FIELDS OF DEFINITION FOR REPRESENTATIONS OF ASSOCIATIVE ALGEBRAS

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ABSTRACT. We examine situations, where representations of a finite-dimensional F-algebra A defined over a separable extension field K/F, have a unique minimal field of definition. Here the base field F is assumed to be a field of dimension $\leqslant 1$. In particular, F could be a finite field or $k(t)$ or $k((t))$, where k is algebraically closed.

We show that a unique minimal field of definition exists if (a) K/F is an algebraic extension or (b) A is of finite representation type. Moreover, in these situations the minimal field of definition is a finite extension of F. This is not the case if A is of infinite representation type or F fails to be of dimension $\leqslant 1$. As a consequence, we compute the essential dimension of the functor of representations of a finite group, generalizing a theorem of N. Karpenko, J. Pevtsova and the second author.

1. INTRODUCTION

Notational conventions. Throughout this paper F will denote a base field and A a finite-dimensional associative algebra over F. If K/F is a field extension (not necessarily algebraic), we will denote the tensor product $K \otimes_F A$ by A_K. Let M be an A_K-module. Unless otherwise specified, we will always assume that M is finitely generated (or equivalently, finite-dimensional as a K-vector space). If L/K is a field extension, we will write M_L for $L \otimes_K M$.

An intermediate field $F \subset K_0 \subset K$ is called a *field of definition* for M if there exists a K_0-module M_0 such that $M \cong (M_0)_K$. In this case we will also say that M *descends* to K_0.

Minimal fields of definition. A field of definition K_0 of M is said to be *minimal* if whenever M descends to a field L with $F \subset L \subset K$, we have $K_0 \subset L$.

Minimal fields of definition do not always exist. For example, let $F = \mathbb{Q}$ and A be the quaternion algebra

$$A = \mathbb{Q}\{i, j, k\}/(i^2 = j^2 = k^2 = ijk = -1).$$

Then A_K has a two dimensional module M given by

$$i \mapsto \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \qquad j \mapsto \begin{pmatrix} b & -a \\ -a & -b \end{pmatrix}, \qquad k \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

2010 *Mathematics Subject Classification.* 16G10, 16G60, 20C05.

Key words and phrases. Modular representation, field of definition, finite representation type, essential dimension.

The research of the first author was supported by the Collaborative Research Group in Geometric and Cohomological Methods in Algebra at the Pacific Institute for the Mathematical Sciences, Vancouver, Canada (2016).

The second author was partially supported by NSERC Discovery Grant 250217-2012.

over any field K of characteristic 0 having two elements a and b such that $a^2 + b^2 = -1$. Examples of such fields include $\mathbb{C}$, $\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-5})$. If we take K to be "the generic field" of this type, i.e., the field of fractions of $\mathbb{Q}[a, b]/(a^2+b^2+1)$, then M has no minimal field of definition; see Proposition 6.3(b).

Fields of dimension $\leqslant 1$. Such examples arise because of the existence of noncommutative finite-dimensional division algebras over F. So, it makes sense to develop a theory over those fields F over which these division algebras do not exist. More precisely, we require that

$$\mathrm{Br}(E) = 0 \text{ for every algebraic field extension } E/F, \tag{1.1}$$

where $\mathrm{Br}(E)$ denotes the Brauer group of E. This class of fields was studied in detail by J.-P. Serre in connection with his celebrated Conjecture I; see [Se, §II.3]. Serre referred to fields satisfying (1.1) as "fields of dimension $\leqslant 1$". If F is perfect, this condition is equivalent to the cohomological dimension of the absolute Galois group $\mathrm{Gal}(F)$ being $\leqslant 1$; see [Se, Proposition II.3.1.6]. In particular, this condition is satisfied by all finite fields, all algebraically closed fields and all field extensions of transcendence degree 1 over an algebraically closed field. For proofs of these assertions and further examples, see [Se, §II.3.3].

Our first main result is as follows.

Theorem 1.2. *Let F be a field satisfying* (1.1)*, A be a finite-dimensional F-algebra, K/F be a separable algebraic field extension and M be an an A_K-module. Then M has a minimal field of definition $F \subset K_0 \subset K$ such that $[K_0 : F] < \infty$.*

To illustrate Theorem 1.2, let us consider a simple case, where $\mathrm{char}(F) = 0$, $A := FG$ is the group algebra of a finite group G, and M is absolutely irreducible KG-module. Denote the character of G associated to M by $\chi\colon G \to K$. We claim that in this case the minimal field of definition is $F(\chi)$, the field generated over F by the character values $\chi(g)$, as g ranges over G. Indeed, it is clear that $F(\chi)$ has to be contained in any field of definition $F \subset K_0 \subset K$ of M. Thus to prove the above assertion, we only need to show that M descends to $F(\chi)$. The minimal degree of a finite field extension $L/F(\chi)$, such that M is defined over L (i.e., there exists an LG-module with character χ), is the Schur index s_M; cf. [CR$_1$, Definition 41.4]. Thus it suffices to show that $s_M = 1$. By [CR$_1$, Theorem (70.15)], s_M is the index of the endomorphism algebra $\mathrm{End}_A(M)$ of M, which is a central simple algebra over $F(\chi)$. Since F satisfies condition (1.1) and $[F(\chi) : F] < \infty$, the index of every central simple algebra over $F(\chi)$ is 1. In particular, $s_M = 1$, and M descends to $F(\chi)$, as claimed.

