective tensor product:

$$\frac{d}{dx}(f\otimes a):=\left(\frac{d}{dx}f(x)\right)\otimes a,\quad \int_G(f\otimes a)d\mu:=\left(\int_G fd\mu\right)\otimes a,$$

where μ denotes the Lebesgue measure on $\mathbb{R}$ and the normalized Lebesgue measure on $\mathbb{T}$.

Definition 2.5. Let A be a Fréchet-Arens-Michael algebra, and let $G = \mathbb{R}$ or $G = \mathbb{T}$. Then the action α of G on A via automorphisms is called

(a) *m-tempered* (as in [14]), if there exists a generating family of submultiplicative seminorms $\{\|\cdot\|_m\}_{m\in\mathbb{N}}$ on A such that for every $m \in \mathbb{N}$ there is a polynomial $p_m(x) \in \mathbb{R}[x]$, satisfying

$$\|\alpha_x(a)\|_m \leq |p_m(x)| \, \|a\|_m \quad (a \in A, x \in G).$$

(b) *C^∞-m-tempered* or *smooth m-tempered* , if the following conditions are satisfied:

(1) for every $a \in A$ the function

$$\alpha_x(a) : G \longrightarrow A, \quad x \mapsto \alpha_x(a),$$

is C^∞-differentiable,

(2) there exists a generating family of submultiplicative seminorms $\{\|\cdot\|_m\}_{m\in\mathbb{N}}$ on A such that for any $k \geq 0$ and $m > 0$ there exists a polynomial $p_{k,m} \in \mathbb{R}[x]$, satisfying

$$\left\|\alpha_x^{(k)}(a)\right\|_m \leq |p_{k,m}(x)| \, \|a\|_m \quad (k \in \mathbb{N}, x \in G, a \in A).$$

The following theorem can be considered as a definition of smooth crossed products.

Theorem 2.6 ([14], Theorem 3.1.7). Let A be a Fréchet-Arens-Michael algebra with an
m-tempered action of one of the groups $G = \mathbb{R}$ or $G = \mathbb{T}$. Then the space $\mathscr{S}(G, A)$ endowed with the following multiplication:

$$(f *_\alpha g)(x) = \int_G f(y)\alpha_y(g(x-y))dy$$

becomes a Fréchet-Arens-Michael algebra.

When $G = \mathbb{R}$, we will denote this algebra by $\mathscr{S}(\mathbb{R}, A; \alpha)$, and in the case $G = \mathbb{T}$ we will write $C^\infty(\mathbb{T}, A; \alpha)$.

Remark. If α is the trivial action, then $\mathscr{S}(G, A; \alpha) = \mathscr{S}(G, A)$ with the usual convolution product.

Proposition 2.7. Let A be a Fréchet-Arens-Michael algebra. Consider an action $\alpha : \mathbb{R} \to \mathrm{Aut}(A)$. Then α is a smooth m-tempered action if and only if the following holds:

1. the derivative $\alpha'_x(a)$ exists at $x = 0$ for every $a \in A$, and, as a corollary, derivatives all of orders at zero exist.
2. there exists a generating family of submultiplicative seminorms $\{\|\cdot\|_m\}_{m\in\mathbb{N}}$ on A such that for every $m \in \mathbb{N}$ and $k \in \mathbb{N}$ there exist polynomials $p_m(x) \in \mathbb{R}[x]$ and $C_{k,m} > 0$, satisfying

$$\|\alpha_x(a)\|_m \leq |p_m(x)| \, \|a\|_m \, , \quad \left\|\alpha_0^{(k)}(a)\right\|_m \leq C_{k,m} \, \|a\|_m$$

for every $a \in A$, $x \in \mathbb{R}$.

Proof. ($\Rightarrow$) If α is C^∞-m-tempered, choose the seminorms $\|\cdot\|_m$ and the polynomials $p_{m,k}(x)$ as in Definition 2.5, and set

$$p_m(x) = p_{0,m}(x), \quad C_{k,m} = p_{k,m}(0).$$

($\Leftarrow$) Notice that

$$\alpha'_x(a) = \lim_{h\to 0} \frac{\alpha_{x+h}(a) - \alpha_x(a)}{h} = \alpha_x \left(\lim_{h\to 0} \frac{\alpha_h(a) - a}{h} \right) = \alpha_x(\alpha'_0(a)) \quad (a \in A). \tag{2.1}$$

Therefore,

$$\alpha_x^{(k)}(a) = \alpha_x^{(k-1)}(\alpha'_0(a)) = \alpha_x^{(k-2)}(\alpha'_0(\alpha'_0(a))) = \cdots = \alpha_x(\underbrace{\alpha'_0(\ldots(\alpha'_0(a)))}_{k \text{ times}}).$$

However,

$$\alpha'_0(\alpha'_0(x)) = \lim_{h\to 0} \frac{\alpha_h(\alpha'_0(x)) - \alpha'_0(x)}{h} = \lim_{h\to 0} \frac{\alpha'_h(x) - \alpha'_0(x)}{h} = \alpha''_0(x).$$

By induction we obtain the following equality:

$$\alpha_x^{(k)}(a) = \alpha_x(\alpha_0^{(k)}(a)) \tag{2.2}$$

for every $a \in A$, $x \in \mathbb{R}$, $k \in \mathbb{Z}_{\geq 0}$.

As an immediate corollary, $\alpha_x(a) \in C^\infty(\mathbb{R}, A)$ for every $a \in A$. This also implies that

$$\left\|\alpha_x^{(k)}(a)\right\|_m = \left\|\alpha_x(\alpha_0^{(k)}(x))\right\|_m \leq |p_m(x)| \left\|\alpha_0^{(k)}(a)\right\|_m \leq |p_m(x)|\, C_{k,m} \|a\|_m .$$

Now set $p_{k,m}(x) = C_{k,m} p_m(x)$. □

The proposition can be restated for $G = \mathbb{T}$:

Proposition 2.8. Let A be a Fréchet-Arens-Michael algebra. Consider an action $\alpha : \mathbb{T} \to \mathrm{Aut}(A)$. Then α is a smooth m-tempered action if and only if the following holds:

1. the derivative $\alpha'_x(a)$ exists at $x = 0$ for every $a \in A$, and, as a corollary, derivatives all of orders at zero exist.
2. there exists a generating family of submultiplicative seminorms $\{\|\cdot\|_m\}_{m\in\mathbb{N}}$ on A such that for every $m \in \mathbb{N}$ and $k \in \mathbb{N}$ there exist $C_m, C_{k,m} > 0$, satisfying

$$\|\alpha_x(a)\|_m \leq C_m \|a\|_m , \quad \left\|\alpha_0^{(k)}(a)\right\|_m \leq C_{k,m} \|a\|_m$$

for every $a \in A$, $x \in \mathbb{T}$.

Proof. The proof is the same as in the previous proposition, we only need keep in mind that

$$|p_{k,m}(x)| \leq \sup_{x\in\mathbb{T}} |p_{k,m}(x)| < \infty.$$

□

2.2. Explicit construction

Remark. In this subsection we only treat the case $G = \mathbb{R}$ here, the case $G = \mathbb{T}$ can be dealt with in the same way.

Definition 2.9. A $\hat{\otimes}$-algebra A is called *self-induced*, if the multiplication map $m_A : A\hat{\otimes}_A A \longrightarrow A$ is an A-$\hat{\otimes}$-bimodule isomorphism.

Until the end of this section, A will denote a self-induced Fréchet-Arens-Michael algebra. We will also consider a smooth m-tempered $\mathbb{R}$-action α of A.

In this subsection we will construct a $\hat{\Omega}^1_A$-like admissible sequence for $\mathscr{S}(\mathbb{R}, A; \alpha)$.

Proposition 2.10. For any $F \in \mathscr{S}(\mathbb{R}, A)$ define $T(F)(x) = \alpha_x(F(x))$. Then the following statements hold:

1. The mapping T is a well-defined continuous linear map $T : \mathscr{S}(\mathbb{R}, A) \to \mathscr{S}(\mathbb{R}, A)$,
2. Moreover, T is invertible, with the inverse, defined for every $F \in \mathscr{S}(\mathbb{R}, A)$ as follows:
$$T^{-1}(F)(x) = \alpha_{-x}(F(x)).$$
In particular, we have
$$\left(T \circ \frac{d}{dx} \circ T^{-1}\right)(F)(x) = F'(x) - \alpha_0'(F(x)) \tag{2.3}$$
for any $F \in \mathscr{S}(\mathbb{R}, A)$.
3. For any $F, G \in \mathscr{S}(\mathbb{R}, A; \alpha)$ we have
$$F' *_\alpha T(G) = F *_\alpha T(G').$$
This equality is equivalent to
$$F' *_\alpha G = F *_\alpha \left(T \circ \frac{d}{dx} \circ T^{-1}\right)(G).$$

Proof.

