CARTAN AND IWASAWA DECOMPOSITIONS IN LIE THEORY

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Abstract. In this article we will be discussing about various decompositions of semisimple Lie algebras, which are very important to understand their structure theories. Throughout the article we will be assuming existence of a compact real form of complex semisimple Lie algebra. We will start with describing Cartan involution and Cartan decompositions of semisimple Lie algebras. Iwasawa decompositions will also be constructed at Lie group and Lie algebra level. We will also be giving some brief description of Iwasawa of semisimple groups.

1. Introduction

The idea for beginning an investigation of the structure of a general semisimple Lie group, not necessarily classical, is to look for same kind of structure in its Lie algebra. We start with a Lie algebra L of matrices and seek a decomposition into symmetric and skew-symmetric parts. To get this decomposition we often look for the occurrence of a compact Lie algebra as a real form of the complexification $L^{\mathbb{C}}$ of L.

If L is a real semisimple Lie algebra, then the use of a compact real form of $L^{\mathbb{C}}$ leads to the construction of a 'Cartan Involution' θ of L. This involution has the property that if $L = H \oplus P$ is corresponding eigenspace decomposition or 'Cartan Decomposition' then, $L^{\mathbb{C}}$ has a compact real form (like $H \oplus iP$) which generalize decomposition of classical matrix algebra into Hermitian and skew-Hermitian parts.

Similarly if G is a semisimple Lie group, then the 'Iwasawa decomposition' $G = NAK$ exhibits closed subgroups A and N of G such that they are simply connected abelian and nilpotent respectively and A normalizes N and multiplication $K \times A \times N \to G$ is a diffeomorphism. Iwasawa decomposition generalizes the Gram-Schimdt orthogonalization process.

2. Cartan Involution and decomposition on Lie Algebra level

Mostly it will be assumed that $\mathfrak{g}$ is a real Lie algebra of matrices over $\mathbb{R}$ or $\mathbb{C}$ and closed under $(\cdot)^*$. However, we can state that, every real semisimple Lie algebra can be realized as a Lie algebra of real matrices closed under transpose (which will be clear after proposition (1)). To detect semisimplicity of some specific Lie algebra of matrices we critically use the conjugate transpose mapping $X \to X^*$. Here we define a new map $\theta(X) = -X^*$

on a real semisimple Lie algebra $\mathfrak{g}$, which is actually an involution, i.e. an automorphism of the Lie algebra with square equals to the identity.

Let B be the Killing form on $\mathfrak{g}$. Define $B_\theta : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$,

$$B_\theta(X, Y) = -B(X, \theta(Y))$$

Proposition 1.

(1) *θ respects the bracket and is an involution.*
(2) *Killing from B is invariant under any automorphism.*
(3) *B_θ symmetric and positive definite.*

Proof.

(1) $\theta[X, y] = -[X, Y]^* = -[Y^*, X^*] = [-X^*, -Y^*] = [\theta(X), \theta(Y)]$, therefore clearly, θ is an automorphism. Also by definition $\theta^2 = Id$ i.e. θ is involution.
(2) We have for any automorphism δ of $\mathfrak{g}$ and $X, Y \in \mathfrak{g}$

$$\begin{aligned} ad(\delta X)Y &= [\delta X, Y] \\ &= \delta[X, \delta^{-1}Y] \\ &= (\delta(adX)\delta^{-1})Y. \end{aligned}$$

Therefore,

$$\begin{aligned} B(\delta X, \delta Y) &= Tr(ad(\delta X)ad(\delta Y)) \\ &= Tr(\delta(adX)\delta^{-1}\delta(adY)\delta^{-1}) \\ &= Tr((adX)ad(Y)) \\ &= B(X, Y). \end{aligned}$$

(3) We will use part (2) to prove (3).

$$\begin{aligned} B_\theta(X, Y) &= -B(X, \theta Y) \\ &= -B(\theta X, \theta^2 Y) \\ &= -B(\theta X, Y) \\ &= -B(Y, \theta X) = B_\theta(Y, X). \end{aligned}$$

So B_θ is symmetric. Also we see,

$$\begin{aligned} B_\theta(X, X) &= -B(X, \theta X) \\ &= -Tr((adX)(ad(\theta X)) \\ &= Tr((adX)(adX^*)) \\ &= Tr((adX)(ad(X)^*) \geqslant 0. \end{aligned}$$

So B_θ is positive definite.

