GROUP ACTIONS AND INVARIANTS IN ALGEBRAS OF GENERIC MATRICES

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ABSTRACT. We show that the fixed elements for the natural GL_m-action on the universal division algebra $UD(m,n)$ of m generic $n \times n$-matrices form a division subalgebra of degree n, assuming $n \geq 3$ and $2 \leq m \leq n^2 - 2$. This allows us to describe the asymptotic behavior of the dimension of the space of SL_m-invariant homogeneous central polynomials $p(X_1, \ldots, X_m)$ for $n \times n$-matrices. Here the base field is assumed to be of characteristic zero.

CONTENTS

1. Introduction 1
2. Preliminaries 4
3. Geometric actions on division algebras 6
4. Dimension counting in the Grassmannian 9
5. Proof of Theorem 1.4 over an algebraically closed field 11
6. SL_m-invariant generic matrices 13
7. Proof of Theorems 1.2 and 1.4 14
8. The case $m \geq n^2 - 1$ 15
9. Proof of Theorem 1.3 18
10. Standard polynomials 19
Acknowledgments 20
References 21

1. INTRODUCTION

Let k be a field of characteristic zero, m and n be integers ≥ 2, and $G_{m,n} = k\{X_1, \ldots, X_m\}$ be the k-algebra of m generic $n \times n$-matrices. That

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Date: September 2, 2005.

2000 *Mathematics Subject Classification.* 16R30, 14L30, 16K20, 16W22, 15A72.

Key words and phrases. Generic matrices, universal division algebra, central polynomial, PI-degree, group action, geometric action, invariants, concomitants, Gelfand-Kirillov dimension.

Z. Reichstein was supported in part by an NSERC research grant.

N. Vonessen gratefully acknowledges the support of the University of Montana and the hospitality of the University of British Columbia during his sabbatical in 2002/2003, when part of this research was done.

is, $G_{m,n}$ is the k-subalgebra of $\mathrm{M}_n(k[x_{ij}^{(h)}])$ generated by

$$X_1 = (x_{ij}^{(1)}), \dots, X_m = (x_{ij}^{(m)}) \,,$$

where the $x_{ij}^{(h)}$ are mn^2 independent commuting variables. By a theorem of Amitsur, $G_{m,n}$ is a domain of PI-degree n. There is a natural action of the general linear group GL_m on $G_{m,n}$ given by

$$g = (g_{ij})\colon X_j \longmapsto \sum_{i=1}^{m} g_{ij} X_i \,. \tag{1.1}$$

In this paper we will prove the following theorem.

1.2. **Theorem.** *For $2 \le m \le n^2 - 2$, the domain $(G_{m,n})^{\mathrm{SL}_m}$ has PI-degree n.*

The trace ring $T_{m,n}$ of $G_{m,n}$ is defined as the subring of $\mathrm{M}_n(k[x_{ij}^{(h)}])$ generated by elements of the form Y and $\mathrm{tr}(Y)$, as Y ranges over $G_{m,n}$. The action (1.1) on $G_{m,n}$ naturally extends to $T_{m,n}$. Note that the algebras $G_{m,n}$ and $T_{m,n}$, and their centers $\mathrm{Z}(G_{m,n})$ and $\mathrm{Z}(T_{m,n})$ have a natural $\mathbb{Z}$-grading inherited from $\mathrm{M}_n(k[x_{ij}^{(h)}])$ (each variable $x_{ij}^{(h)}$ has degree 1) and that this grading is preserved by the action (1.1). As a consequence of Theorem 1.2 we obtain the following result.

1.3. **Theorem.** *Let $2 \le m \le n^2 - 2$, and let R be one of the rings $G_{m,n}$, $T_{m,n}$, $\mathrm{Z}(G_{m,n})$, or $\mathrm{Z}(T_{m,n})$. Denote the degree d homogeneous component of R by $R[d]$. Then*

$$\limsup_{d \to \infty} \frac{\dim_k R^{\mathrm{SL}_m}[d]}{d^{(m-1)n^2 - m^2 + 1}}$$

is a finite nonzero number.

One can think of the center of $G_{m,n}$ as consisting of the m-variable central polynomials for $n \times n$-matrices (over commutative k-algebras). Theorem 1.3 thus describes, for $R = \mathrm{Z}(G_{m,n})$, the asymptotic behavior of the dimension of the space of SL_m-invariant homogeneous central polynomials $p(X_1, \dots, X_m)$ for $n \times n$-matrices.

The GL_m-representations on $G_{m,n}$, $\mathrm{Z}(G_{m,n})$, $T_{m,n}$ and $\mathrm{Z}(T_{m,n})$ have been extensively studied; see, e.g., [1, 2, 7, 9, 17]. Once again, let R be one of these rings. Recall that the irreducible polynomial representations of GL_m are indexed by partitions $\lambda = (\lambda_1, \dots, \lambda_s)$ with $s \le m$ parts; cf., e.g., [9, Section 2]. Denote the multiplicity of the irreducible GL_m-representation corresponding to λ in R by $\mathrm{mult}_\lambda(R)$. If (r^m) is the partition $(r, \dots, r)$ (m times) then it is easy to show that

$$\dim R^{\mathrm{SL}_m}[d] = \begin{cases} \mathrm{mult}_{(r^m)}(R) & \text{if } d = rm, \\ 0 & \text{if } d \text{ is not a multiple of } m; \end{cases}$$

see Remark 9.2. The conclusion of Theorem 1.3 can thus be rephrased by saying that

$$\limsup_{r\to\infty} \frac{\mathrm{mult}_{(r^m)}(R)}{r^{(m-1)n^2-m^2+1}}$$

is a finite nonzero number. We also note that by the Berele-Drensky-Formanek correspondence, $\mathrm{mult}_{(r^m)}(R)$ equals the multiplicity of the S_m-character $\chi^{(d^m)}$ in the cocharacter sequence of R; see [9, Section 4].

The division algebra of quotients of $G_{m,n}$ (or equivalently, of $T_{m,n}$) is called the *universal division algebra* of m generic $n \times n$-matrices; we shall denote it by $UD(m,n)$. Note that the GL_m-action (1.1) on $G_{m,n}$ naturally extends to $UD(m,n)$. We shall deduce Theorem 1.2 from the following related result.

1.4. **Theorem.** *If $2 \le m \le n^2 - 2$ and $n \ge 3$, then $UD(m,n)^{\mathrm{GL}_m}$ is a division algebra of degree n.*

For all other values of $m, n \ge 2$, $UD(m,n)^{\mathrm{GL}_m}$ is a field; see Propositions 8.1 and 8.3. A brief summary of the properties of $UD(m,n)^{\mathrm{GL}_m}$ and $UD(m,n)^{\mathrm{SL}_m}$ is given in the two tables below.

TABLE 1. Properties of $UD(m,n)^{\mathrm{GL}_m}$

Case	PI-Degree	Transcendence Degree/k	Central in $UD(m,n)$?
$m \le n^2-2$, $n \ge 3$	n	$(m-1)n^2-m^2+1$	No
$m = n^2-1$	1	$n-1$	No
$m = n = 2$	1	1	No
$m \ge n^2$	1	0	Yes

TABLE 2. Properties of $UD(m,n)^{\mathrm{SL}_m}$

Case	PI-Degree	Transcendence Degree/k	Central in $UD(m,n)$?
$m \le n^2-2$	n	$(m-1)n^2-m^2+2$	No
$m = n^2-1$	1	n	No
$m = n^2$	1	1	Yes
$m \ge n^2+1$	1	0	Yes

The assertions of the tables in the cases where $m \le n^2 - 2$ and $n \ge 3$ are based on Theorems 1.4 and 5.1, the case where $m = n = 2$ is considered in [20, Section 14], and the cases where $m \ge n^2 - 1$ are treated in detail in Section 8.

