A PROOF OF BOREL-WEIL-BOTT THEOREM

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1. Introduction

In this short note, we prove the Borel-Weil-Bott theorem.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. One basic question in representation theory is to classify all finite dimensional irreducible representations of $\mathfrak{g}$. The answer is provided by the highest weight theorem: For any dominant integral weight λ, there exists a unqiue finite dimsnsional $\mathfrak{g}$-module $V(\lambda)$ with highest weight λ. The Borel-Weil theorem provides an explicit construction of these $\mathfrak{g}$-modules $V(\lambda)$.

Every complex semisimple Lie algebra $\mathfrak{g}$ corresponds to a complex semisimple Lie group G. The homogenous space G/B, where B is a Borel subgroup of G, is a smooth projective variaty. Every integral weight λ corresponds to a holomorphic line bundle L_λ on G/B. $\mathfrak{g}$ acts on the space of global holomorphic sections

$$\Gamma(G/B, \mathscr{O}_\lambda) \cong H^0(G/B, \mathscr{O}_\lambda)$$

by differentiation. When λ is dominant, the Borel-Weil theorem asserts that $H^0(G/B, \mathscr{O}_\lambda)$ is a finite dimensional irreducible $\mathfrak{g}$-module with highest weight λ.

The Borel-Weil-Bott theorem is a generalization of Borel-Weil theorem. It deals with all integral weights, and consider not only global sections, but also higher cohomology groups $H^p(G/B, \mathscr{O}_\lambda)$.

In section 2, we introduce the basic results in Lie groups and Lie algebras. In section 3 we introduce the homogenous space G/B and induced representations. In section 4 we describe how the Casimir element $c_\mathfrak{g}$ serves as a handy tool to decompose $\mathfrak{g}$-modules. In the last two sections, we prove the Borel-Weil theorem and the Borel-Weil-Bott theorem respectively.

If possible, we follow the notations in [3], except that we use $\mathfrak{g}$ instead of L to denote Lie algebras. The proof of Borel-Weil theorem and Borel-Weil-Bott theorem are from [5]. We also use a result from [1], which describe the positivity of the line bundle L_λ.

2. Basic Lie theory

Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. As in [3], we have the root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \ , \tag{2.1}$$

where $\alpha \in \Phi$ are the roots of $\mathfrak{g}$. If $x_\alpha \in \mathfrak{g}_\alpha$ and $h \in \mathfrak{h}$,

$$[h, x_\alpha] = \alpha(h) x_\alpha \ . \tag{2.2}$$

Let E be the real span of Φ in $\mathfrak{h}^*$. It is known that $\big(E, \Phi, (\cdot,\cdot)\big)$ forms a root system, where $(\cdot,\cdot)$ is dual to the Killing form κ.

An **integral weight** is an element in E such that

$$\langle \lambda, \alpha \rangle := \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z} \tag{2.3}$$

for all roots $\alpha \in \Phi$. If a set of positive roots Φ^+ and simple roots $\Delta \subset \Phi^+$ are chosen, the **positive Weyl Chamber** $\mathcal{C}^+$ is the set of elements $v \in E$ such that $\langle v, \alpha \rangle > 0$ for all $\alpha \in \Delta$.

We write $\lambda \leq \mu$ if $\mu - \lambda \in \overline{\mathcal{C}^+}$. An integral weight is called **dominant** if $\lambda \geq 0$, or $\langle \lambda + \delta, \alpha \rangle > 0$ for all $\alpha \in \Delta$, where ([3], section 13)

$$\delta = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = \sum_{i=1}^{l} \lambda_i \tag{2.4}$$

and λ_i are the set of fundamental dominant weight corresponding to Δ.

Let (π, V) be a $\mathfrak{g}$-module. When restricted to $\mathfrak{h}$, (π, V) decomposes into direct sum of weight spaces V_λ defined by

$$V_\lambda = \{ v \in V : \ \pi(h) v = \lambda(v) v \}. \tag{2.5}$$

Such $\lambda \in \mathfrak{h}^*$ is called a **weight** of (π, V). The set of all weights is denoted Λ. We will need the following results:

Theorem 2.1. *Let (π, V) be a finite dimensional irreducible $\mathfrak{g}$-modules with weights Λ. Then*

1. *There is a unique $\lambda \in \Lambda$ with the highest weight. That is $\mu \leq \lambda$ for all $\mu \in \Lambda$.*
2. *The weight space V_λ is one dimensional.*
3. *Λ is closed under the Weyl group action.*
4. *$\lambda \geq 0$.*
5. *$||\mu|| \leq ||\lambda||$ for all $\mu \in \Lambda$.*
6. *$\mu \in \Lambda$ is extremal if and only if $||\mu|| = ||\lambda||$, and the Weyl group acts transitively on the set of extremal weights.*

(7) $||\mu+\delta|| < ||\lambda+\delta||$ *if* $\mu \in \Lambda$ *and* $\mu \neq \lambda$.