Algebras of finite representation type. A finite-dimensional F-algebra A is said to be *of finite representation type* if there are only finitely many indecomposable finitely generated A-modules (up to isomorphism).

Our next result shows that for algebras of finite representation type Theorem 1.2 remains valid even if the field extension K/F is not assumed to be algebraic.

Theorem 1.3. *Let F be a field satisfying* (1.1)*, A be a finite-dimensional F-algebra of finite representation type, K/F be a field extension, and M be an A_K-module. Assume* *further that F is perfectly closed in K. Then M has a minimal field of definition $F \subset K_0 \subset K$ such that $[K_0 : F] < \infty$.*

Essential dimension. Given the A_K-module M, the *essential dimension* $\mathrm{ed}(M)$ of M over F is defined as the minimal value of the transcendence degree $\mathrm{trdeg}(K_0/F)$, where the minimum is taken over all fields of definition $F \subset K_0 \subset K$. The integer $\mathrm{ed}(M)$ may be viewed as a measure of the complexity of M. Note that $\mathrm{ed}(M)$ is well-defined, irrespective of whether M has a minimal field of definition or not. We also remark that this number implicitly depends on the base field F, which is assumed to be fixed throughout. As a consequence of Theorem 1.3, we will deduce the following.

Theorem 1.4. *Let F be a field satisfying* (1.1)*, A be finite-dimensional F-algebra of finite representation type, K/F be a field extension, and M be an A_K-module. Then* $\mathrm{ed}(M) = 0$.

Both Theorem 1.3 and 1.4 fail if we do not require F to satisfy (1.1); see Section 6.

The essential dimension of the functor of A-modules. We will also be interested in the essential dimension $\mathrm{ed}(\mathrm{Mod}_A)$ of the functor Mod_A from the category of field extensions of F to the category of sets, which associates to a field K, the set of isomorphism classes of A_K-modules. By definition,

$$\mathrm{ed}(\mathrm{Mod}_A) := \sup\ \mathrm{ed}(M)\,,$$

where the supremum is taken over all field extensions K/F and all finitely generated A_K-modules M. The value of $\mathrm{ed}(\mathrm{Mod}_A)$ may be viewed as a measure the complexity of the representation theory of A. For generalities on the notion of essential dimension we refer the reader to [BF, Re_1, Re_2, Me_1, Me_2]. Essential dimensions of representations of finite groups and finite-dimensional algebras are studied in [KRP] and [BDH, Section 3].

Note that while $\mathrm{ed}(M) < \infty$, for any given A_K-module M (see Lemma 2.1), $\mathrm{ed}(\mathrm{Mod}_A)$ may be infinite. In particular, in the case, where $A = FG$ is the group algebra of a finite group G over a field F, it is shown in [KRP, Theorem 14.1] that $\mathrm{ed}(\mathrm{Mod}_A) = \infty$, provided that F is a field of characteristic $p > 0$ and G has a subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$. Our final main result is the following amplification of [KRP, Theorem 14.1].

Theorem 1.5. *Let G be a finite group and F be a field of characteristic p. Then the following conditions are equivalent:*

(1) *The p-Sylow subgroup of G is cyclic,*
(2) $\mathrm{ed}(\mathrm{Mod}_{FG}) = 0$*,*
(3) $\mathrm{ed}(\mathrm{Mod}_{FG}) < \infty$*.*

Note that by a theorem of D. Higman [Hi], the condition that the p-Sylow subgroup of G is cyclic is equivalent to the group algebra FG being of finite representation type.

2. Preliminaries on fields of definition

Lemma 2.1. *Let A be a finite-dimensional F-algebra, K/F be a field extension and M be an A_K-module. Then M descends to an intermediate subfied $F \subset E \subset K$, where E/F is finitely generated.*

Proof. Suppose $a_1, \dots, a_r$ generate A as an F-algebra. Choose an F-vector space basis for M. Then the A-module structure of M is completely determined by the matrices representing multiplication by $a_1, \dots, a_r$ in this basis. Each of these matrices has n^2 entries in K, where $n = \dim_F(M)$. Let $E \subset K$ be the field extension of F obtained by adjoining these these rn^2 entries to F. Then M descends to E. □

Next we recall the classical theorem of Noether and Deuring. For a proof, see [CR_1, (29.7)] or [BP, Lemma 5.1].