It appears likely that Theorems 1.2 – 1.4 remain valid in prime characteristic (perhaps, not dividing n); we have not attempted to extend them in this direction. Our arguments rely on the work of Richardson [21] and on our own prior papers [19, 20], all of which make the characteristic zero assumption.[1]

Conventions and Terminology. All central simple algebras in this paper are assumed to be finite-dimensional over their centers. All algebraic varieties, algebraic groups, group actions, morphisms, rational maps, etc., are assumed to be defined over the base field k (which we always assume to be of characteristic zero). By a point of an algebraic variety X we shall always mean a k-point. Throughout, G will denote a linear algebraic group. We shall refer to an algebraic variety X endowed with a regular G-action as a G-variety. We will say that a G-variety X (or the G-action on X) is generically free if $\mathrm{Stab}_G(x) = \{1\}$ for $x \in X$ in general position. Finally, unless otherwise specified, m and n are integers ≥ 2.

2. Preliminaries

Concomitants. Let Γ be an algebraic group and V and W be Γ-varieties. Then we shall denote the set of Γ-equivariant morphisms $V \longrightarrow W$ (also known as *concomitants*) by $\mathrm{Morph}_\Gamma(V, W)$ and the set of Γ-equivariant rational maps $V \dashrightarrow W$ (also known as *rational concomitants*) by $\mathrm{RMaps}_\Gamma(V, W)$.

In the case where W is a finite-dimensional linear representation of Γ, we also define a *relative concomitant* as a morphism $f\colon V \longrightarrow W$ satisfying the following condition (which is slightly weaker than Γ-equivariance): there is a character $\chi\colon \Gamma \longrightarrow k^*$ such that

$$f(g \cdot v) = \chi(g)\ \big(g \cdot f(v)\big)$$

for all $v \in V$ and $g \in \Gamma$. For a rational map $f\colon V \dashrightarrow W$ the notion of a *relative rational concomitant* is defined in a similar manner. If $W = k$, with trivial Γ-action, then the term "invariant" is used in place of "concomitant". For future reference we record the following:

2.1. **Lemma.** *Suppose V and W are finite-dimensional linear representations of Γ. Every rational concomitant $f\colon V \dashrightarrow W$ can be written as $\frac{a}{b}$, where a is a relative concomitant and b is a relative invariant.*

Proof. See the proof of [5, Chapter 1, Proposition 1]. Note that the characters associated to a and b are necessarily equal. □

If W is a k-algebra and Γ acts on W by k-algebra automorphisms, then the algebra structure of W induces algebra structures on $\mathrm{Morph}_\Gamma(V, W)$ and $\mathrm{RMaps}_\Gamma(V, W)$ in a natural way. Namely, given $a, b\colon V \longrightarrow W$ (or

[1] We remark that Richardson [21] worked over $k = \mathbb{C}$, and his proofs are based on analytic techniques. The results we need (in particular, [21, Theorem 9.3.1]), remain valid over any algebraically closed field of characteristic zero by the Lefschetz principle. Extending [21, Theorem 9.3.1] to prime characteristic is an open problem of independent interest.

$a, b \colon V \dashrightarrow W$), one defines $a+b$ and ab by $(a+b)(v) = a(v)+b(v)$ and $ab(v) = a(v)b(v)$ for $v \in V$.

2.2. **Theorem.** (Procesi [16, Theorem 2.1]; cf. also [9, Theorem 10], or [24, Theorem 14.16]) *Let* $(\mathrm{M}_n)^m$ *be the space of* m*-tuples of* $n \times n$*-matrices; the group* PGL_n *acts on it by simultaneous conjugation. Then*

(a) $\mathrm{Morph}_{\mathrm{PGL}_n}((\mathrm{M}_n)^m, \mathrm{M}_n) \simeq T_{m,n}$
(b) $\mathrm{RMaps}_{\mathrm{PGL}_n}((\mathrm{M}_n)^m, \mathrm{M}_n) \simeq UD(m,n)$

Moreover, the two isomorphisms identify the i*-th projection* $(\mathrm{M}_n)^m \longrightarrow \mathrm{M}_n$ *with the* i*-th generic matrix* X_i*.* □

Here $T_{m,n}$ and $UD(m,n)$ are, respectively, the trace ring and the universal division algebra of m generic $n \times n$-matrices, defined in the introduction. Note that part (b) of Theorem 2.2 follows from part (a) by Lemma 2.1, since the simple group PGL_n does not have nontrivial characters (so that relative concomitants and invariants are actually concomitants and invariants, respectively).

We also remark that the construction of $T_{m,n}$ remains well-defined if $m = 1$. Theorem 2.2 also holds in this case, provided that one defines $UD(1,n)$ to be the field of quotients of $T_{1,n}$, rather than $G_{1,n}$. (For $m \geq 2$, $T_{m,n}$ and $G_{m,n}$ have the same division algebra of quotients, but this is not the case for $m = 1$.).

Geometric actions. For the rest of this section we will assume that k is algebraically closed. If X is a PGL_n-variety, then, as we mentioned above, $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$ has an algebra structure naturally induced from M_n. If the PGL_n-action on X is generically free then $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$ is a central simple algebra of degree n, with center $k(X)^{\mathrm{PGL}_n}$; cf. [18, Lemmas 8.5 and 9.1].

Suppose that X is a $G \times \mathrm{PGL}_n$-variety, and that the PGL_n-action on X is generically free. Then the G-action on X naturally induces a G-action on $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$. Following [20] we define the action of an algebraic group G on a central simple algebra A to be *geometric* if A is G-equivariantly isomorphic to $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$ for some $G \times \mathrm{PGL}_n$-variety X as above. The $G \times \mathrm{PGL}_n$-variety X is then called the *associated variety* for the G-action on A; this associated variety is unique (as a $G \times \mathrm{PGL}_n$-variety), up to birational isomorphism; cf. [20, Corollary 3.2].

Note that we defined geometric actions only if k is algebraically closed. Also note that if an algebraic group acts geometrically on a central simple algebra A, then the center of A is necessarily a finitely generated field extension of k.

Of particular interest to us will be the case where $X = (\mathrm{M}_n)^m$ is the space of m-tuples of $n \times n$-matrices. Here PGL_n acts on $(\mathrm{M}_n)^m$ by simultaneous conjugation (since $m \geq 2$, this action is generically free) and $G = \mathrm{GL}_m$ acts on $(A_1, \dots, A_m) \in (\mathrm{M}_n)^m$ by sending $(A_1, \dots, A_m)$ to $(B_1, \dots, B_m)$ where $B_j = \sum_{i=1}^m c_{ij} A_i$ and $g^{-1} = (c_{ij})$. The actions of GL_m and PGL_n commute, and the GL_m-action on $(\mathrm{M}_n)^m$ induces the GL_m-action (1.1) on $UD(m,n)$. So $(\mathrm{M}_n)^m$ is the associated variety for the GL_m-action on $UD(m,n)$; see Theorem 2.2 (cf. also [20, Example 3.4]).