We also need the **Theorem of highest weights:** ([3], theorem 21.2)

Theorem 2.2. *Up to isomorphism, every dominant weight* λ *corresponds to a unique finite dimensional irreducible representation* $V(\lambda)$ *with hightest weight* λ.

A **Borel subalgebra** $\mathfrak{b}$ of $\mathfrak{g}$ is a maximal solvable subalgebra. Examples are

$$\mathfrak{b}^{\pm} = \mathfrak{h} \oplus \mathfrak{n}^{\pm} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^{\pm}} \mathfrak{g}_{\alpha} \tag{2.6}$$

Next we introduce Lie groups. A real (complex) Lie group is a group with a structure of smooth (complex) manifold, such that the group multiplication $(g,h) \mapsto gh$ is smooth (holomorphic). By considering left or right invariant vectors fields, every real (complex) Lie group G corresponds to a real (complex) Lie algebra $\mathfrak{g}$.

G is called semisimple if its Lie algebra is semisimple. In this case we have ([4], Chapter 6 and 7)

Theorem 2.3. *Let* G, G' *be connected complex semisimple Lie groups. Then*

(1) *G has a compact real form K, that is, a compact Lie subgroup K of G with Lie algebra $\mathfrak{k}$ such that*

$$\mathfrak{k}_{\mathbb{C}} = \mathfrak{k} \oplus i\mathfrak{k} \cong \mathfrak{g} \ . \tag{2.7}$$

(2) *Write $\mathfrak{p}_0 = i\mathfrak{k}$, then the map $K \times \mathfrak{p}_0 \to G$ given by $(k,p) \mapsto k \exp p$ is a diffeomorphism. Thus $K \subset G$ induces the isomorphism $\pi_1(K) \cong \pi_1(G)$.*
(3) *Let K' be a compact Lie form of G'. Then every homomorphisms $K \to K'$ extends to a holomorphic homomorphism $G \to G'$.*
(4) *Every representation $K \to GL(V)$ extends to $G \to GL(V)$. In particular if V is a faithful representation of K, then the extension is a faithful holomorphic representation of G.*
(5) *Every finite dimensional holomorphic representation of G is completely reducible.*

It is known that every complex semisimple Lie group has a unique structure of an affine algebraic group ([5], section 15). Whenever it is convenient, we would treat complex semisimple Lie groups as an affine algebraic groups.

Definition 2.1. Let G be an affine connected algebraic group with Lie algebra $\mathfrak{g}$, G is called semisimple if its Lie algebra is $\mathfrak{g}$ semisimple. Let H and B be connected algebraic subgroup of G. H is called a **Cartan subgroup** if its Lie algebra $\mathfrak{h}$ is a Cartan subalgebra. B is called a **Borel subgroup** if its Lie algebra $\mathfrak{b}$ is a Borel subalgebra.

From ([5], chapter 15), when G is semisimple, every Cartan subgroup H is abelian and isomorphic to $(\mathbb{C}^*)^r$ for some r. Every Borel subgroup B is maximal solvable subgroup of G. When $\mathfrak{b} = \mathfrak{b}^\pm$ defined in (2.6), $B = HN$, where N is a an algebraic subgroup with Lie algebra $\mathfrak{n}^\pm$.

We will need the following results. Recall that a character of H is an algebraic homomorphism $H \to \mathbb{C}^*$.

Lemma 2.1. *Every charcter $\sigma : H \to \mathbb{C}^*$ extends uniquely to B.*

Proof. N is a normal subgroup of B and $B = HN$, $H \cap N = \{0\}$. Let σ be a character of H. Then one can extend σ to B by $\sigma(bn) = \sigma(b)$. On the other hand, if σ is a character of B, considering its restriction to N. Then the differential is a Lie algebra homomorphism $\mathfrak{n} \to \mathfrak{gl}_1(\mathbb{C})$. As $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$, we have $\mathfrak{n} \to \mathfrak{sl}_1(\mathbb{C})$. Thus the differential is zero and $\sigma|_N$ is the identity. Hence σ descends to $B/N = H$. □

Theorem 2.4. *Let G be a semisimple simply connected complex Lie group with Lie algebra $\mathfrak{g}$. Then $\lambda \in \mathfrak{h}^*$ is the differential of a character of H if and only if λ is an integral weight.*

Proof. ($\Rightarrow$) Let (π, V) be a faithful representation of G such that $\pi(H)$ lies in the diagonal subgroup $(\mathbb{C}^*)^m \subset GL(V)$ after choosing some basis $\{v_1, \cdots, v_m\}$. Each v_i corresponds to a character γ_i defined by $\pi(h)v_i = \gamma_i(h)v_i$. The differential of γ_i corresponds to the weight of the representation (π, V), thus is an integral weight (All weights from a representation are integral). Identifying $H \cong \pi(H)$, we know that all characters on H are restricition of chacracter in $(\mathbb{C}^*)^m \subset GL(V)$. As $\gamma_i = c_i|_H$, where c_i are the obvious generators of the characters of $(\mathbb{C}^*)^m$, γ_i generates the characters of H. Thus all differential of characters of H are integral weights.