1. Let us write down the derivative of $\alpha_x(F(x))$:
$$\begin{aligned}\frac{d}{dx}(\alpha_x(F(x))) &= \lim_{h\to 0} \frac{\alpha_{x+h}(F(x+h)) - \alpha_x(F(x))}{h} = \\ &= \alpha_x\left(\lim_{h\to 0} \frac{\alpha_h(F(x) + F'(x)h + o(h)) - F(x)}{h}\right) = \\ &= \alpha_x\left(\alpha_0'(F(x)) + F'(x)\right) = \alpha_x'(F(x)) + \alpha_x(F'(x)).\end{aligned}$$
It is easily seen that
$$\frac{d^k}{dx^k}(\alpha_x(F(x))) = \sum_{i=0}^{k} \binom{n}{k} \alpha_x^{(i)}(F^{(k-i)}(x)). \tag{2.4}$$

Now fix a generating system of seminorms on A which satisfies the conditions of Proposition 2.7. Let us show that $\alpha_x^{(m)}(F(x))$ lies in $\mathscr{S}(\mathbb{R},A)$ for any $F\in\mathscr{S}(\mathbb{R},A)$ and $m\geq 0$:

$$\|T(F)\|_{k,l,m}=\sup_{x\in\mathbb{R}}\left\|x^l\frac{d^k}{dx^k}(\alpha_x(F(x)))\right\|_m\leq\sum_{i=0}^k\binom{k}{i}\sup_{x\in\mathbb{R}}\left\|\alpha_x^{(i)}(F^{(k-i)}(x))\right\|\leq$$
$$\leq\sum_{i=0}^k\binom{k}{i}\sup_{x\in\mathbb{R}}\left(|p_{i,m}(x)|\left\|F^{(k-i)}(x)\right\|_m\right)<\infty.$$

2. Notice that the same argument shows works for T^{-1}, as well. As for the equality, notice that

$$\frac{d}{dx}(\alpha_{-x}(F(x)))=-\alpha'_{-x}(F(x))+\alpha_{-x}(F'(x)),\tag{2.5}$$

so we have

$$T\left(\frac{d}{dx}(\alpha_{-x}(F(x)))\right)=-\alpha_x(\alpha'_0(F(x)))+(F'(x))\overset{2.2}{=}-\alpha'_0(F(x))+F'(x)$$

3. This is equivalent to

$$\int_{\mathbb{R}}F'(y)\alpha_y(TG(x-y))dy=\int_{\mathbb{R}}F(y)\alpha_y(TG'(x-y))dy\Leftrightarrow$$
$$\Leftrightarrow\int_{\mathbb{R}}F'(y)\alpha_x(G(x-y))dy=\int_{\mathbb{R}}F(y)\alpha_x(G'(x-y))dy\Leftrightarrow$$
$$\Leftrightarrow\int_{\mathbb{R}}\frac{d}{dy}(F(y))\alpha_x(G(x-y))dy=-\int_{\mathbb{R}}F(y)\frac{d}{dy}(\alpha_x(G(x-y)))dy\Leftrightarrow$$
$$\Leftrightarrow\int_{\mathbb{R}}\frac{d}{dy}(F(y))\alpha_x(G(x-y))dy+\int_{\mathbb{R}}F(y)\frac{d}{dy}(\alpha_x(G(x-y)))dy=0\quad\text{(int. by parts)}$$

□

Let $\mathscr{S}(\mathbb{R},A;\alpha)_\alpha=:S_\alpha$ denote the $\mathscr{S}(\mathbb{R},A;\alpha)$-$\hat{\otimes}$-bimodule and A-$\hat{\otimes}$-bimodule, which coincides with $\mathscr{S}(\mathbb{R},A)$ as a LCS, and the bimodule actions are given below:

$$(F\cdot a)(x)=F(x)\alpha_x(a),\qquad a\cdot F(x)=aF(x)\qquad\text{for any } a\in A, F\in S_\alpha$$
$$(F\cdot G)(x)=(F*_\alpha G)(x),\qquad (G\cdot F)(x)=(G*_\alpha F)(x)\qquad\text{for any } F,G\in S_\alpha.$$

Proposition 2.11.

1. For any $F\in\mathscr{S}(\mathbb{R},A)$ and $a\in A$ the functions $F\cdot a$ and $a\cdot F$ belong to $\mathscr{S}(\mathbb{R},A)$. As a corollary, S_α is well-defined.
2. The following equalities take place:

$$(F\cdot a)'=(F'\cdot a)+(F\cdot\alpha'_0(a)),\tag{2.6}$$
$$\left(T\circ\frac{d}{dx}\circ T^{-1}\right)(F\cdot a)=\left(\left(T\circ\frac{d}{dx}\circ T^{-1}\right)F\right)\cdot a,\tag{2.7}$$
$$\left(T\circ\frac{d}{dx}\circ T^{-1}\right)(a\cdot F)=a\cdot\left(T\circ\frac{d}{dx}\circ T^{-1}\right)(F)-\alpha'_0(a)\cdot F.\tag{2.8}$$

Proof.

1. The argument for $a \cdot F$ is pretty much trivial, we only need to check that $F \cdot a \in \mathscr{S}(\mathbb{R}, A)$. Fix a generating system of seminorms $\{\|\cdot\|_m\}$ on A, satisfying the conditions of Proposition 2.7.

   Notice that for every $k, l \in \mathbb{Z}_{\geq 0}$ and $m \in \mathbb{N}$ we have

$$\|F \cdot a\|_{k,l,m} = \sup_{x \in \mathbb{R}} \left\| x^l \frac{d^k}{dx^k}(F \cdot a)(x) \right\|_m = \sup_{x \in \mathbb{R}} \left\| \sum_{i=0}^{k} x^l \binom{k}{i} F^{(i)}(x) \alpha_0^{(k-i)}(a) \right\|_m \leq$$
$$\leq \sum_{i=0}^{k} \binom{k}{i} \left|x^l\right| \left\|F^{(i)}(x)\right\|_m \left\|\alpha_0^{(k-i)}(a)\right\|_m \leq$$
$$\leq \|a\|_m \sum_{i=0}^{k} \binom{k}{i} C_{k-i,m} \left|x^l\right| \left\|F^{(i)}(x)\right\|_m =$$
$$= \|a\|_m \sum_{i=0}^{k} \binom{k}{i} C_{k-i,m} \|F\|_{i,l,m} < \infty.$$

2. Checking these equalities is pretty straightforward:

$$(F \cdot a)'(x) = F(x)\alpha_x'(a) + F'(x)\alpha_x(a) \overset{2.2}{=} F(x)\alpha_x(\alpha_0'(a)) + F'(x)\alpha_x(a) =$$
$$= (F' \cdot a)(x) + (F \cdot \alpha_0'(a))(x),$$

$$\left(T \circ \frac{d}{dx} \circ T^{-1}\right)(F \cdot a) = \left(\left(T \circ \frac{d}{dx}\right)(T^{-1}F) \circ a\right) = \left(\left(T \circ \frac{d}{dx} \circ T^{-1}\right) F\right) \cdot a.$$

$$\left(T \circ \frac{d}{dx} \circ T^{-1}\right)(a \cdot F)(x) = \left(T \circ \frac{d}{dx}\right)(\alpha_{-x}(a)(T^{-1}F)(x)) =$$
$$= T\left(\alpha_{-x}(a)(T^{-1}F)'(x) - \alpha_{-x}'(a)(T^{-1}F)(x)\right) =$$
$$= \left(a \cdot \left(T \circ \frac{d}{dx} \circ T^{-1}\right)(F) - \alpha_0'(a) \cdot F\right)(x).$$

□

Lemma 2.12. The bimodule S_α belongs to the categories $\mathscr{S}(\mathbb{R}, A; \alpha)$-**mod**-$A$ and A-**mod**-$\mathscr{S}(\mathbb{R}, A; \alpha)$. In particular, the following $\mathscr{S}(\mathbb{R}, A; \alpha)$-$\hat{\otimes}$-bimodule structure on $S_\alpha \hat{\otimes}_A S_\alpha$ is well-defined:

$$H \cdot (F \otimes G) = (H *_\alpha F) \otimes G, \quad (F \otimes G) \cdot H = F \otimes (G *_\alpha H)$$

for any $F, G, H \in \mathscr{S}(\mathbb{R}, A)$.

Proof. We only need to prove that $(H \cdot F) \cdot a = H \cdot (F \cdot a)$ and $(a \cdot F) \cdot H = a \cdot (F \cdot H)$ for any $a \in A; F, H \in S_\alpha$.

$$H \cdot (F \cdot a)(x) = \int_{\mathbb{R}} H(y)\alpha_y(F(x-y))\alpha_x(a)dy = (H \cdot F) \cdot a(x),$$

$$(a \cdot F) \cdot H(x) = \int_{\mathbb{R}} aF(y)\alpha_y(H(x-y))dy = a \cdot (H \cdot F)(x).$$

□

It is also easy to see that $S_\alpha \hat{\otimes}_A S_\alpha$ is a well-defined A-$\hat{\otimes}$-bimodule.

Lemma 2.13. Define the following maps:

$$m : S_\alpha \hat{\otimes}_A S_\alpha \longrightarrow S_\alpha, \qquad m(F \otimes G) = F *_\alpha G,$$

$$j : S_\alpha \hat{\otimes}_A S_\alpha \longrightarrow S_\alpha \hat{\otimes}_A S_\alpha, \qquad j(F \otimes G) = F' \otimes G - F \otimes \left(T \circ \frac{d}{dx} \circ T^{-1}\right) G.$$

These maps are well-defined $\mathscr{S}(\mathbb{R}, A; \alpha)$-$\hat{\otimes}$-bimodule and A-$\hat{\otimes}$-bimodule homomorphisms.

