COMPLEX REDUCTIVE ALGEBRAIC GROUPS

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While an algebraic group over an arbitrary field is said to be reductive if its unipotent radical is trivial, this definition tends to be rather unenlightening on a first glance. Fortunately, if one is willing to work over a field of characteristic zero, this concept can be defined in more transparent ways. For instance, in this case, an algebraic group is reductive if and only if all of its representations are completely reducible. As a topologist, one is often interested in complex algebraic groups. In this setting, one can state this definition under another guise as follows:

A complex algebraic group is reductive if and only if it has a compact real form.

The caveat is that statements such as the one above often appear without reference in the literature and complete proofs are hard to come by. As such, this note is in some sense a result of the author's quest for such a proof whose hardest direction is now provided in Section 2.3 below.

1. Preliminaries

Throughout these preliminaries, $\mathbb{K}$ will denote $\mathbb{C}$ or $\mathbb{R}$. The material below is a mixture of various sources: [OV90], [Hum75], [HN12], and [FH91]. We will indicate precise locations of proofs for the most important statements.

1.1. **Affine Varieties.**

In what follows, $n-$dimensional affine space over $\mathbb{K}$ will be denoted by $\mathbb{A}^n$. Recall that an (affine) algebraic variety in $\mathbb{A}^n$ is the vanishing locus of a set of polynomials in $\mathbb{K}[X_1, \ldots, X_n]$. For a variety $M \subset \mathbb{A}^n$ we denote by $I(M)$ the ideal of functions in $\mathbb{K}[X_1, \ldots, X_n]$ vanishing on M and by $\mathbb{K}[M] := \mathbb{K}[X_1, \ldots, X_n]/I(M)$ the algebra of polynomials on M. A *morphism* of a variety $M \subset \mathbb{A}^n$ into a variety $N \subset \mathbb{A}^m$ is a map $f : M \to N$ that may be determined by polynomials in some coordinates. The algebraic varieties in $\mathbb{A}^n$ form a basis for the closed sets of the *Zariski topology* on $\mathbb{A}^n$ so that we may endow varieties $M \subset \mathbb{A}^n$ with the subspace Zariski topology. A variety is said to be *irreducible* if it cannot be written as a union of two non-empty Zariski closed proper subsets.

Since the distinction is about to become important, let us now denote complex affine space as $\mathbb{A}^n_{\mathbb{C}}$ and real affine space as $\mathbb{A}^n_{\mathbb{R}}$. Complex algebraic varieties may be considered as real algebraic varieties of doubled dimension through an operation we will call *realification* (extension of scalars). Explicitly, let us agree to identify

$(Z_1, \ldots, Z_n) \in \mathbb{A}^n_{\mathbb{C}}$ with $(X_1, \ldots, X_n, Y_1, \ldots, Y_n) \in \mathbb{A}^{2n}_{\mathbb{R}}$ where $X_j + iY_j = Z_j$. Now, if $M \subset \mathbb{A}^n_{\mathbb{C}}$ is an algebraic variety, we may rewrite the polynomial equations defining it in real coordinates to realize it as a variety $M^{\mathbb{R}} \subset \mathbb{A}^{2n}_{\mathbb{R}}$ called the *realification* of M. While the real polynomial algebra of $M^{\mathbb{R}}$ is generated by the real and imaginary parts of polynomials in $\mathbb{C}[M]$, it is more convenient to consider the "complex" polynomial algebra of $M^{\mathbb{R}}$:

$$\mathbb{C}[M^{\mathbb{R}}] = \mathbb{R}[M^{\mathbb{R}}] \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}[z_1, \ldots, z_n, \overline{z_1}, \ldots, \overline{z_n}]$$

where the z_j are the restriction onto M of coordinate function on $\mathbb{A}^n_{\mathbb{C}}$ and $\overline{z_j}$ denotes complex conjugation. From this point of view, Zariski closed subsets of $M^{\mathbb{R}}$ are those defined by equations in the z_j and $\overline{z_j}$; we call them *real subvarieties of* M. A map $f : M \to N$ between complex varieties is an *anitholomorphic morphism* if for every $g \in \mathbb{C}[N]$ we have $g \circ f \in \overline{\mathbb{C}[M]} := \mathbb{C}[\overline{z_1}, \ldots, \overline{z_n}]$.

Let now $M_o \subset \mathbb{A}^n_{\mathbb{R}}$ be a real variety and denote by $M_o(\mathbb{C}) \subset \mathbb{A}^n_{\mathbb{C}}$ its set of complex zeroes. We call $M_o(\mathbb{C})$ the *complexification* of M_o. In this case, there is always a unique antiholomorphic automorphism $x \mapsto \overline{x}$ of $M_o(\mathbb{C})$ (the associated *complex conjugation*) for which $M_o = \{x \in M_o(\mathbb{C}) : \overline{x} = x\}$. A *real form* of a complex affine variety M is a real subvariety $M_o \subset M$ for which the embedding $M_o \subset M$ extends to an isomorphism $M_o(\mathbb{C}) \cong M$. One should be aware that this operation inverse to complexification is not always defined!

