

ADO'S THEOREM

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ABSTRACT. The purpose of this expository note is to give a proof of Ado's Theorem, which states that every finite-dimensional Lie algebra over the field of characteristic 0 has a faithful finite-dimensional representation.

1. INTRODUCTION

Let L be a finite-dimensional Lie algebra over the field F . By a finite-dimensional representation, we mean a Lie algebra homomorphism $\varphi : L \rightarrow \mathfrak{gl}(V)$ for some vector space V with $\dim(V) < \infty$. When φ is injective, φ is said to be a *faithful* representation. If L has a faithful representation, then we can view elements of L as matrices with entries in F . It is natural to ask whether every finite dimensional Lie algebra can be concretely realized in this way. Ado [1] gave an affirmative answer when $\text{char}(F) = 0$. The result was extended by Iwasawa [2] to cover the case $\text{char}(F) = p$ for prime p . We follow Fulton & Harris [3] to give the proof for the case $\text{char}(F) = 0$.

2. UNIVERSAL ENVELOPING ALGEBRAS

Recall that if L is a Lie algebra over the field F , the *universal enveloping algebra* $\mathcal{U}$ of L is an associative F -algebra with 1 together with a linear map $i : L \rightarrow \mathcal{U}$ satisfying $i([x, y]) = i(x)i(y) - i(y)i(x)$ for $x, y \in L$, and has the following universal property: If $\mathfrak{A}$ is any F -algebra with 1, and $j : L \rightarrow \mathfrak{A}$ is a linear map satisfying $j([x, y]) = j(x)j(y) - j(y)j(x)$, then there exists unique F -algebra homomorphism $\psi : \mathcal{U} \rightarrow \mathfrak{A}$ such that $j = \psi \circ i$. In other words, the following diagram commutes:

$$\begin{array}{ccc} L & & \\ \downarrow i & \searrow j & \\ \mathcal{U} & \xrightarrow{\psi} & \mathfrak{A} \end{array}$$

See Humphreys [4] Chapter V for explicit construction of $\mathcal{U}$. One noteworthy result, known as Poincare-Birkhoff-Witt Theorem, will be used in the proof of Ado's Theorem:

Theorem (Poincare-Birkhoff-Witt) Let $(x_1, x_2, \dots)$ be any ordered basis of L . Set $y_n = i(x_n) \in \mathcal{U}$. Then the elements $y_{j(1)} \dots y_{j(m)}$, $m \in \mathbb{N}$ and $j(1) \leq j(2) \leq \dots \leq j(m)$ together with 1 form a basis of $\mathcal{U}$.

Proof. Corollary 17.3.D in Humphreys [4].

There is another important result that lets us identify L with its image $i(L)$ in $\mathcal{U}(L)$:

Theorem (Embedding L into $\mathcal{U}(L)$) The canonical map $i : L \rightarrow \mathcal{U}(L)$ is injective.

Proof. Corollary 17.3.C in Humphreys [4].

This result immediately leads to a weaker version of Ado-Iwasawa Theorem. If L is any Lie algebra, then for each $x \in L$, define $\psi_x : \mathcal{U} \rightarrow \mathcal{U}$ by $\psi_x(t) = i(x)t$. Define $\Psi : L \rightarrow \mathfrak{gl}(\mathcal{U})$ by $\Psi(x) = \psi_x$. Then we get a representation of L . Since $\psi(x) : 1 \mapsto i(x)$ and $\psi(y) : 1 \mapsto i(y)$, it follows that $\psi(x) \neq \psi(y)$ for $x \neq y$ and so Ψ is injective. Therefore, every Lie algebra has a faithful representation. Of course, the space $\mathcal{U}$ is almost

always infinite-dimensional. On the other hand, Ado-Iwasawa theorem guarantees the existence of a faithful *finite-dimensional* representation of L .

3. ADO'S THEOREM

It can be shown (Lemma C.27 in [3]) that if δ is a derivation of L , then there exists a derivation $\tilde{\delta}$ of $\mathcal{U}$ such that $\tilde{\delta} \circ i = i \circ \delta$. We start by proving a lemma that allows us to replace an ideal of finite co-dimension in $\mathcal{U}$ with a smaller ideal of finite co-dimension that is invariant under derivations of L . We will use $\text{Nil}(L)$ and $\text{Rad}(L)$ to denote the maximal nilpotent ideal and maximal solvable ideal of L , respectively.

Lemma 1. Suppose $\mathfrak{S}$ is a finite-dimensional solvable Lie algebra over F where $\text{char}(F) = 0$. Let $\mathcal{U}$ be the universal enveloping algebra of $\mathfrak{S}$, together with its linear map $i : \mathfrak{S} \rightarrow \mathcal{U}$. Suppose I is an ideal of $\mathcal{U}$ satisfying the following conditions:

- (i) $\mathcal{U}/I$ has a finite dimension.
- (ii) $\overline{i(x)}$ is nilpotent in $\mathcal{U}/I$ for every $x \in \text{Nil}(\mathfrak{S})$.

Then there exists an ideal J of $\mathcal{U}$ with $J \subseteq I$ such that J satisfies (1) and (2), and moreover,

- (iii) If δ is a derivation of $\mathfrak{S}$, the corresponding derivation $\tilde{\delta}$ of $\mathcal{U}$ satisfies $D(J) \subseteq J$.