We conclude this section with a simple result which we will use repeatedly.

2.3. **Lemma.** *Assume k is algebraically closed. Let X be a $G \times \mathrm{PGL}_n$-variety which is PGL_n-generically free. Denote by $\pi\colon X \dashrightarrow X/G$ the rational quotient map for the G-action. Then for $x \in X$ in general position, the projection $\mathrm{pr}_2\colon G \times \mathrm{PGL}_n \longrightarrow \mathrm{PGL}_n$ onto the second factor induces an isomorphism from $\mathrm{Stab}_{G\times\mathrm{PGL}_n}(x)/\mathrm{Stab}_G(x)$ onto $\mathrm{Stab}_{\mathrm{PGL}_n}(\pi(x))$.* □

Proof. Recall that by a theorem of Rosenlicht, $\pi^{-1}(\overline{x})$ is a single G-orbit for $\overline{x} \in X/G$ in general position; see [22, Theorem 2] or [14, Section 2.3]. Consequently, for $x \in X$ in general position the projection pr_2 restricts to a surjective morphism $\mathrm{Stab}_{G\times\mathrm{PGL}_n}(x) \longrightarrow \mathrm{Stab}_{\mathrm{PGL}_n}(\pi(x))$ of algebraic groups. The kernel of this morphism is clearly $\mathrm{Stab}_G(x)$, and the lemma follows. □

3. Geometric actions on division algebras

Throughout this section we will assume that k is algebraically closed. The main result of this section is the following theorem; after its proof, we will deduce several corollaries.

3.1. **Theorem.** *Assume k is algebraically closed. Let G be a linear algebraic group acting geometrically on a division algebra D of degree n. Let X be the associated $G \times \mathrm{PGL}_n$-variety. Then for $x \in X$ in general position,*

$$S_x := \mathrm{Stab}_{G\times\mathrm{PGL}_n}(x)/\mathrm{Stab}_G(x)$$

is reductive.

Proof. Let X be the associated $G \times \mathrm{PGL}_n$-variety for the G-action on D. Recall that the PGL_n-action on X is generically free. We want to show that the group $S_x = \mathrm{Stab}_{G\times\mathrm{PGL}_n}(x)/\mathrm{Stab}_G(x)$ is reductive for $x \in X$ in general position. Assume the contrary. Denoting the unipotent radical by R_u, this means that $R_u(\mathrm{Stab}_{G\times\mathrm{PGL}_n}(x))$ is not contained in G. Since unipotent groups are connected, this is equivalent to

$$Lie\big(R_u(\mathrm{Stab}_{G\times\mathrm{PGL}_n}(x))\big) \not\subseteq Lie(G) \tag{3.2}$$

for $x \in X$ in general position. Here and in the sequel *Lie* stands for the Lie algebra. To simplify notation, set $H = G \times \mathrm{PGL}_n$, and for $x \in X$, set $H_x = R_u(\mathrm{Stab}_H(x))$. Now define $U_X \subseteq X \times Lie(H)$ by

$$U_X = \big\{(x,a) \,\big|\, x \in X \text{ and } a \in Lie(H_x)\big\}\,.$$

We first show that U_X is a vector bundle over a dense open subset $X_0 \subset X$. By [21, 6.2.1, 9.2.1, and 6.5.3], there is an H-stable dense open subset X_0 of X such that $\{H_x \mid x \in X_0\}$ is an algebraic family of algebraic subgroups of H. Moreover, $\dim(H_x)$ is constant for $x \in X_0$, say equal to d. Replacing X by X_0, we may assume that $\{H_x \mid x \in X\}$ is an algebraic family of algebraic subgroups of H. By [21, 6.2.2], $x \mapsto Lie(H_x)$ defines a morphism of algebraic varieties from X to the Grassmannian of d-dimensional subspaces of $Lie(H)$. Since the universal bundle over this Grassmannian is a vector bundle (see, e.g., [28, 3.3.1]), its pull-back U_X is a vector bundle over X.

Note also that U_X is, by definition, an H-invariant subbundle of the trivial bundle $X \times Lie(H) \longrightarrow X$; here H acts on its Lie algebra by the adjoint action. Since the PGL_n-action on X is generically free, the no-name lemma tells us that there is a PGL_n-equivariant birational isomorphism $U_X \dashrightarrow X \times k^d$ such that the following diagram commutes

$$\begin{array}{ccc} U_X & \overset{\simeq}{\dashrightarrow} & X \times k^d \\ \downarrow & \swarrow & \\ X & & \end{array}$$

(For a proof and a brief discussion of the history of the no-name lemma, see [4, Section 4.3].) In other words, the vector bundle $U_X \longrightarrow X$ has d PGL_n-equivariant rational sections $\beta_1, \ldots, \beta_d \colon X \dashrightarrow U_X$ such that $\beta_1(x), \ldots, \beta_d(x)$ are linearly independent for $x \in X$ in general position. We identify here $\beta_i(x)$ with a if $\beta_i(x) = (x, a) \in \{x\} \times Lie(H_x)$. In view of (3.2), some k-linear combination $\beta = c_1\beta_1 + \cdots + c_d\beta_d$ has the property that $\beta(x) \notin Lie(G)$ for $x \in X$ in general position.

Now recall that the natural projection $\mathrm{SL}_n \longrightarrow \mathrm{PGL}_n$ induces a Lie algebra isomorphism $sl_n \longrightarrow Lie(\mathrm{PGL}_n)$, allowing us to identify the two Lie algebras. Hence

$$U_X \subseteq X \times Lie(G) \times sl_n .$$

Let $f = pr \circ \beta \colon X \dashrightarrow sl_n$, where $pr \colon U_X \longrightarrow sl_n$ denotes the natural projection. Note that $sl_n \hookrightarrow gl_n = \mathrm{M}_n$, so that f may be viewed as a PGL_n-equivariant rational map $X \dashrightarrow \mathrm{M}_n$, i.e., as an element of D. The condition that $\beta(x) \notin Lie(G)$ ensures that $f \neq 0$. On the other hand, we will show below that for $x \in X$ in general position, $f(x)$ is a nilpotent $n \times n$-matrix, so that $f^n = 0$. This means that D contains a non-zero nilpotent element f, contradicting our assumption that D is a division algebra.

It remains to be shown that for $x \in X$ in general position, $f(x)$ is a nilpotent matrix. The natural projection $G \times \mathrm{PGL}_n \longrightarrow \mathrm{PGL}_n$ maps the unipotent group H_x to a unipotent subgroup U of PGL_n. Denote by K the preimage of U in SL_n. It is a solvable group, so its subset K_u of unipotent elements is a closed subgroup. The surjection $K_u \longrightarrow U$ is finite-to-one, so their Lie algebras are isomorphic. In particular, $f(x)$ belongs to $Lie(K_u) \subset sl_n \subset gl_n = \mathrm{M}_n$. Finally, since K_u is a unipotent subgroup of GL_n, its Lie algebra in M_n consists of nilpotent matrices, see, e.g., [3, I.4.8]. This completes the proof of Theorem 3.1. $\square$

We now proceed with the corollaries. Recall that a subgroup $S \subset \Gamma$ is said to be a *stabilizer in general position* for a Γ-variety X if there exists a dense Γ-invariant subset $U \subset X$ such that $\mathrm{Stab}(x)$ is conjugate to S for any $x \in U$. For a detailed discussion of this notion, see [14, Section 7].

