# EQUIVARIANT RESOLUTION OF POINTS OF INDETERMINACY

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ABSTRACT. We prove an equivariant form of Hironaka's theorem on elimination of points of indeterminacy. Our argument uses canonical resolution of singularities and an extended version of Sumihiro's equivariant Chow Lemma.

## 1. INTRODUCTION

Throughout this note we shall work over an algebraically closed field of characteristic 0. All algebraic varieties, schemes, groups, and all maps between them will be defined over $k$.

Let $G$ be a linear algebraic group (not necessarily connected). The main objects of interest for us will be algebraic varieties with a $G$-action; we will refer to them as $G$-varieties. A $G$-equivariant morphism between two such varieties will be called a morphism of $G$-varieties. The terms "rational map of $G$-varieties", "birational morphism of $G$-varieties", "birational isomorphism of $G$-varieties", etc., are defined in a similar manner.

Hironaka's theorem on elimination of points of indeterminacy (see [Hi, §0.5, Question E and Main Theorem II]) asserts that every rational map $f\colon X \dashrightarrow Y$ can be resolved into a regular map by a sequence of blowups $\pi\colon X_m \longrightarrow \dots \longrightarrow X_0 = X$ with smooth centers. In other words, $\pi$ can be chosen so that the composition $f\pi$ is regular. The purpose of this paper is to prove the following equivariant version of this result.

**Theorem 1.** *Let $f\colon X \dashrightarrow Y$ be a rational map of $G$-varieties where $Y$ is complete. Then there is a sequence of blowups*

$$\pi\colon X_m \longrightarrow X_{m-1} \longrightarrow \dots \longrightarrow X_1 \longrightarrow X_0 = X$$

*with smooth $G$-invariant centers such that the composition $f\pi$ is regular.*

Our proof will rely on canonical resolution of singularities. Along the way we prove an equivariant form of Chow's lemma (Proposition 2), generalizing a theorem of Sumihiro ([Su, Theorem 2]). For applications of Theorem 1 see [$\mathrm{RY}_1$, Remark 5.4] and [$\mathrm{RY}_2$, Remark 6.3].

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## 2. Equivariant Chow Lemma

In this section we will prove the following generalization of Chow's lemma.

**Proposition 2.** *For every $G$-variety $X$, there exists a quasiprojective $G$-variety $Z$ and a proper birational morphism $Z \longrightarrow X$. If $X$ is complete then $Z$ is projective.*

Note that if $G$ is assumed to be connected, this result is a well-known theorem of Sumihiro [Su, Theorem 2]; see also [PV, Theorem 1.3]. The argument below reduces the general case to the case where $G$ is connected.

*Proof.* The second assertion is an immediate consequence of the first: if $X$ is complete, $Z \longrightarrow X$ is proper and $Z$ is quasiprojective then $Z$ is also complete and, hence, projective.

To prove the first assertion let $G_0$ be the connected component of $G$. Applying Sumihiro's theorem to $X$, viewed as a $G_0$-variety, yields a quasiprojective $G_0$-variety $Y$ and a proper $G_0$-equivariant birational morphism $f\colon Y \longrightarrow X$.

For the rest of this proof we shall use set-theoretic notation: by a "point" we will always mean a closed point.

Recall that the homogeneous fiber product $G{*_{G_0}}Y$ is the $G$-variety defined as the geometric quotient $(G\times Y)/G_0$ for the action of $G_0$ given by $g_0(g,y) = (gg_0^{-1}, g_0y)$, where $g \in G$, $g_0 \in G_0$ and $y \in Y$; see [PV, Section 4.8]. We shall write $[g,y]$ for the element of $G{*_{G_0}}Y$ represented by $(g,y) \in G\times Y$. Since $G_0$ has finite index in $G$, $G{*_{G_0}}Y$ admits a more concrete description as a disjoint union of $|G/G_0|$ copies of $Y$. More precisely, if we choose a representative $a_h$ for each $h \in G/G_0$, we can explicitly identify $G/G_0 \times Y$ and $G *_{G_0} Y$ as abstract varieties, via $(h,y) \mapsto [a_h, y]$. Moreover, if we define a $G$-action on $G/G_0 \times Y$ by $g(h,y) \longrightarrow (\overline{g}h, (a_h^{-1}g^{-1}a_{\overline{g}h})y)$ then $(h,y) \mapsto [a_h,y]$ identifies $G/G_0 \times Y$ and $G *_{G_0} Y$ as $G$-varieties. Here $\overline{g}$ is the image of $g$ in $G/G_0$ and $(a_h^{-1}g^{-1}a_{\overline{g}h})y$ is well-defined because $a_h^{-1}g^{-1}a_{\overline{g}h}$ is an element of $G_0$.