□

Definition 1. *An involution θ of a real semisimple Lie algebra $\mathfrak{g}$ such that the Bilinear form*

$$B_\theta(X, Y) = -B(X, \theta Y)$$

is symmetric and positive definite, is called a **Cartan Involution.**

We shall see that any real semisimple Lie algebra has a Cartan involution and that the Cartan involution is unique up to inner automorphism.

Definition 2. *If* $\mathfrak{g}_0$ *is a real Lie algebra, the complex Lie algebra*

$$\mathfrak{g}_0^{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_0$$

is called **Complexification** *of* $\mathfrak{g}_0$*. Similarly, when a complex Lie algebra* $\mathfrak{g}$ *and a real Lie algebra* $\mathfrak{g}_0$ *are related by as vector space over* $\mathbb{R}$ *by*

$$\mathfrak{g}^{\mathbb{R}} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0,$$

we say that $\mathfrak{g}_0$ *is a* **Real Form** *of the complex Lie algebra of* $\mathfrak{g}$*.*

Theorem 1. *If* $\mathfrak{g}$ *is a complex semisimple Lie algebra then* $\mathfrak{g}$ *has a compact real form* $\mathfrak{u}_0$*.*

This is a consequence of a particular normalization of root vectors whose construction uses the Isomorphism Theorems. We will not discuss the proof of this theorem. Following this, we have our next proposition, that $\mathfrak{g}^{\mathbb{R}}$ has a Cartan involution.

Proposition 2. *Let* $\mathfrak{g}$ *be a complex semisimple Lie algebra, let* $\mathfrak{u}_0$ *be a compact real form of* $\mathfrak{g}$*, and let* τ *be the conjugation of* $\mathfrak{g}$ *respect to* $\mathfrak{u}_0$*. If* $\mathfrak{g}$ *is regarded as a real Lie algebra of* $\mathfrak{g}^{\mathbb{R}}$*, then* τ *is a Cartan involution of* $\mathfrak{g}^{\mathbb{R}}$

Proof. Clearly τ is an involution on $\mathfrak{g}^{\mathbb{R}}$ which is semisimple complex Lie algebra. The Killing form $B_{\mathfrak{g}}$ of $\mathfrak{g}$ and $B_{\mathfrak{g}^{\mathbb{R}}}$ of $\mathfrak{g}^{\mathbb{R}}$ are related by

$$B_{\mathfrak{g}^{\mathbb{R}}}(Z_1, Z_2) = 2\Re(B_{\mathfrak{g}}(Z_1, Z_2)).$$

Write $Z \in \mathfrak{g}$ as $Z = X + iY$, with $X, Y \in \mathfrak{u}_0$. Then,

$$\begin{aligned} B_{\mathfrak{g}}(Z, \tau Z) &= B_{\mathfrak{g}}(X + iY, X - iY) \\ &= B_{\mathfrak{g}}(X, X) + B_{\mathfrak{g}}(Y, Y) \\ &= B_{\mathfrak{u}_0}(X, X) + B_{\mathfrak{u}_0}(Y, Y) < 0, \end{aligned}$$

as $\mathfrak{u}_0$ is semisimple $B_{\mathfrak{u}_0}$ is negative definite. It follows that,

$$(B_{\mathfrak{g}^{\mathbb{R}}})_{\tau}(Z_1, Z_2) = -2\Re(B_{\mathfrak{g}}(Z_1, \tau Z_2))$$

is a a positive definite symmetric bilinear form on $\mathfrak{g}^{\mathbb{R}}$, and therefore τ is a Cartan involution on $\mathfrak{g}^{\mathbb{R}}$ □

Now we address the problem of aligning compact real form properly when we start with a real semisimple Lie algebra $\mathfrak{g}_0$ and obtain $\mathfrak{g}$ by complexification.We will also show the existence and uniqueness up to conjugation of Cartan involution.