Theorem 2.2. (Noether-Deuring Theorem) *Let K/E be a field extension, A be a finite-dimensional E-algebra, and M, M' be A-modules. If $M_K = K \otimes_E M$ and $M'_K = K \otimes_E M'$ are isomorphic as A_K-modules, then M and M' are isomorphic as A-modules.* □

Lemma 2.3. *Let F be a field, A be a finite-dimensional F-algebra, $F \subset E \subset K$ be field extensions, N be A_E-module, and $F \subset E_0 \subset E$ be an intermediate field. Then*

(a) N_K descends to E_0 if and only if N descends to E_0.

(b) If $F \subset E_{min} \subset K$ is a minimal field of definition for N_K, then E_{min} is a minimal field of definition for N.

Proof. (a) If N descends to E_0, then clearly so does N_K. Conversely, suppose N_K descends to E_0. That is, there exists a E_0-module M such that $K \otimes_{E_0} M \simeq N_K$ as an A_K-module. The A_E-modules $M_E = E \otimes_{E_0} M$ and N become isomorphic to $M_K = N_K$ over K. By Theorem 2.2, $M_E \simeq N$ as A_E-modules. Thus N descends to E_0, as desired.

(b) Clearly E is a field of definition for N_K. Hence, by definition of E_{min}, $E_{min} \subset E$. On the other hand, by part (a), E_{min} is a field of definition for N, and part (b) follows. □

We finally come to the main result of this section.

Proposition 2.4. *Suppose F is a field satisfying* (1.1)*, A is a finite-dimensional F-algebra, K/F is a field extension, M is an indecomposable A_K-module, and $F \subset K_0 \subset K$ is an intermediate field, such that $[K_0 : F] < \infty$.*

If M^n is defined over K_0 for some positive integer n, then so is M.

Proof. Set $\mathrm{End}^{ss}_{A_K}(M)$ to be the quotient of $\mathrm{End}_{A_K}(M)$ by its Jacobson radical. By our assumption $M^n \simeq K \otimes_{K_0} N$ for some A_{K_0}-module N. By Fitting's Lemma,

$$\mathrm{End}^{ss}_{A_K}(M^n) \simeq \mathrm{M}_n(D),$$

where D is a finite-dimensional division algebra over some field extension K' of K, where $[K' : K] < \infty$. On the other hand,

$$\mathrm{M}_n(D) \simeq \mathrm{End}^{ss}_{A_K}(M^n) \simeq \mathrm{End}^{ss}_{A_K}(K \otimes_{K_0} N) \simeq K \otimes_{K_0} \mathrm{End}^{ss}_{A_{K_0}}(N)\,. \tag{2.5}$$

We conclude that $\mathrm{End}^{ss}_{A_{K_0}}(N)$ is a simple algebra over K_0, i.e.,

$$\mathrm{End}^{ss}_{A_{K_0}}(N) \simeq M_m(D_0) \tag{2.6}$$

over K_0, for some integer $m \geqslant 0$ and some finite-dimensional central division algebra D_0 over a field K'_0 such that K'_0/K_0 is a field extension of finite degree. Now recall that we are assuming that F satisfies (1.1) and

$$F \subset K_0 \subset K'_0$$

are field extensions of finite degree. Hence, every finite-dimensional division algebra over K_0' is commutative. In particular, $D_0 = K_0'$, is a field, and

$$\mathrm{M}_n(D) \simeq K \otimes_{K_0} \mathrm{End}^{ss}_{A_{K_0}}(N) \simeq K \otimes_{K_0} \mathrm{M}_m(K_0') \,.$$

Since $\mathrm{M}_n(D)$ is a simple algebra, we conclude that $K \otimes_{K_0} K_0'$ is a field. Moreover, the index of $\mathrm{M}_m(K \otimes_{K_0} K_0')$ is 1; hence, $D = K'$ is commutative, $K \otimes_{K_0} K_0' = K'$, and $m = n$.

Now (2.6) tells us that $N \simeq M_0^n$ as a A_{K_0}-module, for some indecomposable A_{K_0}-module M_0. Since $K \otimes_{K_0} N \simeq M^n$, by the Krull-Schmidt theorem $K \otimes_{K_0} M_0 \simeq M$. Thus M descends to K_0, as claimed. □

3. Proof of Theorem 1.2

We begin with a simple criterion for the existence of a minimal field of definition.

Lemma 3.1. *Let A be a finite-dimensional F-algebra, and K/F be a field extension, and M be an A_K-module, satisfying conditions (a) and (b) below. Then M has a minimal field of definition.*

(a) Suppose M descends to an intermediate field $F \subset L \subset K$, i.e., $M \simeq K \otimes_L N$ for some A_L-module N. Then N further descends to a subfield $F \subset E \subset L$, where $[E : F] < \infty$.

(b) Suppose M descends to an intermediate field $F \subset E \subset K$ such that $[E : F] < \infty$. That is, $M \simeq K \otimes_E N$ for some A_E-module N. Then N has a minimal field of definition $E_{min} \subset E$.

Proof. Condition (a) implies that M is defined over some $F \subset E \subset K$ with $[E : F] < \infty$. Let the A_E-module N and the field $E_{min} \subset E$ be as in (b).

