A NON-SPLIT TORSOR WITH TRIVIAL FIXED POINT OBSTRUCTION

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Abstract. Let G be a linear algebraic group and X be an irreducible algebraic variety with a generically free G-action, all defined over an algebraically closed base field of characteristic zero. It is well known that X can be viewed as a G-torsor, representing a class $[X]$ in $H^1(K,G)$, where K is the field of G-invariant rational functions on X. We have previously shown that if X has a smooth H-fixed point for some non-toral diagonalizable subgroup of G then $[X] \neq 1$. It is natural to ask if the converse is true, assuming G is connected and X is projective and smooth. In this note we show that the answer is "no".

1. Introduction

Let G be a linear algebraic group defined over an algebraically closed base field k of characteristic zero. By a *G-variety* we shall mean an algebraic variety X with a regular action of G (defined over k). We shall say that X is *generically free* if G acts freely on a dense open subset of X. Birational isomorphism classes of G-varieties X with $k(X)^G = K$ are in 1-1 correspondence with $H^1(K,G)$; see [6, 1.3]. We will call X *split* if one (and thus all) of the following equivalent conditions hold.

- X represents the trivial class in $H^1(K,G)$.
- X is birationally isomorphic to $Y \times G$ as a G-variety. Here Y is an algebraic variety with trivial G-action, and G acts on $Y \times G$ by left translations on the second factor.
- The (rational) quotient map $X \dashrightarrow X/G$ has a rational section;

cf. [6, 1.4]. We shall say that a subgroup of G is *toral* if it lies in a subtorus of G and *non-toral* otherwise. The starting point for this note is the following:

Proposition 1. ([10, Lemma 4.3]) *Let X be a generically free G-variety. If X has a smooth H-fixed point for some non-toral diagonalizable subgroup of H of G, then X is not split.*

In other words, the presence of a smooth H-fixed point on X is an obstruction to X being split; we shall refer to it as the *fixed point obstruction.*

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In the case where H is a non-toral finite abelian subgroup of G, we have described this obstruction in a more quantitative way by giving lower bounds on the essential dimension [9, Theorem 1.2], splitting degree [10, Theorem 1.1], and the size of a splitting group of X [10, Theorem 1.2] in terms of H. (Recall that a split variety has essential dimension 0, splitting degree 1 and splitting group $\{1\}$.)

The question that remained unanswered in [9] and [10] is whether or not the converse to Proposition 1 is also true. Of course, in stating the converse, we need to assume that the G-variety X is smooth and complete; otherwise the fixed point obstruction may not be "visible" because it may "hide" in the 'boundary" or in the singular locus of X. Fortunately, every class in $H^1(K, G)$ can be represented by a smooth complete (and even projective) G-variety; see [10, Proposition 2.2]. Moreover, the fixed point obstruction is detectable on any such model in the following sense. Suppose X is a generically free G-variety and Y is a smooth complete G-variety birationally isomorphic to X. If X has a smooth H-fixed point for some non-toral diagonalizable subgroup $H \subset G$ then so does Y; see [9, Proposition A2]. We also remark that if H is toral then $X^H \neq \emptyset$ by the Borel Fixed Point Theorem [1, Theorem 10.4]; thus only non-toral subgroups H are of interest here. To sum up, we will address the following:

Question 2. Is the fixed point obstruction the only obstruction to splitting? In other words, if X is a smooth projective generically free G-variety such that $X^H = \emptyset$ for every diagonalizable non-toral subgroup $H \subset G$, is X necessarily split?