3.3. **Corollary.** *Assume k is algebraically closed. Let G be a linear algebraic group acting geometrically on a division algebra D of degree n. Let X be the associated $G \times \mathrm{PGL}_n$-variety.*

(a) *The induced PGL_n-action on the rational quotient X/G has a stabilizer S in general position. Moreover, S is reductive, and $S \simeq S_x = \mathrm{Stab}_{G\times \mathrm{PGL}_n}(x)/\mathrm{Stab}_G(x)$ for $x \in X$ in general position.*

(b) *If the G-action on X is generically free, then*

$$\begin{aligned}\mathrm{trdeg}_k(\mathrm{Z}(D^G)) &= \mathrm{trdeg}_k(\mathrm{Z}(D)^G) \\ &= \dim(X) - \dim(G) + \dim(S) - n^2 + 1\,.\end{aligned}$$

Proof. (a) It follows from Theorem 3.1 and Lemma 2.3 that points in X/G in general position have a reductive stabilizer. A theorem of Richardson (see [21, Theorem 9.3.1] or [14, Theorem 7.1]) now implies that the PGL_n-action on X/G has a stabilizer S in general position. By Lemma 2.3, $S \simeq S_x = \mathrm{Stab}_{G\times \mathrm{PGL}_n}(x)/\mathrm{Stab}_G(x)$ for $x \in X$ in general position.

(b) The first equality follows from the fact that $\mathrm{Z}(D^G)$ is an algebraic extension of $\mathrm{Z}(D)^G$. Indeed, the minimal polynomial of any element of D^G over $\mathrm{Z}(D)$ is unique and must therefore have coefficients in $\mathrm{Z}(D)^G$.

To prove the second equality, note that, $\mathrm{Z}(D) = k(X/\mathrm{PGL}_n) = k(X)^{\mathrm{PGL}_n}$ and thus

$$\mathrm{Z}(D)^G \simeq (k(X)^{\mathrm{PGL}_n})^G = k(X)^{G\times \mathrm{PGL}_n} = k(X/(G \times \mathrm{PGL}_n))\,.$$

Since we are assuming that G acts generically freely on X, part (a) implies that $S \simeq \mathrm{Stab}_{G\times \mathrm{PGL}_n}(x)$ for $x \in X$ in general position. Hence the dimension of the general fiber of the rational quotient map $X \dashrightarrow X/(G \times \mathrm{PGL}_n)$ is equal to the dimension of $(G\times \mathrm{PGL}_n)/S$. The fiber dimension theorem now tells us that the transcendence degree of $\mathrm{Z}(D)^G$ is

$$\dim X/(G \times \mathrm{PGL}_n) = \dim(X) - \dim(G) - \dim(\mathrm{PGL}_n) + \dim(S)\,. \quad \square$$

3.4. **Corollary.** *Assume k is algebraically closed. Let G be a unipotent linear algebraic group acting geometrically on a division algebra D of degree n. Then D^G is a division algebra of degree n.*

This was proved for algebraic actions in [20, Proposition 12.1].

Proof. By [20, Lemma 7.1], for $x \in X$ in general position, $\mathrm{Stab}_{G\times \mathrm{PGL}_n}(x)$ is isomorphic to a subgroup of G, so is unipotent. On the other hand, by Theorem 3.1, the projection S_x of this group to PGL_n is reductive. Thus S_x is both unipotent and reductive, which is only possible if $S_x = \{1\}$. In other words,

$$\mathrm{Stab}_{G\times \mathrm{PGL}_n}(x) \subset G \times \{1\}\,.$$

The desired conclusion now follows from [20, Theorem 1.4]. $\square$

4. Dimension counting in the Grassmannian

In preparation for the proof of Theorem 1.4 in the next section, we will now establish the following:

4.1. **Proposition.** *Assume k is algebraically closed. Let V be an N-dimensional k-vector space and let $V = V_1 \oplus \cdots \oplus V_r$, where $\dim(V_i) = N_i \geq 1$. Let Z be the subset of the Grassmannian $\mathrm{Gr}(m, N)$ consisting of m-dimensional subspaces W of V of the form*

$$W = W_1 \oplus \cdots \oplus W_r ,$$

where $W_i \subseteq V_i$. (Here we allow $W_i = (0)$ for some i.) Then Z is a closed subvariety of $\mathrm{Gr}(m, N)$. If $2 \leq m \leq N-2$, then each irreducible component of Z has codimension $\geq N - \max_{i=1,\dots,r}(N_i)$ in $\mathrm{Gr}(m, N)$. Moreover, equality holds (for some irreducible component of Z) only if (i) *$r = 1$ or* (ii) *$r = 2$, $m = 2$ and $N = 4$.*

Proof. Let $m_1, \dots m_r$ be non-negative integers such that $m_1 + \cdots + m_r = m$ and such that $m_i \leq N_i$ for all i. Let $Z_{m_1,\dots,m_r}$ be the image of the map

$$\phi_{m_1,\dots,m_r} \colon \mathrm{Gr}(m_1, N_1) \times \cdots \times \mathrm{Gr}(m_r, N_r) \longrightarrow \mathrm{Gr}(m, N)$$

given by $(W_1, \dots, W_r) \mapsto W_1 \oplus \cdots \oplus W_r$. (Here $\mathrm{Gr}(m_i, N_i)$ is the Grassmannian of m_i-dimensional vector subspaces of V_i.) Since the domain of the map $\phi_{m_1,\dots,m_r}$ is projective, its image is closed in $\mathrm{Gr}(m, N)$. Thus each $Z_{m_1,\dots,m_r}$ is a closed irreducible subvariety of $\mathrm{Gr}(m, N)$ birationally isomorphic to

$$\mathrm{Gr}(m_1, N_1) \times \cdots \times \mathrm{Gr}(m_r, N_r)$$

and Z is the union of the $Z_{m_1,\dots,m_r}$. It remains to show that

$$\dim\, \mathrm{Gr}(m, N) - \sum_{i=1}^{r} \dim\, \mathrm{Gr}(m_i, N_i) \geq N - \max_{i=1,\dots,r}(N_i) , \tag{4.2}$$

and that equality is only possible if $r = 1$ or $r = 2$, $N_1 = N_2 = 2$ and $m_1 = m_2 = 1$ (and thus $N = N_1 + N_2 = 4$ and $m = m_1 + m_2 = 2$). Recall that $\dim \mathrm{Gr}(m, N) = (N - m)m$. Letting $l_i = N_i - m_i$ and $l = N - m = l_1 + \cdots + l_r$, we can rewrite (4.2) as

$$lm - \sum_{i=1}^{r} l_i m_i \geq l + m - \max_{i=1,\dots,r}(l_i + m_i)$$

or, equivalently,

$$(l-1)(m-1) - 1 \geq \sum_{i=1}^{r} l_i m_i - \max_{i=1,\dots,r}(l_i + m_i) .$$

Proposition 4.1 is thus a consequence of the following elementary lemma. □

4.3. **Lemma.** *Let* $(l_1, m_1), \dots, (l_r, m_r)$ *be* r *pairs of non-negative integers and let* $l = \sum_{i=1}^r l_i$ *and* $m = \sum_{i=1}^r m_i$. *Assume that* $l_i + m_i \geq 1$ *for every* $i = 1, \dots, r$ *and* $l, m \geq 2$. *Then*

$$(l-1)(m-1) - 1 \geq \sum_{i=1}^{r} l_i m_i - \max_{i=1,\dots,r} (l_i + m_i)\,. \tag{4.4}$$

Moreover, equality holds if and only if either (i) $r = 1$ *or* (ii) $r = 2$ *and* $(l_1, m_1) = (l_2, m_2) = (1, 1)$.