Let $\alpha\colon G *_{G_0} Y \longrightarrow X$ and $\beta\colon G *_{G_0} Y \longrightarrow G/G_0$ be the maps of $G$-varieties given by $\alpha\colon [g,y] \longrightarrow gf(y)$ and $\beta\colon [g,y] \mapsto \overline{g}$. (Here $G$ acts on $G/G_0$ by left multiplication.) These maps are shown in the diagram below.

$$\begin{tikzcd} & G *_{G_0} Y \simeq G/G_0 \times Y \arrow[dl, "\alpha"'] \arrow[dr, shift left, "\beta"] & \\ X & & G/G_0 \arrow[ul, shift left, "s"] \end{tikzcd}$$

Let $S$ be the set of all sections $s$ of $\beta$. Note that if we identify $G *_{G_0} Y$ with $G/G_0 \times Y$ as above, then $\beta\colon G/G_0 \times Y \longrightarrow G/G_0$ is the projection to the first factor. Thus $S \simeq Y^{|G/G_0|}$ as an abstract variety. Moreover, since $\beta$ is $G$-equivariant, $G$ acts on this variety by $g\colon s \mapsto t$, where $s,t \in S$ and $t(h) = g \cdot s(\overline{g}^{-1}h)$ for any $h \in G/G_0$.

Let $Z$ be the closed $G$-invariant subvariety of $S$ consisting of those sections $s\colon G/G_0 \longrightarrow G *_{G_0} Y \simeq G/G_0 \times Y$ with the property that $\alpha \circ s(G/G_0)$ is a single point of $X$; we shall denote this point by $x_s$.

We claim that the morphism $\phi\colon Z \longrightarrow X$ given by $s \longrightarrow x_s$ has the properties asserted in the proposition. Indeed, since $Y$ is quasiprojective, and $Z$ is a closed subvariety of $S \simeq Y^{|G/G_0|}$, $Z$ is quasiprojective as well.

To show that $\phi$ is a birational morphism, assume the birational morphism $f\colon Y \longrightarrow X$ is an isomorphism over a dense open subset $U \subset X$. Then $V = \bigcap_{h \in G/G_0} a_h U$ is also a dense open subset of $X$, and for every $x \in V$, $\alpha^{-1}(x) = \{[a_h, f^{-1}(a_{h^{-1}}x)] \,:\, h \in G/G_0\}$; it is the image of the unique section $s_x \in S$ satisfying $x_{s_x} = x$. This section is given by $s_x(h) = [a_h, f^{-1}(a_h^{-1}x)]$, and the morphism $V \longrightarrow Z$, $x \mapsto s_x$, is a two-sided rational inverse to $\phi$. $\square$

## 3. Birational morphisms as blowups

Our proof of Theorem 1 in the next section will rely on the following equivariant analogue of [Ha, Theorem 7.17].

**Proposition 3.** *Let $f\colon X' \longrightarrow X$ be a birational proper morphism of $G$-varieties, where $X$ is smooth and $X'$ is quasiprojective. Then there exists a $G$-invariant sheaf of ideals $\mathcal{I}$ on $X$ such that $X'$ is the blowup of $\mathcal{I}$.*

*Proof.* Let $\sigma\colon G \times X \longrightarrow X$ be the given action of $G$ on $X$ and $\mathrm{pr}_2\colon G \times X \longrightarrow X$ be the projection onto the second factor.