Example 3. If G is a finite group then the answer is "no", because G can be made to act freely on an irreducible smooth projective curve X. Over $\mathbb{C}$ such a curve can be constructed as follows. Suppose G is generated by n elements, $g_1, \dots, g_n$. Let Y be a curve of genus n. Then the fundamental group $\pi_1(Y)$ is given by $2n$ generators $a_1, \dots, a_n, b_1, \dots, b_n$ and one relation

$$\prod_{i=1}^{n} a_i b_i a_i^{-1} b_i^{-1} = 1 \, .$$

The surjective homomorphism $\pi_1(Y) \longrightarrow G$, sending a_i to g_i and b_i to 1, gives rise to an unramified G-cover $X \longrightarrow Y$ of Riemann surfaces. By the Riemann Embedding Theorem, X is a smooth projective algebraic curve with a free G-action. The same argument goes through over any algebraically closed base field k of characteristic zero, provided that $\pi_1(Y)$ is interpreted as Grothendieck's algebraic fundamental group of Y; see [4, Expose XIII, Corollaire 2.12]. □

Question 2 becomes more delicate if we G is assumed to be connected. The purpose of this note is to show that under this assumption the answer is still "no". Our main result is the following:

Theorem 4. *Let p be an odd prime. Then there exists a smooth projective generically free PGL_p-variety X with the following properties:*

(a) X is not split,

(b) $X^H = \emptyset$ for every diagonalizable non-toral subgroup H of PGL_p,

(c) $k(X)^{\mathrm{PGL}_p}$ is a purely transcendental extension of k.

The rest of this paper is devoted to proving Theorem 4. In Sections 2 and 3 we reduce the proof to the question of existence of a certain division algebra of degree p; see Proposition 7. Our construction of this algebra in Section 4 relies on a criterion of Fein, Saltman and Schacher [2].

2. Nontoral subgroups of PGL_p

Consider the $p \times p$-matrices

$$(1) \qquad \sigma = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \zeta & \dots & 0 \\ & & \ddots & 0 \\ 0 & 0 & 0 & \zeta^{p-1} \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

where ζ is a primitive pth root of unity in k. Note that

$$\sigma\tau = \zeta\tau\sigma \ .$$

Thus the elements $\overline{\sigma}, \overline{\tau} \in \mathrm{PGL}_p$ represented, respectively, by σ and τ, generate an abelian subgroup; we shall denote this subgroup by A. Clearly $A \simeq (\mathbb{Z}/p\mathbb{Z}) \times (\mathbb{Z}/p\mathbb{Z})$. It is well known that, up to conjugacy, A is the unique non-toral elementary abelian subgroup of PGL_p; cf., e.g., [3, Theorem 3.1]. In the sequel we will need to know that A is in fact the unique *diagonalizable* subgroup with this property. For lack of a suitable reference, we give a direct elementary proof of this fact below.

Lemma 5. *Let H be a non-toral diagonalizable subgroup of PGL_p, where p is a prime. Then H is conjugate to A.*

In the sequel we will only need this lemma for odd p; however, for the sake of completeness, we will treat the case $p = 2$ as well.

Proof. Let $\widetilde{H}$ be the preimage of H in SL_p. Then for every $x, y \in \widetilde{H}$, $xyx^{-1}y^{-1}$ is a scalar matrix in SL_p, i.e., a matrix of the form $f(x,y)I$, where I is the $p \times p$ identity matrix and $f(x,y)$ is a pth root of unity. If $f(x,y) = 1$ for every $x, y \in \widetilde{H}$ then $\widetilde{H}$ is a commutative subgroup of SL_p consisting of semisimple elements. This implies that $\widetilde{H}$ is toral in SL_p (see, e.g., [1, Proposition 8.4]) and thus H is toral in PGL_p, contradicting our assumption. Therefore, $f(x,y)$ is a primitive pth root of unity for some $x, y \in \widetilde{H}$. Replacing x by x^i for an appropriate i, we may assume $f(x,y) = \zeta$, i.e.,

$$(2) \qquad xy = \zeta yx \ .$$

Suppose v is an eigenvector of x with associated eigenvalue $\lambda \neq 0$. Then (2) shows that $v_i = y^i(v)$ is an eigenvector of x with eigenvalue $\lambda\zeta^i$. These eigenvalues are distinct for $i = 0, 1, \dots, p-1$, and hence, the eigenvectors $v = v_0, v_1, \dots, v_{p-1}$ form a basis of k^p. Moreover, since $y^p(v)$ is an eigenvector for x with eigenvalue λ and the λ-eigenspace of x is 1-dimensional, $y^p(v) = cv_0$ for some $c \in k$. Writing x and y in the basis $v_0, \dots, v_{p-1}$, we see that

$$x = \lambda\sigma \quad \text{and} \quad y = \operatorname{diag}(c, 1, \dots, 1)\tau \ ,$$

where σ and τ are as in (1). Since $\det(y) = 1$, we see that $c = (-1)^{p+1}$. We now consider two cases:

(i) p is odd. Then $c = 1$ and $x, y \in \mathrm{SL}_p$ represent, respectively, $\overline{\sigma}$ and $\overline{\tau}$ in PGL_p.