1.2. **Algebraic Groups.** An (affine) *algebraic group* G is an affine variety endowed with the structure of a group for which the multiplications and inverse maps are morphisms of algebraic varieties.

Prototypical example: algebraic matrix groups. Consider the set $\mathrm{GL}_{n,\mathbb{K}}$ of $n \times n$ invertible matrices with entries in $\mathbb{K}$. This is a group under matrix multiplication called the *general linear group*. The set of $n \times n$ matrices $M_n\mathbb{K}$ can be identified with $\mathbb{K}^{n^2}$ where $\mathrm{GL}_{n,\mathbb{K}}$ is the Zariski open set defined by the nonvanishing of the determinant polynomial. To realize it as a variety, we can identify $\mathrm{GL}_{n,\mathbb{K}}$ with a Zariski closed subset of $\mathbb{K}^{n^2+1}$ by the map $(T_{ij}) \mapsto (T_{11}, T_{12}, \ldots, T_{nn}, 1/det(T_{ij}))$ where the T_{ij} are the restrictions of the n^2 coordinate functions to $\mathrm{GL}_{n,\mathbb{K}}$. The formulas for matrix multiplication and inversion (via Cramer's rule) then make it clear that $\mathrm{GL}_{n,\mathbb{K}}$ is an algebraic group.

Many other examples can be constructed using the fact that a Zariski closed subgroup of an algebraic group is again an algebraic group. As such, we will say that a subgroup is *algebraic* if it is Zariski closed. Strikingly, *affine algebraic groups can all be realized as an algebraic subgroup of some* $\mathrm{GL}_{n,\mathbb{K}}$ [Hum75, Theorem 8.6]. For instance, the *special linear group* $\mathrm{SL}_{n,\mathbb{K}} \subset \mathrm{GL}_{n,\mathbb{K}}$ is the group of matrices of determinant one. It is clearly a group by the product rule for the determinant and it is Zariski closed because it coincides with the vanishing of the polynomial $det(T_{ij}) - 1$.

Similarly we have the algebraic subgroup of *upper triangular matrices* which is the set of zeros of the polynomials $\{T_{ij} : i > j\}$, the group of *upper triangular unipotent matrices* which is the set of zeros of the polynomials $\{T_{ij}, T_{ii} - 1 : i > j\}$ and the group of *diagonal matrices* which is the set of zeroes of the polynomials $\{T_{ij} : i \neq j\}$.

Two of the most frequently occurring algebraic groups in this note are the *multiplicative group* $\mathbb{G}_m \cong \mathrm{GL}_{1,\mathbb{K}}$ and the *additive group* $\mathbb{G}_a \cong \mathbb{K}$. An algebraic group consisting of the direct product of n copies of the additive group (resp. the multiplicative group) is called an *n-dimensional vector group* (resp. an *n-dimensional algebraic torus*).

We say that the algebraic group G is *connected* if it is connected in the Zariski topology, i.e., if it is irreducible as a variety. The connected component of the identity element is denoted by G^o; it is always a normal subgroup of finite index in G and its cosets coincide with the connected components of G.

Definition. As with varieties, we may consider the realification $G^{\mathbb{R}}$ of any complex algebraic group G to realize it as a real algebraic group of double the dimension. A real algebraic subgroup $G_o \subset G$ is a *real form* of G if the inclusion $G_o \subset G$ extends to an isomorphism $G_o(\mathbb{C}) \cong G$.

Example. Consider the $n-$dimensional complex algebraic torus $(\mathbb{C}^\times)^n$:

(1) The real structure given by the antiholomorphic automorphism
$$(z_1, \ldots, z_n) \mapsto (\overline{z_1}, \ldots, \overline{z_n})$$
determines the real form $(\mathbb{R}^\times)^n \subset (\mathbb{C}^\times)^n$.

(2) The real structure given by the antiholomorphic automorphism
$$(z_1, \ldots, z_n) \mapsto ((\overline{z_1})^{-1}, \ldots, (\overline{z_n})^{-1})$$
determines the real form $\mathbb{T}^n := \{(z_1, \ldots, z_n) : |z_j| = 1,\ 1 \leq j \leq n\} \subset (\mathbb{C}^\times)^n$.

1.3. **Semisimple and Reductive Algebraic Groups.** Let V be a finite dimensional complex vector space and recall the following elementary definitions from linear algebra: an endomorphism σ of V is said to be

(1) *Nilpotent* if $\sigma^n = 0$ for some $n \in \mathbb{N}$.
(2) *Unipotent* if $\sigma - Id$ is nilpotent.
(3) *Semisimple* if V is spanned by eigenvectors of σ.

If $G \subset \mathrm{GL}_n \mathbb{C}$ is an algebraic group, we say that $g \in G$ is semisimple (resp. unipotent) if it is semisimple (resp. unipotent) as an endomorphism of $\mathbb{C}^n$. One can show that this definition does not depend on the chosen embedding of G in $\mathrm{GL}_n \mathbb{C}$ [Hum75, Chapter VI]. We say that an algebraic group G is *unipotent* if it consists of unipotent elements. The prototypical example of this type is the previously mentioned group of unipotent upper triangular matrices.