We claim that E_{min} is independent of the choice of E. That is, suppose $F \subset E' \subset K$ is another field of definition of M with $[E' : F] < \infty$, $M := K \otimes_{E'} N'$ for some $A_{E'}$-module N'. Let $E'_{min} \subset E'$ be the minimal field of definition of N', so that $N' := E' \otimes_{E'_{min}} N'_{min}$. Then our claim asserts that $E_{min} = E'_{min}$. If we can prove this claim, then clearly E_{min} is the minimal field of definition for M. Our proof of the claim will proceed in two steps.

First assume $E \subset E'$. By Lemma 2.3(b), E'_{min} is a minimal field of definition for N. By uniqueness of the minimal field of definition for N, $E_{min} = E'_{min}$.

Now suppose $F \subset E \subset K$ and $F \subset E' \subset K$ are fields of definition for M such that $[E : F] < \infty$ and $[E' : F] < \infty$. Let E'' be the composite of E and E' in K and E''_{min} be the minimal field of definition of $N_{E''} \simeq N'_{E''}$. (Note that $N_{E''}$ and $N'_{E''}$ become isomorphic over K; hence, by Theorem 2.2, they are isomorphic over E''.) Then, $[E'' : F] < \infty$, and $E, E' \subset E''$. As we just showed, $E_{min} = E''_{min}$ and $E'_{min} = E''_{min}$. Thus $E_{min} = E'_{min}$, as desired. □

We now proceed with the proof of Theorem 1.2.

Reduction 3.2. *For the purpose of proving Theorem 1.2, we may assume without loss of generality that*

(i) *K is a finite extension of F.*

(ii) *K is a Galois extension of F.*

Proof. (i) follows from Lemma 3.1. Indeed, we are assuming that Theorem 1.2 holds whenever K is a finite extension of F. That is, condition (b) of Lemma 3.1 holds. On the other hand, condition (a) of Lemma 3.1 follows from Lemma 2.1.

(ii) By part (i), we may assume that K/F is finite. Let L be the normal closure of K over F. Then L/F is finite Galois. Lemma 2.3(b) now tells us that if $M_L := L \otimes_K M$ has a minimal field of definition then so does M. □

We are now ready to finish the proof of Theorem 1.2. In view of Reduction 3.2, it remains to establish the following.

Lemma 3.3. *Let F be a field satisfying* (1.1)*, A be a finite-dimensional F-algebra, K/F be a finite Galois extension, and M be an A_K-module. The Galois group $G := \mathrm{Gal}(K/F)$ acts on the set of isomorphism classes of A_K-modules via*

$$g\colon N \to {}^gN := K \otimes_g N\,.$$

Let G_M be the stabilizer of M under this action. Then the fixed field K^{G_M} of G_M is the minimal field of definition for M.

Proof. Suppose M is defined over K_0, where $F \subset K_0 \subset K$. Then clearly ${}^gM \simeq M$ for every $g \in \mathrm{Gal}(K/K_0)$. Hence, $\mathrm{Gal}(K/K_0) \subset G_M$ and consequently, $K^{G_M} \subset K_0$. This shows that K^{G_M} is contained in every field of definition of M.

It remans to show that M descends to $K_0 := K^{G_M}$. Write $M = M_1^{d_1} \oplus \cdots \oplus M_r^{d_r}$, where $M_1, \ldots, M_r$ are distinct indecomposables. The condition that ${}^gM \simeq M$ for any $g \in G_M$ is equivalent to the following: if $M_j \simeq {}^gM_i$ for some $g \in \mathrm{Gal}(K/K_0)$, then $d_i = d_j$. Grouping G_M-conjugate indecomposables together, we see that $M \simeq S_1 \oplus \cdots \oplus S_m$, where each $S_1, \ldots, S_m$ is the G_M-orbit sum of one of the indecomposable modules M_i. (Here the orbit sums $S_1, \ldots, S_m$ may not be distinct.) It thus suffices to show that each orbit sum is defined over K_0.

Consider a typical G_M-orbit sum $S := M_1 \oplus \cdots \oplus M_s$, where we renumber the indecomposable factors of M so that $M_1, \ldots, M_s$ are the G_M-translates of M_1. Let H be the stabilizer of M_1 in G_M. That is,

$$H := \{h \in G_M \mid {}^hM_1 \simeq M_1\}\,.$$

Let $K_1 := K^H$. Then

$$K \otimes_{K_1} (M_1)_{\downarrow K_1} = \bigoplus_{h \in H} {}^hM_1 = M_1^{|H|}\,.$$

In particular, this tells us that $M_1^{|H|}$ descends to K_1. By Proposition 2.4, so does M_1. In other words, $M_1 \simeq K \otimes_{K_1} N_1$ for some K_1-module N_1. We claim that

$$K \otimes_{K_0} (N_1)_{\downarrow K_0} \simeq S. \tag{3.4}$$

If we can prove this claim, then S descends to K_0, and we are done.

