COMPRESSIONS OF GROUP ACTIONS

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Let G be a finite group. A *G-variety* X is an algebraic variety with a regular G-action; X is *faithful* if every $1 \neq g \in G$ acts non-trivially. I will refer to a dominant G-equivariant rational (respectively, regular) map of faithful G-varieties as a rational (respectively, *regular*) compression. All varieties, actions, vector spaces, maps, etc., are assumed to be defined over a fixed algebraically closed base field k of characteristic zero; all varieties are assumed to be irreducible.

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1. Essential dimension

Let V be a faithful linear representation of G and let d be the minimal value of $\dim(X)$, where the minimum is taken over all rational compressions $f : V \dashrightarrow X$. Note that

(a) (see [1, Theorem 3.1] or [6, Theorem 3.4(b)]) d depends only on the group G and not on the choice of V, and

(b) (cf. [6, Proposition 7.1]) in the definition of d we may assume that X is a G-invariant subvariety of V, i.e., X is the closure of the image of a rational covariant $f : V \dashrightarrow V$.

The number d is called the *essential dimension* of G and is usually denoted by $\mathrm{ed}(G)$. This number has interesting connections with the algebraic form of Hilbert's 13th problem, cohomological invariants, generic polynomials and other topics; these connections are described in [1] and [2]. The case where $G = \mathrm{S}_n$ is of particular interest. (The notion of essential dimension is also of interest in the context of algebraic groups; see [6] and [7].)

Problem 1. *Find* $\mathrm{ed}(G)$ *and, in particular,* $\mathrm{ed}(\mathrm{S}_n)$.

The value of $\mathrm{ed}(G)$ is known if G is an abelian group; see [1, Theorem 6.1]. For symmetric groups, $\mathrm{ed}(\mathrm{S}_n) \geq [n/2]$; this is proved, in different ways, in [1, Theorem 6.5(c)], [2, Section 8], [6, Example 12.8] and [7, Section 9]. (For $n = 5$ this inequality goes back to Felix Klein [3].) The best known upper bound on $\mathrm{ed}(\mathrm{S}_n)$ is $\mathrm{ed}(\mathrm{S}_n) \leq n-3$; see [1, Theorem 6.5(c)], and exact values of $\mathrm{ed}(\mathrm{S}_n)$ are only known for $n \leq 6$; see [1, Theorems 6.2 and 6.5(d)].

We now consider an analogous notion in the context of regular, rather than rational compressions. Let V be a faithful linear representation of G

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and let D be the minimal value of $\dim(X)$, where the minimum is taken over all regular compressions $F\colon V \longrightarrow X$ from V to a faithful affine G-variety X. Properties (a) and (b) remain valid in this setting:

(A) D depends only on the group G and not on the choice of V and

(B) in the definition of D we may assume that X is a G-invariant subvariety of V, i.e., X is the closure of the image of a (regular) covariant $F\colon V \longrightarrow V$.

Part (B) is an immediate consequence of [5, Theorem 1.7.12]. To prove (A), assume that V_1 and V_2 are faithful representations of G, and $F_i\colon V_i \longrightarrow X_i$ is a regular compression with minimal possible values of $\dim(X_i) = D_i$, $i = 1, 2$. We want to show that $D_1 = D_2$. Choose a point $v_2 \in V_2$ such that $F_2(v_2)$ has a trivial stabilizer in X_2. By [5, Theorem 1.7.12] there exists a covariant $\phi\colon V_1 \longrightarrow V_2$ such that v_2 lies in the image of ϕ. Let X_1' be the closure of the image of $F_2 \circ \phi\colon V_1 \longrightarrow X_2$. Then X_1' is a faithful G-variety (because it contains $F(v_2)$) and thus, by the definition of D_1,

$$D_1 \leq \dim(X_1') \leq \dim(X_2) = D_2\,.$$

By symmetry, we also have $D_2 \leq D_1$, so that $D_1 = D_2$, as claimed. □

In view of (A), it is natural to write $\mathrm{eD}(G)$ in place of D.

Problem 2. *Find* $\mathrm{eD}(G)$ *and, in particular,* $\mathrm{eD}(\mathrm{S}_n)$.