Definition 3. *Let* $\mathfrak{g}$ *be a real Lie algebra. We know the* $Aut_{\mathbb{R}}\mathfrak{g} = Der_{\mathbb{R}}\mathfrak{g}$*. If* $\mathfrak{g}$ *is semisimple we define* $Inn\ \mathfrak{g}$ *to be the* **Identity Component** *of* $Aut_{\mathbb{R}}\mathfrak{g}$*, so that we can have the inclusion* $Inn\ \mathfrak{g} \hookrightarrow Aut_{\mathbb{R}}\mathfrak{g}$ *which is smooth and everywhere regular.*

Lemma 1. *Let $\mathfrak{g}$ be a real finite-dimensional semisimple Lie algebra, and let ρ be an automorphism of $\mathfrak{g}$ that is diagonalizable with positive eigenvalues $d_1, \ldots, d_m$ and eigenspaces are $\mathfrak{g}_{d_j}$. Then we define for $r \in \mathbb{R}$ ρ^r to be the linear transformation on $\mathfrak{g}$ that is d_j^r on $\mathfrak{g}_{d_j}$. Then for any $r \in \mathbb{R}$ we have that $\rho^r \in Inn\ \mathfrak{g}$.*

Proof. If $X, Y \in \mathfrak{g}_{d_j}$, then

$$\rho[X, Y] = [\rho X, \rho Y] = d_i d_j [X, Y],$$

since ρ is an automorphism. Hence $[X, Y] \in \mathfrak{g}_{d_i d_j}$, and we obtain that

$$\rho^r[X, Y] = (d_i d_j)^r [X, Y] = [d_i^r X, d_j^r Y] = [\rho^r X, \rho^r Y].$$

Consequently ρ^r is an automorphism, therefore is an one parameter subgroup in $Aut\ \mathfrak{g}$, hence in the identity component $(Aut\ \mathfrak{g})_0$. As $\mathfrak{g}$ is semisimple $\rho^r \in Inn\ \mathfrak{g}$. □

Theorem 2. *Let $\mathfrak{g}$ be a real semisimple Lie algebra, θ be a Cartan involution, and σ be any involution. Then $\exists\ \phi \in Inn\ \mathfrak{g}$ such that $\phi\theta\phi^{-1}$ commutes with σ.*

Proof. Since θ is given as a Cartan involution, B_θ is an inner product for $\mathfrak{g}$. Put $\omega = \sigma\theta$, which makes $\omega^{-1} = \theta\sigma$. This is an automorphism of $\mathfrak{g}$, and as in proof of proposition 1, we have that it leaves B invariant. From $\sigma^2 = \theta^2 = 1$, we therefore have

$$B(\omega X, \theta Y) = B(X, \omega^{-1}\theta Y) = B(X, \theta\omega Y)$$
$$\implies B_\theta(\omega X, Y) = B_\theta(X, \omega Y).$$

Thus ω is symmetric and its square $\rho = \omega^2$ is positive definite. By previous lemma ρ^r lies in $Inn\ \mathfrak{g}$. Now we see,

$$\rho\theta = \omega^2\theta = \sigma\theta\sigma\theta^2 = \sigma\theta\sigma$$
$$= \theta^2\sigma\theta\sigma = \theta\omega^{-2} = \theta\rho^{-1}.$$

In terms of a basis of $\mathfrak{g}$ that diagonalizes ρ the matrix from of the equation is

$$\rho_{ii}\theta_{ij} = \theta_{ij}\rho_{jj}^{-1},\ \forall i, j.$$

Considering separately the cases $\theta_{ij} = 0 \& \theta_{ij} \neq 0$, we see that

$$\rho_{ij}^r\theta_{ij} = \theta\rho_{ij}^{-r} \implies \rho^r\theta = \theta\rho^{-1}.$$