To prove the claim, note that on the one hand,

$$K \otimes_{K_0} (M_1)_{\downarrow K_0} = \prod_{g \in G_M} {}^gM_1 = S^{|H|}\,. \tag{3.5}$$

On the other hand, since $M_1 \simeq K \otimes_{K_1} N_1$, we have

$$(M_1)_{\downarrow K_0} \simeq ((M_1)_{\downarrow K_1})_{\downarrow K_0} \simeq (N_1^{|H|})_{\downarrow K_0}\,,$$

and thus

$$K \otimes_{K_0} (M_1)_{\downarrow K_0} = (K \otimes_{K_0} ((N_1)_{\downarrow K_0})^{|H|}) \simeq (K \otimes_{K_0} (N_1)_{\downarrow K_0}))^{|H|}\,. \tag{3.6}$$

Comparing (3.5) and (3.6), we obtain

$$(K \otimes_{K_0} (N_1)_{\downarrow K_0})^{|H|} \simeq S^{|H|}\,. \tag{3.7}$$

The desired isomorphism (3.4) follows from (3.7) by the Krull-Schmidt theorem. This completes the proof of Lemma 3.3 and thus of Theorem 1.2. □

4. Algebras of finite representation type

A finite-dimensional F-algebra A is said to be *of finite representation type* if there are only finitely many indecomposable finitely generated A-modules (up to isomorphism).

Theorem 4.1. *Let F be a field satisfying* (1.1)*, A be finite-dimensional F-algebra of finite representation type, and K/F be a field extension (not necessarily algebraic) such that F is perfectly closed in K. (That is, for every subextension $F \subset E \subset K$ with $[E : F] < \infty$, E is separable over F.) Suppose M is an indecomposable A_K-module. Then*

(a) M descends to an intermediate subfield $F \subset E \subset K$ such that $[E : F] < \infty$.

(b) M is a direct summand of $K \otimes_F N$ for some indecomposable A_F-module N.

Proof. (a) Consider the A-module $M_{\downarrow F}$. Generally speaking this module is not finitely generated over A. Nevertheless, since A has finite representation type, thanks to a theorem of H. Tachikawa [Ta, Corollary 9.5], $M_{\downarrow F}$ can be written as a direct sum of finitely generated indecomposable A-modules. Denote one of these modules by N. That is,

$$M_{\downarrow F} \simeq N \oplus N'\,, \tag{4.2}$$

for some A-module N' (not necessarily finitely generated).

Let us now take a closer look at N. By Fitting's lemma, $E := \mathrm{End}^{ss}_A(N)$ is a finite-dimensional division algebra over F. Since F is a field satisfying (1.1), E is a field extension of F. Now set $F' := E \cap K$ and $m = [F' : F]$. Since F is perfectly closed in K, F' is finite and separable over F. Thus

$$\mathrm{End}^{ss}_A(F' \otimes_F N) \simeq F' \otimes_F \mathrm{End}^{ss}_A(N) \simeq E \times \cdots \times E\,.$$

This tells us that over F', N decomposes into a direct sum of m indecomposables,

$$F' \otimes_F N = N_1 \oplus \cdots \oplus N_m. \tag{4.3}$$

By the definition of F', $K \otimes_{F'} E$ is a field. Hence, each indecomposable $A_{F'}$-module N_i remains indecomposable over K.

Tensoring both sides of (4.2) with K, we obtain an isomorphism of A_K-modules

$$\begin{aligned} K \otimes M_{\downarrow F} &\simeq (K \otimes_F N) \oplus (K \otimes_F N') \\ &= (\bigoplus_{i=1}^{m} K \otimes_{F'} N_i) \oplus (K \otimes_F N') \\ &= (K \otimes_F N_1) \oplus N''\,, \end{aligned}$$

where $N'' := (\bigoplus_{i=2}^m K \otimes_{F'} N_i) \oplus (K \otimes_F N')$. Note that

$$K \otimes M_{\downarrow F'} \simeq \bigoplus_B M ,$$

where B is a basis of K as an F'-vector space. As we mentioned above, $K \otimes_{F'} N_1$ is an indecomposable A_K-module. Since $K \otimes_{F'} N_1$ is finitely generated and is contained in $\bigoplus_B M$, it lies in the direct sum of finitely many copies of M, say, in $M^r := M \oplus \cdots \oplus M$ (r copies). Thus we have maps

$$K \otimes_F N_1 \hookrightarrow M^r \hookrightarrow \bigoplus_B M \twoheadrightarrow K \otimes_F N_1$$

whose composite is the identity, and so $K \otimes_F N_1$ is isomorphic to a direct summand of M^r. By the Krull-Schmidt Theorem, $K \otimes_{F'} N_1 \simeq M$. In particular, M descends to F', as claimed.