Proof. First of all, let us prove that m and j are well-defined:

$$m((F \cdot a) \otimes G)(x) = \int_{\mathbb{R}} F(y)\alpha_y(a)\alpha_y(G(x-y))dy =$$
$$= \int_{\mathbb{R}} F(y)\alpha_y(aG(x-y))dy = m(F \otimes (a \cdot G))(x, y).$$

$$j(F \cdot a \otimes G) = (F \cdot a)' \otimes G - (F \cdot a) \otimes \left(T \circ \frac{d}{dx} \circ T^{-1}\right)(G) \overset{2.6}{=}$$
$$\overset{2.6}{=} (F' \cdot a) \otimes G + (F \cdot \alpha_0'(a)) \otimes G - (F \cdot a) \otimes \left(T \circ \frac{d}{dx} \circ T^{-1}\right)(G) =$$
$$= F' \otimes a \cdot G + F \otimes \alpha_0'(a) \cdot G - F \otimes a \cdot \left(T \circ \frac{d}{dx} \circ T^{-1}\right)(G).$$

$$j(F \otimes a \cdot G) = F' \otimes a \cdot G - F(x) \otimes \left(T \circ \frac{d}{dx} \circ T^{-1}\right)(a \cdot G) \overset{2.8}{=}$$
$$\overset{2.8}{=} F' \otimes a \cdot G - F \otimes a \cdot \left(T \circ \frac{d}{dx} \circ T^{-1}\right)(G) + F \otimes \alpha_0'(a) \cdot G =$$
$$= j(F \cdot a \otimes G).$$

The algebra $\mathscr{S}(\mathbb{R}, A; \alpha)$ is associative, therefore, m is an $\mathscr{S}(\mathbb{R}, A; \alpha)$-$\hat{\otimes}$-bimodule homomorphism.

It is relatively easy to show that j is a left $\mathscr{S}(\mathbb{R}, A; \alpha)$-$\hat{\otimes}$-module homomorphism:

$$j((H *_\alpha F) \otimes G) = (H *_\alpha F)' \otimes G - (H *_\alpha F) \otimes \left(T \circ \frac{d}{dx} \circ T^{-1}\right)(G) =$$
$$= (H *_\alpha F') \otimes G - (H *_\alpha F) \otimes \left(T \circ \frac{d}{dx} \circ T^{-1}\right)(G) =$$
$$= H *_\alpha j(F \otimes G).$$

And it is slightly more difficult to show that it is a right $\mathscr{S}(\mathbb{R}, A; \alpha)$-$\hat{\otimes}$-module homomorphism.

$$j(F \otimes (G *_\alpha H)) = F' \otimes (G *_\alpha H) - F \otimes T((T^{-1}(G * H))')$$

$$T^{-1}(G * H)(x) = \int_{\mathbb{R}} \alpha_{-x}(G(y))\alpha_{-x+y}(H(x-y))dy =$$
$$= \int_{\mathbb{R}} \alpha_{-x}(G(y+x))\alpha_y(H(-y))dy$$

$$(T^{-1}(G*H))'(x) = \int_{\mathbb{R}} \frac{d}{dx}\left(\alpha_{-x}(G(y+x))\right)\alpha_y(H(-y))dy \overset{2.5}{=}$$
$$\overset{2.5}{=} \int_{\mathbb{R}} (\alpha_{-x}(G'(x+y)) - \alpha'_{-x}(G(x+y)))\alpha_y(H(-y))dy.$$
$$T((T^{-1}(G*H))')(x) = \int_{\mathbb{R}} (G'(x+y) - \alpha_x(\alpha'_{-x}(G(x+y))))\alpha_{x+y}(H(-y))dy \overset{2.2}{=}$$
$$\overset{2.2}{=} \int_{\mathbb{R}} (G'(x+y) - \alpha'_0(G(x+y)))\alpha_{x+y}(H(-y))dy =$$
$$= \int_{\mathbb{R}} (G'(y) - \alpha'_0(G(y)))\alpha_y(H(x-y))dy \overset{2.3}{=}$$
$$\overset{2.3}{=} \left(\left(T \circ \frac{d}{dx} \circ T^{-1}\right)(G) *_\alpha H\right)(x).$$

Therefore, we have

$$j(F \otimes (G *_\alpha H)) = F' \otimes (G *_\alpha H) - F \otimes \left(T \circ \frac{d}{dx} \circ T^{-1}\right)(G) *_\alpha H = j(F \otimes G) *_\alpha H.$$

Now let us check that j and m are A-$\hat{\otimes}$-bimodule homomorphisms:

$$m(a \cdot F \otimes G)(x) = \int_R aF(y)\alpha_y(G(x-y))dy = (a \cdot m(F \otimes G))(x)$$
$$m(F \otimes G \cdot a)(x) = \int_R F(y)\alpha_y(G(x-y))\alpha_x(a)dy = (m(F \otimes G) \cdot a)(x)$$
$$j(a \cdot F \otimes G) = (a \cdot F)' \otimes G - a \cdot F \otimes \left(T \circ \frac{d}{dx} \circ T^{-1}\right)(G) = a \cdot j(F \otimes G)$$
$$j(F \otimes G \cdot a) = F' \otimes G \cdot a - F \otimes \left(T \circ \frac{d}{dx} \circ T^{-1}\right)(G \cdot a) \overset{2.7}{=}$$
$$\overset{2.7}{=} F' \otimes G \cdot a - F(x) \otimes \left(T \circ \frac{d}{dx} \circ T^{-1}\right)(G) \circ a = j(F \otimes G) \circ a.$$

□

As a corollary from Proposition 2.10 we have $m \circ j = 0$.

Proposition 2.14. The tensor product $S_\alpha \hat{\otimes}_A S_\alpha$ is isomorphic to $\mathscr{S}(\mathbb{R}^2, A)$ as a locally convex space:

$$I_1 : S_\alpha \hat{\otimes}_A S_\alpha \longrightarrow \mathscr{S}(\mathbb{R}^2, A), \quad I_1(F \otimes G)(x,y) = \alpha_{-x}(F(x))G(y).$$

Proof. First of all, we can replace $\mathscr{S}(\mathbb{R}^2, A)$ with $\mathscr{S}(\mathbb{R}^2, A \hat{\otimes}_A A)$, because A is isomorphic to $A \hat{\otimes}_A A$ as a locally convex space. This is precisely where we use the fact that A is a self-induced algebra.

Let X be a complete LCS and consider a continuous A-balanced map $Q : S_\alpha \times S_\alpha \longrightarrow X$. Define the map

$$\widetilde{Q} : \mathscr{S}(\mathbb{R}^2, A \hat{\otimes}_A A) \longrightarrow X, \quad \widetilde{Q}(f(x)g(y)a \otimes b) = I_1(f(x)\alpha_x(a), g(x)b).$$

This map is a well-defined continuous linear map, because I_1 is A-balanced and the linear span of $\{f(x)g(y)a \otimes b : f, g \in \mathscr{S}(\mathbb{R}), a, b \in A\}$ is dense in $\mathscr{S}(\mathbb{R}^2, A\hat{\otimes}_A A)$. From the construction of $\widetilde{Q}$ it follows that the following diagram is commutative:

$$\begin{array}{ccc} \mathscr{S}(\mathbb{R}^2, A\hat{\otimes}_A A) & \xrightarrow{\widetilde{Q}} & X \\ {\scriptstyle I_1}\uparrow & \nearrow {\scriptstyle Q} & \\ S_\alpha \times S_\alpha & & \end{array}$$

Moreover, $\widetilde{Q}$ is a unique mapping which makes this diagram commute. □

Therefore, the isomorphism I_1 induces the structure of an $\mathscr{S}(\mathbb{R}, A; \alpha)$-$\hat{\otimes}$-module on $\mathscr{S}(\mathbb{R}^2, A)$, which we will denote by $\mathscr{S}(\mathbb{R}^2, A)_\alpha$. Let us describe the action of $\mathscr{S}(\mathbb{R}, A; \alpha)$ and A on $\mathscr{S}(\mathbb{R}^2, A)_\alpha$ explicitly.

Lemma 2.15. The algebra A acts on $\mathscr{S}(\mathbb{R}^2, A)_\alpha$ as follows:

$$(a \cdot F)(x,y) = \alpha_{-x}(a)F(x,y) \qquad \text{for any } F \in \mathscr{S}(\mathbb{R}^2, A)_\alpha, a \in A$$
$$(F \cdot a)(x,y) = F(x,y)\alpha_y(a) \qquad \text{for any } F \in \mathscr{S}(\mathbb{R}^2, A)_\alpha, a \in A$$

The algebra $\mathscr{S}(\mathbb{R}, A; \alpha)$ acts on $\mathscr{S}(\mathbb{R}^2, A)_\alpha$ as follows:

$$(H \cdot F)(x,y) = \int_{\mathbb{R}} \alpha_{-x}(H(z))F(x-z,y)dz \qquad \text{for any } F \in \mathscr{S}(\mathbb{R}^2, A)_\alpha, H \in \mathscr{S}(\mathbb{R}, A; \alpha)$$
$$(F \cdot H)(x,y) = \int_{\mathbb{R}} F(x,z)\alpha_z(H(y-z))dz \qquad \text{for any } F \in \mathscr{S}(\mathbb{R}^2, A)_\alpha, H \in \mathscr{S}(\mathbb{R}, A; \alpha).$$

Proof. In all cases we will check every relation on a dense subset, then we will use the continuity arguments to finish the proof. Let $F(x,y) = G_1(x)G_2(y)$.