Proof. We consider two cases.

Case 1: Suppose that for every $i = 1, \dots, r$, either $l_i = 0$ or $m_i = 0$. Since $l, m \geq 2$, we have $(l-1)(m-1) - 1 \geq 0$. On the other hand, $\sum_{i=1}^r l_i m_i - \max_{i=1,\dots,r}(l_i + m_i) = -\max_{i=1,\dots,r}(l_i + m_i) < 0$. Hence, in this case (4.4) holds and is a strict inequality.

Case 2: Now suppose that $l_i, m_i \geq 1$ for some $i \geq 1, \dots, r$. After renumbering the pairs $(l_1, m_1), \dots, (l_r, m_r)$, we may assume $i = 1$. Now set

$$l'_j = \begin{cases} l_1 - 1, \text{ if } j = 1, \\ l_j, \text{ otherwise;} \end{cases} \quad \text{and} \quad m'_j = \begin{cases} m_1 - 1, \text{ if } j = 1 \\ m_j, \text{ otherwise.} \end{cases}$$

Note that $l'_j, m'_j \geq 0$ for every $j = 1, \dots, r$. Thus

$$\begin{aligned} (l-1)(m-1) - 1 &= (\sum_{i=1}^r l'_i)(\sum_{j=1}^r m'_j) - 1 = \sum_{i=1}^r l'_i m'_i + \sum_{i \neq j} l'_i m'_j - 1 \\ &\geq \sum_{i=1}^r l'_i m'_i - 1 = \sum_{i=1}^r l_i m_i - (l_1 + m_1) \\ &\geq \sum_{i=1}^r l_i m_i - \max_{i=1,\dots,r} (l_i + m_i)\,. \end{aligned}$$

This completes the proof of the inequality (4.4).

It is easy to see that equality holds in cases (i) and (ii). It remains to show that the inequality (4.4) is strict for all other choices of $(l_1, m_1), \dots, (l_r, m_r)$. Indeed, a closer look at the above argument shows that equality in (4.4) can hold if and only if we are in Case 2 and

(a) $l'_i m'_j = 0$ whenever $i \neq j$ and

(b) $l_1 + m_1 = \max_{i=1,\dots,r}(l_i + m_i)$.

Assume that conditions (a) and (b) are satisfied. Since $\sum_{i=1}^r l'_i = l - 1 \geq 1$, we cannot have $l'_i = 0$ for all $i = 1, \dots, r$. In other words, $l'_{i_0} \geq 1$ for some $i_0 \in \{1, \dots, r\}$. Then condition (a) says that $m'_j = 0$ for every $j \neq i_0$. On the other hand, $m'_{i_0} = \sum_{j=1}^r m'_j = m - 1 \geq 1$, and applying condition (a) once again, we conclude that that $l'_i = 0$ for every $i \neq i_0$. To sum up, there exists an $i_0 \in \{1, \dots, r\}$ such that $l'_{i_0} = l - 1$, $m_{i_0} = m - 1$ and $l'_i = m'_i = 0$ for every $i \neq i_0$.

In particular, for every $i \neq 1, i_0$, we have $l_i = l'_i = 0$ and $m_i = m'_i = 0$, contradicting our assumption that $l_i + m_i \geq 1$. Thus $i \in \{1, i_0\}$ for every $i = 1, \ldots, r$. In other words, either $i_0 = 1$ and $r = 1$ (in which case (i) holds, and we are done) or $i_0 = 2$ and $r = 2$. In the latter case $l'_1 = m'_1 = 0$, $l'_2 = l-1$ and $m'_2 = m-1$, i.e., $(l_1, m_1) = (1, 1)$ and $(l_2, m_2) = (l-1, m-1)$. Condition (b) now tells us that $l = m = 2$, so that (ii) holds.

This completes the proof of Lemma 4.3 and thus of Proposition 4.1. □

5. Proof of Theorem 1.4 over an algebraically closed field

Recall from Section 2 that $X = (\mathrm{M}_n)^m$ is the associated $\mathrm{GL}_m \times \mathrm{PGL}_n$-variety for the GL_m-action on $UD(m,n)$. Here PGL_n acts on $(\mathrm{M}_n)^m$ by simultaneous conjugation (since $m \geq 2$, this action is generically free) and GL_m acts on $(\mathrm{M}_n)^m$ by sending $(A_1, \ldots, A_m)$ to $(B_1, \ldots, B_m)$, where $B_j = \sum_{i=1}^m c_{ij} A_i$ and $g^{-1} = (c_{ij})$.

We shall assume throughout this section that k is an algebraically closed field of characteristic zero. Our goal is to prove Theorem 1.4 over such k. In view of [20, Theorem 1.4(a)], we only need to establish the following.

5.1. **Theorem.** *Assume k is algebraically closed. If $n \geq 3$ and $2 \leq m \leq n^2 - 2$, then the $\mathrm{GL}_m \times \mathrm{PGL}_n$-action on $(\mathrm{M}_n)^m$ is generically free.*

Proof. The linear action of GL_m on $(\mathrm{M}_n)^m$ is easily seen to be the direct sum of n^2 copies of the natural m-dimensional representation of GL_m, i.e., to be isomorphic to the GL_m-action on n^2-tuples of vectors in k^m. Since $n^2 > m$, this action is generically free. Corollary 3.3(a) with $G = \mathrm{GL}_m$ and $X = (\mathrm{M}_n)^m$ tells us that the PGL_n-action on $(\mathrm{M}_n)^m/\mathrm{GL}_m$ has a reductive stabilizer S in general position, and that $S \simeq \mathrm{Stab}_{\mathrm{GL}_m \times \mathrm{PGL}_n}(x)/\mathrm{Stab}_{\mathrm{GL}_m}(x) = \mathrm{Stab}_{\mathrm{GL}_m \times \mathrm{PGL}_n}(x)$ for $x \in X$ in general position.

Recall that $(\mathrm{M}_n)^m/\mathrm{GL}_m$ is PGL_n-equivariantly birationally isomorphic to the Grassmannian $\mathrm{Gr}(m, n^2)$ of m-dimensional subspaces of M_n. Thus the PGL_n-action on $\mathrm{Gr}(m, n^2)$ has a stabilizer S in general position, where S is a reductive subgroup of PGL_n. (Recall that S is only well-defined up to conjugacy in PGL_n). To prove Theorem 5.1, it suffices to show that S is trivial.