Definition. The following two definitions shall be most important for our purposes:

(1) Any complex algebraic group G possesses a unique maximal normal solvable subgroup $S \subset G$ which is automatically Zariski closed. Its identity component $\mathrm{Rad}(G) := S^o$ is then the maximal connected normal solvable subgroup of G; we call it the *radical* of G.
(2) The subgroup $\mathrm{Rad}_u(G) \subset \mathrm{Rad}(G)$ consisting of all its unipotent elements is normal in G; we call it the *unipotent radical* of G.

An algebraic group G is *semisimple* (resp. *reductive*) if $\mathrm{Rad}(G)$ (resp. $\mathrm{Rad}_u(G)$) is trivial. The prototypical examples are the special linear group $\mathrm{SL}_{n,\mathbb{K}}$ which is semisimple and the general linear group $\mathrm{GL}_{n,\mathbb{K}}$ which is reductive.

We summarize the basic properties of connected reductive groups that we shall need as follows [Hum75, Sections 19.5 and 27.5]:

Basic Structure of Connected Reductive Groups. *If G^o is a connected reductive complex algebraic group, then the following holds:*

(1) The radical $\mathrm{Rad}(G^o) = Z(G^o)^o$ is an algebraic torus.
(2) The derived subgroup $[G^o, G^o]$ is connected, semisimple and normal in G^o.
(3) These two subgroups yield the decomposition

$$G^o = [G^o, G^o] \cdot Z(G^o)^o.$$

(4) The intersection $Z(G^o)^o \cap [G^o, G^o]$ is finite.

□

In lieu of a proof, we offer an example:

Example. Consider the complex general linear group $\mathrm{GL}_n \mathbb{C}$. It is easily seen to be connected while its centre $Z(\mathrm{GL}_n \mathbb{C})$ and radical $\mathrm{Rad}(\mathrm{GL}_n \mathbb{C})$ coincide with the algebraic subgroup of non-zero scalar multiples of the identity matrix. On the other hand, its derived subgroup $[\mathrm{GL}_n \mathbb{C}, \mathrm{GL}_n \mathbb{C}]$ coincides with the connected semisimple algebraic subgroup $\mathrm{SL}_n \mathbb{C}$. The intersection of these subgroups is therefore the centre of $\mathrm{SL}_n \mathbb{C}$ which can be described explicitly as the finite group of $n-$th root of unity scalar multiples of the identity matrix.

1.4. **Lie Groups and Lie Algebras.** A real Lie group G is a smooth real manifold endowed with the structure of a group for which the group operations are smooth mappings. If G is a Lie group and $g, h \in G$, we define the mappings

(1) $L_g : G \to G$, $L_g(h) := gh$
(2) $R_g : G \to G$, $R_g(h) := hg$

which are always diffeomorphisms. Since G is a smooth manifold, it has a well defined tangent space at the identity element $e \in G$ which we denote by the appropriate lower case gothic letter $\mathfrak{g}$. We can now define the *adjoint linear representation* of G on the vector space $\mathfrak{g}$, $Ad : G \to \mathrm{Aut}(\mathfrak{g})$ by the rule

$$Ad(g) = d_e(L_g \circ (R_g)^{-1}) : \mathfrak{g} \to \mathfrak{g}.$$

We can then consider the differential of the map Ad to obtain a new map:

$$ad : \mathfrak{g} \to End(\mathfrak{g}).$$

and define $[X, Y] := ad(X)(Y)$ to endow $\mathfrak{g}$ with the structure of a real Lie algebra. Similar definitions hold in the complex case.

Remark. This can be put to good use for algebraic groups:

(1) A real (resp. complex) algebraic group G is always a nonsingular variety. Consequently, G admits a unique real (resp. complex) analytic manifold structure and we may consider G as a real (resp. complex) Lie group of the same dimension. As such, we can define the tangent Lie algebra of any real or complex algebraic group G as the tangent algebra $\mathfrak{g}$ of the associated real or complex Lie group.

(2) One should be aware that, although a complex algebraic group is connected in the Zariski topology if and only if the corresponding complex Lie group is connected in the Euclidean topology, this no longer holds for real algebraic groups. Nevertheless, the corresponding real Lie group always has finitely many connected components in the Euclidean topology. The simplest occurrence of this phenomenon is perhaps the multiplicative group $\mathbb{R}^\times$ (compare with the multiplicative group $\mathbb{C}^\times$).

We now introduce a special class of Lie algebras which gives us additional control on the structure of their associated Lie groups. It will play a key rôle in Section 2.3 where we address the main result in this note.

Definition. A bilinear form $b(\cdot, \cdot)$ on a Lie algebra $\mathfrak{g}$ is said to be *invariant* if $b([x, y], z) + b(y, [x, z]) = 0$ for all $x, y, z \in \mathfrak{g}$. A Lie algebra $\mathfrak{g}$ is called *compact* if there exists a positive-definite and symmetric invariant bilinear form on $\mathfrak{g}$. Notice that any direct sum of compact Lie algebras is compact.

