STABLE AFFINE MODELS FOR ALGEBRAIC GROUP ACTIONS

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Abstract. Let G be a reductive linear algebraic group defined over an algebraically closed base field k of characteristic zero. A G-variety is an algebraic variety with a regular action of G, defined over k. An affine G-variety is called stable if its points in general position have closed G-orbits. We give a simple necessary and sufficient condition for a G-variety to have a stable affine birational model.

1. Introduction

Let G be a linear algebraic group, defined over an algebraically closed base field k of characteristic zero. We shall refer to a reduced but not necessarily irreducible algebraic variety X (defined over k), with a regular action of G (also defined over k) as a G-variety. By a morphism $X \longrightarrow Y$ of G-varieties, we shall mean a G-equivariant morphism. The notions of isomorphism, rational map, birational isomorphism, etc. of G-varieties are defined in a similar manner. As usual, given a G-action on X, we shall denote the orbit of $x \in X$ by Gx and the stabilizer subgroup of x by $G_x \subseteq G$. Finally, we shall say that a property holds for $x \in X$ in general position if it holds for every point x of some dense open subset of X.

In this note we will be interested in studying G-varieties up to birational isomorphism. In this context it is natural to ask whether or not a given G-variety X has an affine model. Indeed, there are numerous results and constructions in invariant theory that are available for affine G-varieties but not in general, especially if G is reductive; cf. [PV].

Recall that an affine G-variety X is called *stable*, if the orbit Gx is closed for $x \in X$ in general position. If G is reductive, these varieties have many nice properties; for a summary, see, e.g., [V, Section 8]. The question we will address in this note is: Which G-varieties have a stable affine birational model? Our main result is the following:

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Theorem 1. *Let G be a reductive linear algebraic group and X be a G-variety. Then the following are equivalent:*

(a) *X is birationally equivalent to a stable affine G-variety.*

(b) *The stabilizer G_x is reductive for x in general position in X.*

In the case where $X = G/H$ is a homogeneous space, Theorem 1 reduces to a theorem of Matsushima [Ma] which says that G/H is affine if and only if H is reductive. Moreover, the implication (a) $\Longrightarrow$ (b) of Theorem 1 is an immediate consequence of Matsushima's theorem. Indeed, after replacing X by a stable affine model, we see that for $x \in X$ in general position the orbit $Gx \simeq G/G_x$ is affine, so that G_x is reductive.

Our proof of the implication (b) $\Longrightarrow$ (a) will be based on the following more general result:

Theorem 2. *Let G be a linear algebraic group and X a G-variety. Denote by G_x the stabilizer of $x \in X$ in G. Assume that either*

(i) *$G_x = \{1\}$ for $x \in X$ in general position (i.e., the G-action on X is generically free), or*

(ii) *the normalizer $\mathrm{N}_G(G_x)$ is reductive for $x \in X$ in general position.*

Then X is birationally isomorphic to a stable affine G-variety.

Note that if G and G_x are both reductive then so is the normalizer $\mathrm{N}_G(G_x)$; see [LR, Lemma 1]. Thus Theorem 2(ii) proves the implication (b) $\Longrightarrow$ (a) of Theorem 1.

The rest of this note will be devoted to proving Theorem 2. Our proof of part (ii) will be based on part (i) and a theorem of Richardson [Ri, Theorem 9.3.1] about the existence of stabilizers in general position.

We remark that the theorems of Matsushima and Richardson mentioned above were originally proved only for $k = \mathbb{C}$ (by analytic methods). An algebraic proof of Matsushima's theorem over an algebraically closed field k of characteristic zero can be found in [L, Section 2]. Richardson's theorem is also valid over such k by the Lefschetz principle; it is stated in this form in [PV, Theorem 7.1]. Nevertheless, it would be interesting to find a direct algebraic proof.

2. Proof of Theorem 2(i)

We begin with a simple lemma.

Lemma 3. *Every linear algebraic group G has a stable generically free linear representation.*

Proof. After embedding G as a closed subgroup in SL_n for some $n \geq 1$, we may assume that $G = \mathrm{SL}_n$. The action of SL_n on $\mathrm{M}_n(k)$ by left multiplication is easily seen to be generically free and stable. □

We are now ready to proceed with the proof of Theorem 2(i). Recall that a G-variety is called *primitive* if G transitively permutes the irreducible components of X. It is easy to see that every X is birationally isomorphic to a disjoint union of primitive G-varieties; cf. [Re, Lemma 2.2]. Hence we may assume that X is primitive.