By a theorem of Kambayashi [Ka], there is an action of $G$ on the projective space $\mathbb{P}^n$ (via a representation $G \longrightarrow \mathrm{PGL}_{n+1}$) and a $G$-equivariant embedding $X' \hookrightarrow \mathbb{P}^n$; this yields a $G$-equivariant embedding $i\colon X' \hookrightarrow \mathbb{P}^n \times X$.

Here $\mathbb{P}^n \times X$ is a projective space over $X$; set $\mathcal{L} = i^*\mathcal{O}_{\mathbb{P}^n \times X}(1)$ and $\mathcal{T} = \bigoplus_{d=0}^{\infty} f_*(\mathcal{L}^d)$, where $\mathcal{L}^0 = \mathcal{O}_{\mathbb{P}^n \times X}$. Let $\mathcal{T}_1 = f_*\mathcal{L}$ be the component of $\mathcal{T}$ of degree one. The action of $G$ on $\mathbb{P}^n \times X$ yields a $G$-linearization of the sheaf $\mathcal{T}_1$, i.e., an isomorphism $\sigma^*\mathcal{T}_1 \cong \mathrm{pr}_2^*\,\mathcal{T}_1$ which satisfies the same cocycle condition as in the definition of $G$-linearization of an invertible sheaf (see, e.g., [MFK, Definition 1.6]); informally speaking, $G$ acts on the pair $(X, \mathcal{T}_1)$.

We refer to the proof of [Ha, Theorem 7.17] for the following facts:

1. After replacing the embedding $i$ by its $e$-fold embedding for some positive integer $e$ (thus replacing $\mathcal{L}$ by $\mathcal{L}^e$), we may assume that the graded $\mathcal{O}_X$-algebra $\mathcal{T}$ is generated by $\mathcal{T}_1$.
2. $X' \cong \mathrm{Proj}\,\mathcal{T}$.
3. Assume $\mathcal{T}$ is generated by $\mathcal{T}_1$ as in (1). If there is an invertible sheaf $\mathcal{M}$ on $X$ and a sheaf of ideals $\mathcal{I}$ on $X$ such that $\mathcal{I} \cong \mathcal{T}_1 \otimes \mathcal{M}$, then $X'$ is isomorphic to the blowup of $\mathcal{I}$.

The variety $X$ is smooth, and hence, for any sheaf of ideals $\mathcal{F}$ on $X$ of rank one without torsion, its dual $\mathcal{F}^* = \mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{O}_X)$ is an invertible sheaf. (To see this, note that locally at any point $x \in X$, the generator of $\mathcal{F}_x^*$ is given by the homomorphism $\mathcal{F}_x \longrightarrow \mathcal{O}_{X,x}$ which maps the generators of $\mathcal{F}_x$ as an $\mathcal{O}_{X,x}$-module of rank one, into elements of $\mathcal{O}_{X,x}$ not having a nontrivial common multiple; such homomorphism is unique up to an invertible multiple since the local ring $\mathcal{O}_{X,x}$ is regular, and hence, factorial.)

Thus the second dual $\mathcal{T}_1^{**}$ is an invertible sheaf, and we have an embedding $\mathcal{T}_1 \hookrightarrow \mathcal{T}_1^{**}$. The $G$-linearization of $\mathcal{T}_1$ yields a $G$-linearization of $\mathcal{T}_1^{**}$, and the above embedding is, in fact, an embedding of $G$-linearized sheaves. Taking $\mathcal{M} = (\mathcal{T}_1^{**})^{-1}$, we see that $X'$ is isomorphic to the blowup of the sheaf of ideals $\mathcal{I} = \mathcal{T}_1 \otimes (\mathcal{T}_1^{**})^{-1}$.