This definition is motivated by the fact that the Lie algebra of a compact Lie group is always compact and that every compact Lie algebra can be realized as the Lie algebra of a compact Lie group. However, one should keep in mind that a non-compact Lie group may very well have a compact Lie algebra. The simplest instance of this phenomenon is perhaps the real Lie group $\mathbb{R}$ or, more generally, the vector Lie group $\mathbb{R}^n$.

Since the main topic of this note concerns the interaction between complex groups and their real subgroups via realification and complexifications, we now introduce the analogous notion for Lie algebras:

Definition. To any Lie algebra $\mathfrak{k}$ over $\mathbb{R}$ there corresponds a Lie algebra $\mathfrak{k}^{\mathbb{C}} := \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$ over $\mathbb{C}$ via an operation called *complexification*. The complex vector space $\mathfrak{k}^{\mathbb{C}}$ consists of the symbols $X + iY$ where $X, Y \in \mathfrak{k}$ and its Lie algebra structure is given by the following operation:

$$[X + iY, Z + iT] = [X, Z] - [Y, T] + i([Y, Z] + [X, T]).$$

If a complex Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{k}^{\mathbb{C}} \cong \mathfrak{k} \oplus i\mathfrak{k}$ we say that $\mathfrak{k}$ is a *real form* of $\mathfrak{g}$. It turns out that, just as for real forms of varieties, the real forms of a complex Lie algebra $\mathfrak{g}$ are in one-to-one correspondence with involutive antilinear automorphisms of the underlying complex vector space up to conjugacy.

Example. Consider once again the $n-$dimensional algebraic torus $(\mathbb{C}^\times)^n$ as a complex Lie group, along with the two real forms previously shown:

(1) The Lie algebra $\mathbb{R}^n$ of the real form $(\mathbb{R}^\times)^n \subset (\mathbb{C}^\times)^n$ is a real form of the Lie algebra $\mathbb{C}^n$ of $(\mathbb{C}^\times)^n$.
(2) The Lie algebra $i\mathbb{R}^n$ of the (compact) real form $\mathbb{T}^n := \{(z_1, \ldots, z_n) : |z_j| = 1,\ 1 \leq j \leq n\} \subset (\mathbb{C}^\times)^n$ is a real form of the Lie algebra $\mathbb{C}^n$ of $(\mathbb{C}^\times)^n$.

To give the reader more of a sense for the preceding definitions we conclude our preliminaries with two striking result that will be used later on which can be found in [OV90, Sections 4.3.4, 5.1.3 and 5.1.4]:

Weyl's Theorem. *Every connected semisimple complex algebraic group G has a compact real form K. The Lie algebra $\mathfrak{k}$ of K is a compact real form of the Lie algebra $\mathfrak{g}$ of G.*

□

Knowing that real forms exist in a semisimple complex Lie algebra, one might wonder how they are related to each other. In fact, the following holds:

Conjugacy of Compact Real Forms. *If $\mathfrak{g}$ is the Lie algebra of a connected semisimple complex algebraic group G, e.g., if $\mathfrak{g}$ is a complex semisimple Lie algebra, then any two compact real forms of $\mathfrak{g}$ are conjugate. More precisely, if $\mathfrak{k}$ and $\mathfrak{k}'$ are two real forms of $\mathfrak{g}$, then there is some $g \in G$ for which $Ad(g)(\mathfrak{k}) = \mathfrak{k}'$.*

□

2. Compact Real Forms

We now enter the main core of this note. After introducing polar decompositions and structural results for Lie groups with compact Lie algebras we will embark on the proof of the main theorem and show that every complex reductive algebraic group has a compact real form. The material in this section is mostly based on [OV90].

2.1. **Polar Decompositions.** Let V denote a finite dimensional Hermitian vector space over $\mathbb{C}$. Recall that any matrix $g \in \mathrm{GL}(V)$ may be decomposed uniquely as a product $g = u \cdot p$ where $u \in \mathrm{U}(V)$ and $p \in \mathrm{P}(V)$, the spaces of unitary and positive-definite Hermitian matrices. In fact, $\mathrm{P}(V)$ coincides with the exponential image of the space of symmetric matrices $\mathrm{S}(V)$ and the map $\mathrm{U}(V) \times \mathrm{S}(V) \to \mathrm{GL}(V)$ given by sending

$$(\dagger) \qquad (u, s) \mapsto u \cdot \exp(s)$$

is a diffeomorphism. This is the usual *polar decomposition* in $\mathrm{GL}(V)$. Such decompositions exists in some amount of generality and turn out to be a valuable tool in the study of reductive algebraic groups.

The first case of interest is the setting of so-called *self-adjoint groups* which was explored by Mostow in [Mos55]. Essentially all variants of the polar decomposition are consequences of this version.

Mostow's Theorem. *Let V be a finite-dimensional Hermitian vector space and let $G \subset \mathrm{GL}(V)$ be a self-adjoint complex algebraic group (i.e. for every g in G the corresponding adjoint operator g^* is also in G). If $K := G \cap \mathrm{U}(V)$, $P := G \cap \mathrm{P}(V)$ and $\mathfrak{p} := \mathfrak{g} \cap \mathrm{S}(V)$, then the map*

$$K \times \mathfrak{p} \to G$$

given by ($\dagger$) is a diffeomorphism, e.g., $G = K \cdot P$.