Assume the contrary. Since S is reductive, it contains a non-trivial element g of finite order. Then every $L \in \mathrm{Gr}(m, n^2)$ in general position is fixed by some conjugate of g. In other words, the map

$$
\begin{array}{ccc}
\mathrm{PGL}_n \times \mathrm{Gr}(m, n^2)^g & \longrightarrow & \mathrm{Gr}(m, n^2) \\
(h, L) & \mapsto & h(L)
\end{array}
\tag{5.2}
$$

is dominant; here $\mathrm{Gr}(m, n^2)^g$ denotes the fixed points of g in $\mathrm{Gr}(m, n^2)$. Denote by $C(g)$ the centralizer of g in PGL_n. Note that $\mathrm{Gr}(m, n^2)^g$ is $C(g)$-stable. Hence the fiber of (5.2) over $h(L)$ contains $(hc, c^{-1}(L))$ for every $c \in C(g)$. So by the fiber dimension theorem,

$$
\dim \mathrm{Gr}(m, n^2) + \dim C(g) \leq \dim \mathrm{PGL}_n + \dim \mathrm{Gr}(m, n^2)^g \, . \tag{5.3}
$$

Since g has finite order, it is diagonalizable. So we may assume that

$$g = \operatorname{diag}(\underbrace{\lambda_1, \dots, \lambda_1}_{l_1 \text{ times}}, \dots, \underbrace{\lambda_s, \dots, \lambda_s}_{l_s \text{ times}}) = \operatorname{diag}(\alpha_1, \dots, \alpha_n)\,,$$

where $\lambda_1, \dots, \lambda_s$ are the (distinct) eigenvalues of g. Note that $s \geq 2$, because $g \neq 1$ in PGL_n and that g acts on the matrix units E_{ij} by $g \cdot E_{ij} = \alpha_i \alpha_j^{-1} E_{ij}$. The matrix algebra M_n naturally decomposes as a direct sum of character spaces

$$V_\mu = \operatorname{Span}(E_{ij} \mid \alpha_i \alpha_j^{-1} = \mu)\,.$$

In particular, V_1 is the commutator of g in M_n. Now (5.3) implies

$$\begin{aligned}\dim\, \mathrm{Gr}(m, n^2) - \dim\, \mathrm{Gr}(m, n^2)^g &\leq \dim(\mathrm{PGL}_n) - \dim C(g) \\ &= n^2 - \dim(V_1)\,.\end{aligned}$$

So if $n \geq 3$, part (b) of the following lemma gives the desired contradiction, which completes the proof of Theorems 5.1, and thus of Theorem 1.4 in the case that k is algebraically closed. □

5.4. **Lemma.** *Let $n \geq 2$, and $2 \leq m \leq n^2 - 2$.*

(a) $\dim V_1 \geq \dim\, V_\mu$ *for any $\mu \neq 1$.*

(b) *If $n \geq 3$ (or $n = 2$ and there are more than two distinct nonzero V_μ), then* $\dim\, \mathrm{Gr}(m, n^2) - \dim\, \mathrm{Gr}(m, n^2)^g > n^2 - \dim(V_1)$.

Proof. (a) Note that $\dim\, V_1 = l_1^2 + \cdots + l_s^2$ and

$$\dim\, V_\mu = \sum_{\lambda_i \lambda_j^{-1} = \mu} l_i l_j$$

Since the eigenvalues $\lambda_1, \dots, \lambda_s$ of g are distinct, the last sum has at most one term for each $i = 1, \dots, s$. Thus there is a permutation σ of $\{1, \dots, s\}$ such that

$$\dim\, V_\mu \leq l_1 l_{\sigma(1)} + \cdots + l_s l_{\sigma(s)}\,.$$

So for $v = (l_1, \dots, l_s)$ and $w = (l_{\sigma(1)}, \dots, l_{\sigma(s)})$, $\dim V_\mu \leq v \cdot w$. Hence by the Cauchy-Schwarz inequality,

$$\dim\, V_\mu \leq v \cdot w \leq |v|\,|w| = |v|^2 = l_1^2 + \cdots + l_s^2 = \dim(V_1)\,.$$

(b) Since g is semisimple, every $L \in \mathrm{Gr}(m, n^2)^g$ is a direct sum of its character subspaces spaces, i.e., a direct sum of vector subspaces of the V_μ. Part (b) now follows from Proposition 4.1 with $V = \mathrm{M}_n$, $N = n^2$, $N_\mu = \dim(V_\mu)$, $Z = \mathrm{Gr}(m, n^2)^g$, and r the number of distinct nonzero V_μ. □

5.5. **Remark.** We assumed throughout this section that $n \geq 3$. If $n = 2$ then the above argument still goes through provided there are more than two distinct non-zero character subspaces V_μ; see Lemma 5.4(b). In particular, this will always be the case if $g^2 \neq 1$ in PGL_n; indeed, in this case $g = (\lambda_1, \lambda_2)$, where $\mu = \lambda_1/\lambda_2 \neq \pm 1$ and the three spaces V_1, V_μ and $V_{\mu^{-1}}$ are distinct. Thus the above argument also shows that for $n = m = 2$, either

$|S| = 1$ or S has exponent 2. It turns out that, in fact, in this case $|S| = 2$; see [20, Lemma 14.2].

5.6. **Remark.** An alternative approach to proving Theorem 5.1 would be to appeal to the classification, due to A. G. Elashvili [8] and A. M. Popov [13], of pairs (G, ϕ), where G is a semisimple algebraic group and $\phi\colon G \hookrightarrow \mathrm{GL}(V)$ is an irreducible linear representation of G such that the G-action on V has a non-trivial stabilizer in general position. Since this classification is rather involved, and since additional work would be required to apply it in our situation (note that the group $\mathrm{GL}_m \times \mathrm{PGL}_n$ is not semisimple, and that its representation on $(\mathrm{M}_n)^m$ is not irreducible), we opted instead for the self-contained direct proof presented in this section.

6. SL_m-invariant generic matrices

The goal of this section is to relate the rings of SL_m-invariants in $G_{m,n}$ and $UD(m,n)$.

6.1. **Lemma.** (a) *Every element of $UD(m,n)^{\mathrm{GL}_m}$ can be written in the form $\frac{a}{b}$, where a is a homogeneous element of $(T_{m,n})^{\mathrm{SL}_m}$, and b is a non-zero homogeneous element of $\mathrm{Z}(T_{m,n})^{\mathrm{SL}_m}$ of the same degree as a.*

(b) *Assume that a subgroup G of GL_m has no non-trivial characters. Then every element of $UD(m,n)^G$ can be written as $\frac{a}{b}$ where $a \in (T_{m,n})^G$ and $0 \neq b \in \mathrm{Z}(T_{m,n})^G$.*

Proof. Both parts follows from Lemma 2.1. In part (a), we take $\Gamma = \mathrm{GL}_m \times \mathrm{PGL}_n$, $V = (\mathrm{M}_n)^m$ (with the Γ-action defined in the beginning of Section 5) and $W = \mathrm{M}_n$ (where GL_m acts trivially on W and PGL_n acts by conjugation). Here the relative concomitants $(\mathrm{M}_n)^m \longrightarrow \mathrm{M}_n$ are the homogeneous elements of $(T_{m,n})^{\mathrm{SL}_m}$ and the relative invariants $(\mathrm{M}_n)^m \longrightarrow k$ are the homogeneous elements of $\mathrm{Z}(T_{m,n})^{\mathrm{SL}_m}$; cf. Theorem 2.2.