Proof. It is a standard fact that $\mathcal{U}$ is generated (as an associative algebra) by $\{i(x) : x \in \mathfrak{S}\}$. Hence, $\mathcal{U}/I$ is generated by $\{\overline{i(x)} : x \in \mathfrak{S}\}$. Let $\mathfrak{N}$ be the ideal in $\mathcal{U}/I$ generated by $\{\overline{i(x)} : x \in \text{Nil}(\mathfrak{S})\}$. We claim that $\mathfrak{N}^k = 0$ for some positive integer k . Define the Lie algebra homomorphism $f : \text{Nil}(\mathfrak{S}) \rightarrow \mathfrak{gl}(\mathcal{U}/I)$ by $f(x) = \pi_x$ where $\pi_x : t \mapsto \overline{i(x)}t$. Since $\text{Nil}(\mathfrak{S})$ is nilpotent, it follows that $f(\text{Nil}(\mathfrak{S}))$ is nilpotent. By Engel's Theorem applied to $f(\text{Nil}(\mathfrak{S})) \subseteq \mathfrak{gl}(\mathcal{U}/I)$, there exists $k \geq 1$ such that any product of k elements from $f(\text{Nil}(\mathfrak{S}))$ is 0, i.e. $f(\pi_{x_1}) \circ f(\pi_{x_2}) \circ \cdots \circ f(\pi_{x_k}) = 0$ that is, $\overline{i(x_1)i(x_2) \cdots i(x_k)} = 0$ for any $x_i \in \text{Nil}(\mathfrak{S})$. To prove $\mathfrak{N}^k = 0$, it is enough to show that the products in $\mathcal{U}/J$ containing at least k elements of the form $\overline{i(x)}$ with $x \in \text{Nil}(\mathfrak{S})$ is zero. If $x \in \text{Nil}(\mathfrak{S})$ and $y \in \mathfrak{S}$, then $\overline{i(x)i(y)} = \overline{i([x, y])} + \overline{i(y)i(x)}$. Since $\text{Nil}(\mathfrak{S})$ is an ideal, $[x, y] \in \text{Nil}(\mathfrak{S})$. By repeated application of this identity, we can move all $\overline{i(x)}$ with $x \in \text{Nil}(\mathfrak{S})$ to the right. So any product containing at least k elements of the form $\overline{i(x)}$ with $x \in \text{Nil}(\mathfrak{S})$ can be rewritten as a linear combination of products each of whose last (right-most) k factors are $\overline{i(x)}$ with $x \in \text{Nil}(\mathfrak{S})$, which is zero. Thus, $\mathfrak{N}^k = 0$. Since $\mathfrak{N}$ is an ideal in $\mathcal{U}/I$, we can write $\mathfrak{N} = \mathcal{B}/I$ for some ideal $\mathcal{B}$ in $\mathcal{U}$ (by correspondence theorem). Hence, $\mathfrak{N}^k = 0 \Rightarrow \mathcal{B}^k \subseteq I$. Set $J = \mathcal{B}^k$. We will show that J satisfies condition (i), (ii) and (iii).

(i) Note that $\dim(\mathcal{U}/\mathcal{B}) < \infty$ because $\dim(\mathcal{U}/\mathcal{B}) + \dim(\mathcal{B}/I) = \dim(\mathcal{U}/I) < \infty$. Let $\alpha_1, \dots, \alpha_n$ be basis for $\{i(x) : x \in \mathfrak{S}\}$. Since $\mathcal{U}/\mathcal{B}$ is finite-dimensional, there exists monic polynomials $f_1, f_2, \dots, f_n \in F[X]$ such that $f_i(\overline{\alpha_i}) = 0$ in $\mathcal{U}/\mathcal{B}$. Then $f_i(\alpha_i) \in \mathcal{B}$ for each $1 \leq i \leq n$. Set $g_i(X) = f_i(X)^k \in F[X]$. Then $g(\alpha_i) \in J$ for each $1 \leq i \leq n$. By Poincare-Birkhoff-Witt Theorem, $\{\alpha_1^{j_1}, \alpha_2^{j_2}, \dots, \alpha_n^{j_n} : \text{each } j_i \geq 0\}$ is a basis for $\mathcal{U}$, and so $S = \{\overline{\alpha_1^{j_1}}, \overline{\alpha_2^{j_2}}, \dots, \overline{\alpha_n^{j_n}} : \text{each } j_i \geq 0\}$ is a basis for $\mathcal{U}/J$. But $\overline{\alpha_1}, \dots, \overline{\alpha_n}$ each satisfies a monic polynomial in $\mathcal{U}/J$ (the polynomials $g_1, \dots, g_n$ respectively). Thus S is finite, and $\mathcal{U}/J$ is finite-dimensional.

(ii) Suppose $x \in \text{Nil}(\mathfrak{S})$. Since I satisfies condition (ii), $\overline{i(x)}^m = 0$ in $\mathcal{U}/I$ for some $m \in \mathbb{N}$, i.e. $i(x)^m \in I$. This gives $i(x)^{mk} = (i(x)^m)^k \in I^k \subseteq \mathcal{B}^k = J$. Hence, $\overline{i(x)}$ is nilpotent in $\mathcal{U}/J$ for every $x \in \text{Nil}(\mathfrak{S})$.

(iii) We use the fact that if δ is a derivation of a Lie algebra L over the field of characteristic 0, then $\delta(\text{Rad}(L))$ is contained in a nilpotent ideal (The proof is given in the last section "Derivations of Lie Algebras"). Here $\mathfrak{S} = \text{Rad}(\mathfrak{S})$ as $\mathfrak{S}$ is solvable; so $\delta(\mathfrak{S}) \subseteq \text{Nil}(\mathfrak{S})$ for every derivation δ of $\mathfrak{S}$. The corresponding derivation $\tilde{\delta} : \mathcal{U} \rightarrow \mathcal{U}$ satisfies $\tilde{\delta} \circ i = i \circ \delta$. Hence, $\tilde{\delta}(i(\mathfrak{S})) = i(\delta(\mathfrak{S})) \subseteq i(\text{Nil}(\mathfrak{S})) \subseteq \mathcal{B}$. Since $i(\mathfrak{S})$ generates $\mathcal{U}$, it follows that $\tilde{\delta}(\mathcal{U}) \subseteq \mathcal{B}$. Since $\tilde{\delta}$ is a derivation, $\tilde{\delta}(\mathcal{B}^k) \subseteq \tilde{\delta}(\mathcal{B}^k)$, that is, $\tilde{\delta}(J) \subseteq J$. $\square$

We call a representation $\rho : L \rightarrow \mathfrak{gl}(V)$ a *nilrepresentation* if $\rho(x)$ is a nilpotent endomorphism for each $x \in \text{Nil}(L)$. The following result plays a pivotal role in the proof:

Proposition 2. Let $L = \mathfrak{S} \oplus H$ be a finite-dimensional Lie algebra where $\mathfrak{S}$ is a solvable ideal and H is a subalgebra of L . Suppose $\sigma : \mathfrak{S} \rightarrow \mathfrak{gl}(V)$ be a finite-dimensional nilrepresentation. Then there exists a finite-dimensional representation $\rho : L \rightarrow \mathfrak{gl}(V')$ such that $\ker(\rho) \cap \mathfrak{S} \subseteq \ker(\sigma)$. Moreover, if $\text{Nil}(L) = \text{Nil}(\mathfrak{S})$ or $\text{Nil}(L) = L$, then ρ can be chosen to be a nilrepresentation.

Proof. We break down the proof into 6 steps, and highlight the main idea in each step.