The $G$-linearizations of $\mathcal{T}_1$ and $\mathcal{T}_1^{**}$ yield a $G$-linearization of $\mathcal{I}$ and a $G$-linearized embedding $\mathcal{I} = \mathcal{T}_1 \otimes (\mathcal{T}_1^{**})^{-1} \hookrightarrow \mathcal{T}_1^{**} \otimes (\mathcal{T}_1^{**})^{-1} = \mathcal{O}_X$. This shows that $\mathcal{I}$ is a $G$-invariant sheaf of ideals on $X$. $\square$

## 4. Proof of Theorem 1

First of all, we may assume without loss of generality that $X$ is smooth. Indeed, let

$$X_l \xrightarrow{\pi_l} \dots \xrightarrow{\pi_1} X_0 = X \ , \tag{1}$$

be the canonical resolution of singularities of $X$, as in [V, Theorem 7.6.1] or [BM, Theorem 13.2]. Here $X_l$ is smooth, the centers $C_i \subset X_i$ are smooth and $G$-invariant, and the action of $G$ lifts to the entire resolution sequence (1). Replacing $X$ by $X_l$, we may assume that $X$ is smooth.

Now let $\Gamma$ be the closure of the graph of $f\colon X \dashrightarrow Y$ in $X \times Y$. By Proposition 2 there exists a proper birational morphism $\phi\colon Z \longrightarrow \Gamma$ of $G$-varieties, where $Z$ is quasi-projective. Thus we have the following diagram of $G$-equivariant maps:

$$\begin{tikzcd} & Z \arrow[d, "\phi"] & \\ & \Gamma \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X \arrow[rr, dashed, "f"] & & Y \end{tikzcd} ,$$

where the maps $\alpha\colon \Gamma \longrightarrow X$ and $\beta\colon \Gamma \longrightarrow Y$ are the natural projections to the first and the second factors. Since $Y$ is complete, $\alpha$ is a proper birational morphism, and consequently so is $\alpha\phi\colon Z \longrightarrow X$. Proposition 3 now says that $\alpha\phi$ is the blow up of a $G$-invariant ideal sheaf $\mathcal{I}$ on $X$. By a theorem of Hironaka [Hi, Main Theorem II] there is a sequence

$$\pi\colon X_m \xrightarrow{\pi_n} X_{n-1} \cdots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = X$$

of blowups with smooth centers such that $\pi^*\mathcal{I}$ is locally principal. Moreover, by a theorem of Bierstone and Milman [BM, Theorem 1.10] the sequence $\pi$ can be chosen in a canonical way; in particular, the center of each $\pi_i$ can be chosen to be smooth and $G$-invariant; see [BM, Remark 1.5]. Now by the universal property of blowing up (see [Ha, Proposition II.7.14]), the composition $(\alpha\phi)^{-1}\pi\colon X_m \dashrightarrow Z$ is a regular map. Hence, $f\pi = \beta\phi(\alpha\phi)^{-1}\pi\colon X_m \dashrightarrow Y$ is also regular, as claimed. $\square$

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## References

[BM] E. Bierstone, P. D. Milman, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Invent. math. **128** (1997), no. 2, 207–302.

[Ha] R. Hartshorne. Algebraic geometry. Springer, 1977.

[Hi] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic 0, I and II*, Annals of Math. **79** (1964), 109–326.

[Ka] T. Kambayashi, *Projective representations of algebraic groups of transformations*, Amer. J. Math. **88** (1966), 199–205.

[MFK] D. Mumford, J. Fogarty and F. Kirwan. Geometric invariant theory. Third enlarged edition, Springer, 1994.

[PV] V. L. Popov, E. B. Vinberg, *Invariant Theory*, in Encyclopedia of Math. Sciences **55**, Algebraic Geometry IV, edited by A. N. Parshin and I. R. Shafarevich, Springer-Verlag, 1994.

[$RY_1$] Z. Reichstein, B. Youssin, *Essential dimensions of algebraic groups and a resolution theorem for G-varieties*, with an appendix by J. Kollár and E. Szabó, Canadian J. Math., **52**, no. 5 (2000).(*)

[$RY_2$] Z. Reichstein, B. Youssin, *A birational invariant for algebraic group actions*, preprint.(*)

[Su] H. Sumihiro, *Equivariant completion*, J. Math. Kyoto Univ., **14**, no. 1 (1974), 1–28

[V] O. E. Villamayor U., *Patching local uniformizations,* Ann. scient. Éc. Norm. Sup., 4e série, **25** (1992), 629–677.

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