Proof. Let us show that $G = K \cdot P$. If $g \in G$, then $g^*g \in P$ and by (what has now become) a typical exercise in undergraduate linear algebra we have that $\sqrt{(g^*g)} \in P$. Now,

$$g = g \cdot \sqrt{(g^*g)}^{(-1)} \cdot \sqrt{(g^*g)}$$

and $k := g \cdot \sqrt{(g^*g)}^{(-1)}$ satisfies $k^*k = Id$ so it is an element of K and we are done. $\square$

For our purposes, we will need to apply polar decompositions in the realm of complex reductive algebraic groups so we elaborate on this variant below. To do so, it is convenient to introduce the concept of a *topological real form* of a complex algebraic group G. This is a real Lie subgroup $K \subset G$ satisfying the following two properties:

(1) $G = K \cdot G^o$ (i.e. K intersects every component of G)
(2) $\mathfrak{k}^{\mathbb{C}} \cong \mathfrak{g}$.

Polar Decomposition. *Let V be a complex vector space and let $G \subset \mathrm{GL}(V)$ be a complex algebraic group. If $K \subset G$ is a compact topological real form of G, then the map*

$$K \times i\mathfrak{k} \to G^{\mathbb{R}}$$

defined by

$$(k, iy) \mapsto k \cdot \exp(iy)$$

is a diffeomorphism of real manifolds. Moreover, K is a compact real form of G.

Proof. Since K is compact, it is no loss of generality to assume that V is endowed with a $K-$invariant Hermitian inner-product. In this case, $\mathfrak{k}$ consists of skew-Hermitian operators and $\mathfrak{p} := i\mathfrak{k}$ consist of self-adjoint operators, e.g., $\mathfrak{p} = \mathfrak{g} \cap \mathrm{S}(V)$.

In order to apply Mostow's Theorem, we need to show that G is self-adjoint. To do this, let us first show that $\mathfrak{g}$ is self-adjoint. Suppose that $Z \in \mathfrak{g}$ and consider the real Lie group automorphism $S : \mathrm{GL}(V) \to \mathrm{GL}(V)$ given by $g \mapsto (g^*)^{-1}$ whose differential is given by $Z \mapsto -Z^*$. Since $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$ we may write $Z = X + iY$ where $X, Y \in \mathfrak{k}$ to see that

$$(X + iY) \mapsto -(X + iY)^* = -X^* - (iY)^* = -(-X) - (iY) = X - iY.$$

Here, we have used the facts that $\mathfrak{k}$ consists of skew-Hermitian operators and $\mathfrak{p} = i\mathfrak{k}$ consists of self-adjoint operators. Clearly, $X - iY \in \mathfrak{k} \oplus i\mathfrak{k}$ so $\mathfrak{g}$ is indeed self-adjoint and consequently $S(G^o) = G^o$, e.g., G^o is self-adjoint. Now $G = KG^o$ so any $g \in G$ may be written as $g = kh$ where $k \in K$ is unitary and $h \in G^o$. Therefore, $g^* = h^*k^* = h^*k^{-1} \in G$ since $h^* \in G^o$ and $k^{-1} \in K$ so G is also self-adjoint.

Applying Mostow's Theorem to G yields a diffeomorphism $\tilde{K} \times \mathfrak{p} \to G$ where $\tilde{K} := G \cap \mathrm{U}(V)$. At this point, we would like to claim that $K = \tilde{K}$. To see this, notice first that since K consists of unitary operators we have $K \subset G \cap \mathrm{U}(V) = \tilde{K}$ and $K^o = \tilde{K}^o$ since their Lie algebras coincide. On the other hand,

$$G = K \cdot G^o = K \cdot (\tilde{K}^o \cdot P) = K \cdot P$$

where the first equality follows because K is a topological real form of G, the second equality follows by Mostow's Theorem and the third equality follows because $K^o = \tilde{K}^o$. Now, $G = K \cdot P = \tilde{K} \cdot P$ so in fact $K \supset \tilde{K}$. This completes the proof of the first assertion in the statement of the theorem.

Finally, since G is self-adjoint, the real Lie group automorphism $S : \mathrm{GL}(V) \to \mathrm{GL}(V)$ restricts to an antiholomorphic automorphism of the algebraic group $G \subset \mathrm{GL}(V)$ whose fixed point set is $\tilde{K} = K$ so K is indeed a compact real form of G. □

Remark. Weyl's Theorem implies the existence of the Polar Decomposition described above for all connected semisimple algebraic groups.