Now putting $\phi = \rho^{1/4}$ we see that,

$$(\phi\theta\phi^{-1})\sigma = \rho^{1/4}\theta\rho^{-1/4}\sigma = \rho^{1/2}\theta\sigma$$
$$= \rho^{1/2}\omega^{-1} = \rho^{-1/2}\rho\omega$$
$$= \rho^{-1/2}\omega = \omega\rho^{-1/2}$$
$$= \sigma\theta\rho^{-1/2} = \sigma\rho^{1/4}\theta\rho^{-1/4} = \sigma(\phi\theta\phi^{-1}),$$

as required. □

Theorem 3. *If $\mathfrak{g}_0$ is a real semisimple Lie algebra, then $\mathfrak{g}_0$ has a Cartan involution. Any two Cartan involutions are conjugate via Inn $\mathfrak{g}_0$.*

Proof. Let $\mathfrak{g}$ be the complexification of $\mathfrak{g}_0$, and choose a compact real form $\mathfrak{u}_0$ of $\mathfrak{g}$. Let σ and τ be the complex conjugation (not the conjugation by *Inn* $\mathfrak{g}$) of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$ and $\mathfrak{u}_0$ respectively. If we regard $\mathfrak{g}$ as a real lie algebra $\mathfrak{g}^{\mathbb{R}}$, then proposition 2 shows that τ is Cartan involution. By theorem 2 we can find a $\phi \in Inn\ \mathfrak{g}^{\mathbb{R}} = Inn\ \mathfrak{g}$ such that $\phi\theta\phi^{-1}$ commutes with σ.

Here $\phi\theta\phi^{-1}$ is the conjugation of $\mathfrak{g}$ with respect to $\phi(\mathfrak{u}_0)$, which is another compact real form of $\mathfrak{g}$. Thus,

$$(B_{\mathfrak{g}^{\mathbb{R}}})_{\phi\tau\phi^{-1}}(Z_1, Z_2) = -2\Re B_{\mathfrak{g}}(Z_1^{-1}, \phi\tau\phi^{-1}Z_2^{-1})$$

is positive definite on $\mathfrak{g}^{\mathbb{R}}$.

The Lie algebra $\mathfrak{g}_0$ is characterized as he fixed set of σ. If $\sigma(X) = X$, then

$$\sigma(\phi\tau\phi^{-1}X) = \phi\tau\phi^{-1}\sigma X = \phi\tau\phi^{-1}X.$$

Hence $\phi\tau\phi^{-1}$ restricts to an involution θ of $\mathfrak{g}_0$. We have,

$$B_\theta(X, Y) = -B_{\mathfrak{g}_0}(X, \theta Y)$$

$$= -B\mathfrak{g}(X, \phi\tau\phi^{-1}Y) = \frac{1}{2}(B_{\mathfrak{g}^{\mathbb{R}}})_{\phi\tau\phi^{-1}}(X, Y).$$

Thus B_θ is positive definite on $\mathfrak{g}_0$ and hence θ is a Cartan involution.

Now let θ and θ' be two Cartan involutions. Taking $\sigma = \theta'$ in theorem 2, we can find a $\phi \in Inn\ \mathfrak{g}_0$ such that $\phi\theta\phi^{-1}$ commutes with θ'. Here $\phi\theta\phi^{-1}$ is another Cartan involution of $\mathfrak{g}_0$. So we may assume that θ and θ' commute as well and prove that $\theta = \theta'$.