Step 1. Use Lemma 1 to create a suitable ideal inside $\mathcal{U}(\mathfrak{S})$.

Let $\mathcal{U} = \mathcal{U}(\mathfrak{S})$ be the universal enveloping algebra of L , together with its linear map $i : \mathfrak{S} \rightarrow \mathcal{U}$. Given $\sigma : \mathfrak{S} \rightarrow \mathfrak{gl}(V)$, we apply universal property of $\mathcal{U}$ to obtain a F -algebra homomorphism $\tilde{\sigma} : \mathcal{U} \rightarrow \text{End}(V)$ such that $\sigma = \tilde{\sigma} \circ i$. Note that $\mathfrak{gl}(V)$ and $\text{End}(V)$ are the same set; different notation has been used to suggest the respective algebraic structure under consideration (i.e. Lie algebra structure and associative algebra structure in this case). Let $I = \ker(\tilde{\sigma})$ which is an ideal in $\mathcal{U}$. By First Isomorphism Theorem, $\mathcal{U}/I$ is isomorphic to a subalgebra of $\text{End}(V)$. Since $\dim(V) < \infty$, it follows that $\dim(\text{End}(V)) < \infty$, and so $\dim(\mathcal{U}/I) < \infty$. If $x \in \text{Nil}(\mathfrak{S})$, then $\tilde{\sigma}(i(x)) = \sigma(x)$ is nilpotent in $\text{End}(V)$ (since σ is a nilrepresentation), and hence $\overline{i(x)}$ is nilpotent in $\mathcal{U}/I$. By Lemma 1, there exists an ideal J in $\mathcal{U}$ with $J \subseteq I$ such that $\dim(\mathcal{U}/J) < \infty$, $\overline{i(x)}$ is nilpotent in $\mathcal{U}/J$ for every $x \in \text{Nil}(J)$, and J is stable under derivations induced by the derivations of $\mathfrak{S}$.

Step 2. For each $x \in L$, construct the map $T_x : \mathcal{U} \rightarrow \mathcal{U}$ which satisfies $T_x(J) \subseteq J$.

For each $y \in \mathfrak{S}$, let $\psi_y : \mathcal{U} \rightarrow \mathcal{U}$ be the multiplication by $i(y) \in U$, i.e. $\psi_y(t) = i(y)t$. For each $z \in H$, the map $\text{ad } z : \mathfrak{S} \rightarrow \mathfrak{S}$ that sends $z \mapsto [z, w]$ for $w \in \mathfrak{S}$ is well-defined because $\mathfrak{S}$ is an ideal of L . Since $\text{ad } z$ is a derivation, there is a corresponding derivation $\delta_z : \mathcal{U} \rightarrow \mathcal{U}$ with $\delta_z \circ i = i \circ \text{ad}(z)$. For $x \in L$, write $x = y + z$ with $y \in \mathfrak{S}$, $z \in H$ and set $T_x := \phi_y + \delta_z : \mathcal{U} \rightarrow \mathcal{U}$. Since J is an ideal in $\mathcal{U}$, $\phi(J) \subseteq J$ and because J is stable under derivations, $\delta_z(J) \subseteq J$. Hence, $T_x(J) \subseteq J$.

Step 3. Prove that $T_{[x_1, x_2]} = T_{x_1} \circ T_{x_2} - T_{x_2} \circ T_{x_1}$.

Let $x_1 = y_1 + z_1$ and $x_2 = y_2 + z_2$ be elements of L , where $y_1, y_2 \in \mathfrak{S}$, and $z_1, z_2 \in H$. We have

$$\begin{aligned} T_{x_1} \circ T_{x_2} - T_{x_2} \circ T_{x_1} &= (\psi_{y_1} + \delta_{z_1}) \circ (\psi_{y_2} + \delta_{z_2}) - (\psi_{y_2} + \delta_{z_2}) \circ (\psi_{y_1} + \delta_{z_1}) \\ (*) \quad &= (\psi_{y_1} \circ \psi_{y_2} - \psi_{y_2} \circ \psi_{y_1}) + (\psi_{y_1} \circ \delta_{z_2} - \delta_{z_2} \circ \psi_{y_1}) + (\delta_{z_1} \circ \psi_{y_2} - \psi_{y_2} \circ \delta_{z_1}) + (\delta_{z_1} \circ \delta_{z_2} - \delta_{z_2} \circ \delta_{z_1}) \end{aligned}$$

For each $t \in U$,

$$\begin{aligned} (\psi_{y_1} \circ \psi_{y_2} - \psi_{y_2} \circ \psi_{y_1})(t) &= \psi_{y_1}(i(y_2)t) - \psi_{y_2}(i(y_1)t) \\ &= i(y_1)i(y_2)t - i(y_2)i(y_1)t \\ &= i([y_1, y_2])t = \psi_{[y_1, y_2]}(t) \end{aligned}$$

Thus, $\psi_{y_1} \circ \psi_{y_2} - \psi_{y_2} \circ \psi_{y_1} = \psi_{[y_1, y_2]}$. Next,

$$\begin{aligned} (\psi_{y_1} \circ \delta_{z_2} - \delta_{z_2} \circ \psi_{y_1})(t) &= i(y_1)\delta_{z_2}(t) - \delta_{z_2}(i(y_1)t) \\ &= i(y_1)\delta_{z_2}(t) - i(y_1)(\delta_{z_2}(t)) - \delta_{z_2}(i(y_1))t && \text{because } \delta_z \text{ is a derivation.} \\ &= -\delta_{z_2}(i(y_1))t = -i(\text{ad}(z_2)(y_1))t && \text{because } \delta_z \circ i = i \circ \text{ad}(z) \\ &= -i([z_2, y_1])t = i([y_1, z_2])t = \psi_{[y_1, z_2]}(t) \end{aligned}$$

Thus, $\psi_{y_1} \circ \delta_{z_2} - \delta_{z_2} \circ \psi_{y_1} = \psi_{[y_1, z_2]}$. Similarly, $\delta_{z_1} \circ \psi_{y_2} - \psi_{y_2} \circ \delta_{z_1} = \psi_{[z_1, y_2]}$. Now, for every $w \in \mathfrak{S}$, we have

$$\begin{aligned} (\delta_{z_1} \circ \delta_{z_2} - \delta_{z_2} \circ \delta_{z_1})(i(w)) &= \delta_{z_1} \circ \delta_{z_2}(i(w)) - \delta_{z_2} \circ \delta_{z_1}(i(w)) \\ &= \delta_{z_1} \circ i(\text{ad}_{z_2}(w)) - \delta_{z_2} \circ i(\text{ad}_{z_1}(w)) \\ &= i(\text{ad}(z_1) \circ \text{ad}(z_2)(w)) - i(\text{ad}(z_2) \circ \text{ad}(z_1)(w)) \\ &= i \circ (\text{ad}(z_1) \circ \text{ad}(z_2) - \text{ad}(z_2) \circ \text{ad}(z_1))(w) \\ &= i \circ \text{ad}([z_1, z_2])(w) = \delta_{[z_1, z_2]}(i(w)) \end{aligned}$$