Normalizers in Polar Decompositions. *If K and G are as above and $P :=$ exp$(i\mathfrak{k})$, then*

$$N_G(K) = K \times (Z(G) \cap P).$$

In particular, if G is semisimple then $Z(G) \subset K$ so in this case

$$N_G(K) = K.$$

Proof. Intersecting $N_G(K)$ with the Polar Decomposition, we have

$$N_G(K) = (N_G(K) \cap K) \cdot (N_G(K) \cap P) = K \cdot (N_G(K) \cap P)$$

where we claim that $N_G(K) \cap P = Z(G) \cap P$. Clearly, $N_G(K) \cap P \supset Z(G) \cap P$, so it suffices to establish the reverse inclusion. Let $h \in N_G(K) \cap P$, i.e., let $h \in P$ be such that $hKh^{-1} = K$. This means that for any $k \in K$ we have

$$(\ddagger) \qquad hkh^{-1} = \tilde{k} \in K.$$

On the other hand, for any $g \in G$ we have that $gPg^* = P$, in particular:

$$(\ddagger\,\ddagger) \qquad k^{-1}hk = k^*hk = \tilde{h} \in P.$$

Combining (‡) and (‡ ‡) we see that $\tilde{k}h = hk = k\tilde{h}$ so the uniqueness of the polar decomposition ensures that $h = \tilde{h}$ and $k = \tilde{k}$, i.e., h commutes with K. Finally, since $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}} = \mathfrak{k} \oplus i\mathfrak{k}$, this means that the adjoint action of $h \in N_G(K) \cap P$ on $\mathfrak{g}$ is trivial and consequently h is central in $P \subset G^o$. Since $G = K \cdot P$ and h commutes with both K and P, we conclude that $h \in Z(G)$. □

2.2. **Compact Lie Algebras.** Although Lie groups with a compact Lie algebra are not necessarily compact, they do have a nice structure theory that we explore below. To begin, let us revisit the simplest non-compact instance of a Lie group with a compact Lie algebra: $\mathbb{R}^n$. The reason why such a Lie group has a compact Lie algebra is that its Lie algebra is *abelian* so any positive-definite and symmetric bilinear form will satisfy the definition. As such, the first natural generalization of groups such as $\mathbb{R}^n$ in this setting consists of connected abelian Lie groups. Such a group T always splits as a direct product of a vector group and a compact torus $T \cong A \times B$ where $A \cong \mathbb{R}^p$ and $B \cong \mathbb{T}^q$. As it turns out, B is always the unique maximal compact subgroup of T. Since T is abelian, so is its Lie algebra $\mathfrak{t}$. Therefore, the ideal $\mathfrak{b} \subset \mathfrak{t}$ corresponding to B has a complement which we call $\mathfrak{a}$ resulting in the splitting $\mathfrak{t} = \mathfrak{a} \oplus \mathfrak{b}$. The subgroup $A \subset T$ may then be chosen to coincide with $\exp(\mathfrak{a})$. We summarize these facts as follows [OV90, Section 5.2.2]:

Decomposition of Connected Abelian Lie Groups. *Let T be a connected abelian Lie group. Then, its Lie algebra admits a splitting $\mathfrak{t} = \mathfrak{a} \oplus \mathfrak{b}$ where $B := \exp(\mathfrak{b})$ is the unique maximal compact subgroup of T and $A := \exp(\mathfrak{a})$ is a vector group. We call A (resp. B) the* non-compact *(resp.* compact*) parts of T.*

Example. Consider once again the algebraic torus $(\mathbb{C}^\times)^n$ but let us shift our perspective this time and view it as a real connected abelian Lie group T. The decomposition $\mathfrak{t} = \mathfrak{a} \oplus \mathfrak{b}$ as above corresponds to a splitting $\mathbb{C}^n = \mathbb{R}^n \oplus i\mathbb{R}^n$ resulting in the decomposition $(\mathbb{C}^\times)^n \cong (\mathbb{R}^+)^n \times \mathbb{T}^n$ where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle and $\mathbb{R}^+$ denotes the (multiplicative) Lie group of strictly positive real numbers.

More generally, the following structural result holds [OV90, Section 5.2.2]:

Structure of Lie Groups with Compact Lie Algebras. *Let G be a Lie group with finitely many connected components and suppose that its Lie algebra $\mathfrak{g}$ is compact. Then, there is a decomposition $Z(G^o)^o = A \times B$ of the connected abelian Lie group $Z(G^o)^o$ into a non-compact and compact part for which A is a normal subgroup of G. In this case, G admits a decomposition of the form*

$$G = A \rtimes K,\ G^o = A \times K^o$$

where K is a maximal compact subgroup of G for which $K^o = B \cdot [G^o, G^o]$.

□

2.3. **Complex Reductive Algebraic groups.** We are now ready to tackle the harder direction of the theorem mentioned in the introduction by generalizing Weyl's Theorem to the reductive case. Our proof is based on an argument found in [OV90, Section 5.2.3].

Theorem. *A complex reductive algebraic group has a compact real form.*

Proof. Given a complex reductive algebraic group G, we seek to produce a compact real form $K \subset G$. In order to do this, we will proceed in several steps. The idea guiding our approach will be to apply Weyl's Theorem to a connected semisimple subgroup $H \subset G$ and use the compact real form $C \subset H$ it provides as a building block for K.