$$(a \cdot F)(x,y) = I_1(a \cdot TG_1 \otimes G_2)(x,y) = \alpha_{-x}(a)G_1(x)G_2(y) = \alpha_{-x}(a)F(x,y)$$
$$(F \cdot a)(x,y) = I_1(TG_1 \otimes G_2 \cdot a)(x,y) = G_1(x)G_2(y)\alpha_y(a) = F(x,y)\alpha_y(a)$$

$$(H \cdot F)(x,y) = I_1\left(\left(\int_{\mathbb{R}} H(z)\alpha_z(TG_1(x-z))dz \otimes G_2(x)\right) \otimes G_2(x) = \right.$$
$$= I_1\left(\left(\int_{\mathbb{R}} H(z)\alpha_x(G_1(x-z))dz \otimes G_2(x)\right) = \right.$$
$$= \left(\int_{\mathbb{R}} \alpha_{-x}(H(z))(G_1(x-z))G_2(y)dz\right) =$$
$$= \int_{\mathbb{R}} \alpha_{-x}(H(z))F(x-z,y)dz.$$

$$(F \cdot H)(x,y) = I_1\left(TG_1(x) \otimes \int_{\mathbb{R}} G_2(z)\alpha_z(H(x-z))dz\right) =$$
$$= \int_{\mathbb{R}} G_1(x)G_2(z)\alpha_z(H(y-z))dz =$$
$$= \int_{\mathbb{R}} F(x,z)\alpha_z(H(y-z))dz.$$

□

Now we want to construct two right inverse maps to m and and two left inverse maps to j. First of all, let us describe the action of these maps on $\mathscr{S}(\mathbb{R}^2, A)$.

Lemma 2.16. The following diagrams are commutative:

$$\begin{array}{ccc} S_\alpha \hat{\otimes}_A S_\alpha & \xrightarrow{j} & S_\alpha \hat{\otimes}_A S_\alpha \\ \downarrow I_1 & & \downarrow I_1 \\ \mathscr{S}(\mathbb{R}^2, A)_\alpha & \xrightarrow{\iota} & \mathscr{S}(\mathbb{R}^2, A)_\alpha \end{array} \qquad \begin{array}{ccc} S_\alpha \hat{\otimes}_A S_\alpha & \xrightarrow{m} & S, \\ \downarrow I_1 & \nearrow \pi & \\ \mathscr{S}(\mathbb{R}^2, A)_\alpha & & \end{array}$$

where

$$\iota(F)(x,y) = \left(\frac{\partial F}{\partial x} - \frac{\partial F}{\partial y}\right)(x,y) + \alpha_0'(F(x,y)) \quad \text{for any } F \in \mathscr{S}(\mathbb{R}^2, A)$$

$$\pi(F)(x) = \int_{\mathbb{R}} \alpha_y(F(y, x-y))dy \quad \text{for any } F \in \mathscr{S}(\mathbb{R}^2, A)$$

Proof. It is obvious that ι and ρ are continuous, so we can assume that $F(x,y) = G(x)H(y)$ for some $G, H \in S$:

$$\begin{aligned} I_1 \circ j \circ I_1^{-1}(F)(x,y) &= I_1 \circ j(TG \otimes H)(x,y) = \\ &= I_1\left((TG)'(x)H(y) - (TG)(x)(T(T^{-1}H)(y))\right) = \\ &= T^{-1}((TG)'(x))H(y) - G(x)T((T^{-1}H)'(y)) \overset{2.3}{=} \\ &\overset{2.3}{=} (G'(x) + \alpha_0'(G(x)))H(y) - G(x)(H'(y) - \alpha_0'(H(y))) = \\ &= \left(\frac{\partial F}{\partial x} - \frac{\partial F}{\partial y}\right)(x,y) + \alpha_0'(G(x))H(y) + G(x)\alpha_0'(H(y)) = \\ &= \left(\frac{\partial F}{\partial x} - \frac{\partial F}{\partial y}\right)(x,y) + \frac{d}{dt}\, \alpha_t(G(x)H(y))\big|_{t=0}. \end{aligned}$$

$$m \circ I_1^{-1}(F)(x) = \int_R TG(y)\alpha_y(H(x-y))dy = \int_R \alpha_y(G(y)H(x-y))dy = \pi(F)(x).$$

□

Now we can construct two right inverse maps for π.

Lemma 2.17. Fix a function $\varphi \in C_c^\infty(\mathbb{R})$ with $\int_{\mathbb{R}} \varphi(t)dt = 1$. Define the maps

$$\rho_x : S_\alpha \longrightarrow \mathscr{S}(\mathbb{R}^2, A)_\alpha, \quad \rho_x(F)(x,y) = \varphi(y)\alpha_{-x}(F(x+y)),$$
$$\rho_y : S_\alpha \longrightarrow \mathscr{S}(\mathbb{R}^2, A)_\alpha, \quad \rho_y(F)(x,y) = \varphi(x)\alpha_{-x}(F(x+y)).$$

Then ρ_x is an $\mathscr{S}(\mathbb{R}, A; \alpha)$-$A$-$\hat{\otimes}$-bimodule homomorphism, and ρ_y is an A-$\mathscr{S}(\mathbb{R}, A; \alpha)$-$\hat{\otimes}$-bimodule homomorphism. Moreover, we have

$$\pi \circ \rho_x = \pi \circ \rho_y = \mathrm{Id}_{S_\alpha}.$$

Proof. For any $F, H \in \mathscr{S}(\mathbb{R}, A; \alpha), a \in A$ we have

$$\rho_x(H *_\alpha F)(x,y) = \varphi(y) \int_{\mathbb{R}} \alpha_{-x}(H(z))\alpha_{z-x}(F(x+y-z))dz$$

$$H \cdot \rho_x(F)(x,y) = \int_{\mathbb{R}} \alpha_{-x}(H(z))\rho_x(F)(x-z,y)dz =$$

$$= \int_{\mathbb{R}} \varphi(y)\alpha_{-x}(H(z))\alpha_{z-x}(F(x+y-z))dz$$

$$\rho_x(F \cdot a)(x,y) = \varphi(y)\alpha_{-x}((F \cdot a)(x+y)) = \varphi(y)F(x+y)\alpha_y(a) = (\rho_x(F)) \cdot a$$

$$(\pi \circ \rho_x)(F)(x) = \int_{\mathbb{R}} \alpha_y(\rho_x(F)(y, x-y))dy =$$

$$= \int_{\mathbb{R}} \varphi(x-y)F(x)dy = F(x) \int_{\mathbb{R}} \varphi(x-y)dy = F(x).$$

$$\rho_y(F *_\alpha H)(x,y) = \varphi(x) \int_{\mathbb{R}} \alpha_{-x}(F(z))\alpha_{z-x}(H(x+y-z))dz =$$

$$= \int_{\mathbb{R}} \varphi(x)\alpha_{-x}(F(z+x))\alpha_z(H(y-z))dz$$

$$\rho_y(F) \cdot H(x,y) = \int_{\mathbb{R}} \rho_y(F)(x,z)\alpha_z(H(y-z))dz =$$

$$= \int_{\mathbb{R}} \varphi(x)\alpha_{-x}(F(x+z))\alpha_z(H(y-z))dz$$

$$\rho_y(a \cdot F)(x,y) = \varphi(x)\alpha_{-x}((a \cdot F)(x+y)) =$$

$$= \varphi(x)\alpha_{-x}(a)F(x+y) = (a \cdot \rho_y(F))(x,y)$$

$$(\pi \circ \rho_y)(F)(x) = \int_{\mathbb{R}} \alpha_y(\rho_y(F)(y, x-y))dy = \int_{\mathbb{R}} \varphi(y)F(x)dy = F(x).$$

□

Lemma 2.18. Fix a function $\varphi \in C_c^\infty(\mathbb{R})$ with $\int_{\mathbb{R}} \varphi(t)dt = 1$. Define the maps

$$\beta_x : \mathscr{S}(\mathbb{R}^2, A)_\alpha \longrightarrow \mathscr{S}(\mathbb{R}^2, A)_\alpha,$$
$$\beta_y : \mathscr{S}(\mathbb{R}^2, A)_\alpha \longrightarrow \mathscr{S}(\mathbb{R}^2, A)_\alpha,$$
$$\beta_x(F)(x,y) = \int_{-\infty}^{x} \left(\alpha_{t-x}(F(t, x+y-t)) - \varphi(x+y-t)\int_{\mathbb{R}} \alpha_{z-x}(F(z, x+y-z))dz\right) dt$$
$$\beta_y(F)(x,y) = \int_{-\infty}^{x} \left(\alpha_{t-x}(F(t, x+y-t)) - \varphi(t)\int_{\mathbb{R}} \alpha_{z-x}(F(z, x+y-z))dz\right) dt$$

Then β_x is an $\mathscr{S}(\mathbb{R}, A; \alpha)$-$A$-$\hat{\otimes}$-bimodule homomorphism, and β_y is an A-$\mathscr{S}(\mathbb{R}, A; \alpha)$-$\hat{\otimes}$-bimodule homomorphism. Moreover, we have

$$\beta_x \circ \iota = \beta_y \circ \iota = \mathrm{Id}_{\mathscr{S}(\mathbb{R}^2, A)_\alpha}.$$