By a theorem of Rosenlicht there exists a rational quotient map

$$\pi_{\text{rat}} \colon X \dashrightarrow Z\,,$$

separating the G-orbits in general position in X; see [Ro_1] (for the case where X is irreducible) and [Ro_2] (for general X). Here Z is only defined up to birational isomorphism, so we may assume without loss of generality that Z is affine. After replacing X by a dense open G-invariant subset, we may assume that $G_x = \{1\}$ for every $x \in X$, and that π_{rat} is regular and separates the G-orbits in X. Since X is primitive, Z is irreducible.

By Lemma 3 there exists a stable generically free linear representation V of G. Let V_0 be a G-stable dense open subset of V such that every point $v \in V_0$ has a closed orbit (in V) and trivial stabilizer. By [Re, Proposition 7.1] there is a G-equivariant rational map $f \colon X \dashrightarrow V$ whose image contains a point $v \in V_0$. Let Y be the closure of the image of $f \times \pi_{\text{rat}} \colon X \dashrightarrow V \times Z$. Note that Y is G-primitive and affine. Moreover, $U = Y \cap (V_0 \times Z)$ is a G-invariant non-empty (and hence, dense) open subset of Y, and every point of U has a trivial stabilizer in G and a closed G-orbit in $V \times Z$. Thus Y is a stable affine generically free G-variety.

It remains to show that $f \times \pi_{\text{rat}}$ is a birational isomorphism between X and Y. Since we are working in characteristic zero, since X is primitive, and since $f \times \pi_{\text{rat}} \colon X \dashrightarrow Y$ is dominant, it suffices to check that $f \times \pi_{\text{rat}}$ is injective on a dense open subset of X. Indeed, let $W = (f \times \pi_{\text{rat}})^{-1}(U)$. Then W is a G-stable nonempty (and thus dense) open subset of X. Now assume that $y = (f \times \pi_{\text{rat}})(x_1) = (f \times \pi_{\text{rat}})(x_2)$ for some $x_1, x_2 \in W$. Since π_{rat} separates the orbits in X, $x_2 = g(x_1)$ for some $g \in G$. But then $g \in G_y = \{1\}$. We conclude that $x_1 = x_2$. □

3. Proof of Theorem 2(ii)

We begin with several preliminary reductions. First note that if $\mathrm{N}_G(G_x)$ is reductive then G_x itself must be reductive. Indeed, the unipotent radical $R_u(G_x)$ is trivial, because it is a normal unipotent subgroup of $\mathrm{N}_G(G_x)$.

Secondly, we may assume, as we did in the previous section, that X is primitive, i.e., G transitively permutes the irreducible components of X.

Thirdly, by a theorem of Richardson (see [Ri, Theorem 9.3.1] or [PV, Theorem 7.1]), we may assume that X has a stabilizer $S \subseteq G$ in general position. In other words, after replacing X by a G-invariant dense open subset, we may assume that G_x is conjugate to S for every $x \in X$. As we remarked above, S is reductive. Set $N = \mathrm{N}_G(S)$, and denote by X^S the set of S-fixed points in X. By comparing stabilizers, we see that

$$GX^S = X \tag{4}$$

and

$$\text{if } gx_1 = x_2 \text{ for some } x_1, x_2 \in X^S \text{ and } g \in G, \text{ then } g \in N. \tag{5}$$

Now let Y be the union of irreducible components of X^S of maximal dimension. Since S acts trivially on Y, we can think of Y as an N/S-variety. By our assumption G_x is conjugate to S for every $x \in X$. In particular, $G_x = S$ for every $x \in X^S$. Hence, the N/S-action on Y is generically free and, by Theorem 2(i), there is a stable affine N/S-variety Z, birationally equivalent to Y.

Our goal is to show that X is birationally isomorphic to the G-variety $X' = G *_N Z$. The remainder of the proof will amount to checking that X' is affine and stable and constructing a birational isomorphism between X and X'. Some of our arguments are closely related to those in [P, 1.7]; however, for the sake of completeness (and because we are assuming that X is primitive but not necessarily irreducible), our proof will be self-contained.

First we observe that

$$GY \text{ is dense in } X. \tag{6}$$

Indeed, consider the map $f\colon G \times X^S \longrightarrow X$ given by $(g, x) \longrightarrow gx$. By (4), f is surjective. By (5), the fibers of f are precisely the N-orbits in $G \times X^S$, where N acts by $n \cdot (g, x) \longrightarrow (gn^{-1}, nx)$. Since this action is free (i.e., the stabilizer of every point is trivial), every fiber has the same dimension $\dim N$, and (6) follows from the fiber dimension theorem.