Our first step is to find a suitable semisimple subgroup $H \subset G$. Recall from the Basic Structure of Connected Reductive Groups that $\mathrm{Rad}(G^o) = Z(G^o)^o$ and that we have a decomposition

$$G^o = [G^o, G^o] \cdot Z(G^o)^o \tag{1}$$

where $H := [G^o, G^o]$ is a connected semisimple algebraic subgroup of G. As such, by Weyl's Theorem, H has a compact real form C.

In order to build a compact real form of G from the compact real form $C \subset H$, we need to understand how C interacts with elements of G. From this perspective, a natural subgroup to investigate is the normalizer $N_G(C)$. Our first key observation for this subgroup is that the following variant of decomposition (1) holds:

$$G = [G^o, G^o] \cdot N_G(C) \text{ where } C \text{ is a compact real form of } H := [G^o, G^o]. \tag{2}$$

To see this, consider the adjoint action of G on the set of real forms in $\mathfrak{h}$ and recall from the Conjugacy of Compact Real forms that the action of $H \subset G$ on this set is transitive. Decomposition (2) follows at once since the stabilizer of the real form $\mathfrak{c} \subset \mathfrak{h}$ coincides with $N_G(C)$ and $G/\,\mathrm{Stab}(\mathfrak{l})$ acts freely on the orbit $G \cdot \mathfrak{l} = H \cdot \mathfrak{l}$.

Now that we have a better picture of how C sits as a subgroup in G via the normalizer $N_G(C)$, we ought to see how $N_G(C)$ relates back to C. Once again, our first observation will be about the identity component; it admits the following decomposition:

$$N_G(C)^o = Z(G^o)^o \cdot C. \tag{3}$$

Recall from Normalizers in Polar Decompositions for H with respect to C that $N_H(C) = C$ and, consequently:

$$N_G(C) \cap G^o = N_G(C) \cap (Z(G^o)^o \cdot H) = Z(G^o)^o \cdot C.$$

Since $Z(G^o)^o$ is connected and C is connected (it is homotopy equivalent to H by the polar decomposition), we conclude that $N_G(C) \cap G^o$ is connected and therefore $N_G(C)^o = N_G(C) \cap G^o$.

Having found a decomposition of the identity component $N_G(C)^o$ into simpler groups, we can use it to decompose its Lie algebra. In fact, this decomposition leads us to the key feature of the normalizer that will allow us to complete our analysis:

(4) the Lie group $N_G(C)$ has a compact Lie algebra.

To see how this follows from (3), consider the induced surjective group homomorphism

$$\varphi : Z(G^o)^o \times C \to N_G(C)^o,\ \varphi(z, c) := z \cdot c.$$

Since $\ker(\varphi) = Z(G^o)^o \cap C \subset Z(G^o)^o \cap H$ and this intersection is finite by the Basic Structure of Connected Reductive Groups, $\ker(\varphi)$ is discrete and it follows that $Z(G^o)^o \times C$ is a covering space of $N_G(C)^o$. As such, the Lie algebra of $N_G(C)$ is isomorphic to $\mathfrak{t} \oplus \mathfrak{c}$ where $\mathfrak{t}$ is the (abelian) Lie algebra of the algebraic torus $Z(G^o)^o$ and $\mathfrak{c}$ is the Lie algebra of C. Finally, (4) is established since $\mathfrak{t}$ and $\mathfrak{c}$ are compact and the direct sum of compact Lie algebras is compact.

Now that we know that $N_G(C)$ has a compact Lie algebra, we seek to apply the Structure of Lie Groups with Compact Lie Algebras. To do this, we first need to show that $N_G(C)$ has finitely many connected components. This turns out to be a consequence of the fact that, in any algebraic group G, the identity component G^o is a normal subgroup of finite index. Indeed, since $[G^o, G^o] \subset G^o$, it follows from decomposition (2) that $G = N_G(C) \cdot G^o$ and by the Second Isomorphism Theorem for groups:

$$G/G^o \cong (N_G(C) \cdot G^o)/G^o \cong N_G(C)/(N_G(C) \cap G^o) \cong N_G(C)/N_G(C)^o.$$

At last, we can apply the Structure of Lie Groups with Compact Lie Algebras. Namely, if we consider the splitting of the connected abelian Lie group $Z(N_G(C)^o)^o$ into its non-compact and compact parts $Z(N_G(C)^o)^o \cong A \times B$, we obtain the following decomposition:

$$N_G(C) = A \rtimes K \tag{5}$$

where K is a compact subgroup of $N_G(C)$ for which $K^o = B \cdot [C, C] = B \cdot C$.

At this point, we would like to claim that K is a compact real form of G. To see why this is the case, recall from the Polar Decomposition that we need only check that K is a topological real form of G, e.g., we need to check that the following two conditions hold:

$$(a)\ G = G^o K \quad \text{and} \quad (b)\ \mathfrak{g} \cong \mathfrak{k}^{\mathbb{C}}.$$

Now, condition (a) follows easily from decomposition (2). Indeed,

$$G = [G^o, G^o] \cdot N_G(C) = [G^o, G^o] \cdot (A \rtimes K)$$

so any $g \in G$ may be written as a product $g = h \cdot a \cdot k$ where $h \in [G^o, G^o]$, $a \in A$ and $k \in K$. Since $[G^o, G^o] \subset G^o$ and $A \subset Z(N_G(C)^o)^o \subset G^o$, we have that $h \cdot a \in G^o$ as claimed.