So $\delta_{z_1} \circ \delta_{z_2} - \delta_{z_2} \circ \delta_{z_1}$ and $\delta_{[z_1, z_2]}$ agree on $\{i(w) : w \in \mathfrak{S}\}$. Since $\{i(w) : w \in \mathfrak{S}\}$ generate $\mathcal{U}$, we deduce that $\delta_{z_1} \circ \delta_{z_2} - \delta_{z_2} \circ \delta_{z_1} = \delta_{[z_1, z_2]}$. Putting all these pieces into (*), we get the result:

$$\begin{aligned} T_{x_1} \circ T_{x_2} - T_{x_2} \circ T_{x_1} &= \psi_{[y_1, y_2]} + \psi_{[y_1, z_2]} + \psi_{[z_1, y_2]} + \delta_{[z_1, z_2]} \\ &= \psi_{[y_1, y_2] + [y_1, z_2] + [z_1, y_2]} + \delta_{[z_1, z_2]} \\ &= T_{[y_1, y_2] + [y_1, z_2] + [z_1, y_2] + [z_1, z_2]} \\ &= T_{[y_1 + z_1, y_2 + z_2]} = T_{[x_1, x_2]} \end{aligned}$$

Step 4. Construct the desired representation $\rho : L \rightarrow \mathfrak{gl}(\mathcal{U}/J)$, and verify $\ker(\rho) \cap \mathfrak{S} \subseteq \ker(\sigma)$.

For each $x \in J$, $T_x(J) \subseteq J$. Thus, there is an induced map $\bar{T}_x : U/J \rightarrow U/J$ defined by $\bar{T}_x(\bar{y}) = \overline{T_x(y)}$. Define $\rho : L \rightarrow \mathfrak{gl}(\mathcal{U}/J)$ by $x \mapsto \bar{T}_x$. From step 3, we obtain

$$\bar{T}_{[x_1, x_2]} = \bar{T}_{x_1} \circ \bar{T}_{x_2} - \bar{T}_{x_2} \circ \bar{T}_{x_1}$$

Therefore, $\rho([x_1, x_2]) = \rho(x_1) \circ \rho(x_2) - \rho(x_2) \circ \rho(x_1)$, and so ρ is a representation. If $x \in \ker(\rho) \cap \mathfrak{S}$, then $x = x + 0$ is the decomposition in $\mathfrak{S} \oplus H$, and so $T_x = \psi_x$. Hence, $\rho(x) : \bar{t} \mapsto \overline{i(x)t}$. Since $\rho(x) = 0$, it follows that $\overline{i(x)t} = \bar{0}$, or equivalently, $i(x)t \in J$ for all $t \in \mathcal{U}$. In particular when $t = 1$, we get $i(x) \in J \subseteq I = \ker(\tilde{\sigma})$. Consequently, $\sigma(x) = \tilde{\sigma}(i(x)) = 0$ which proves $x \in \ker \sigma$. Thus, $\ker(\rho) \cap \mathfrak{S} \subseteq \ker(\sigma)$.

Step 5. Analyze the case $\text{Nil}(L) = \text{Nil}(\mathfrak{S})$.

As in Step 4, if $x \in \text{Nil}(L) = \text{Nil}(\mathfrak{S}) \subseteq \mathfrak{S}$ then $\rho(x) : \bar{t} \mapsto \overline{i(x)t}$. But from Step 1, $\overline{i(x)}$ is nilpotent in U/J for each $x \in \text{Nil}(\mathfrak{S})$. Since $\rho^n : \bar{t} \mapsto (\overline{i(x)})^n \bar{t}$, it follows that ρ is a nilpotent endomorphism for all $x \in \text{Nil}(L)$, implying that $\rho(x)$ is a nilrepresentation.

Step 6. Analyze the case $\text{Nil}(L) = L$.

By the same idea in Step 5, $p(x)$ is nilpotent for each $x \in \text{Nil}(\mathfrak{S})$. We would like to prove something stronger, namely, $\rho(x)$ is nilpotent for each $x \in L$. Let $\mathfrak{A}$ be the associative algebra in $\mathcal{U}/J$ generated by the set $\rho(L) = \{\rho(x) : x \in L\}$, and let $\mathfrak{N}$ be the ideal in $\mathfrak{A}$ generated by $\rho(\mathfrak{S}) = \{\rho(x) : x \in \mathfrak{S}\}$. We will show that $\mathfrak{N}^k = 0$ for some $k \geq 1$ in almost exactly the same way as in the proof of Lemma. First, by applying Engel's theorem to $\rho(\mathfrak{S}) \subseteq \mathfrak{gl}(U/J)$, there exists some $k \geq 1$ such that product of any k elements from $\rho(\mathfrak{S})$ is zero. If $x \in \rho(\mathfrak{S})$ and $y \in \rho(L)$, then using the identity $xy = [xy] + yx$ we can shift each $x \in \rho(\mathfrak{S})$ to the right of the product. Hence, every product in $\mathfrak{A}$ in which at least k factors are from $\rho(\mathfrak{S})$ is 0. Thus $\mathfrak{N}^k = 0$. Since L is nilpotent, $\text{ad}(z)$ is nilpotent for every $z \in L$. For each $z \in H \subseteq L$, let $\delta_z : U \rightarrow U$ be the derivation corresponding to $\text{ad}(z) : \mathfrak{S} \rightarrow \mathfrak{S}$ as before. Then, $\bar{\delta}_z : U/J \rightarrow U/J$ is nilpotent because U/J is finite-dimensional. Let $x \in L$ be arbitrary. Suppose $x = y + z$ with $y \in \mathfrak{S}$ and $z \in H$. Choose ℓ such that $\rho(z)^\ell = 0$. Then, $\rho(x)^{\ell k} = (\rho(y) + \rho(z))^{\ell k}$ and after expanding, each product will contain either k consecutive terms from $\mathfrak{N}$, or $\rho(z)^\ell$ will occur in the product (by the pigeonhole principle). Hence, $(\rho(x))^{\ell k} = 0$ and ρ is a nilrepresentation. This completes the proof of Proposition 2. $\square$

The key idea in the proof of Ado's theorem is the following: We start with a nilrepresentation for the centre of the Lie algebra. Using the proposition above, we successively build a representation for the Lie algebra. We achieve the desired faithful representation with the help of adjoint representation.