($\Leftarrow$) Every integral weight λ lies in the closure of some Weyl chamber and this chamber is positive with respect to some choice of positive roots. Hence it suffices to assume that λ is dominant. By Theorem 2.2, let $V(\lambda)$ be the irreducible $\mathfrak{g}$-modules with highest weight λ and highest weight vector v_λ. The representation $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ corresponds to a homomorphism $G \to GL(V)$ (still call it π), as G is simply connected. Let $\xi \in \mathfrak{h}$ and $h \in H$ such that $h = \exp(\xi)$. Using $\pi(\xi)v_\lambda = \lambda(\xi)v_\lambda$, we obtain

$$\pi(h)v_\lambda = \pi(\exp(\xi))v_\lambda = e^{\pi(\xi)}v_\lambda = e^{\lambda(\xi)}v_\lambda \ .$$

Then $(\pi|_H, H)$ acts on V_λ and λ is the differential of this character.

□

3. Induced representation

Let G be a complex semisimple Lie group with Lie algebra $\mathfrak{g}$, and B be a Borel subgroup with Lie algebra $\mathfrak{b} \cong \mathfrak{h} \oplus \mathfrak{n}^-$. As in ([5], 14.3), the left coset space G/B can be given a structure of complex manfold such that the action of G is holomorphic. We denote this manifold as $\mathcal{M}$. $\mathcal{M}$ is compact as $\mathcal{M} \cong K/T$, where T is the maximal torus such that $T = K \cap B$.

Let (σ, Q) be a holomorphic representation of B. Define

$$G \times_B Q = G \times Q / \sim,$$

where $(g, q) \sim (gb^{-1}, \sigma(b)q)$. $G \times_B Q$ is given the quotient topology and quotient structure sheaf fromt the quotient map. Let $\pi : G \times B \to \mathcal{M}$ be the map $\pi([g, q]) = gB$, then $\pi^{-1}(gB) \cong gB \times Q$ has a vector space structure. The following results are proved in ([5], 16.1). The first one follows essentially from the holomorphic version of implicit function theorem.

Proposition 3.1. *With the projection map $\pi : G \times_B Q \to \mathcal{M}$, $G \times_B Q$ is a G-equivariant holomorphic vector bundle on $\mathcal{M}$.*

Lemma 3.1. *$G \times_B Q$ is holomorphically trivial if and only if Q is the restriction of a holomorphic representation of G.*

The sheaf of holomorphic section on $G \times_B Q$ is denoted $\mathscr{O}_\sigma$. For each open set $U \subset \mathcal{M}$,

$$\mathscr{O}_\sigma(U) \cong \{f \in \mathscr{O}\big(\pi^{-1}(U)\big) \otimes Q : \ f(gb^{-1}) = \sigma(b) f(g), \ \ \forall b \in B\} \ .$$

Note that $\mathscr{O}_\sigma(U)$ has a structure of $\mathfrak{g}$-module: view $\mathfrak{g}$ as the set of right invariant holomorphic vector fields on G, $\mathfrak{g}$ acts on $\mathscr{O}_\sigma(U)$ by differentiation.

As $\mathcal{M}$ is compact and $\mathscr{O}_\sigma$ is a coherent sheaf, $H^p(\mathcal{M}, \mathscr{O}_\sigma)$ is finite dimensional for all p. Moreover, the action on $\mathscr{O}_\sigma$ induces an $\mathfrak{g}$-action on each $H^p(\mathcal{M}, \mathscr{O}_\sigma)$.

Let λ be an integral weight. By Theorem 2.4 and Lemma 2.4, λ is the differential of a character of B. The induced line bundle with respect to the character is denoted L_λ. The sheaf of holomorphic section on L_λ is denoted $\mathscr{O}_\lambda$.

4. The Casimir operator $c_{\mathfrak{g}}$

Let $\mathfrak{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. By definition, any Lie algebra morphism $\psi : \mathfrak{g} \to A$ of $\mathfrak{g}$ into an associative algebra extends to a morphism $\tilde{\psi} : \mathfrak{U}(\mathfrak{g}) \to A$ of associative algebras. The Casimir element $c_{\mathfrak{g}} \in \mathfrak{U}(\mathfrak{g})$ is given by

$$c_{\mathfrak{g}} = \sum_i \eta_i^2 + \sum_{\alpha\in\Phi^+} h_\alpha + 2\sum_{\alpha\in\Phi^+} z_\alpha x_\alpha \ , \tag{4.1}$$

where $\{\eta_i\}$ is a self dual basis of $\mathfrak{h}$, and h_α, x_α and z_α are defined as in ([3], section 22). It is known that $c_{\mathfrak{g}}$ is well-defined independent of the choice of $\mathfrak{h}$, Φ^+ and $c_{\mathfrak{g}} \in Z(\mathfrak{g})$. As a result, we can use $c_{\mathfrak{g}}$ to decompose any finite dimensional $\mathfrak{g}$-modules into its eigenspaces.