Proof. We'll start by proving that β_x is well-defined: it is not entirely obvious from the construction that these integrals define functions which belong to $\mathscr{S}(\mathbb{R}^2, A)$. Let us prove that the corresponding integral over $\mathbb{R}$ equals zero, then we can use the vector-valued version of the Haramard's lemma to prove that the antiderivative lies in $\mathscr{S}(\mathbb{R}^2, A)$, as well.

$$\int_{\mathbb{R}} \left(\alpha_{t-x}(F(t, x+y-t)) - \varphi(x+y-t)\int_{\mathbb{R}} \alpha_{z-x}(F(z, x+y-z))dz\right) dt =$$
$$= \int_{\mathbb{R}} \alpha_{t-x}(F(t, x+y-t))dt - \int_{\mathbb{R}} \left(\varphi(x+y-t)\int_{\mathbb{R}} \alpha_{z-x}(F(z, x+y-z))dz\right) dt =$$
$$= \int_{\mathbb{R}} \alpha_{t-x}(F(t, x+y-t))dt - \int_{\mathbb{R}} \varphi(x+y-t)dt \int_{\mathbb{R}} \alpha_{z-x}(F(z, x+y-z))dz =$$
$$= \int_{\mathbb{R}} \alpha_{t-x}(F(t, x+y-t))dt - \int_{\mathbb{R}} \alpha_{z-x}(F(z, x+y-z))dz = 0$$

Now we can prove that β_x is a $\hat{\otimes}$-bimodule homomorphism. We notice that for every $H \in S, F \in \mathscr{S}(\mathbb{R}^2, A)$ we have

$$\alpha_{t-x}((H \cdot F)(t, x+y-t)) = \alpha_{t-x}\left(\int_{\mathbb{R}} \alpha_{-t}(H(s))F(t-s, x+y-t)ds\right) =$$
$$= \int_{\mathbb{R}} \alpha_{-x}(H(s))\alpha_{t-x}(F(t-s, x+y-t))ds,$$

therefore, we have

$$\beta_x(H \cdot F)(x, y) = \int_{-\infty}^{x} \left(\alpha_{t-x}(H \cdot F(t, x+y-t)) - \varphi(x+y-t)\int_{\mathbb{R}} \alpha_{z-x}(H \cdot F(z, x+y-z))dz\right) dt =$$
$$= \int_{-\infty}^{x}\int_{\mathbb{R}} \alpha_{-x}(H(s))\alpha_{t-x}(F(t-s, x+y-t))dsdt -$$
$$- \int_{-\infty}^{x} \varphi(x+y-t)\int_{\mathbb{R}}\int_{\mathbb{R}} \alpha_{-x}(H(s))\alpha_{z-x}(F(z-s, x+y-z))dsdzdt =$$
$$= \int_{-\infty}^{x}\int_{\mathbb{R}} \alpha_{-x}(H(s))\alpha_{t-x}(F(t-s, x+y-t))dsdt -$$
$$- \int_{-\infty}^{x} \varphi(x+y-t)\int_{\mathbb{R}} \alpha_{-x}(H(s))\int_{\mathbb{R}} \alpha_{z-x}(F(z-s, x+y-z))dzdsdt =$$
$$= \int_{-\infty}^{x}\left(\int_{\mathbb{R}} \alpha_{-x}(H(s))\left(\alpha_{t-x}(F(t-s, x+y-t)) - \varphi(x+y-t)\int_{\mathbb{R}} \alpha_{z-x}(F(z-s, x+y-z))dz\right) ds\right) dt =$$
$$= \int_{\mathbb{R}}\left(\int_{-\infty}^{x} \alpha_{-x}(H(s))\left(\alpha_{t-x}(F(t-s, x+y-t)) - \varphi(x+y-t)\int_{\mathbb{R}} \alpha_{z-x}(F(z-s, x+y-z))dz\right) dt\right) ds =$$
$$= \int_{\mathbb{R}} \alpha_{-x}(H(s))\left(\int_{-\infty}^{x}\left(\alpha_{t-x}(F(t-s, x+y-t)) - \varphi(x+y-t)\int_{\mathbb{R}} \alpha_{z-x}(F(z-s, x+y-z))dz\right) dt\right) ds =$$
$$= \int_{\mathbb{R}} \alpha_{-x}(H(s))\int_{-\infty}^{x} \alpha_{t+s-x}(F(t, x+y-t-s))dtds -$$
$$- \int_{\mathbb{R}} \alpha_{-x}(H(s))\int_{-\infty}^{x} \varphi(x+y-t-s)\int_{\mathbb{R}} \alpha_{z+s-x}(F(z, x+y-z-s))dzdtds =$$
$$= \int_{\mathbb{R}} \alpha_{-x}(H(s))\beta_x(F)(x-s, y)ds = (H \cdot \beta_x(F))(x, y),$$

$$\beta_x(F \cdot a)(x, y) =$$
$$= \int_{-\infty}^{x}\left(\alpha_{t-x}((F \cdot a)(t, x+y-t)) - \varphi(x+y-t)\int_{\mathbb{R}} \alpha_{z-x}((F \cdot a)(z, x+y-z))dz\right) dt =$$
$$= \int_{-\infty}^{x}\left(\alpha_{t-x}(F(t, x+y-t))\alpha_y(a) - \varphi(x+y-t)\int_{\mathbb{R}} \alpha_{z-x}(F(z, x+y-z))\alpha_y(a)dz\right) dt =$$
$$= \left(\int_{-\infty}^{x}\left(\alpha_{t-x}(F(t, x+y-t)) - \varphi(x+y-t)\int_{\mathbb{R}} \alpha_{z-x}(F(z, x+y-z))dz\right) dt\right)\alpha_y(a) =$$
$$= (\beta_x(F) \cdot a)(x, y).$$

To check that β_x is the left inverse to ι, we have to assume that $F(x,y) = G(x)H(y)$. First of all, notice that

$$\begin{aligned}
&\frac{d}{dt}(\alpha_{t-x}(F(t,x+y-t))) = \frac{d}{dt}(\alpha_{t-x}(G(t))\alpha_{t-x}(H(x+y-t))) = \\
&= (\alpha_{t-x}(G'(t)) + \alpha'_{t-x}(G(t)))\alpha_{t-x}(H(x+y-t)) + \\
&+ \alpha_{t-x}(G(t))(\alpha'_{t-x}(H(x+y-t)) - \alpha_{t-x}(H'(x+y-t))) = \\
&= \alpha_{t-x}(G'(t)H(x+y-t) + \alpha'_0(G(t))H(x+y-t) + \\
&+ G(t)\alpha'_0(H(x+y-t)) - G(t)H'(x+y-t)) \overset{2.3}{=} \\
&\overset{2.3}{=} \alpha_{t-x}(T^{-1}((TG)')(t)H(x+y-t) - G(t)T((T^{-1}H)')(x+y-t)) = \\
&= \alpha_{t-x}(\iota(F)(t,x+y-t)).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&(\beta_x \circ \iota(F))(x,y) = \\
&= \int_{-\infty}^{x} \left(\alpha_{t-x}(\iota(F)(t,x+y-t)) - \varphi(x+y-t) \int_{\mathbb{R}} \alpha_{z-x}(\iota(F)(z,x+y-z))dz \right) dt = \\
&= \int_{-\infty}^{x} \left(\frac{d}{dt}(\alpha_{t-x}(F(t,x+y-t))) - \varphi(x+y-t) \int_{\mathbb{R}} \frac{d}{dz}(\alpha_{z-x}(F(z,x+y-z)))dz \right) dt = \\
&= F(x,y).
\end{aligned}$$

The necessary computations for β_y are, essentially, the same. □

By combining the Lemmas 2.12 – 2.18, we obtain the following theorem:

Theorem 2.19. Let A be a self-induced Fréchet-Arens-Michael algebra with a smooth m-tempered action α of $\mathbb{R}$ on A. Then the following diagram is commutative, moreover, the rows are short exact sequences of $\mathscr{S}(\mathbb{R}, A; \alpha)$-bimodules which split in the categories $\mathscr{S}(\mathbb{R}, A; \alpha)$-**mod**-$A$ and A-**mod**-$\mathscr{S}(\mathbb{R}, A; \alpha)$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{S}(\mathbb{R}^2, A)_\alpha & \xrightarrow{\iota} & \mathscr{S}(\mathbb{R}^2, A)_\alpha & \xrightarrow{\pi} & \mathscr{S}(\mathbb{R}, A; \alpha)_\alpha & \longrightarrow & 0, \\
 & & \wr\uparrow & & \wr\uparrow & & \uparrow\mathrm{Id} & & \\
0 & \longrightarrow & S_\alpha \hat{\otimes}_A S_\alpha & \xrightarrow{j} & S_\alpha \hat{\otimes}_A S_\alpha & \xrightarrow{m} & S_\alpha & \longrightarrow & 0,
\end{array}$$

where

$$\iota(F)(x,y) = \left(\frac{\partial F}{\partial x} - \frac{\partial F}{\partial y} \right)(x,y) + \alpha'_0(F(x,y)) \qquad \text{for any } F \in \mathscr{S}(\mathbb{R}^2, A)$$

$$\pi(F)(x) = \int_{\mathbb{R}} \alpha_y(F(y,x-y))dy \qquad \text{for any } F \in \mathscr{S}(\mathbb{R}^2, A)$$

$$j(F \otimes G) = F' \otimes G - F \otimes T((T^{-1}G)') \qquad \text{for any } F, G \in \mathscr{S}(\mathbb{R}, A; \alpha)$$

$$m(F \otimes G) = F *_\alpha G \qquad \text{for any } F, G \in \mathscr{S}(\mathbb{R}, A; \alpha).$$

Proof. In the previous lemmas we have constructed the sections $\rho_x, \rho_y, \beta_x, \beta_y$. The only thing that is left to check that $\iota \circ \beta_x + \rho_x \circ \pi = \iota \circ \beta_y + \rho_y \circ \pi = \mathrm{Id}_{\mathscr{S}(\mathbb{R}^2, A)_\alpha}$, then we use [5, Proposition 3.1.8].