Theorem 3 (Ado). Let L be a finite-dimensional Lie algebra over a field of characteristic 0. Then L has a faithful finite-dimensional representation.

Proof. Let $Z(L) = \{z \in L : [x, z] = 0 \text{ for all } x \in L\}$ denote the centre of L . Let $z_1, z_2, \dots, z_n$ be a basis for $Z(L)$. We claim that the map $\rho_0 : Z(L) \rightarrow \mathfrak{gl}(F^{2n})$ defined by

$$\rho_0(c_1 z_1 + c_2 z_2 + \dots + c_n z_n) : (a_1, b_1, a_2, b_2, \dots, a_n, b_n) \mapsto (c_1 b_1, 0, c_2 b_2, 0, \dots, c_n b_n, 0)$$

is a faithful finite-dimensional nilrepresentation of $Z(L)$. Indeed if $\rho_0(c_1 z_1 + c_2 z_2 + \dots + c_n z_n) = \rho_0(d_1 z_1 + d_2 z_2 + \dots + d_n z_n)$, then $(c_1 b_1, 0, c_2 b_2, 0, \dots, c_n b_n, 0) = (d_1 b_1, 0, d_2 b_2, 0, \dots, d_n b_n, 0)$ for every $b_1, b_2, \dots, b_p \in F$. Setting $b_i = 1$ and $b_j = 0$ for all $i \neq j$, we get $c_i = d_i$. So ρ_0 is faithful. To check that ρ_0 is a representation, we need to prove $\rho_0([x, y]) = [\rho_0(x), \rho_0(y)]$ for every $x, y \in Z(L)$. Since $Z(L)$ is commutative, this is equivalent to showing $[\rho_0(x), \rho_0(y)] = 0$, i.e. $\rho_0(x) \circ \rho_0(y) = \rho_0(y) \circ \rho_0(x)$. Let $x = c_1 z_1 + \dots + c_n z_n$ and $y = d_1 z_1 + \dots + d_n z_n$. Then,

$$\rho_0(x) \circ \rho_0(y) : (a_1, b_1, a_2, b_2, \dots, a_n, b_n) \xrightarrow{\rho_0(y)} (d_1 b_1, 0, d_2 b_2, 0, \dots, d_n b_n, 0) \xrightarrow{\rho_0(x)} (0, 0, 0, 0, \dots, 0, 0)$$

and similarly,

$$\rho_0(y) \circ \rho_0(x) : (a_1, b_1, a_2, b_2, \dots, a_n, b_n) \xrightarrow{\rho_0(x)} (c_1 b_1, 0, c_2 b_2, 0, \dots, c_n b_n, 0) \xrightarrow{\rho_0(y)} (0, 0, 0, 0, \dots, 0, 0)$$

So $\rho_0(x) \circ \rho_0(y) = \rho_0(y) \circ \rho_0(x) = 0$ as desired. In particular, $(\rho_0(x))^2 = 0$ for all $x \in Z(L)$, which shows that ρ_0 is a nilrepresentation.

Because $\text{Rad}(L)$ is solvable, and $Z(L) \subseteq \text{Rad}(L)$ there exists a sequence of solvable subalgebras in L :

$$Z(L) = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_k = \text{Nil}(L) \subseteq L_{k+1} \subseteq \cdots \subseteq L_m = \text{Rad}(L) \subseteq L_{m+1} = L$$

such that L_i is an ideal in L_{i+1} for $0 \leq i \leq m$, and $\dim(L_i) + 1 = \dim(L_{i+1})$ for $0 \leq i \leq m-1$. Hence, $L_{i+1} = L_i \oplus Fv_i$ where $v_i \in L_{i+1} \setminus L_i$ for each $0 \leq i \leq m-1$. And for the case $i = m$, we use Levi's Theorem to get $L = \text{Rad}(L) \oplus H$ for some semisimple subalgebra H of L .

Starting with the nilrepresentation ρ_0 of $Z(L) = L_0$, we apply Proposition 2 inductively to get a representation ρ_i of L_1 such that $\ker(\rho_1) \cap L_0 \subseteq \ker(\rho_0)$. For the inductive step to work, we need to make sure that each ρ_i is nilrepresentation. This can be done because $\text{Nil}(L_i) = L_i$ for $1 \leq i \leq k$ (as $L_i \subseteq \text{Nil}(L)$ for $0 \leq i \leq k$) and $\text{Nil}(L_i) = \text{Nil}(L)$ for $i \geq k$ (as $\text{Nil}(L)$ is the maximal nilpotent ideal of L). Note that from $Z(L) = Z(L) \cap L_i$ for all $0 \leq i \leq m+1$ we get

$$Z(L) \cap \ker(\rho_i) = Z(L) \cap L_{i-1} \cap \ker(\rho_i) \subseteq Z(L) \cap \ker(\rho_{i-1})$$

for all $1 \leq i \leq m+1$. Since $Z(L) \cap \ker(\rho_0) = 0$ (because ρ is faithful), it follows that $Z(L) \cap \ker(\rho_i) = 0$ for all $1 \leq i \leq m+1$. In particular $Z(L) \cap \ker(\rho_{m+1}) = 0$. We have the representation $\rho_{m+1} : L_{m+1} = L \rightarrow \mathfrak{gl}(V)$ for some vector space with $\dim(V) < \infty$. Define $\rho = \rho_{m+1} \oplus \text{ad} : L \rightarrow \mathfrak{gl}(V \oplus L)$ by $\rho(x) = (\rho_{m+1}, \text{ad}(x))$. Then ρ is a finite-dimensional representation of L satisfying

$$\ker(\rho) = \ker(\rho_{m+1}) \cap \ker(\text{ad}) = \ker(\rho_{m+1}) \cap Z(L) = 0$$

so ρ is faithful. This concludes the proof of Ado's Theorem. $\square$

4. DERIVATIONS OF LIE ALGEBRAS

The purpose of this section is to collect some properties of derivations, and in particular to prove the theorem that if L is a Lie algebra over F where $\text{char}(F) = 0$, the image of $\text{Rad}(L)$ under any derivation is contained in a nilpotent ideal of L . The hypothesis $\text{char}(F) = 0$ will be explicitly used when we appeal to Lie's theorem.