Problems 1 and 2 are closely related, because

$$\mathrm{ed}(G) \leq \mathrm{eD}(G) \leq \mathrm{ed}(G) + 1\ .$$

The inequality $\mathrm{ed}(G) \leq \mathrm{eD}(G)$ is obvious from the definition. To prove that $\mathrm{eD}(G) \leq \mathrm{ed}(G) + 1$, note that if $f\colon V \dashrightarrow V$ is a rational covariant then $F = \alpha f\colon V \longrightarrow V$ is a regular covariant for some $0 \neq \alpha \in k[V]^G$. (That is, $F(v) = \alpha(v) f(v)$ for every $v \in V$.) Then

$$\dim\, f(V) \leq \dim\, F(V) \leq \dim\, f(V) + 1\,,$$

and the proof is complete. □

2. Incompressible varieties

I will call a G-variety X *incompressible* if every rational compression $X \dashrightarrow Y$ is a birational isomorphism. (Note that this definition bears some resemblance to the definition of a relatively minimal model in algebraic geometry; cf., e.g., [4, p. 418].)

Problem 3. *Classify incompressible G-varieties.*

To motivate this problem, I will discuss three simple examples.

Example 4. If $G = \{1\}$ then the only incompressible G-variety is a point.

Example 5. If $G = <c>$ is a cyclic group of order $n \geq 2$ then there are no incompressible G-varieties.

Indeed, assume, to the contrary, that X is an incompressible G-variety. Diagonalizing the G-action on $k(X)$ (viewed as a finite-dimensional $k(X)^G$-vector space), we can find a non-zero rational function $f \in k(X)$ such that

$f(c \cdot x) = \zeta f(x)$, where ζ is a primitive nth root of unity. Then $x \mapsto f(x)$ is a G-compression $X \dashrightarrow \mathbb{A}^1$, where c acts on $\mathbb{A}^1$ by $x \longrightarrow \zeta x$. Thus X is birationally isomorphic to $\mathbb{A}^1$ (as a G-variety). On the other hand, $\mathbb{A}^1$ is not incompressible; indeed, $x \longrightarrow x^{n+1}$ is a nontrivial compression $\mathbb{A}^1 \longrightarrow \mathbb{A}^1$, a contradiction. □

Example 6. Let G be a finite group that does not embed in $\mathrm{Aut}(X)$ for any curve X of genus ≤ 1. There are many such groups; see below. Let X be a faithful G-curve of the smallest possible genus $g(X)$. (It is easy to see that faithful G-curves exist for any finite group G; cf., e.g., [1, Remark 4.5].) I claim that X is incompressible. Indeed, suppose that $X \dashrightarrow Y$ is a G-compression. By our assumption, $g(Y) \geq 2$. Then by the Hurwitz formula (see, e.g., [4, Corollary 4.2.4] or [9, Theorem II.5.9]), $g(Y) \leq g(X)$, a contradiction.

To construct a group G that does not embed in $\mathrm{Aut}(X)$ for any curve X of genus ≤ 1, recall the following classical results on automorphisms of rational and elliptic curves.

- The only finite subgroups of $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$, that are not cyclic or dihedral, are A_4, S_4 and A_5; see [3, Chapter 1] or [8, Theorem 2.6.1].
- Let E be an elliptic curve. Then there exists an exact sequence

$$\{1\} \longrightarrow E \longrightarrow \mathrm{Aut}(E) \longrightarrow H \longrightarrow \{1\},$$

  where E acts on itself by translations (and thus is embedded in $\mathrm{Aut}(E)$) and $|H| = 2$, 4 or 6; see, e.g., [9, Theorems III.10.1 and Proposition X.5.1]. (Note that [9] uses the symbols $\mathrm{Isom}(E)$ and $\mathrm{Aut}(E)$ in place of my $\mathrm{Aut}(E)$ and H, respectively.)

It is now easy to see that many finite groups cannot be embedded in $\mathrm{Aut}(\mathbb{P}^1)$ or $\mathrm{Aut}(E)$ for any elliptic curve E. Examples of such groups include (i) the symmetric group S_5, (ii) the alternating group $G = \mathrm{A}_6$, and perhaps more surprisingly (in view of Example 5), the abelian groups (iii) $G = (\mathbb{Z}/p)^3$ for $p \geq 5$, (iv) $G = (\mathbb{Z}/3)^4$, and (v) $G = (\mathbb{Z}/2)^5$. Of course, we can also take G to be any group containing a subgroup isomorphic to (i), (ii), (iii), (iv) or (v). □

In conclusion, note that Problem 3 can be posed for any algebraic (possibly infinite) group G; here a compression should be defined as a dominant map of *generically free*, rather than faithful, G-varieties (cf. [6, Definition 2.15]).

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