For any $F(x,y) \in \mathscr{S}(\mathbb{R}^2, A)_\alpha$ we have

$$
\begin{aligned}
&(\iota \circ \beta_x)(F)(x,y) = \iota\left(\int_{-\infty}^{x} \left(\alpha_{t-x}(F(t,x+y-t)) - \varphi(x+y-t)\int_{\mathbb{R}} \alpha_{z-x}(F(z,x+y-z))dz\right) dt\right) = \\
&= F(x,y) - \varphi(y)\int_{\mathbb{R}} \alpha_{z-x}(F(z,x+y-z))dz + \\
&+ \int_{-\infty}^{x} \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)\left(\alpha_{t-x}(F(t,x+y-t)) - \varphi(x+y-t)\int_{\mathbb{R}} \alpha_{z-x}(F(z,x+y-z))dz\right) dt + \\
&+ \int_{-\infty}^{x} \left(\alpha'_{t-x}(F(t,x+y-t)) - \varphi(x+y-t)\int_{\mathbb{R}} \alpha'_{z-x}(F(z,x+y-z))dz\right) dt = \\
&= F(x,y) - \varphi(y)\int_{\mathbb{R}} \alpha_{z-x}(F(z,x+y-z))dz + \\
&+ \int_{-\infty}^{x} \left(-\alpha'_{t-x}(F(t,x+y-t)) + \varphi(x+y-t)\int_{\mathbb{R}} \alpha'_{z-x}(F(z,x+y-z))dz\right) dt + \\
&+ \int_{-\infty}^{x} \left(\alpha'_{t-x}(F(t,x+y-t)) - \varphi(x+y-t)\int_{\mathbb{R}} \alpha'_{z-x}(F(z,x+y-z))dz\right) dt = \\
&= F(x,y) - \varphi(y)\int_{\mathbb{R}} \alpha_{z-x}(F(z,x+y-z))dz,
\end{aligned}
$$

$$
(\rho_x \circ \pi)(F)(x) = \varphi(y)\alpha_{-x}(\pi(F)(x+y)) = \varphi(y)\int_{\mathbb{R}} \alpha_{z-x}(F(z,x+y-z))dz,
$$

therefore, we have

$$
\iota \circ \beta_x + \rho_x \circ \pi(F)(x,y) = F(x,y).
$$

The argument for $\iota \circ \beta_y + \rho_y \circ \pi$ is similar. □

The case $G = \mathbb{T}$ can be treated in the same fashion, to formulate the result, we will denote the module $C^\infty(\mathbb{T}, A; \alpha)$ (the definition is the same as for S_α) by T_α.

Theorem 2.20. Let A be a self-induced Fréchet-Arens-Michael algebra with a smooth m-tempered action α of $\mathbb{T}$ on A. Then the following diagram is commutative, moreover, the rows are short exact sequences of $C^\infty(\mathbb{T}, A; \alpha)$-bimodules which split in the categories $C^\infty(\mathbb{T}, A; \alpha)$-**mod**-$A$ and A-**mod**-$C^\infty(\mathbb{T}, A; \alpha)$:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C^\infty(\mathbb{T}^2, A)_\alpha & \xrightarrow{\iota} & C^\infty(\mathbb{T}^2, A)_\alpha & \xrightarrow{\pi} & C^\infty(\mathbb{T}, A; \alpha)_\alpha & \longrightarrow & 0, \\
 & & \wr\big\uparrow & & \wr\big\uparrow & & \mathrm{Id}\big\uparrow & & \\
0 & \longrightarrow & T_\alpha \hat{\otimes}_A T_\alpha & \xrightarrow{j} & T_\alpha \hat{\otimes}_A T_\alpha & \xrightarrow{m} & T_\alpha & \longrightarrow & 0,
\end{array}
$$

where

$$
\begin{aligned}
\iota(F)(x,y) &= \left(\frac{\partial F}{\partial x} - \frac{\partial F}{\partial y}\right)(x,y) + \alpha'_0(F(x,y)) && \text{for any } F \in C^\infty(\mathbb{T}^2, A) \\
\pi(F)(x) &= \int_{\mathbb{T}} \alpha_y(F(y, x-y))dy && \text{for any } F \in C^\infty(\mathbb{T}^2, A) \\
j(F \otimes G) &= F' \otimes G - F \otimes T((T^{-1}G)') && \text{for any } F, G \in C^\infty(\mathbb{T}, A; \alpha) \\
m(F \otimes G) &= F *_\alpha G && \text{for any } F, G \in C^\infty(\mathbb{T}, A; \alpha).
\end{aligned}
$$

3. Obtaining upper estimates for homological dimensions of smooth crossed products by $\mathbb{R}$ and $\mathbb{T}$

Remark. Again, we provide the proofs only for the case $G = \mathbb{R}$, but the same arguments work for $G = \mathbb{T}$, as well.

Here we adapt the arguments in [6], which were used to obtain the upper estimates, to the non-unital case.

Definition 3.1. Let A be a $\hat{\otimes}$-algebra. Then a $\hat{\otimes}$-algebra S together with an A-$\hat{\otimes}$-bimodule structure is called an A-$\hat{\otimes}$-algebra if:

1. $S \in A$-**mod**-S and $S \in S$-**mod**-A,
2. We have
$$(s \cdot a) *_S t = s *_S (a \cdot t) \tag{3.1}$$
for every $s, t \in S$, $a \in A$, where $\cdot$ defines the outer module multiplication, and $*_S$ defines the ring multiplication in S.

This definition works as expected in the unital case.

Proposition 3.2. Let A be a unital $\hat{\otimes}$-algebra. An A-$\hat{\otimes}$-algebra structure on a unital A-$\hat{\otimes}$-algebra S is uniquely defined by a (unital) algebra homomorphism $\eta : A \to S$:

$$a \cdot s = \eta(a)s, \quad s \cdot a = s\eta(a)$$

for every $a \in A, s \in S$, where $\cdot$ denotes the outer A-module multiplication.

Proof. Define η as follows: $\eta(a) = 1_S \cdot a \cdot 1_S \overset{(3.1)}{=} a \cdot 1_S = 1_S \cdot a$. It is easy to see that η is an algebra homomorphism. Also, we have

$$\eta(a)s = (a \cdot 1_S)s = a \cdot (1_S s) = a \cdot s,$$
$$s\eta(a) = s(a \cdot 1_S) = s(1_S \cdot a) = (s1_S) \cdot a = s \cdot a$$

for any $a \in A$, $s \in S$. □

As a corollary from Lemma 2.12 we get that the A-$\hat{\otimes}$-algebra structure on S_α makes $\mathscr{S}(\mathbb{R}, A; \alpha)$ into an A-$\hat{\otimes}$-algebra, and checking (3.1) can be done like this:

$$\begin{aligned}((F \cdot a) *_\alpha G)(x) &= \int_{\mathbb{R}} F(y)\alpha_y(a)\alpha_y(G(x-y))dy = \\ &= \int_{\mathbb{R}} F(y)\alpha_y((a \cdot G)(x-y))dy = \\ &= (F *_\alpha (a \cdot G))(x).\end{aligned}$$

Proposition 3.3. Let A be a Fréchet-Arens-Michael algebra, and let α be a smooth m-tempered action of $\mathbb{R}$ on A. Consider the following multiplication on $\mathscr{S}(\mathbb{R}, A)$:

$$(f *'_\alpha g)(x) = \int_{\mathbb{R}} \alpha_{-y}(f(x-y))g(y)dy.$$

Then the following locally convex algebra isomorphism takes place:

$$i : \mathscr{S}(\mathbb{R}, A; \alpha) \to (\mathscr{S}(\mathbb{R}, A), *'_\alpha), \quad i(f)(x) = \alpha_{-x}(f(x)).$$

Proof. The mapping i is, obviously, a topological isomorphism of locally convex spaces. Now notice that

$$\begin{aligned}(i(f) *' i(g))(x) &= \int_{\mathbb{R}} \alpha_{-y}(i(f)(x-y))i(g)(y)dy = \\ &= \int_{\mathbb{R}} \alpha_{-x}(f(x-y))\alpha_{-y}(g(y))dy = \\ &= \int_{\mathbb{R}} \alpha_{-x}(f(-y))\alpha_{-x-y}(g(y+x))dy = \\ &= \alpha_{-x}\left(\int_{\mathbb{R}} f(-y)\alpha_{-y}(g(y+x))dy\right) = \\ &= i(f * g)(x),\end{aligned}$$

therefore, i is an algebra homomorphism. □

Corollary 3.4. Define the A-$\hat{\otimes}$-bimodule and $(\mathscr{S}(\mathbb{R}, A), *'_\alpha)$-$\hat{\otimes}$-bimodule ${}_{\alpha^{-1}}\mathscr{S}(\mathbb{R}, A; \alpha) := {}_{\alpha^{-1}}S$ as follows: ${}_{\alpha^{-1}}S$ coincides with $\mathscr{S}(\mathbb{R}, A)$ as a LCS, and

$$\begin{aligned}&(F \cdot a)(x) = F(x)a, && a \cdot F(x) = \alpha_{-x}(a)F(x) && \text{for any } a \in A, F \in {}_{\alpha^{-1}}S \\ &(F \cdot G)(x) = (F *'_\alpha G)(x), && (G \cdot F)(x) = (G *'_\alpha F)(x) && \text{for any } F \in {}_{\alpha^{-1}}S, G \in (\mathscr{S}(\mathbb{R}, A), *'_\alpha).\end{aligned}$$

Then the map

$$S_\alpha \longrightarrow {}_{\alpha^{-1}}S, \quad F(x) \longmapsto \alpha_{-x}(F(x)),$$

is an isomorphism of A-$\hat{\otimes}$-bimodules.