The following lemma will help us conclude that $[L, \text{Rad}(L)]$ is a nilpotent ideal of L .

Lemma 4. If $\rho : L \rightarrow \mathfrak{gl}(V)$ is any representation, then each element of $\rho([L, L] \cap \text{Rad}(L))$ is a nilpotent endomorphism of V .

Proof. We may assume that V is irreducible: because if V had a proper subrepresentation W , then arguing by induction on dimension of V , the conclusion of the lemma would hold for W and W/V , which in turn would imply the assertion for V . We may also assume that ρ is injective: because we can write $\rho : L \xrightarrow{\pi} L/\ker(\rho) \xrightarrow{\bar{\rho}} \mathfrak{gl}(V)$ where $\pi : L \rightarrow L/\ker(\rho)$ is canonical projection, and $\bar{\rho} : L/\ker(\rho) \rightarrow \mathfrak{gl}(V)$ is defined by $\bar{\rho}(\bar{x}) = \rho(x)$. It is clear that π is surjective while $\bar{\rho}$ is injective. So if the conclusion of the lemma holds for the representation $\bar{\rho} : L/\ker(\rho) \rightarrow \mathfrak{gl}(V)$, then it will hold for $\rho : L \rightarrow \mathfrak{gl}(V)$ as well.

So assuming that V is an irreducible L -module, and $\rho : L \rightarrow \mathfrak{gl}(V)$ is injective, we will actually prove that $[L, L] \cap \text{Rad}(L) = 0$, which trivially implies the lemma. Before we proceed further, let's recall the following fact, which is available to us from the proof of Engel's Theorem:

Fact: If $\rho : L \rightarrow \mathfrak{gl}(V)$ is a faithful representation, and that N is an ideal of L consisting of nilpotent endomorphisms (i.e. $\rho(x)$ is nilpotent for each $x \in N$), then the subspace

$$W = \{v \in V : \rho(x)(v) = 0 \text{ for all } x \in N\}$$

is invariant under the action of L . Since V is irreducible, $W = V$ which shows that $\rho(x) = 0$ for all $x \in N$.

We now prove $[L, L] \cap \text{Rad}(L) = 0$. If $\text{Rad}(L) = 0$, there is nothing to prove. Otherwise, choose the largest integer m such that $K = \text{Rad}(L)^{(m)} \neq 0$. Here we are using the notation $L^{(0)} = L$ and $L^{(i+1)} = [L^{(i)}, L^{(i)}]$ for all $i \geq 0$. By maximality of m , $[K, K] = 0$, i.e. K is abelian. To prove $[L, L] \cap \text{Rad}(L) = 0$, it is enough to prove $[L, L] \cap K = 0$: if we can show $[L, L] \cap K = 0$, then $m = 0$ follows (i.e. $K = \text{Rad}(L)$ and so $[L, L] \cap \text{Rad}(L) = 0$), because otherwise $m > 1 \implies K \subseteq [L, L] \implies K = [L, L] \cap K = 0$, contradiction.

We will first prove $[L, K] = 0$. Suppose $x \in L, y \in K$. Since $[x, y] \in K$ and K is abelian, y commutes with $[x, y]$; hence, by Jacobi's identity, y commutes with all powers of $[x, y]$, i.e. $[y, [x, y]^{(n)}] = 0$ for every $n \in \mathbb{N}$. Let $z = [x, y]^{(n-1)}$. In particular, $[y, z] = 0$. We have

$$\rho([x, y])^n = \rho([x, y]) \cdot \rho([x, y])^{n-1} = \rho([x, y]) \cdot \rho(z) = [\rho(x), \rho(y)] \cdot \rho(z)$$

Taking trace of both sides, and using the identity $\text{Tr}([A, B] \cdot C) = \text{Tr}(A \cdot [B, C])$,

$$\text{Tr}(\rho([x, y])^n) = \text{Tr}([\rho(x), \rho(y)] \cdot \rho(z)) = \text{Tr}(\rho(x) \cdot [\rho(y), \rho(z)]) = \text{Tr}(\rho(x) \cdot \rho([y, z])) = \text{Tr}(\rho(x) \cdot 0) = 0$$

So $\text{Tr}(\rho([x, y])^n) = 0$ for every $n \in \mathbb{N}$. Thus, $\rho([x, y])$ is a nilpotent endomorphism. By the **Fact** above, $\rho([x, y]) = 0$. Since ρ is injective, $[x, y] = 0$. This proves $[L, K] = 0$.

Now assume $z \in [L, L] \cap K$. Then $z = [x, y]$ for some $x, y \in L$. Now, $[x, y] \in K$ implies $[y, [x, y]] \in [L, K] = 0$ by previous step. So y commutes with $[x, y]$, and therefore with all powers of $[x, y]$, i.e. $[y, [x, y]^{(n)}] = 0$ for every $n \in \mathbb{N}$. By the exact same argument used above, $z = [x, y] = 0$. Therefore, $[L, L] \cap K = 0$. $\square$

Corollary 5. $[L, \text{Rad}(L)]$ is a nilpotent ideal of L .

Proof. Consider the adjoint representation $\text{ad} : L \rightarrow \mathfrak{gl}(L)$. Since $[L, \text{Rad}(L)] \subseteq [L, L] \cap \text{Rad}(L)$, we can apply Lemma 4 to deduce that every element of $\text{ad}([L, \text{Rad}(L)])$ is nilpotent. By Engel's Theorem, $\text{ad}([L, \text{Rad}(L)]) = [\text{ad}(L), \text{ad}(\text{Rad}(L))]$ is a nilpotent ideal of $\mathfrak{gl}(L)$. Note that $\ker(\text{ad}) = Z(L)$. By the first isomorphism theorem, $[L, \text{Rad}(L)]/Z(L) \cong [\text{ad}(L), \text{ad}(\text{Rad}(L))]$ is nilpotent. By Proposition 3.2 in Humphreys [4], it follows that $[L, \text{Rad}(L)]$ is nilpotent. $\square$

Theorem 6. Let L be a Lie algebra over the field F , and $\delta : L \rightarrow L$ be a derivation of L , i.e. δ is a linear map satisfying $\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$. Then L can be embedded (as an ideal) in a Lie algebra L' such that for some $w \in L'$, $\text{ad}(w)$ restricted to L is δ .