(ii) $p = 2$. Here $c = -1$, and in the basis v_0, v_1,

$$x = \lambda\sigma = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Let $g = \operatorname{diag}(1, i)$, where i is a primitive 4th root of unity. Then gxg^{-1} and gyg^{-1} represent, respectively, $\overline{\sigma}$ and $\overline{\tau}$ in PGL_p.

Thus, after conjugation, we may assume that $A \subset H$. Since A is self-centralizing in PGL_p (cf. [9, Lemma 8.12(b)]), we conclude that $H = A$. □

3. Division algebras

Let F be a finitely generated field extension of k. Recall that elements of $H^1(F, \mathrm{PGL}_n)$ may be interpreted in two ways:

- as central simple algebras of degree n with center F; see [11, Section 10.5] or [5, p. 396], and
- as birational isomorphism classes of irreducible generically free PGL_n-varieties X such that $k(X)^{\mathrm{PGL}_n} = F$; see [6, Section 1.3] (cf. also [12, Section I.5.2]).

Thus to every central simple algebra D of degree n over F we can associate a generically free PGL_n-variety X_D with $k(X_D)^{\mathrm{PGL}_n} = F$. Moreover, X_D is uniquely defined up to birational isomorphism of PGL_n-varieties, and D can be recovered from X_D as the algebra of PGL_n-equivariant rational maps $X_D \dashrightarrow \mathrm{M}_n$; see [7, Proposition 8.6 and Lemma 9.1]. We shall write $D = RMaps_{\mathrm{PGL}_n}(X_D, \mathrm{M}_n)$. Note that $D \simeq \mathrm{M}_n(F)$ if and only if the PGL_n-variety X_D is split.

Proposition 6. *Let D be a division algebra of degree p with center K and X_D be an algebraic variety representing the class of D in $H^1(K, \mathrm{PGL}_n)$. Let A be the subgroup of PGL_p defined in Section 2. If D has an element of (reduced) trace 0 and norm 1 then X_D does not have a smooth A-fixed point.*

Proof. The proposition is proved in [8]; however, since it is not stated there in the exact form we need, we supply a short explanation. Let $x \in D$ be an element of trace zero and norm 1. Then the system

$$\begin{cases} \mathrm{Nrd}(x_1) = \cdots = \mathrm{Nrd}(x_p) \\ \mathrm{Trd}(x_1 \dots x_p) = 0 \end{cases} \tag{3}$$

has a nontrivial solution in D, namely $(x_1, \dots, x_p) = (x, 1, \dots, 1)$. (Here, as usual, Nrd and Trd denote, respectively, the reduced norm and the reduced trace in D.) On the other hand, by [8, Proposition 3.3 and Lemma 5.3], if X_D has a smooth A-fixed point then the system (3) has only the trivial solution $(x_1, \dots, x_p) = (0, \dots, 0)$. This shows that X_D does not have a smooth A-fixed point. □

We now observe that in order to prove Theorem 4 it is enough to establish the following:

Proposition 7. *There exists a division algebra D of degree p with center F such that*

(i) F is a purely transcendental extension of k, and

(ii) there exists an element $a \in D$ such that $\mathrm{Trd}(a) = 0$ *and* $\mathrm{Nrd}(a) = 1$.

Indeed, suppose D is a division algebra satisfying the conditions of Proposition 7. Let $X = X_D$ be a smooth projective PGL_p-variety representing the class of D in $H^1(K, \mathrm{PGL}_p)$; such a model exists by [10, Proposition 2.2]. We now check that $X = X_D$ has properties (a) - (c) claimed in the statement of Theorem 4:

(a) X is not split; otherwise $D \simeq \mathrm{M}_p(K)$ would not be a division algebra.