Since θ and θ' commute, they have compatible eigenspace decompositions into $+1$ and -1 eigenspaces. By symmetry it is enough to show that no nonzero $X \in \mathfrak{g}_0$ is in the $+1$ eigenspace for θ and in the -1 eigenspace of θ' simultaneously. Assuming the contrary, suppose $\theta(X) = X$ and $\theta'(X) = -X$. Then we have,

$$0 < B_\theta(X, X) = -B(X, \theta X) = -B(X, X)$$

$$0 < B'_\theta(X, X) = -B(X, \theta' X) = +B(X, X),$$

contradiction. So we conclude that $\theta = \theta'$ and the proof is complete. □

Corollary 1. *If $\mathfrak{g}$ is a complex semisimple Lie algebra, then the only Cartan involutions of $\mathfrak{g}^{\mathbb{R}}$ are the conjugations with respect to the compact real form of $\mathfrak{g}$.*

Proof. Theorem 1 and proposition 2 produce a Cartan involution of $\mathfrak{g}^{\mathbb{R}}$ that is conjugation with respect to some compact real form of $\mathfrak{g}$. Any other Cartan involution is conjugate to this one, according to theorem 3 , and hence is also conjugation with respect to a compact real form of $\mathfrak{g}$. □

A Cartan involution θ of $\mathfrak{g}_0$ yields an eigenspace decomposition

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$$

of $\mathfrak{g}_0$ into $+1$ and -1 eigenspaces, and also we will have the rules,

$$[\mathfrak{k}_0, \mathfrak{k}_0] \subseteq \mathfrak{k}_0, \; [\mathfrak{k}_0, \mathfrak{p}_0] \subseteq \mathfrak{p}_0, \; [\mathfrak{p}_0, \mathfrak{p}_0] \subseteq \mathfrak{k}_0$$

since θ is an involution. From these it follows that

$$\mathfrak{k}_0 \text{ and } \mathfrak{p}_0 \text{ are orthogonal under } B_{\mathfrak{g}_0} \text{ and } B_\theta.$$

In fact , if $X \in \mathfrak{k}_0$ and $Y \in \mathfrak{p}_0$, then $adXadY$ carries $\mathfrak{k}_0 \to \mathfrak{p}_0$ and $\mathfrak{p}_0 \to \mathfrak{k}_0$. Thus it has trace 0, and $B_{\mathfrak{g}_0}(X, Y) = 0$; since $\theta(Y) = -Y$, $B_\theta(X, Y) = 0$ also.

Since B_θ is positive definite, eigenspaces $\mathfrak{k}_0$ and $\mathfrak{p}_0$ have the property that

$$B_{\mathfrak{g}_0} \text{ is} \begin{cases} \text{positive definite on } \mathfrak{p}_0 \\ \text{negative definite on } \mathfrak{k}_0 \end{cases}$$

Definition 4. *A decomposition of $\mathfrak{g}_0$ that satisfies above bracket relations and condition on $B_{\mathfrak{g}_0}$ is called a* **Cartan Decomposition** *of $\mathfrak{g}_0$.*

Conversely a Cartan decomposition determines a Cartan involution θ by the formula,

$$\theta = \begin{cases} -1 \text{ on } \mathfrak{p}_0 \\ +1 \text{ on } \mathfrak{k}_0. \end{cases}$$

Here above bracket relation shows that θ respects bracket. Also from the definition of $B_{\mathfrak{g}_0}$, we have that B_θ is symmetric (as θ has order 2) and positive definite.

3. Cartan Decomposition on Lie Group Level

In this section we will make analogous statement in consideration of groups. Let G be a semisimple Lie group, and $\mathfrak{g}$ be its Lie algebra. The results established in previous section that $\mathfrak{g}$ has a Cartan involution and any two Cartan involutions are conjugate by an inner automorphism. The theorem we are going to state here (no proof!) lifts corresponding Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ to a decomposition of G.

Theorem 4. *Let G be a semisimple Lie group, let θ be a Cartan involution of its Lie algebra $\mathfrak{g}_0$, let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the corresponding Cartan decomposition, and let K be the analytic subgroup with Lie algebra $\mathfrak{k}_0$. Then,*

1. *$\exists$ a Lie group automorphism Θ of G with differential θ and $\Theta^2 = 1$*
2. *the subgroup of G fixed by Θ is K*
3. *the mapping $K \times \mathfrak{p}_0 \to G$ given by $(k, X) \to k\exp(X)$ is a diffeomorphism*
4. *K is closed and contains center Z of G*
5. *K is compact $\iff$ Z is finite*
6. *if Z is finite K is a maximal compact subgroup of G.*

Definition 5. *The automorphism* Θ *in the theorem is called the* **Global Cartan Involution** *and the decomposition in* (3) *is called the* **Global Cartan Decomposition**.