Proposition 4.1. *Let $V(\lambda)$ is the finite dimensional irreducible $\mathfrak{g}$-module with highest weight λ. Then $c_{\mathfrak{g}}$ acts as the scalar $\langle\lambda, \lambda + 2\delta\rangle$.*

Proof. By Schur's lemma, we know that $c_{\mathfrak{g}}$ acts as a scalar. To find out this scalar, we apply $\pi(c_{\mathfrak{g}})$ to the highest weight vector v_λ. As $\pi(x_\alpha)v_\lambda = 0$ for all $\alpha \in \Phi^+$,

$$\begin{aligned}\pi(c_{\mathfrak{g}})v_\lambda &= \sum_i \lambda(\eta_i)^2 v_\lambda + \sum_{\alpha\in\Phi^+} \lambda(h_\alpha)v_\lambda \\ &= ||\lambda||^2 v_\lambda + \sum_{\alpha\in\Phi^+} \langle\lambda, \alpha\rangle v_\lambda \\ &= \langle\lambda, \lambda + 2\delta\rangle v_\lambda \ .\end{aligned}$$

□

Note that $c_{\mathfrak{g}}$ also acts on the sheaf $\mathscr{O}_\lambda$ for all integral weight λ.

Proposition 4.2. *$c_{\mathfrak{g}}$ acts as the scalar $\langle\lambda, \lambda + 2\delta\rangle$ on $\mathscr{O}_\lambda$.*

Proof. As G acts holomorphically and transitively on $\mathcal{M}$, it suffices to check at the point $B \in \mathcal{M}$. Let U be an open neigborhood of B and $f \in \mathscr{O}_\lambda(U)$. Let $b, b_1 \in B$, then

$$f(b^{-1}b_1) = \lambda(b_1^{-1})\lambda(b)f(e) = \lambda(b)f(b_1) \ .$$

Let $b = \exp(t\zeta)$, where $\zeta \in \mathfrak{b}$ and differentiate, we obtain $(\zeta f)(b_1) = \lambda(\zeta)f(b_1)$. In particular,

$$z_\alpha f = 0, \qquad \eta f = \lambda(\eta) f$$

on B, where $\eta \in \mathfrak{h}$. As x_α corresponds to a holomorphic vector field, $x_\alpha f \in \mathscr{O}_\lambda(U)$ for all $\alpha \in \Phi^+$. Thus $y_\alpha x_\alpha f = 0$. Using equation (4.1), as in the proof of proposition 4.1, one can show that $c_{\mathfrak{g}}$ acts as the scalar $\langle\lambda, \lambda + 2\delta\rangle$. □

Corollary 4.1. *$c_{\mathfrak{g}}$ acts trivially on $\mathscr{O}_\mu$ for all $\mu = \omega\delta - \delta$ for $\omega \in W$.*

Proof. Note that for all integral weights λ,

$$\langle \lambda, \lambda + 2\delta \rangle = ||\lambda + \delta||^2 - ||\delta||^2 \ . \tag{4.2}$$

If $\mu = \omega\delta - \delta$ for some $\omega \in W$, then $||\mu + \delta|| = ||\omega\delta|| = ||\delta||$. Thus the corollary follows from proposition (4.2). □

For a general representation Q on B, we have

Proposition 4.3. *Let (θ, Q) be a finite dimensional representation of B with weight Λ. Then $\mathscr{O}_\theta$ decompose into direct sums of $\mathfrak{U}(\mathfrak{g})$-modules S_t*

$$\mathscr{O}_\theta = \bigoplus_t S_t \ ,$$

where t ranges over distinct eigenvalues for the action of $c_{\mathfrak{g}}$ on $\mathscr{O}_\theta$. If $t = \langle \nu, \nu + 2\delta \rangle$ has multiplicity one, then $S_t \cong \mathscr{O}_\nu$.