In other words, every derivation can be viewed as an inner derivation if one embeds the Lie algebra inside a larger Lie algebra.

Proof. Consider $L' = L \oplus F$ and define the Lie bracket on L' by

$$[(x, a), (y, b)] = ([x, y] + a\delta(y) - b\delta(x), 0)$$

for all $x, y \in L, a, b \in F$. We claim that L' is a Lie algebra. We have

$$\begin{aligned} [(x_1, a_1) + (x_2, a_2), (y, b)] &= [(x_1 + x_2, a_1 + a_2), (y, b)] = ([x_1 + x_2, y] + (a_1 + a_2)\delta(y) - b\delta(x_1 + x_2), 0) \\ &= ([x_1, y] + [x_2, y] + a_1\delta(y) + a_2\delta(y) - b\delta(x_1) - b\delta(x_2), 0) \\ &= ([x_1, y] + a_1\delta(y) - b\delta(x_1), 0) + ([x_2, y] + a_2\delta(y) - b\delta(x_2), 0) \\ &= [(x_1, a_1), (y, b)] + [(x_2, a_2), (y, b)] \end{aligned}$$

Similarly, $[(x, a), (y_1, b_1) + (y_2, b_2)] = [(x, a), (y_1, b_1)] + [(x, a), (y_2, b_2)]$. And

$$\begin{aligned} [c(x, a), (y, b)] &= [(cx, ca), (y, b)] = ([cx, y] + ca\delta(y) - b\delta(cx), 0) \\ &= (c[x, y] + ca\delta(y) - cb\delta(x), 0) \\ &= c([x, y] + a\delta(y) - b\delta(x), 0) = c[(x, a), (y, b)] \end{aligned}$$

Similarly, $[(x, a), c(y, b)] = c[(x, a), (y, b)]$. So the Lie bracket on L' is bilinear. Next,

$$[(x, a), (x, a)] = ([x, x] + a\delta(x) - a\delta(x), 0) = ([x, x], 0) = (0, 0)$$

for every $x \in L, a \in F$. Finally,

$$\begin{aligned} [[(x, a), (y, b)], (z, c)] &= [([x, y] + a\delta(y) - b\delta(x), 0), (z, c)] \\ &= ([x, y] + a\delta(y) - b\delta(x), z] + 0\delta(z) - c\delta([x, y] + a\delta(y) - b\delta(x)), 0) \\ &= ([x, y], z] + a[\delta(y), z] - b[\delta(x), z] - c\delta([x, y]) - ca\delta^2(y) + cb\delta^2(x), 0) \\ &= ([x, y], z] + a[\delta(y), z] - b[\delta(x), z] - c\delta([x, y]) - ca\delta^2(y) + cb\delta^2(x), 0) \\ &= ([x, y], z] + a[\delta(y), z] - b[\delta(x), z] - c[\delta(x), y] - c[x, \delta(y)] - ca\delta^2(y) + cb\delta^2(x), 0) \\ &= ([x, y], z] + a[\delta(y), z] - b[\delta(x), z] - c[\delta(x), y] + c[\delta(y), x] - ca\delta^2(y) + cb\delta^2(x), 0) \end{aligned}$$

We have shown that

$$[(x, a), (y, b)], (z, c) = ([x, y], z) + a[\delta(y), z] - b[\delta(x), z] - c[\delta(x), y] + c[\delta(y), x] - ca\delta^2(y) + cb\delta^2(x), 0)$$

Similarly,

$$[(z, c), (x, a)], (y, b) = [[[z, x], y] + c[\delta(x), y] - a[\delta(z), y] - b[\delta(z), x] + b[\delta(x), z] - bc\delta^2(x) + ba\delta^2(z), 0]$$

$$[(y, b), (z, c)], (x, a) = [[[y, z], x] + b[\delta(z), x] - c[\delta(y), x] - a[\delta(y), z] + a[\delta(z), y] - ab\delta^2(z) + ac\delta^2(y), 0]$$

Adding the three equations, we obtain

$$[(x, a), (y, b)], (z, c) + [(z, c), (x, a)], (y, b) + [(y, b), (z, c)], (x, a) = ([x, y], z) + [z, x], y + [y, z], x, 0 = (0, 0)$$

where in the last step we used the Jacobi identity in L . Thus, Jacobi identity holds for L' . We conclude that L' is a Lie algebra. Define $\nu : L \rightarrow L'$ by $x \mapsto (x, 0)$. It is clear that ν is injection. Furthermore,

$$[\nu(x), \nu(y)] = [(x, 0), (y, 0)] = ([x, y] + 0\delta(y) - 0\delta(x), 0) = ([x, y], 0) = \nu([x, y])$$

showing that ν is a Lie algebra homomorphism. So we have an embedding of Lie algebras $L \hookrightarrow L'$.

Let us make the identification $L \cong \{(x, 0) : x \in L\} \subseteq L'$. We will show that L is an ideal of L' . In fact, something stronger is true, namely $[L', L'] \subseteq L$. Indeed,

$$[(x, a), (y, b)] = ([x, y] + a\delta(y) - b\delta(x), 0) \in L$$

Let $w = (0, 1)$. Then, consider the inner derivation $\text{ad}(w) : L' \rightarrow L'$ by $\text{ad}(w)(x') = [w, x']$ for every $x' \in L'$. When $x' = (x, 0) \in L$, we get

$$\text{ad}(w)((x, 0)) = [w, (x, 0)] = [(0, 1), (x, 0)] = ([0, x] + 1\delta(x) - 0\delta(0), 0) = (\delta(x), 0)$$

Therefore, the restriction of the inner derivation $\text{ad}(w)$ to L is equal to the original derivation δ . $\square$

Corollary 7. If κ is the Killing form on L and δ is any derivation of L , then

$$\kappa(\delta(x), y) + \kappa(x, \delta(y)) = 0$$

for all $x, y \in L$.