4. Iwasawa Decomposition on Lie Algebra and Lie Group

The Iwasawa decomposition is a second global decomposition of a semisimple Lie group. Unlike the Cartan decomposition the factors in Iwasawa decomposition are closed subgroups. The prototype is the Gram-Schimdt orthogonalization process in linear algebra. Let us start with examples and motivation.

Let $G = GL(n, \mathbb{C})$, which is the complexification of $K = U(n)$, which is a maximal compact subgroup. Let T be a subgroup of K consisting of diagonal matrices whose eigenvalues have absolute value 1, i.e. the maximal torus inside K. The complexificiation $T_{\mathbb{C}}$ of T can be factored as TA, where A is the group of diagonal matrices whose eigenvalues are positive real numbers. Let B be the group of upper triangular matrices in G, and B_0 be the subgroup of B with elements whose diagonal entries are postive real numbers. Finally, let N be the subgroup of unipotent elements of B. Recalling that a matrix X is called unipotent if its only eigenvalues are 1 i.e. $I - X$ is nilpotent. The elements of N are upper triangualr matrices whose diagonal entries are all 1. We may factor $B = TN$ and $B_0 = AN$. As the subgroup N is normal in both B and B_0 so these decompositions are semidirect products.

Proposition 3. *With* $G = GL(n, \mathbb{C}), K = U(n)$, *and* B_0 *as above, every element of* $g \in G$ *can be factored uniquely as* bk *where* $b \in B_0$ *and* $k \in K$, *or as* $a\nu k$ *where* $a \in A, \nu \in N, k \in K$. *The multiplication map* $A \times N \times K \to G$ *is a diffeomorphism.*

Proof. Let $g \in G$ have the rows $v_1, \ldots, v_n$. Let a be the diagonal matrix whose elements are $|v_i|$. Then the rows of $a^{-1}g$ have length 1. Let $u_i = \frac{v_i}{|v_i|}$ be these rows.

By Gram-Schimdt orthogonalization algorithm, we find constants θ_{ij} with $i < j$ such that $u_n, u_{n-1} + \theta_{n-1,n}u_n, \ldots$ are orthonormal. This means that if

$$\nu := \begin{pmatrix} 1 & \theta_{12} & \cdots & \theta_{1n} \\ & 1 & \cdots & \theta_{2n} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix}$$

then $k = \nu^{-1}a^{-1}g$ is unitary and so $g = a\nu k = b_0 k$ with $b_0 = a\nu$. This proves the existence of the required factorizations. We have $B_0 \cap K = I = A \cap N$, so the factorizations are unique. It is easy to see that the matrices a, ν, and k depend continuously on g, so the multiplication map $A \times N \times K \to G$ is a diffeomorphism. □

Definition 6. *The decomposition* $G \cong A \times N \times K$ *is called the* **Iwasawa Decomposition**.

The decomposition above extends to all semisimple Lie groups. To prove such a theorem , we first obtain a Lie algebra decomposition and then lift to the Lie groups.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. θ be a Cartan involution of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition.

Lemma 2. *If* $\mathfrak{g}$ *is real, then*

$$(adX)^* = -ad(\theta X) \ \forall X \in \mathfrak{g},$$

where adjoint $(\cdot)^*$ *is defined relative to the inner product* B_θ.