(b) By Lemma 5, we may assume $H = A$, and by Proposition 6, A acts on X without fixed points.

(c) $k(X)^{\mathrm{PGL}_p} = F$ is purely transcendental over k by Proposition 7(i). □

4. Conclusion of the proof

Our strategy for proving Proposition 7 will be to find an element a of norm 1 and trace 0 in a suitable field extension L/K of degree p, then embed this field extension into a division algebra.

Lemma 8. *For any $n \geq 3$ there exists a field extension L/K of degree n such that*

(i) K is a purely transcendental extension of k of transcendence degree 1 *and*

(ii) $\mathrm{Tr}_{L/K}(a) = 0$ *and* $\mathrm{N}_{L/K}(a) = 1$ *for some $a \in L$. Here* $\mathrm{Tr}_{L/K}(a)$ *and* $\mathrm{N}_{L/K}(a)$ *are the trace and the norm of a in L/K.*

Proof. Consider the polynomial

$$P(s,t) = s^n + ts + (-1)^n \in k[t,s]\,, \tag{4}$$

where t and s are independent commuting variables over k. Since we can write $P = P_0 t + P_1$, where $P_0 = s$ and $P_1 = s^n + (-1)^n$ are relatively prime in $k[s]$, we conclude that P is irreducible in $k[t,s]$, and hence, in $k(t)[s]$.

Now let $K = k(t)$, $L = K[s]/(P(t,s))$ and let a be the image of s in L. Then condition (i) is clearly satisfied. Moreover, since L/K is a field extension of degree n and P is the minimal polynomial of a over K, $-\operatorname{Tr}_{L/K}(a)$ and $(-1)^n \operatorname{N}_{L/K}(a)$ are, respectively, the coefficient of s^{n-1} and the constant term of P. Thus $\operatorname{Tr}_{L/K}(a) = 0$ and $\operatorname{N}_{L/K}(a) = 1$, as claimed. □

We are now ready to prove Proposition 7. Let L/K be as in Lemma 8, with $n = p$. It is sufficient to show that there exists a division algebra D with center $F = K(\lambda_1, \dots, \lambda_r)$ and maximal subfield $L(\lambda_1, \dots, \lambda_r)$, where $\lambda_1, \dots, \lambda_r$ are algebraically independent variables over K. Then D is the algebra we want: F is a purely transcendental extension of k and an element $a \in D$ with desired properties can be found in $L \subset D$.

To show that such a D exists, we appeal to a result of Fein, Saltman and Schacher [2, Corollary 5.4]. Let G be a finite group, H be a subgroup of G and q be a prime dividing $|G|$. Following [2], we define $m_q(G,H)$ to be the maximal value of $|T|$, taken over all q-subgroups T of G which are contained in $\bigcup_{g \in G} gHg^{-1}$.

Returning to the setting of Lemma 8, let E be the Galois closure of L over K, $G = \operatorname{Gal}(E/K)$ and $H = \operatorname{Gal}(E/L)$. [2, Corollary 5.4] guarantees the existence of D if $m_q(G,H) = |H_q|$ for every q dividing $[L:K]$; here H_q is a Sylow q-subgroup of H. In our case $[L:K] = p$, so we only need to check that $m_p(G,H) = |H_p|$.

Note that E is the splitting field and G is the Galois group of the irreducible polynomial (4) over $K = k(t)$, with $n = p$. Thus G is naturally a subgroup of S_p and consequently $|G|$ is not divisible by p^2. On the other hand, $[G:H] = [L:K] = p$. We conclude that $|H|$ is not divisible by p, i.e., $|H_p| = 1$. Moreover, the order of every element of $\bigcup_{g \in G} gHg^{-1}$ is prime to p; thus $m_p(G,H) = 1$. To sum up, $m_p(G,H) = 1 = |H_p|$, and [2, Corollary 5.4] applies.

This completes the proof of Proposition 7 and thus of Theorem 4. □

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