Proof. Apply Theorem 6 to get a Lie algebra L' with $L \hookrightarrow L'$, and $w \in L'$ satisfying $\text{ad}(w)|_L = \delta$. Consider the Killing form κ' on L' . Since the Killing form is associative, we have

$$\begin{aligned} \kappa'(\text{ad}(w)(x'), y') &= \kappa'([w, x'], y') = -\kappa'([x', w], y') = -\kappa'(x', [w, y']) = -\kappa'(x', \text{ad}(w)(y')) \\ \implies \kappa'(\text{ad}(w)(x'), y') + \kappa'(x', \text{ad}(w)(y')) &= 0 \end{aligned}$$

for all $x', y' \in L'$. Since $\text{ad}(w)|_L = \delta$, we get

$$\kappa'(\delta(x), y) + \kappa'(x, \delta(y)) = 0$$

for all $x, y \in L$. But L is an ideal of L' , so $\kappa'|_L = \kappa$. Thus, $\kappa(\delta(x), y) + \kappa(x, \delta(y)) = 0$. $\square$

Definition. If K is an ideal of L , we say K is **characteristic** if $\delta(K) \subseteq K$ for every derivation δ of L .

By definition, ideals are vector subspaces of L invariant under inner derivations. Being a characteristic ideal is a stronger condition. The analogy in group theory are the notions of normal subgroup and characteristic subgroup. Note that if I is an ideal of L , then any characteristic ideal K of I is an ideal of L . This is the same observation that a characteristic subgroup inside a normal subgroup of G is necessarily a normal subgroup of G . The proof is essentially the same: Any inner derivation $\text{ad}(x)$ where $x \in L$ leaves I invariant (as I is an ideal of L). So the restriction of $\text{ad}(x)$ to K is a derivation of I . As K is characteristic in I , we have $\text{ad}(x)(K) \subseteq K$. Hence, K is an ideal of L .

Corollary 8. If I is a characteristic ideal of L , then its orthogonal complement with respect to the Killing form is also a characteristic ideal.

Proof. Let δ be a derivation of L . By Corollary 7, $\kappa(\delta(x), y) + \kappa(x, \delta(y)) = 0$ for all $x, y \in L$. In particular when $x \in I$, $y \in I^\perp$, we get $\delta(x) \in I$ (as I is characteristic) and so $\kappa(\delta(x), y) = 0$. Thus, $\kappa(\delta(x), y) + \kappa(x, \delta(y)) = 0 \implies \kappa(x, \delta(y)) = 0$. As a result, $\delta(y) \in I^\perp$, proving that $I^\perp$ is characteristic. $\square$

Proposition 9. $\text{Rad}(L)$ is the orthogonal complement of $[L, L]$ with respect to the Killing form on L .

Proof. Let $x, y \in L$ and $z \in \text{Rad}(L)$. We claim that $\kappa([x, y], z) = 0$. By associativity of the Killing form, it is enough to show $\kappa(x, [y, z]) = 0$. Let H subalgebra of L generated by x and $\text{Rad}(L)$. Since $[H, H] \subseteq \text{Rad}(L)$, H is solvable. Applying Lie's theorem to the adjoint map $\text{ad} : H \rightarrow \mathfrak{gl}(L)$, H acts on L by upper-triangular matrices. Since $[y, z] \in [L, L] \cap \text{Rad}(L)$, by Lemma 4 the adjoint action of $[y, z]$ on L is nilpotent. Hence, $\text{ad}(x) \circ \text{ad}([y, z])$ is nilpotent, and

$$\kappa(x, [y, z]) = \text{Tr}(\text{ad}(x) \cdot \text{ad}([y, z])) = 0$$

This proves $\kappa([x, y], z) = 0$, and so $\text{Rad}(L) \subseteq [L, L]^\perp$.

Note that

$$\kappa([g, g]^\perp, [[g, g]^\perp, [g, g]^\perp]) \subseteq \kappa([g, g]^\perp, [g, g]) = 0$$

By Cartan's Criterion, $[g, g]^\perp$ is solvable. This shows $[g, g]^\perp \subseteq \text{Rad}(L)$. Therefore, $[g, g]^\perp = \text{Rad}(L)$. $\square$

Since $\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$, we have $\delta([L, L]) \subseteq [L, L]$. Hence, $[L, L]$ is a characteristic ideal. From Corollary 8 and Proposition 9, $\text{Rad}(L)$ is a characteristic ideal of L .

Lemma 10. If I is an ideal of L , then $\text{Rad}(I) = \text{Rad}(L) \cap I$.

Proof. As we have observed $\text{Rad}(I)$ is characteristic ideal of I . Hence, $\text{Rad}(I)$ is an ideal of L . Since $\text{Rad}(I)$ is solvable, $\text{Rad}(I) \subseteq \text{Rad}(L)$ implying that $\text{Rad}(I) \subseteq \text{Rad}(L) \cap I$. For the reverse direction, $\text{Rad}(L) \cap I$ is a solvable ideal of I , showing that $\text{Rad}(L) \cap I \subseteq \text{Rad}(I) \implies \text{Rad}(I) = \text{Rad}(L) \cap I$. $\square$

We can now prove the main objective of this section.

Theorem 11. Suppose L is a Lie algebra, and δ is a derivation of L . Then $\delta(\text{Rad}(L))$ is contained in a nilpotent ideal of L .

Proof. Apply the construction in Theorem 6 to get an embedding $L \hookrightarrow L' = L \oplus F$ where the Lie bracket is $[(x, a), (y, b)] \subseteq ([x, y] + a\delta(y) - b\delta(x), 0)$. If we make the identification $L \cong \{(x, 0) : x \in L\} \subseteq L'$, then L is an ideal of L' , and $\delta = \text{ad}(w)|_L$ where $w = (0, 1)$. Using Lemma 10 and $\text{Rad}(L) \subseteq \text{Rad}(L')$, we deduce that

$$\delta(\text{Rad}(L)) = [w, \text{Rad}(L)] = [w, \text{Rad}(L') \cap L] \subseteq [L', \text{Rad}(L')] \cap L$$

By Corollary 5, $[L', \text{Rad}(L')]$ is a nilpotent ideal of L' , and so $[L', \text{Rad}(L')] \cap L$ is nilpotent ideal of L . $\square$

REFERENCES

- [1] I. D. Ado, *The representation of Lie algebras by matrices.* (in Russian) *Uspehi Matem. Nauk* (N.S.) **2**, (1947). no. 6(22), 159-173.
- [2] K. Iwasawa, *On the representation of Lie algebras.* *Jap. J. Math.* **19**, (1948). 405-426.
- [3] W. Fulton, J. Harris, *Representation theory: A first course.* Graduate Texts in Mathematics, 129. Readings in Mathematics. Springer-Verlag, New York, 1991.
- [4] J. E. Humphreys, *Introduction to Lie algebras and representation theory.* Graduate Texts in Mathematics, Vol. 9. Springer-Verlag, New York-Berlin, 1972.