Proof. We have,

$$\begin{aligned} B_\theta((ad(\theta X))Y, Z) &= -B([\theta X, Y], \theta Z) \\ &= B(Y, [\theta X, \theta Z) \\ &= B(Y, \theta[X, Z]) \\ &= -B_\theta(Y, (adX)Z) = -B_\theta((adX)^* Y, Z) \end{aligned}$$

□

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. By above lemma $\{adH | H \in \mathfrak{a}\}$ is a commuting family of self-adjoint transformations of $\mathfrak{g}$. Then $\mathfrak{g}$ is the orthogonal direct sum of simultaneous character spaces with all real-valued characters and member of $\mathfrak{a}^*$. For $\lambda \in \mathfrak{a}^*$ we write,

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} | (adH)X = \lambda(H)X \ \forall H \in \mathfrak{a}\}.$$

Definition 7. *If* $\mathfrak{g}_\lambda \neq 0$ *and* $\lambda \neq 0$, *we call* λ *a* **Restricted Root** *of* $\mathfrak{g}$. *The set of restricted roots is denoted by* Σ. *Any nonzero* $\mathfrak{g}_\lambda$ *is called a* **Restricted Root Space**.

Proposition 4. *The restricted roots and restricted root spaces have the following properties:*

(1) $\mathfrak{g}$ *is the orthogonal direct sum* $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda$
(2) $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu}$
(3) $\theta \mathfrak{g}_\lambda = \mathfrak{g}_{-\lambda}$ *hence* $\lambda \in \Sigma \implies -\lambda \in \Sigma$
(4) $\mathfrak{g}_0 = \mathfrak{a} \oplus Z_{\mathfrak{l}}(\mathfrak{a})$ *orthogonally.*

Proof. $(1), (2)$, and (3) are standard.

For (4) we have $\theta \mathfrak{g}_0 = \mathfrak{g}_0$. Hence

$$\mathfrak{g}_0 = (\mathfrak{l} \cap \mathfrak{g}_0) \oplus (\mathfrak{p} \cap \mathfrak{g}_0).$$

Since $\mathfrak{a} \subseteq \mathfrak{p} \cap \mathfrak{g}_0$ and is maximal abelian in $\mathfrak{p}$ so $\mathfrak{a} = \mathfrak{p} \cap \mathfrak{g}_0$. Also $\mathfrak{l} \cap \mathfrak{g}_0 = Z_{\mathfrak{l}}(\mathfrak{a})$, which proves (4). □

Definition 8. *The decomposition in* (4) *is called the* **Restricted Root Space Decomposition**.

Let choose a notion of positivity for $\mathfrak{a}^*$ for instance lexicographic ordering. By the notion of positivity we mean: a set of vectors will be called positive if it satisfies (1) for any nonzero vector ϕ, exactly one of ϕ and $-\phi$ will be positive; and (2) the sum of positive elements is positive, and any positive multiple of a positive element is positive. One way to define positivity in a general vector space V with inner product $\langle , \rangle$ is by means of a Lexicographic ordering. Fix a spanning set of $\phi_1, \ldots, \phi_k$ of V, and define positivity as follows: we say that $\phi > 0$ if there exists an index k such that $\langle \phi, \phi_i \rangle = 0$ for $1 \le i \le k-1$ and $\langle \phi, \phi_k \rangle > 0$

Let Σ^+ is set of positive roots and define $\mathfrak{n} = \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda$. By previous proposition $\mathfrak{n}$ is a nilpotent Lie subalgebra of $\mathfrak{g}$.

Theorem 5. *With the notion above $\mathfrak{g}$ is a vector space direct sum $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Here $\mathfrak{a}$ is abelian, $\mathfrak{n}$ is nilpotent, $\mathfrak{a} \oplus \mathfrak{n}$ is solvable, and $[\mathfrak{a} \oplus \mathfrak{n}, \mathfrak{a} \oplus \mathfrak{n}] = \mathfrak{n}$.*

Proof. We know that $\mathfrak{a}$ is abelian and $\mathfrak{n}$ is nilpotent. Since, $[\mathfrak{a}, \mathfrak{g}_\lambda] = \mathfrak{g}_\lambda$ for each nonzero λ, we see that $[\mathfrak{a}, \mathfrak{n}] = \mathfrak{n}$ and $\mathfrak{a} \oplus \mathfrak{n}$ is solvable with $[\mathfrak{a} \oplus \mathfrak{n}] = \mathfrak{n}$.