If G has no non-trivial characters then relative concomitants are (absolute) concomitants, i.e., elements of $(T_{m,n})^G$. Similarly, relative invariants are elements of $\mathrm{Z}(T_{m,n})^G$, and part (b) is thus simply a restatement of Lemma 2.1 in this special case. □

6.2. **Proposition.** *Let G be a subgroup of GL_m such that G has no non-trivial characters. Then the following conditions are equivalent:*

(a) *$UD(m,n)^G$ has PI-degree n.*
(b) *$(T_{m,n})^G$ has PI-degree n.*
(c) *$(G_{m,n})^G$ has PI-degree n.*

Proof. The equivalence of (a) and (b) follows from Lemma 6.1(b). The implication (c) $\Longrightarrow$ (b) is obvious, since $G_{m,n} \subset T_{m,n}$. It thus remains to prove that (b) $\Longrightarrow$ (c).

Let g_n be the multilinear central polynomial for $n \times n$-matrices in [12, 13.5.11] (or [23, p. 26]). If R is a prime PI-algebra of PI-degree n, denote by $g_n(R)$ the set of all evaluations of g_n in R, and denote by $R\,g_n(R)$ the nonzero ideal of R generated by $g_n(R)$. Denote by T the trace ring of R. (Since we are working in characteristic zero, the (noncommutative) trace ring in [12, 13.9.2] is the same as the one we are using, see [12, 13.9.4].) Then $R\,g_n(R)$ is a common ideal of R and T, see [12, 13.9.6] (or [23, 4.3.1]).

Now let $R = G_{m,n}$. Then its trace ring is $T = T_{m,n}$. Recall that we are assuming (b) holds, i.e., T^G has PI-degree n. Let s be a non-zero evaluation of g_n on T^G. Then s is a nonzero G-invariant, and a central element of T (since it is also an evaluation of g_n on T). Since g_n is multilinear, and since T is generated as an R-module by central elements, s belongs to the ideal of T generated by $g_n(R)$, so that $s\,T \subseteq R\,g_n(R) \subseteq R$. Since s is a G-invariant, it follows that $s\,T^G \subseteq R^G$. Consider the central localization $R^G[s^{-1}] \subseteq UD(m,n)$. Since it contains T^G, $R^G[s^{-1}]$ must have PI-degree n, implying that also R^G must have PI-degree n. This completes the proof of the implication (b) $\Longrightarrow$ (c) and thus of Proposition 6.2. □

6.3. **Remark.** The same argument also shows that if the three equivalent conditions in Proposition 6.2 are true, then the division algebras of fractions of $(G_{m,n})^G$ and $(T_{m,n})^G$ are both equal to $UD(m,n)^G$.

7. Proof of Theorems 1.2 and 1.4

Proof of Theorem 1.2. Proposition 6.2 tells us that $(G_{m,n})^{\mathrm{SL}_m}$ has PI-degree n if and only if so does $UD(m,n)^{\mathrm{SL}_m}$. Thus in order to prove Theorem 1.2 it suffices to show that $UD(m,n)^{\mathrm{SL}_m}$ has PI-degree n whenever $2 \le m \le n^2 - 2$.

For $n = m = 2$ we showed this in [20, Remark 14.4] (in fact, the argument we gave there remains valid over any base field k of characteristic $\neq 2$). For $n \ge 3$ and $2 \le m \le n^2 - 2$, Theorem 1.4 tells us that $UD(m,n)^{\mathrm{GL}_m}$ has PI-degree n (and consequently, so does $UD(m,n)^{\mathrm{SL}_m}$). In summary, we have shown that *Theorem 1.2 follows from Theorem 1.4*. □

Proof of Theorem 1.4. We have already proved Theorem 1.4 in the case where the base field k is algebraically closed; see Section 5. We will now reduce the general case to this one by using Lemma 6.1(a).

We begin with a simple lemma.

7.1. **Lemma.** *Let K be an extension field of k, let V be a finite-dimensional k-vector space, and $V_K = V \otimes_k K$. Given a linear representation of $\mathrm{SL}_m(k)$ on V, we have*

$$(V_K)^{\mathrm{SL}_m(K)} = V^{\mathrm{SL}_m(k)} \otimes_k K\,.$$

Proof. Since $\mathrm{SL}_m(k)$ is dense in $\mathrm{SL}_m(K)$, the subspace $(V_K)^{\mathrm{SL}_m(K)}$ is defined inside V_K by a system of homogeneous linear equations with coefficients in k. Clearly finitely many of these equations suffice. Since the dimension of the solution space of such a system is the rank of the corresponding matrix (which has coefficients in k), $(V_K)^{\mathrm{SL}_m(K)}$ has a K-basis consisting of elements of $V^{\mathrm{SL}_m(k)}$. □

For the remainder of this section, we will write $G_{m,n}(K)$, $T_{m,n}(K)$ and $UD(m,n)(K)$ to denote the generic matrix algebra, trace ring and universal division algebra defined over the field K. Denote the algebraic closure of k by $\overline{k}$. Since the SL_m-action on $UD(m,n)$ preserves degree, Lemma 7.1 immediately implies the following fact, which we record for later use.

7.2. **Corollary.** *$G_{m,n}(\overline{k})^{\mathrm{SL}_m(\overline{k})} = G_{m,n}(k)^{\mathrm{SL}_m(k)} \otimes_k \overline{k}$, and $T_{m,n}(\overline{k})^{\mathrm{SL}_m(\overline{k})} = T_{m,n}(k)^{\mathrm{SL}_m(k)} \otimes_k \overline{k}$.* □

We are now ready to complete the proof of Theorem 1.4 over an arbitrary field k of characteristic zero. In Section 5 we showed that Theorem 1.4 holds over the algebraic closure $\overline{k}$ of k. That is, if $2 \leq m \leq n^2 - 2$ then there exist elements $c_1, \dots, c_r \in UD(m,n)(\overline{k})^{\mathrm{GL}_m(\overline{k})}$ which span $UD(m,n)(\overline{k})$ as a vector space over its center. By Lemma 6.1 we can write $c_i = a_i/b_i$, where $a_i \in T_{m,n}(\overline{k})[d_i]^{\mathrm{SL}_m}$ and $0 \neq b_i \in \mathrm{Z}(T_{m,n}(\overline{k}))[d_i]^{\mathrm{SL}_m}$ for some $d_i \geq 0$, $i = 1, \dots, r$. By Lemma 7.1, with $K = \overline{k}$ and $V = \mathrm{Z}(T_{m,n}(k))[d_i]$, we have $\mathrm{Z}(T_{m,n}(k))[d_i]^{\mathrm{SL}_m} \neq 0$. We may now replace b_i by a non-zero element of $\mathrm{Z}(T_{m,n}(k))[d_i]^{\mathrm{SL}_m}$. The new $c_i = a_i/b_i$ are still GL_m-invariant elements of $UD(m,n)(\overline{k})$, and they still generate $UD(m,n)(\overline{k})$ as a vector space over its center.

We now apply Lemma 7.1 once again (this time with $V = T_{m,n}(k)[d_i]$) to write each a_i as a finite sum $\sum \gamma_{ij} a_{ij}$, where each $\gamma_{ij} \in \overline{k}$ and each $a_{ij} \in T_{m,n}(k)[d_i]^{\mathrm{SL}_m}$. Now replace our collection of GL_m-invariant elements $\{c_i = a_i/b_i\}$ in $UD(m,n)(\overline{k})$ by $\{c_{ij} = a_{ij}/b_i\}$. By construction, the elements c_{ij} lie in $UD(m,n)(k)^{\mathrm{GL}_m}$ and span $UD(m,n)(\overline{k})$ over its center. Hence, these elements generate a k-subalgebra of $UD(m,n)(k)^{\mathrm{GL}_m}$ of PI-degree n. Consequently, $UD(m,n)(k)^{\mathrm{GL}_m}$ itself has PI-degree n. This completes the proof of Theorem 1.4 (and of Theorem 1.2). □

8. The case $m \geq n^2 - 1$

Theorems 1.2 and 1.4 assume that $m \leq n^2 - 2$. We will now describe $UD(m,n)^{\mathrm{GL}_m}$ and $UD(m,n)^{\mathrm{SL}_m}$ for $m \geq n^2 - 1$.