(b) By (4.3), N is an indecomposable A-module, and N_1 is a direct summand of $F' \otimes_F N$. Hence, $M \simeq K \otimes_{F'} N_1$ is a direct summand of $K \otimes_F N$, as desired. □

Corollary 4.4. *Let F be a field satisfying* (1.1)*, A be finite-dimensional F-algebra of finite representation type, and K/F be a field extension such that F is perfectly closed in K. Then A_K is also of finite representation type.*

Proof. By our assumption A has finitely many indecomposable modules $N^{(1)}, \dots, N^{(d)}$. By Theorem 4.1(b) every indecomposable A_K-module is isomorphic to a direct summand of $K \otimes_F N^{(i)}$ for some i. By the Krull-Schmidt Theorem, each $K \otimes_F N^{(i)}$ has finitely many direct summands (up to isomorphism), and the corollary follows. □

5. Proof of Theorems 1.3 and 1.4

We will deduce Theorem 1.3 from Lemma 3.1. M satisfies condition (b) of Lemma 3.1 by Theorem 1.2. It thus remains to show that M satisfies condition (a) of Lemma 3.1. For notational simplicity, we may assume that $K = L$ and $M = N$. That is, we want to show that M descends to some intermediate field $F \subset E \subset K$ with $[E : F] < \infty$. Note that in the case, where M is indecomposable, this is precisely the content of Theorem 4.1(a).

In general, write $M = M_1 \oplus \cdots \oplus M_r$ as a direct product of (not necessarily distinct) indecomposables. By Theorem 4.1(a), each M_i descends to an intermediate field $F \subset K_i \subset K$ such that $[K_i : F] < \infty$. Let E be the compositum of $K_1, \dots, K_r$ inside K. Then $[E : F] < \infty$, and M descends to E. This completes the proof of Theorem 1.3. □

We now proceed with the proof of Theorem 1.4. Denote the perfect closure of F in K by F^{pf}. By Theorem 1.3, M descends to an intermediate field $F^{pf} \subset K_0 \subset K$ such that $[K_0 : F^{pf}] < \infty$. Hence, K_0 is algebraic over F, and consequently, $\mathrm{ed}(M) \leqslant \mathrm{trdeg}_F(K_0) = 0$, as desired. □

6. An example

In this section we will show by example that both Theorems 1.3 and 1.4 fail if we do not require F to be a field satisfying (1.1). Let $F = \mathbb{Q}$ and A be the quaternion algebra

$$A = \mathbb{Q}\{x, y\}/(x^2 = y^2 = -1,\ xy = -yx).$$

and K/F be any field having two elements a and b satisfying $a^2 + b^2 = -1$. Then A has a two dimensional A_K-module M given by

$$x \mapsto \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \qquad y \mapsto \begin{pmatrix} b & -a \\ -a & -b \end{pmatrix}. \tag{6.1}$$

Note that the multiplicative subgroup of A generated by x and y is isomorphic to the quaternion group Q_8. Thus A is naturally a quotient of the group algebra $\mathbb{Q}Q_8$ of Q_8 over $\mathbb{Q}$. Since $\mathbb{Q}Q_8$ is of finite representation type, one readily concludes that so is A.

Lemma 6.2. *The following conditions on an intermediate field $\mathbb{Q} \subset E \subset K$ are equivalent:*

(a) φ descends to E,

(b) A splits over E,

(c) there exist elements a_0, $b_0 \in E$ such that $a_0^2 + b_0^2 = -1$.

Proof. (a) $\Longrightarrow$ (b). Suppose M descends to an A_E-module N. Since $A_E := E \otimes_{\mathbb{Q}} A$ is a central simple 4-dimensional algebra over E, the homomorphism of algebras given by

$$A_E \to \mathrm{End}_E(N) \simeq \mathrm{M}_2(E)$$

is an isomorphism. In other words, E splits A.

(b) $\Longrightarrow$ (a). Conversely, suppose E splits A. Then the representation of $A \to \mathrm{End}_K(M)$ factors as follows:

$$A \to E \otimes_{\mathbb{Q}} A \simeq \mathrm{M}_2(E) \to \mathrm{M}_2(K).$$

This shows that φ descends to E.

The equivalence of (b) and (c) a special case of Hilbert's criterion for the splitting of a quaternion algebra; see the equivalence of conditions (1) and (7) in [Lam, Theorem III.2.7] as well as Remark (B) on [Lam, p. 59]. □

Proposition 6.3. *Let a and b be independent variables over $F = \mathbb{Q}$, E be the field of fractions of $\mathbb{Q}[a, b]/(a^2 + b^2 + 1)$, and M be the 2-dimensional A_E-module given by* (6.1). *Then*

(a) $\mathrm{ed}(M) = 1$,

(b) M does not have a minimal field of definition.

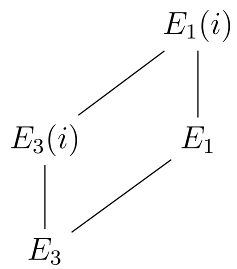
Proof. (a) The assertion of part (a), follows from [KRP, Example 6.1]. For the sake of completeness, we will give an independent proof.