To prove $\mathfrak{l} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is a direct sum, let $X \in \mathfrak{l} \cap (\mathfrak{a} \oplus \mathfrak{n})$. Then $\theta X = X$ with $\theta X \in \mathfrak{a} \oplus \theta \mathfrak{n}$. Since $\mathfrak{a} \oplus \mathfrak{n} \oplus \theta \mathfrak{n}$ is direct sum by previous proposition, $X \in \mathfrak{a}$. But then $X \in \mathfrak{l} \cap \mathfrak{p} = 0$.

The sum $\mathfrak{l} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is all of $\mathfrak{g}$ because for any $X \in \mathfrak{g}$ using some $H \in \mathfrak{a}$, some $X_0 \in Z_{\mathfrak{l}}(\mathfrak{a})$ and elements $X_\lambda \in \mathfrak{g}_\lambda$, as

$$X = H + X_0 + \sum_{\lambda \in \Sigma} X_\lambda$$

$$= (X_0 + \sum_{\lambda \in \Sigma^+} X_{-\lambda} + \theta X_{-\lambda}) + H + (\sum_{\lambda \in \Sigma^+} X_\lambda - \theta X_{-\lambda}),$$

where the right side is in $\mathfrak{l} \oplus \mathfrak{a} \oplus \mathfrak{n}$. □

Definition 9. $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{a} \oplus \mathfrak{n}$ *is called the* **Iwasawa Decomposition** *for complex semisimple Lie algebra* $\mathfrak{g}$.

The following is the corresponding Iwasawa decomposition for Lie group (no proof!).

Theorem 6. *Let G be a complex semisimple Lie group, let $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is be an Iwasawa decomposition of the Lie algebra. Also let K, A and N be connected subgroups of G with smooth inclusions in G and their Lie algebras are $\mathfrak{l}_0$, $\mathfrak{a}$ and $\mathfrak{n}$ respectively. Then the multiplication map*

$$K \times A \times N \to G$$

$$(k, a, n) \to kan$$

is a diffeomorphism and the groups A and N are simply connected.

Iwasawa decomposition is a very important theorem in Harmonic analysis on Hyperbolic planes and in the theory of Automorphic Forms. For instance; it is sometimes convenient to consider homogeneous space model of the hyperbolic plane rather than the Poincare upper half-plane

$$\mathbb{H} = \{x + iy : x \in \mathbb{R}, y > 0\}.$$

Here we give a brief description of the model.

The group $G = SL(2, \mathbb{R})$ acts on $\mathbb{H}$ transitively, so $\mathbb{H}$ can be obtained as a orbit of a point:

$$\mathbb{H} = \{gz : g \in G\}.$$

The stabilizer of the point i is the orthogonal group:

$$K = C_G(i) = \left\{ k(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} : \theta \in \mathbb{R} \right\}$$

The Iwasawa decomposition of G is NAK, where,

$$A = \left\{ \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} : a \in \mathbb{R}^+ \right\},$$

and

$$N = \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} : x \in \mathbb{R} \right\}.$$

We can think

$$\mathbb{H} \cong G/K \cong NA = AN$$

by the map

$$x + iy \to a(y)n(x)$$

where

$$a(y) = \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} \text{ and } n(x) = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}.$$

This helps us to construct left invariant measure, as we see

$$dn = dx,\ da = y^{-1}dy.$$

And by this we can have our Riemannian metric on $\mathbb{H}$ which is $\frac{dxdy}{y^2}$ and also,

$$\begin{aligned} \int_P f(p)dp &= \int_A \int_N f(an)dadn \\ &= \int_A \int_N f(a(y)n(x)\frac{dxdy}{y} \\ &= \int_N \int_A f(a(y)n(x)\frac{dxdy}{y^2}. \end{aligned}$$

Acknowledgements

I would like to thank Professor Lior Silberman for introducing Cartan Involution to me and showing importance of Iwasawa decomposition in the theory of Automorphic forms. I would also like to thank Professor Zinovy Reichstein for helping me through out this project.

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