Definition 3.5. Let A be a $\hat{\otimes}$-algebra. A left A-$\hat{\otimes}$-module M is called essential, if the canonical morphism $\eta_M : A\hat{\otimes}_A M \longrightarrow M$ is an isomorphism of left A-$\hat{\otimes}$-modules.

Example. If A is a self-induced $\hat{\otimes}$-algebra, then for any left A-$\hat{\otimes}$-module M the modules $A\hat{\otimes}_A M$ and $A\hat{\otimes}M$ are essential. We can prove this by noticing that the maps of left A-$\hat{\otimes}$-modules

$$(A\hat{\otimes}_A A)\hat{\otimes}_A M \xrightarrow{m\otimes Id_M} A\hat{\otimes}_A M$$

and

$$(A\hat{\otimes}_A A)\hat{\otimes} M \xrightarrow{m\otimes Id_M} A\hat{\otimes} M$$

are isomorphisms and they do coincide with the canonical projections $\eta_{A\hat{\otimes}_A M}$ and $\eta_{A\hat{\otimes}M}$, respectively.

Lemma 3.6. Let A be a self-induced Fréchet-Arens-Michael algebra together with a smooth m-tempered $\mathbb{R}$-action α of $\mathbb{R}$. Then

1. the module $\mathscr{S}(\mathbb{R}, A; \alpha)_\alpha$ is an essential left and right A-$\hat{\otimes}$-module,
2. and if A is projective as a left and right A-$\hat{\otimes}$-module then the module $\mathscr{S}(\mathbb{R}, A; \alpha)_\alpha$ is projective as a left and right A-$\hat{\otimes}$-module and a as a left and right S-$\hat{\otimes}$-module.

Proof. 1. By definition, $S_\alpha \simeq A\hat{\otimes}\mathscr{S}(\mathbb{R})$, so we can use the proof in the Example 3 to get that S_α is essential as a left A-$\hat{\otimes}$-module. Now the existence of the isomorphism ${}_{\alpha^{-1}}\mathscr{S}(\mathbb{R}, A; \alpha) \cong \mathscr{S}(\mathbb{R}, A; \alpha)_\alpha$ ensures that S_α is essential as a right A-$\hat{\otimes}$-module..

2. Due to our assumption, A is projective as a left A-module, therefore, for any Fréchet space E the left A-$\hat{\otimes}$-module $A\hat{\otimes}E$ is projective, as well. However, by definition, S_α is isomorphic as a left A-$\hat{\otimes}$-module to $A\hat{\otimes}\mathscr{S}(\mathbb{R})$. Therefore, S_α is a projective left A-$\hat{\otimes}$-module. By using the isomorphism ${}_{\alpha^{-1}}\mathscr{S}(\mathbb{R}, A; \alpha) \cong \mathscr{S}(\mathbb{R}, A; \alpha)_\alpha$, we also prove that S_α is projective as a right A-$\hat{\otimes}$-module.

   To prove that S_α is projective as a left S-$\hat{\otimes}$-module, we need to prove that the multiplication map $m_S^+ : S_\alpha\hat{\otimes}(S_\alpha)_+ \to S_\alpha$ is a retraction, due to [5, Proposition 4.1.1]. However, we can prove a stronger statement by noticing that the restriction $m : S_\alpha\hat{\otimes}S_\alpha \to S_\alpha$ is a retraction.

   Unfortunately, for now we only know that the multiplication map $S_\alpha\hat{\otimes}_A S_\alpha \to S_\alpha$ is a retraction of left and right S-$\hat{\otimes}$-modules, because we have already proven that $m\circ(I_1^{-1}\circ\rho_x) = \mathrm{Id}_{S_\alpha}$ and $m\circ(I_1^{-1}\circ\rho_y) = \mathrm{Id}_{S_\alpha}$ in Lemma 2.16 and Lemma 2.17.

   Nevertheless, due to A-$\hat{\otimes}$-module projectivity, we can utilize the canonical projection $\rho : S_\alpha\hat{\otimes}A \to S_\alpha$, which is a retraction of right A-$\hat{\otimes}$-modules. By applying the functor $(-)\hat{\otimes}_A S_\alpha$ to ρ we get the following:

$$S_\alpha\hat{\otimes}S_\alpha \xrightarrow{\sim} S_\alpha\hat{\otimes}A\hat{\otimes}_A S_\alpha \xrightarrow{\rho\otimes 1_{S_\alpha}} S_\alpha\hat{\otimes}_A S_\alpha.$$

   Finally, we can compose the two retractions we have just obtained, $S_\alpha\hat{\otimes}S_\alpha \to S_\alpha\hat{\otimes}_A S_\alpha$, and $S_\alpha\hat{\otimes}_A S_\alpha \to S_\alpha$, to get that S_α is a right projective S-$\hat{\otimes}$-module. A similar argument shows that S_α is a left projective S-$\hat{\otimes}$-module.

□

Lemma 3.7. Consider a $\hat{\otimes}$-algebra A and an A-$\hat{\otimes}$-algebra S which is projective as a left and right A-$\hat{\otimes}$-module, and also is projective as a left and right S-$\hat{\otimes}$-module.

(1) Let X be a projective right A-$\hat{\otimes}$-module. Then the module $X\hat{\otimes}_A S$ is a projective right S-$\hat{\otimes}$-module. Similarly, if X is a projective left A-$\hat{\otimes}$-module, then $S\hat{\otimes}_A X$ is a projective left S-$\hat{\otimes}$-module.

(2) Let X be a projective A-$\hat{\otimes}$-bimodule. Then the module $S\hat{\otimes}_A X\hat{\otimes}_A S$ is a projective S-$\hat{\otimes}$-bimodule.

Proof. (1) If X is a projective right A-$\hat{\otimes}$-module, then there is a retraction $\sigma : E\hat{\otimes}A_+ \to X$ for some l.c.s E. Then we can apply the functor $(-)\hat{\otimes}_A S$ to both parts, thus obtaining the following retraction:

$$E\hat{\otimes}S \simeq E\hat{\otimes}A_+\hat{\otimes}_A S \to X\hat{\otimes}_A S.$$

Moreover, because S itself is projective over S, we can immediately derive that $E\hat{\otimes}S$ is a projective right S-$\hat{\otimes}$-module, and $X\hat{\otimes}_A S$ is projective due to being a retract of a projective module. A similar argument works for left modules and bimodules, therefore, we also get (2) as well.

□

Lemma 3.8. Let A be a $\hat{\otimes}$-algebra. Also let $\{M, d\}$ denote an admissible sequence of right A-$\hat{\otimes}$-modules. If X is a projective left A-$\hat{\otimes}$-module, then the complex $\{M\hat{\otimes}_A X, d \otimes \mathrm{Id}\}$ splits in **LCS**.

Similarly, if $\{M, d\}$ is an admissible sequence of left A-$\hat{\otimes}$-modules, then for every projective right A-$\hat{\otimes}$-module X the complex $\{X\hat{\otimes}_A M, \mathrm{Id} \otimes d\}$ splits in **LCS**.

Proof. If X were a free left A-$\hat{\otimes}$-module, then the statement of the lemma would follow from the canonical isomorphism $M\hat{\otimes}_A A_+ \hat{\otimes} E \cong M\hat{\otimes}E$ for some $E \in$ **LCS**. However, a retract of an admissible sequence is admissible, as well. □

Lemma 3.9. Let A be a $\hat{\otimes}$-algebra and let S be an A-$\hat{\otimes}$-algebra, which is projective as a left A-$\hat{\otimes}$-module. Then we have

$$\mathrm{dh}_{S^{op}}(M\hat{\otimes}_A S) \leq \mathrm{dh}_{A^{op}}(M) \quad \text{for all } M \in \textbf{mod-}A.$$

If S is projective as a right A-$\hat{\otimes}$-module, then

$$\mathrm{dh}_S(S\hat{\otimes}_A M) \leq \mathrm{dh}_A(M) \quad \text{for all } M \in A\textbf{-mod}.$$

Proof. Suppose we have a projective resolution of M in **mod**-A:

$$0 \longleftarrow M \xleftarrow{d_0} P_0 \xleftarrow{d_1} \cdots \longleftarrow P_n \longleftarrow 0 \longleftarrow \ldots$$

Then due to Lemma 3.7 and 3.8 the following sequence is a projective resolution for $M\hat{\otimes}_A S$ in **mod**-S:

$$0 \longleftarrow M\hat{\otimes}_A S \xleftarrow{d_0 \otimes \mathrm{Id}} P_0\hat{\otimes}_A S \xleftarrow{d_1 \otimes \mathrm{Id}} \cdots \longleftarrow P_n\hat{\otimes}_A S \longleftarrow 0 \longleftarrow \ldots$$

Therefore, $\mathrm{dh}_{S^{op}}(M\hat{\otimes}_A S) \leq \mathrm{dh}_{A^{op}}(M)$. □

From now on, we will, in addition, assume that A is projective as a left and as a right A-$\hat{\otimes}$-module. For brevity, we will just say that A is **projective**.