Proof. Let $\dim Q = n$. By considering $\theta_* : \mathfrak{b} \to \mathfrak{gl}(Q)$ and using Lie's theorem, Q has a filtration $\{Q_i\}_{i=0}^n$ of B-submodules such that Q_j/Q_{j-1} are one dimensional. It is easy to see that B acts on Q_j/Q_{j-1} as a character $\nu_j \in \Lambda$. On the sheaf level, one has a sequence of subsheafs $\mathscr{O}_{\theta_j}$ such that $\mathscr{O}_{\theta_j}/\mathscr{O}_{\theta_{j-1}} \cong \mathscr{O}_{\nu_j}$. Here θ_j is the restriction of θ to Q_j. All these are $\mathfrak{g}$ submodules and $c_{\mathfrak{g}}$ acts on them. By Proposition 4.2, $c_{\mathfrak{g}}$ acts on $\mathscr{O}_{\nu_j}$ as $\langle \nu_j, \nu_j + 2\delta \rangle$. Thus

$$\Pi_j \big(c_{\mathfrak{g}} - \langle \nu_j, \nu_j + 2\delta \rangle \big) \tag{4.3}$$

acts as zero on $\mathscr{O}_\theta$. Express (4.3) as $\Pi_t (c_{\mathfrak{g}} - t)^{k_t}$, where k_t is the multiplicity of $t = \langle \nu_j, \nu_j + 2\delta \rangle$. Write $S_t = \ker(c_{\mathfrak{g}} - t)^{k_t}$, then

$$\mathscr{O}_\theta = \bigoplus_t S_t \ .$$

If $k_t = 1$ for some t, then there is an eigenspace E_t of Q with eigenvalue t. As $c_{\mathfrak{g}}$ lies in the center of $\mathfrak{U}(\mathfrak{g})$, E_t is a $\mathfrak{U}(\mathfrak{g})$-module. Since $E_t \cong Q_j/Q_{j-1}$ for some j, $S_t \cong \mathscr{O}_{\nu_j}$ as sheaf of $\mathfrak{U}(\mathfrak{g})$-modules. □

5. Borel-Weil theorem

We are ready to prove the Borel-Weil theorem. The key argument is in the following lemma.

Lemma 5.1. *Let $\omega \in W$ and write $\mu = w\delta - \delta$. Then for all p,*

1. *$H^p(\mathcal{M}, \mathscr{O}_\mu)$ is a trivial $\mathfrak{g}$-module.*

(2) *Let $V(\lambda)$ be a finite dimensional irreducible representation of $\mathfrak{g}$ with highest weight λ, then*

$$H^p(\mathcal{M}, \mathscr{O}_{\omega\lambda+\mu}) \cong H^p(\mathcal{M}, \mathscr{O}_\mu) \otimes V(\lambda) \tag{5.1}$$

as $\mathfrak{g}$-modules.

Proof. First we show (1). By Corollary 4.1, $c_{\mathfrak{g}}$ acts trivially on $\mathscr{O}_\mu$. Hence $c_{\mathfrak{g}}$ acts trivially on $H^p(\mathcal{M}, \mathscr{O}_\mu)$ for all p. As $H^p(\mathcal{M}, \mathscr{O}_\mu)$ is finite dimensional, it decomposes into irreducible $\mathfrak{g}$-modules. Let A_ϵ be one of them with highest weight ϵ (thus $A_\epsilon \cong V(\epsilon)$). By proposition 4.1, $c_{\mathfrak{g}}$ acts on A_ϵ by the scalar $\langle \epsilon, \epsilon + 2\delta\rangle = 0$. As ϵ is dominant,

$$\langle \epsilon, \delta\rangle \geq 0 \Rightarrow \epsilon = 0 \ .$$

Hence A_ϵ is the trivial $\mathfrak{g}$-module. Hence $H^p(\mathcal{M}, \mathscr{O}_\mu)$ is a trivial $\mathfrak{g}$-module.

Let $V = V(\lambda)$ and Λ be the set of weight of V. Let G be the simply connected complex Lie group with Lie algebra $\mathfrak{g}$. We use the same notation V to denote the irreducible representation on G. Let (θ, V) be the restriction of the G-modules V to B. By Lemma 3.1, the induced bundle is trivial $\mathcal{M} \times V$. Thus $\mathscr{O}_\theta \cong \mathscr{O} \otimes V$. On the other hand, we twist θ with the weight μ to obtain a representation $\theta \otimes \mu$ on B. The induced bundle is $(\mathcal{M} \times V) \otimes L_\mu$ and

$$\mathscr{O}_{\theta\otimes\mu} \cong \mathscr{O}_\mu \otimes V \ .$$

Thus, as $\mathfrak{g}$-modules,

$$H^p(\mathcal{M}, \mathscr{O}_{\theta\otimes\mu}) \cong H^p(\mathcal{M}, \mathscr{O}_\mu) \otimes V \ . \tag{5.2}$$