Suppose M descends to an intermediate subfield $\mathbb{Q} \subset E_0 \subset E$. Since $\mathrm{trdeg}_{\mathbb{Q}}(E) = 1$, $\mathrm{trdeg}_{\mathbb{Q}}(E_0) = 0$ or 1. Our goal is to show that $\mathrm{trdeg}_{\mathbb{Q}}(E_0) \neq 0$. Assume the contrary, i.e., E_0 is algebraic over $\mathbb{Q}$.

Note that E is the function field of the conic curve $a^2 + b^2 + c^2 = 0$ in $\mathbb{P}^2$. Since this curve is absolutely irreducible, $\mathbb{Q}$ is algebraically closed in E. Since E_0 is algebraic over $\mathbb{Q}$, we conclude that $E_0 = \mathbb{Q}$. On the other hand, M does not descend to $\mathbb{Q}$ by Lemma 6.2, a contradiction.

(b) Suppose M descends to $E_1 \subset E$. Our goal is to show that M descends to a proper subfield $E_3 \subset E_1$. By Lemma 6.2(c) there exist a_1 and b_1 in E_1 such that $a_1^2 + b_1^2 = -1$. If $\mathbb{Q}(a_1, b_1)$ is properly contained in E_1, then we are done. Thus we may assume without loss of generality that $E_1 = \mathbb{Q}(a_1, b_1)$. Set $E_3 := \mathbb{Q}(a_3, b_3)$, where $a_3 := a_1^3 - 3a_1b_1^2$ and $b_3 = 3a_1^2b_1 - b_1^3$. We claim that (i) A splits over E_3, and (ii) $E_3 \subsetneq E_1$.

In order to establish (i) and (ii), let us consider the following diagram



of field extensions. Here as usual, i denotes a primitive 4th root of 1. It is easy to see that $E_1(i) = \mathbb{Q}(i)(a_1, b_1) = \mathbb{Q}(i)(z)$ is a purely transcendental extension of $\mathbb{Q}(i)$, where $z = a_1 + b_1 i$ and $\dfrac{1}{z} = -a_1 + b_1 i$. Similarly $E_3(i) = \mathbb{Q}(i)(z^3)$, where $z^3 = a_3 + b_3 i$ and $\dfrac{1}{z^3} = -a_3 + b_3 i$. In particular, this shows $a_3^2 + b_3^2 = -1$, thus proving (i). Moreover, since z is transcendental over $\mathbb{Q}(i)$, we have

$$[E_1(i) : E_3(i)] = [\mathbb{Q}(i)(z) : \mathbb{Q}(i)(z^3)] = 3$$

and thus

$$[E_1 : E_3] = \frac{[E_3(i) : E_3] \cdot [E_1(i) : E_3(i)]}{[E_1(i) : E_1]} = \frac{2 \cdot 3}{2} = 3\,.$$

This proved (ii). □

Remark 6.4. Write $z^n = a_n + b_n i$ for suitable $a_n, b_n \in E_1$ and set $E_n = \mathbb{Q}(a_n, b_n)$. We showed above that $[E_1 : E_3] = 3$ and thus $E_3 \subsetneq E_1$. The same argument yields $[E_1 : E_n] = n$ for any positive integer n.

7. Proof of Theorem 1.5

We shall actually prove a stronger, more natural theorem, about blocks of finite group algebras. Theorem 1.5 will follow from the fact that p-Sylow subgroups of a finite group G are cyclic if and only if every block over a field F of characteristic p has cyclic defect; see [Ha] or [CR$_2$, Theorem 62.21].

Theorem 7.1. *Let B be a block of a finite group algebra FG, where F is a field of characteristic p. Then the following are equivalent:*

(1) *B has cyclic defect,*
(2) $\mathrm{ed}(\mathrm{Mod}_B) = 0$*,*
(3) $\mathrm{ed}(\mathrm{Mod}_B) < \infty$*.*

The implication (1) $\Longrightarrow$ (2) is a direct consequence of Theorem 1.4. The implication (2) $\Longrightarrow$ (3) is obvious.

The remainder of this section will be devoted to proving that (3) $\Longrightarrow$ (1). We shall show that if B has non-cyclic defect, then $\mathrm{ed}(\mathrm{Mod}_B) = \infty$. Let K be an extension field of F, let e be the block idempotent of B, let D be a defect group of B, and let $N = \Phi(D)$, the Frattini subgroup of D. If D is not cyclic, D/N is elementary abelian of rank $r \geq 2$, with basis the images of elements $g_1, \ldots, g_r \in D$. Since D is a defect group of B, any KD-module M is a summand of $\mathrm{Res}_{G,D}(e.\,\mathrm{Ind}_{D,G}(M))$.