Next we recall the definition of the G-variety $G *_N Z$; cf., e.g., [PV, Section 4.8]. Consider the action of $G \times N$ on the affine variety $G \times Z$ given by

$$(g, n) \cdot (h, z) \mapsto (ghn^{-1}, nz)\,. \tag{7}$$

The variety $X' = G *_N Z$ is, by definition, the categorical quotient of $G \times Z$ for the N-action given by the above formula (where we identify N with the subgroup $\{1\} \times N$ of $G \times N$). In particular, X' is an affine variety. For future reference, we denote the categorical quotient map for the action of N by

$$\pi_{\text{cat}}\colon G \times Z \longrightarrow X'\,.$$

Since the actions of G and N on $G \times Z$ commute, the G-action on $G \times Z$ descends to a G-action on X', thus giving X' the structure of a G-variety. Theorem 2(ii) is now a consequence of the following:

Lemma 8. (a) *The $G \times N$-action on $G \times Z$, given by* (7) *is stable.*

(b) *The G-action on X' is stable.*

(c) *Every N-orbit in $G \times Z$ is closed. Here we identify N with the subgroup $\{1\} \times N$ of $G \times N$, and the N-action on $G \times Z$ is given by* (7).

(d) *X and X' are birationally isomorphic as G-varieties.*

Proof. (a) The $G \times N$-orbit of $(g, z) \in G \times Z$ is $G \times (Nz)$. Since the N-action on Z is stable, this orbit is closed for z in general position in Z.

(b) G-orbits in X' are images, under $\pi_{\rm cat}$, of $G \times N$-orbits in $G \times Z$. The desired conclusion now follows from part (b) and the fact that $\pi_{\rm cat}$ maps N-invariant closed sets in $G \times Z$ to closed sets in X'; cf. [PV, p. 188, Corollary].

(c) Assume the contrary: there is a non-closed N-orbit in $G \times Z$. Then the closure of this orbit contains an orbit of lower dimension. On the other hand, it is easy to see that the stabilizer $N_{(g,z)} = \{1\}$ for every $(g, z) \in G \times Z$. Consequently, every orbit has dimension dim N, a contradiction.

(d) Let $\phi\colon Z \overset{\simeq}{\dashrightarrow} Y$ be a birational isomorphism between the N/S-varieties Z and Y. Define a G-equivariant rational map $\Psi\colon G \times Z \dashrightarrow X$ by $\Psi(g, z) \longrightarrow g\phi(z)$. The N-action on $G \times Z$ is stable by part (c); hence, the categorical quotient map $\pi_{\rm cat}\colon G \times Z \longrightarrow X'$ separates closed orbits in $G \times Z$; see, e.g., [PV, p. 189, Corollary]. This implies that $\pi_{\rm cat}$ is the rational quotient map for the N-variety $G \times Z$ (see, e.g, [Re, Remark 2.5]). Since Ψ sends N-orbits in $G \times Z$ to points in X, the universal property of rational quotients of N-varieties (see e.g., [Re, Remark 2.4]) says that Ψ descends to a rational map $\psi\colon X' \dashrightarrow X$ of G-varieties. We claim that ψ is a birational isomorphism.

To prove the claim, first observe that Ψ (and hence, ψ) is dominant by (6). Secondly, since the irreducible components of Y have the same dimension and the N-action on $G \times Z$ is stable and free (i.e., the stabilizer $N_{(g,z)} = \{1\}$ for every $(g, z) \in G \times Z$), we conclude that the irreducible components of $G *_N Z$ are also of the same dimension (namely, of dimension, dim G + dim $Y -$ dim N). Thus in order to show that ϕ is a birational isomorphism, we only need to check that ψ is generically one-to-one. More precisely, we will show that if z_1 and z_2 belong to a dense open subset of Z on which ϕ is defined and one-to-one, and if $\Psi(g_1, z_1) = \Psi(g_2, z_2)$, then (g_1, z_1) and (g_2, z_2) lie in the same N-orbit in $G \times Z$.

Indeed, $\Psi(g_1, z_1) = \Psi(g_2, z_2)$ can, by definition, be rewritten as $\phi(z_1) = g_1^{-1} g_2 \phi(z_2)$. By (5), $g_1^{-1} g_2 \in N$. Setting $n = g_1^{-1} g_2$, we see that $(g_1, z_1) = (g_2 n^{-1}, n z_2)$, so that (g_1, z_1) and (g_2, z_2) are, indeed, in the same N-orbit. This completes the proof of Lemma 8 and thus of Theorem 2(ii). □

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