In order to verify condition (b), it is convenient to reinterpret the connected abelian Lie group used to obtain (5) by observing that:

$$Z(G^o)^o = Z(N_G(C)^o)^o. \tag{6}$$

To prove this equality, we proceed in two steps. First, recall from (3) that $N_G(C)^o = Z(G^o)^o \cdot C$ so we immediately obtain that $Z(G^o)^o \subset Z(N_G(C)^o)^o$. For the reverse inclusion, note that if $g \in Z(N_G(C)^o)$ then g commutes with C and therefore g commutes with the Zariski closure of C which is $H = [G^o, G^o]$. Moreover, since $N_G(C)^o \subset G^o$, such a g also commutes with $Z(G^o)^o$. Finally since $G^o = [G^o, G^o] \cdot Z(G^o)^o$ by (1) and since g commutes with both factors, g also commutes with G^o and thus $Z(G^o) \supset Z(N_G(C)^o)$.

We can now determine the Lie algebras of K and G in a suitable form to verify that $\mathfrak{k}^{\mathbb{C}} \cong \mathfrak{g}$. Let us start with G; consider the surjective group homomorphism induced from (1):

$$\psi : [G^o, G^o] \times Z(G^o)^o \to G^o,\ \psi(g, z) := g \cdot z.$$

Since $\ker(\psi) = [G^o, G^o] \cap Z(G^o)^o$ and this intersection is finite by the Basic Structure of Connected Reductive Groups, $\ker(\varphi)$ is discrete and it follows that $[G^o, G^o] \times Z(G^o)^o$ is a covering space of G^o. As such $\mathfrak{g} \cong \mathfrak{h} \oplus \mathfrak{t}$ where $\mathfrak{h}$ is the Lie algebra of $H = [G^o, G^o]$ and $\mathfrak{t}$ is the Lie algebra of $Z(G^o)^o$.

On the other hand, as a consequence of the Structure of Lie Groups with Compact Lie Algebras in (5) we saw that $K^o = C \cdot B$. Here, (6) ensures that $B \subset Z(G^o)^o$ so that $C \cap B \subset H \cap Z(G^o)^o$ and the intersection is finite. Proceeding with the same covering space argument as above, we obtain a splitting $\mathfrak{k} \cong \mathfrak{c} \oplus \mathfrak{b}$ where $\mathfrak{b}$ denotes the Lie algebra of B. Now, by assumption, $\mathfrak{c}^{\mathbb{C}} \cong \mathfrak{h}$ so it suffices to show that $\mathfrak{b}^{\mathbb{C}} \cong \mathfrak{t}$.

Recall from (6) that $Z(G^o)^o = Z(N_G(C)^o)^o$ so that the latter is a connected algebraic torus isomorphic to $T = (\mathbb{C}^\times)^n$ for some n. As we have indicated in our sequence of examples for algebraic tori, the natural decomposition of this torus into a non-compact and compact parts is of the form $T \cong (\mathbb{R}^+)^n \times \mathbb{T}^n = A' \times B$ in which case $\mathfrak{t} \cong i\mathfrak{b} \oplus \mathfrak{b}$, e.g., $\mathfrak{t} \cong \mathfrak{b}^{\mathbb{C}}$. This completes the proof. □

To conclude, let us mention that once we know a complex reductive algebraic group G has a compact real form, the argument in the proof of the Polar Decomposition shows that given a realization $G \subset \mathrm{GL}(V)$ where V is a finite dimensional complex vector space, a positive definite Hermitian form may be introduced on V relative to which G is self-adjoint. Since a self-adjoint family of linear transformations is easily seen to be completely reducible this indicates one path to showing that every finite dimensional representation of a complex reductive algebraic group is completely reducible. On the other hand, it is also true that a complex algebraic group for which every finite dimensional representation is completely reducible is necessarily reductive [OV90]. It is the author's understanding that the first proof of the existence of a compact real form for complex reductive algebraic groups was obtained by Mostow in [Mos55] where he adopts this dual point of view. However, his main goal appears to have been to prove that if G is any algebraic group of linear transformations on a real or complex vector space V for which every finite dimensional representation is completely reducible, then a positive definite Hermitian form may be introduced on V relative to which G is self-adjoint. In closing, let us mention that the following very nice theorem also holds [OV90, Theorem 5.2.12]:

Characterization Theorem for Complex Reductive Algebraic Groups. *On any compact Lie group K there exists a unique real algebraic group structure and the complexification $K(\mathbb{C})$ is a complex reductive algebraic group. Any complex reductive algebraic group possesses an algebraic compact real form. Two compact Lie groups are isomorphic (as Lie groups or as algebraic groups over $\mathbb{R}$) if and only if the corresponding reductive algebraic groups over $\mathbb{C}$ are isomorphic.*

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