Now we show $H^p(\mathcal{M}, \mathscr{O}_{\theta\otimes\mu}) \cong H^p(\mathcal{M}, \mathscr{O}_{\omega\lambda+\mu})$. By Proposition 4.3, $\mathscr{O}_{\theta\otimes\mu}$ decompose into direct sum of subsheaves S_t, where $t = \nu + \mu$ for some $v \in \Lambda$. We know that $||\lambda + \delta|| > ||\nu + \delta||$ for all $\nu \neq \lambda$, $\nu \in \Lambda$. As $\omega\Lambda = \Lambda$, $\omega(\Lambda + \delta) = \Lambda + \omega\delta = \Lambda + \mu + \delta$. Thus $\omega\lambda + \mu$ will be the unique element in $\Lambda + \mu$ such that $||(w\lambda + \mu) + \delta|| = ||\lambda + \delta||$. By (4.2),

$$\langle (\omega\lambda + \mu), (\omega\lambda + \mu) + 2\delta\rangle = \langle \lambda, \lambda + 2\delta\rangle =: t$$

and the multiplicity of $\omega\lambda + \mu$ is one. By Proposition 4.3 again,

$$S_t \cong \mathscr{O}_{\omega\lambda+\mu}$$

and $\mathscr{O}_{\theta\otimes\mu} \cong \mathscr{O}_{\omega\lambda+\mu} \oplus \tilde{S}$, where $\Omega - \langle \lambda, \lambda + 2\delta\rangle$ is injective on $\tilde{S}$. Also we have

$$H^p(\mathcal{M}, \mathscr{O}_{\theta\otimes\mu}) = H^p(\mathcal{M}, \mathscr{O}_{\omega\lambda+\mu}) \oplus H^p(\mathcal{M}, \tilde{S}) \ .$$

However, by (5.2), Proposition 4.1 and part one of this lemma, $c_{\mathfrak{g}}$ acts as the scalar $\langle \lambda, \lambda + 2\delta\rangle$ on $H^p(\mathcal{M}, \mathscr{O}_{\theta\otimes\mu})$. Thus $H^p(\mathcal{M}, \tilde{S}) = \{0\}$ for all p and (2) is shown. □

Corollary 5.1. *The set of integers p such that $H^p(\mathcal{M}, \mathscr{O}_\lambda) \neq 0$ is constant as $\lambda+\delta$ varies over the set of integral weights in a given Weyl chamber.*

Theorem 5.1. *(Borel-Weil theorem) Let λ be a dominant integral weight. Then*

(1) $H^0(\mathcal{M}, \mathscr{O}_\lambda) \cong V(\lambda)$ *as $\mathfrak{g}$-modules.*
(2) $H^p(\mathcal{M}, \mathscr{O}_\lambda) = \{0\}$ *for $p \neq 0$.*

Proof. Putting $\omega = \mathrm{id}$ in (5.1), as $\mu = 0$,

$$H^p(\mathcal{M}, \mathscr{O}_\lambda) \cong H^p(\mathcal{M}, \mathscr{O}) \otimes V(\lambda) \ . \tag{5.3}$$

When $p = 0$, $H^0(\mathcal{M}, \mathscr{O}) \cong \mathbb{C}$ as $\mathcal{M}$ is compact. Thus part one is shown. When $p > 0$ ($p < 0$ is trivial), we use a result in ([1], Proposition 10.1), which says that L_λ is positive if $\lambda \in \mathcal{C}^+$. Fix $\lambda \in \mathcal{C}^+$, by Kodaira Vanishing theorem [2], $H^p(\mathcal{M}, \mathscr{O}_{m\lambda}) = \{0\}$ for all $p > 0$ for m large enough. Replacing λ by $m\lambda$ in (5.3), we obtain $H^p(\mathcal{M}, \mathscr{O}) = \{0\}$ for all $p > 0$. Put this back in (5.3), we conclude $H^p(\mathcal{M}, \mathscr{O}_\lambda) = \{0\}$ for all $p > 0$. □

6. Borel-Weil-Bott theorem

In this last section we prove the Borel-Weil-Bott theorem, which describes the cohomology groups $H^p(\mathcal{M}, \mathscr{O}_\lambda)$ for all integral weight λ. First we deal with the case where $\lambda + \delta$ lies in a wall.

Lemma 6.1. *If λ is an integral weight and $\lambda + \delta$ lies in a wall, then $H^p(\mathcal{M}, \mathscr{O}_\lambda) = \{0\}$ for all p.*

Proof. If not, $H^p(\mathcal{M}, \mathscr{O}_\lambda)$ would have an irreducible $\mathfrak{g}$-module with highest dominant weight γ. Using Proposition 4.1, Proposition 4.2 and (4.2), $||\lambda + \delta|| = ||\gamma + \delta||$. Then $\lambda + \delta$ and $\gamma + \delta$ are in the same Weyl group orbit, by Theorem 2.1. This implies that $\gamma + \delta$ also lies in some wall, which is impossible as γ is dominant. □

The next lemma relates two integral weights lying in adjacent Weyl Chambers, separated by a wall P_α.