Lemma 3.10. Let A be a projective self-induced Fréchet-Arens-Michael algebra together with a smooth m-tempered $\mathbb{R}$-action α. Set $S = \mathscr{S}(\mathbb{R}, A; \alpha)$. For any right S-$\hat{\otimes}$-module M we have the following estimate:

$$\mathrm{dh}_{S^{\mathrm{op}}}(M\hat{\otimes}_S S_\alpha) \leq \mathrm{dh}_{A^{\mathrm{op}}}(M\hat{\otimes}_S S_\alpha) + 1 \leq \mathrm{dgr}(A) + 1.$$

And for any left S-$\hat{\otimes}$-module M we have

$$\mathrm{dh}_S(S_\alpha\hat{\otimes}_S M) \leq \mathrm{dh}_A(S_\alpha\hat{\otimes}_S M) + 1 \leq \mathrm{dgl}(A) + 1.$$

Proof. Due to Theorem 2.19 we have the following sequence:

$$0 \longrightarrow S_\alpha\hat{\otimes}_A S_\alpha \longrightarrow S_\alpha\hat{\otimes}_A S_\alpha \longrightarrow S_\alpha \longrightarrow 0. \tag{3.2}$$

By applying the functor $M\hat{\otimes}_S(-)$ to (3.2), we get

$$0 \longrightarrow M\hat{\otimes}_S S_\alpha\hat{\otimes}_A S_\alpha \longrightarrow M\hat{\otimes}_S S_\alpha\hat{\otimes}_A S_\alpha \longrightarrow M\hat{\otimes}_S S_\alpha \longrightarrow 0.$$

Since (3.2) splits due to Theorem 2.19, this sequence is admissible, therefore, we can apply Lemma 3.9, because S_α is a projective A-$\hat{\otimes}$-module (Lemma 3.6).

$$\mathrm{dh}_{S^{\mathrm{op}}}(M\hat{\otimes}_S S_\alpha) \le \mathrm{dh}_{S^{\mathrm{op}}}(M\hat{\otimes}_S S_\alpha\hat{\otimes}_A S_\alpha)+1 \overset{L3.9}{\le} \mathrm{dh}_{A^{\mathrm{op}}}(M\hat{\otimes}_S S_\alpha)+1 \le \mathrm{dgr}(A)+1.$$

□

So, we have just obtained the upper bound for projective dimension of essential modules. To obtain an estimate for an arbitrary right S-$\hat{\otimes}$-module, we use the method, described in the Lemmas 1-3 of the paper [10].

Theorem 3.11. [5, Theorem 5.2.1] Let A be a $\hat{\otimes}$-algebra and let X be a left A-$\hat{\otimes}$-module. Then there exists an admissible complex in A-**mod**:

$$0 \longleftarrow X \longleftarrow (A_+\hat{\otimes}X)\oplus(A\hat{\otimes}_A X) \overset{\delta_0}{\longleftarrow} (A_+\hat{\otimes}(A\hat{\otimes}_A X))\oplus A\hat{\otimes}X \overset{\delta_1}{\longleftarrow}$$
$$\overset{\delta_1}{\longleftarrow} A\hat{\otimes}(A\hat{\otimes}_A X) \longleftarrow 0.$$

Moreover, this sequence is isomorphic to the Yoneda product of the following short admissible complexes:

$$0 \longleftarrow \operatorname{Im}\delta_0 \overset{\delta_0}{\longleftarrow} (A_+\hat{\otimes}(A\hat{\otimes}_A X))\oplus A\hat{\otimes}X \overset{\delta_1}{\longleftarrow} A\hat{\otimes}(A\hat{\otimes}_A X) \longleftarrow 0, \tag{3.3}$$

$$0 \longleftarrow X \longleftarrow (A_+\hat{\otimes}X)\oplus(A\hat{\otimes}_A X) \hookleftarrow \operatorname{Im}\delta_0 \longleftarrow 0. \tag{3.4}$$

Remark. It is important to observe that this theorem works in a fairly general setting, without any strong assumptions on A and X.

Consider a projective self-induced Fréchet-Arens-Michael algebra A equipped with a smooth m-tempered $\mathbb{R}$-action α, and let us, once again, denote $S = \mathscr{S}(\mathbb{R}, A; \alpha)$.

To obtain the main result, it remains to observe that the modules $(A_+\hat{\otimes}(A\hat{\otimes}_A X))\oplus A\hat{\otimes}X$ and $A\hat{\otimes}(A\hat{\otimes}_A X)$ are projective left A-modules, because A itself is projective. Therefore, $\mathrm{dh}_A(\operatorname{Im}\delta_0) \le 1$.

But then we also have

$$\mathrm{dh}_A(X) \le \max\{\mathrm{dh}_A((A_+\hat{\otimes}X)\oplus(A\hat{\otimes}_A X)), \mathrm{dh}_A(\operatorname{Im}\delta_0)+1\} \le \max\{\mathrm{dh}_A(A\hat{\otimes}_A X), 2\}, \tag{3.5}$$

for any projective $\hat{\otimes}$-algebra A and a left A-$\hat{\otimes}$-module X. Combining (3.5) with Lemma 3.10, we get the following: for every left S-$\hat{\otimes}$-module M we have

$$\mathrm{dh}_S(M) \overset{(3.5)}{\le} \max\{\mathrm{dh}_S(S_\alpha\hat{\otimes}_S M), 2\} \overset{L3.10}{\le} \max\{\mathrm{dgl}(A)+1, 2\} = \max\{\mathrm{dgl}(A), 1\}+1.$$

Theorem 3.12. Let A be a projective self-induced Fréchet-Arens-Michael algebra equipped with a smooth m-tempered $\mathbb{R}$-action α. Then the following estimate takes place:

$$\mathrm{dgl}(\mathscr{S}(\mathbb{R}, A; \alpha)) \le \max\{\mathrm{dgl}(A), 1\} + 1.$$

The same result holds for $G = \mathbb{T}$:

Theorem 3.13. Let A be a projective self-induced Fréchet-Arens-Michael algebra equipped with a smooth m-tempered $\mathbb{T}$-action α. Then the following estimate takes place:

$$\mathrm{dgl}(C^\infty(\mathbb{T}, A; \alpha)) \le \max\{\mathrm{dgl}(A), 1\} + 1.$$

References

[1] J. Cuntz and D. Quillen. "Algebra Extensions and Nonsingularity". In: *Journal of the American Mathematical Society* 8.2 (1995).

[2] Gerald B. Folland. *Real analysis: modern techniques and their applications*. 2nd ed. PAM. Wiley, 1999. ISBN: 0471317160,9780471317166.

[3] O. Gabriel and M. Grensing. "Six-Term Exact Sequences for Smooth Generalized Crossed Products". In: *ArXiv e-prints* (Nov. 2011). arXiv: `1111.2154 [math.KT]`.

[4] Alexandre Grothendieck. "Produits tensoriels topologiques et espaces nucléaires". In: *Mem. Amer. Math. Soc.* 16 (1955), Chapter 1: 196 pp., Chapter 2: 140. ISSN: 0065-9266.

[5] A. Ya. Helemskii. *The homology of Banach and topological algebras*. Mathematics and Its Applications. Kluwer Academic Publishers, 1986. ISBN: 978-94-010-7560-2.

[6] P. Kosenko. "Homological dimensions of analytic Ore extensions". In: *ArXiv e-prints* (Dec. 2017). arXiv: `1712.06177 [math.FA]`.

[7] A. Mallios. *Topological Algebras Selected Topics*. North-Holland Mathematics Studies 124. Elsevier Science Ltd, 1986.

[8] R. Meyer. "Embeddings of derived categories of bornological modules". In: *ArXiv Mathematics e-prints* (Oct. 2004). eprint: `math/0410596`.

[9] Sergey Neshveyev. "Smooth crossed products of Rieffel's deformations". In: *Letters in Mathematical Physics* 104.3 (2014), pp. 361–371.

[10] OS Ogneva and A Ya Khelemskii. "Homological dimensions of certain algebras of principal (test) functions". In: *Mathematical notes of the Academy of Sciences of the USSR* 35.2 (1984), pp. 93–99.

[11] N. Christopher Phillips and Larry B. Schweitzer. "Representable K-theory of smooth crossed products by R and Z". In: *Transactions of the American Mathematical Society* 344.1 (1994), pp. 173–201.

[12] A. Yu. Pirkovskii. "Arens-Michael envelopes, homological epimorphisms, and relatively quasi-free algebras". In: *Transactions of the Moscow Mathematical Society* 69 (2008).

[13] A. Yu. Pirkovskii. "Homological dimensions and Van den Bergh isomorphisms for nuclear Fréchet algebras". In: *Izvestiya: Mathematics* 76.4 (2012).

[14] Larry B. Schweitzer. "Dense m-convex Fréchet subalgebras of operator algebra crossed products by Lie groups". In: *International Journal of Mathematics* 4.04 (1993), pp. 601–673.

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