Now let $n > 0$, and let $K = F(t_{1,1}, \ldots, t_{n,r})$ be a function field in nr indeterminates, and let M_i ($1 \leq i \leq n$) be the two dimensional KD-module

$$g_j \mapsto \begin{pmatrix} 1 & t_{i,j} \\ 0 & 1 \end{pmatrix}.$$

Then $J^2(KD)$ is in the kernel of M_i, so M_i is really a module for $KD/J^2(KD)$, which has a basis $1, (g_1 - 1), \ldots, (g_r - 1)$. The last r elements of this list form a basis for $J(KD)/J^2(KD)$, and we form a vector space V with basis $(g_1 - 1), \ldots, (g_r - 1)$. The kernel of M_i as a module for $KD/J^2(KD)$ is the codimension one subspace H_i of

$$J(KD)/J^2(KD) \cong V$$

given by

$$H_i := \{\lambda_j(g_j - 1) \mid \sum_j t_{i,j}\lambda_j = 0\}. \tag{7.2}$$

By the Mackey decomposition theorem, the module $M_i' = \mathrm{Res}_{G,D}(e.\,\mathrm{Ind}_{D,G}(M_i))$ is a direct sum of at least one copy of M_i, some conjugates of M_i by elements of $N_G(D)$, and some modules of the form $\mathrm{Ind}_{D\cap {}^gD,D} \mathrm{Res}_{{}^gD,D\cap {}^gD} {}^gM$. It follows that the Jordan canonical form of elements of V on M_i' is constant, except on a set S_i, which is a finite union of hyperplanes $N_G(D)$-conjugates of H_i and linear subspaces of smaller dimension.

Now let $M := \bigoplus_i M_i$. Our goal is to show that

$$\mathrm{ed}(e.\,\mathrm{Ind}_{D,G}(M)) \geqslant n(r-1)\,.$$

This will imply that $\mathrm{ed}(\mathrm{Mod}_B) \geq n(r-1)$ for every $n > 0$ and thus $\mathrm{ed}(\mathrm{Mod}_B) = \infty$, as desired.

Note that $e.\,\mathrm{Ind}_{D,G}(M)$ is a module whose restriction to D is $\bigoplus_i M_i'$. If $e.\,\mathrm{Ind}_{D,G}(M)$ descends to an intermediate subfield $F \subset K_0 \subset K$, then so does the set $\bigcup_i S_i \subset V$ and its natural image in $\mathbb{P}(V) = \mathbb{P}^{r-1}$, which we will denote by S. To complete the proof of Theorem 7.1, it remains to show that if S descends to K_0, then

$$\mathrm{trdeg}_F(K_0) \geqslant n(r-1)\,. \tag{7.3}$$

Lemma 7.4. *Let $S \subset \mathbb{P}^{r-1}$ be a projective variety defined over a field K. Assume that a hyperplane H given by $a_1x_1 + a_2x_2 + \cdots + a_rx_r = 0$ is an irreducible component of S for some $a_1, \ldots, a_r \in K$ (not all zero). Suppose S descends to a subfield $K_0 \subset K$. Then each ratio a_j/a_l is algebraic over K_0, as long as $a_l \neq 0$.*

To deduce the inequality (7.3) from Lemma 7.4, recall that in our case S is the union of the hyperplanes $H_1, \ldots, H_n$, a finite number of other hyperplanes (translates of $H_1, \ldots, H_n$ by elements of $N_G(D)$) and lower-dimensional linear subspaces of $\mathbb{P}(V) = \mathbb{P}^{r-1}$. In the basis $(g_1 - 1), \ldots, (g_r - 1)$ of V, H_i is given by $t_{i,1}x_1 + t_{i,2}x_2 + \cdots + t_{i,r}x_r = 0$; see (7.2). Thus by Lemma 7.4 the elements $t_{i,j}/t_{i,1}$ are algebraic over K_0 for every $i = 1, \ldots, n$ and every $j = 2, \ldots, r$. In other words, if K_1 is the algebraic cosure of K_0 in K, then each $t_{i,j}/t_{i,1} \in K_1$, and thus $\mathrm{trdeg}_F(K_0) = \mathrm{trdeg}_F(K_1) \geqslant n(r-1)$, as desired.

Proof of Lemma 7.4. We may assume without loss of generality that K_0 is algebraically closed. To reduce to this case, we replace K_0 by its algebraic closure $\overline{K_0}$ and K by a compositum of K and $\overline{K_0}$. If we know that each $a_{i,j}$ is algebraic over $\overline{K_0}$ (or equivalently, is contained in $\overline{K_0}$), then $a_{i,j}$ is algebraic over K_0.

Now assume that K_0 is algebraically closed. Since S is defined over K_0, every irreducible component of S is defined over K_0. In particular, H is defined over K_0. That is, the point $(a_1 : \cdots : a_r)$ of the dual projective space $\check{\mathbb{P}}^{r-1}$ is defined over K_0. Equivalently, $a_i/a_j \in K_0$ whenever $a_l \neq 0$. This completes the proof of the claim and thus of Lemma 7.4 and Theorem 7.1. □

Acknowledgements

The authors are grateful to Mathieu Florence, Roberto Pirisi and Julia Pevtsova for helpful comments.

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