Lemma 6.2. *Let V be a finite dimensional irreducible representation of $\mathfrak{g}$ with weight Λ and highest weight λ. Let $\alpha \in \Phi$. If $\mu \in E$ satisfies $\langle \mu, \alpha \rangle = 0$ and $\langle \mu, \beta \rangle > 0$ for all $\beta \in \Phi^+ \setminus \{\pm\alpha\}$, then the maximal value $||\mu + \gamma||$ for $\gamma \in \Lambda$ is achieved at exactly two points λ and $s_\alpha\lambda$.*

Proof. Since $\nu \mapsto ||\nu||^2$ is a convex function in E, the maximum $||\mu+\beta||$ can only occur when $\beta \in \Lambda$ is an extremal weight. Given two extremal weights γ and ν, $||\gamma|| = ||\nu||$ by Theorem 2.1. This implies

$$||\mu + \gamma||^2 - ||\mu + \nu||^2 = 2\langle \mu, \gamma - \nu\rangle \ .$$

Let $\gamma = \lambda$. Then $||\mu + \lambda|| = ||\mu + \nu||$ only when $\lambda = \nu + n\alpha$ for some n. But the only extremal weight of this form is $s_\alpha(\lambda)$. □

Theorem 6.1. *(Borel-Weil-Bott theorem) Let λ be an integral weight.*

(1) *If $\lambda + \delta = \omega(\nu + \delta)$ for some $\omega \in W$ and some dominant weight ν, then*

$$H^{\ell(\omega)}(\mathcal{M}, \mathscr{O}_\lambda) \cong V(\nu) \tag{6.1}$$

as $\mathfrak{g}$-modules, where $\ell(\omega)$ is the length of ω.

(2) *$H^p(\mathcal{M}, \mathscr{O}_\lambda) = \{0\}$ for all $p \neq \ell(\omega)$.*

Proof. We will proceed by induction on $\ell(\omega)$. When $\ell(\omega) = 0$, it reduces to Borel-Weil theorem. Assume that the theorem is true for all Weyl group element of length $\leq k - 1$. Let $\omega \in W$ and $\ell(w) = k$. Let ν be any dominant weight. Let $\alpha \in \Phi^+$ such that $\ell(s_\alpha \omega) = k - 1$. Write $\nu' = \nu + \delta \in \mathcal{C}^+$ and

$$\eta = (s_\alpha\omega)\nu', \quad s_\alpha\eta = \omega\nu', \quad \rho = (\omega\delta) + s_\alpha(\omega\delta), \quad \tau = \rho - \delta \ .$$

The exact formula for ρ is not essential. All we want is an integral weight which satisfies the hypothesis of Lemma 6.2 with respect to $\omega\Phi^+$.

Let V be the finite dimensional irreducible G-module with highest weight ν. Let (θ, V) be the restriction of V to B. As in the proof of Lemma 5.1,

$$\mathscr{O}_{\theta\otimes\tau} \cong \mathscr{O}_\tau \otimes V$$

and

$$H^p(\mathcal{M}, \mathscr{O}_{\theta\otimes\tau}) \cong H^p(\mathcal{M}, \mathscr{O}_\tau) \otimes V \ .$$

As $\tau + \delta = \rho$ lies in a wall, Lemma 6.1 imply that $H^p(\mathcal{M}, \mathscr{O}_\tau) = \{0\}$ for all p. Thus

$$H^p(\mathcal{M}, \mathscr{O}_{\theta\otimes\tau}) = \{0\}, \quad \forall p \ . \tag{6.2}$$

As $\ell(s_\alpha\omega) < \ell(\omega)$, we have $\langle \alpha, s_\alpha\eta\rangle < 0$ and $\langle \alpha, \eta\rangle > 0$. In particular, $s_\alpha\eta < \eta$ with respect to the ordering defined using Φ^+. Let V' be the B-submodule of V containing all weights γ with $\gamma < \eta$ and $V'' = V/V'$, Then we have a short exact sequence of B-modules

$$0 \to (\theta', V') \to (\theta, V) \to (\theta'', V'') \to 0 \ .$$

Tensoring with τ (treated as B-modules) gives

$$0 \to (\theta' \otimes \tau, V') \to (\theta \otimes \tau, V) \to (\theta'' \otimes \tau, V'') \to 0 \ .$$

By considering the sheaf of holomorphic section on the induced bundles, we obtain a short exact sequence of sheaf on $\mathcal{M}$:

$$0 \to \mathscr{O}_{\theta'\otimes\tau} \to \mathscr{O}_{\theta\otimes\tau} \to \mathscr{O}_{\theta''\otimes\tau} \to 0 \ . \tag{6.3}$$

By Proposition 4.3 again, $\mathscr{O}_{\theta\otimes\tau}$ can be written as direct sum of subsheaves S_t, where t is the generalized eigenspace of $c_{\mathfrak{g}}$. t might be of the form

$$t = \langle \gamma + \tau, \gamma + \tau + 2\delta \rangle = ||\gamma + \rho||^2 - ||\delta||^2 \ ,$$

where $\gamma \in \Lambda$. By Lemma 6.2, $s_\alpha \eta$ and η are the only two weights in Λ that maximize t. Projecting (6.3) to its t eigenspace gives

$$0 \to (\mathscr{O}_{\theta'\otimes\tau})_t \to (\mathscr{O}_{\theta\otimes\tau})_t \to (\mathscr{O}_{\theta''\otimes\tau})_t \to 0$$

As $s_\alpha \eta \in V'$ and $\eta \in V''$, both $(\mathscr{O}_{\theta'\otimes\tau})_t$ and $(\mathscr{O}_{\theta''\otimes\tau})_t$ are one dimensional. Thus by Proposition 4.3,

$$(\mathscr{O}_{\theta'\otimes\tau})_t \cong \mathscr{O}_{s_\alpha\eta+\tau}, \quad (\mathscr{O}_{\theta''\otimes\tau})_t \cong \mathscr{O}_{\eta+\tau} \ .$$

Using (6.2) and the fact that $(\mathscr{O}_{\theta\otimes\tau})_t$ is a subsheaf of $\mathscr{O}_{\theta\otimes\tau}$,

$$H^p(\mathcal{M}, (\mathscr{O}_{\theta\otimes\tau})_t) = \{0\}$$

for all p. Thus the long exact sequence on cohomology induces an isomorphism

$$H^{p+1}(\mathcal{M}, \mathscr{O}_{s_\alpha\eta+\tau}) \cong H^p(\mathcal{M}, \mathscr{O}_{\eta+\tau}) \ . \tag{6.4}$$

As $\eta + \tau + \delta = \eta + \rho$ and ρ lies in the hyperplane P_α, $\eta + \tau + \delta \in s_\alpha C$. So

$$(\eta + \tau) + \delta = (s_\alpha \omega)(\chi + \delta)$$

for some dominant integral weight χ. Note that

$$(s_\alpha \eta + \tau) + \delta = s_\alpha \eta + \rho = s_\alpha(\eta + \rho) = \omega(\chi + \delta) \ .$$

By induction hypothesis,

$$H^{\ell(\omega)-1}(\mathcal{M}, \mathscr{O}_{\eta+\tau}) \cong V(\chi), \ H^p(\mathcal{M}, \mathscr{O}_{\eta+\tau}) = 0 \ \text{ when } p \neq \ell(\omega) - 1 \ .$$

Using (6.4), we have

$$H^{\ell(\omega)}(\mathcal{M}, \mathscr{O}_{s_\alpha\eta+\tau}) \cong V(\chi), \ H^p(\mathcal{M}, \mathscr{O}_{s_\alpha\eta+\tau}) = 0 \ \text{ if } p \neq \ell(\omega) \ . \tag{6.5}$$

As a result, the induction step $\ell(\omega) = k$ have been shown for at least one integral weight λ, where $\lambda = s_\alpha \eta + \tau = \omega(\nu + \delta) + \tau$.

But this is good enough: By Lemma 5.1, for all integral weight λ such that $\lambda + \delta = \omega(\chi + \delta)$ (or $\lambda = \omega\chi + \mu$)

$$H^p(\mathcal{M}, \mathscr{O}_\lambda) \cong H^p(\mathcal{M}, \mathscr{O}_\mu) \otimes V(\chi) \tag{6.6}$$

as $\mathfrak{g}$-modules and $H^p(\mathcal{M}, \mathscr{O}_\mu)$ are trivial $\mathfrak{g}$-modules for all p. Using (6.5) we have

$$H^{\ell(\omega)}(\mathcal{M}, \mathscr{O}_\mu) \cong \mathbb{C}, \quad H^p(\mathcal{M}, \mathscr{O}_\mu) = \{0\} \text{ when } p \neq \ell(\omega) .$$

Putting this back to (6.6), the induction step is verified. □

References

1. Raoul Bott *Homogeneous Vectors Bundles*, The Annals of Mathematics, 2nd Ser., Vol. 66, No. 2. (Sep., 1957), pp. 203-248
2. Phillip Griffiths, Joseph Harris *Principles of Algebraic Geometry*, Wiley Classics Library, 1994, John Wiley and Sons, Inc.
3. James E. Humphreys *Introduction to Lie Algebras and Representation Theory*, Graduate Texts in Mathematics 9, (1972), Springer.
4. Anthony W. Knapp, *Lie groups beyond an introduction*, second edition, Progress in Mathematics 140, Birhäuser
5. Joseph L.Taylor, *Several Complex Variables with connections to Algebraic Geometry and Lie Groups*, Graduate Studies in Mathematics - Volumn 46, American Mathematical Society