Recall the definition of the discriminant of n^2 matrices of size $n \times n$, say $A_1, \dots, A_{n^2}$: it is the determinant of the $n^2 \times n^2$-matrix whose i-th row consists of the entries of A_i, cf. (8.5). When viewed as a function $(\mathrm{M}_n)^{n^2} \longrightarrow k$, Δ is the unique multilinear alternating function such that $\Delta(e_{11}, e_{12}, \dots, e_{nn}) = 1$; cf., e.g., [10, Lemma 3]. Here the e_{ij} are the matrix units.

8.1. **Proposition.** (a) *If $m > n^2$, then $UD(m,n)^{\mathrm{SL}_m} = UD(m,n)^{\mathrm{GL}_m} = k$. Now let $m = n^2$, and denote by Δ the discriminant of the generic matrices $X_1, \dots, X_m$.*

(b) *$UD(m,n)^{\mathrm{GL}_m} = k$.*
(c) *$(T_{m,n})^{\mathrm{SL}_m} = k[\Delta]$.*
(d) *$UD(m,n)^{\mathrm{SL}_m} = k(\Delta)$.*

Proof. (a) We may clearly assume that k is algebraically closed. In this case SL_m has a dense orbit in the associated variety $X = (\mathrm{M}_n)^m$. Thus the rational quotient X/SL_m is a single point (with trivial PGL_n-action), and

$$UD(m,n)^{\mathrm{SL}_m} = \mathrm{RMaps}_{\mathrm{PGL}_n}(pt, \mathrm{M}_n) = k\,.$$

Now suppose $m = n^2$. Then GL_m has a dense orbit in $X = (\mathrm{M}_n)^m$. Arguing as in part (a), we prove (b); cf. [20, Proposition 13.1(a)]. (c) is proved in [9, p. 210], and (d) follows from (c) by Lemma 6.1(b). □

8.2. **Remark.** Let $m = n^2$. Formanek showed that $\Delta \notin (G_{m,n})^{\mathrm{SL}_m}$ ([9, p. 214]) but $\Delta^i \in G_{m,n}$ for every integer $i \geq 2$ (this follows from [10, Theorem 16]). Consequently for $m = n^2$,

$$(G_{m,n})^{\mathrm{SL}_m} = k[\Delta^2, \Delta^3]\,.$$

8.3. **Proposition.** *Suppose $m = n^2 - 1$, and let*

$$Y = \sum_{i,j=1}^{n} \Delta(X_1, \ldots, X_m, e_{ji})\, e_{ij}\,,$$

where the e_{ij} are the matrix units.

(a) $Y \in (T_{m,n})^{\mathrm{SL}_m}$.
(b) *The eigenvalues of Y are algebraically independent over k (and, in particular, distinct).*
(c) $(T_{m,n})^{\mathrm{SL}_m} = k[c_1, \ldots, c_{n-1}, Y]$ *is a polynomial ring in n independent variables over k. Here $c_1 = -\operatorname{tr}(Y), \ldots, c_n = (-1)^n \det(Y)$.*
(d) $UD(m,n)^{\mathrm{SL}_m} = k\big(c_1, \ldots, c_{n-1}, Y\big)$.
(e) $UD(m,n)^{\mathrm{GL}_m} = k\Big(\frac{c_2}{(c_1)^2}, \ldots, \frac{c_{n-1}}{(c_1)^{n-1}}, \frac{1}{c_1}Y\Big)$.

Proof. For the proof of (a)—(c), we may assume that k is algebraically closed (cf. Corollary 7.2). (a) We view Y as a regular map $(\mathrm{M}_n)^m \longrightarrow \mathrm{M}_n$. We want to show that this map is PGL_n-equivariant, i.e., $Y \in T_{m,n}$. Since Y is clearly SL_m-equivariant (recall that SL_m acts trivially on M_n), this will imply part (a).

We begin by observing that for any $(A_1, \ldots, A_m) \in (\mathrm{M}_n)^m$, and $Z \in \mathrm{M}_n$,

$$\operatorname{tr}(Y(A_1, \ldots, A_m)Z) = \Delta(A_1, \ldots, A_m, Z)\,. \tag{8.4}$$

Indeed, both sides are linear in Z, so we only need to check (8.4) for the elementary matrices $Z = e_{ij}$, where it is easy to do directly from the definition of Y.

Fix an m-tuple $(A_1, \ldots, A_m) \in (\mathrm{M}_n)^m$ of $n \times n$-matrices. Since the trace form on M_n is non-singular, $Y(A_1, \ldots, A_m)$ is the unique matrix satisfying (8.4) for every $Z \in \mathrm{M}_n$. The PGL_n-equivariance of $Y\colon (\mathrm{M}_n)^m \longrightarrow \mathrm{M}_n$ is an easy consequence of this and the fact that Δ is PGL_n-invariant (see [9, p. 209]). This concludes the proof of (a).

Our proof of parts (b) and (c) relies on the following claim: $Y\colon (\mathrm{M}_n)^m \longrightarrow \mathrm{M}_n$ is the categorical quotient map for the SL_m-action on $(\mathrm{M}_n)^m$. In other words, we claim that the n^2 elements $\Delta(X_1, \ldots, X_m, e_{ij})$ $(i, j = 1, \ldots, n)$ generate $k[(\mathrm{M}_n)^m]^{\mathrm{SL}_m}$ as a k-algebra. To prove this claim we will temporarily write $(A_1, \ldots, A_m) \in (\mathrm{M}_n)^m$ as an $m \times n^2$-matrix

$$A = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \ldots & a_{ij}^{(1)} & \ldots & a_{nn}^{(1)} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{11}^{(m)} & a_{12}^{(m)} & \ldots & a_{ij}^{(m)} & \ldots & a_{nn}^{(m)} \end{pmatrix}. \tag{8.5}$$

That is, we write each $n \times n$ matrix $A_h = (a_{ij}^{(h)})$ as a single row of A. In this notation, $g \in \mathrm{SL}_m$ acts on $(\mathrm{M}_n)^m$ by multiplication by the transpose of g^{-1} on the left; that is, $g(A) = (g^{-1})^{\text{transpose}} \cdot A$ for every $g \in \mathrm{SL}_m$. Let $\delta_{ij}(A_1, \ldots, A_m)$ be the $m \times m$-minor of this matrix obtained by removing the ij-column from A and taking the determinant of the resulting $m \times m$-matrix. The first theorem of classical invariant theory (see [27, Theorem II.6.A] or [6, Theorem 2.1]) says that the elements $\delta_{ij}(X_1, \ldots, X_m)$ generate $k[(\mathrm{M}_n)^m]^{\mathrm{SL}_m}$ as a k-algebra. On the other hand, it is easy to see that $\delta_{ij}(X_1, \ldots, X_m) = \pm\Delta(X_1, \ldots, X_m, e_{ij})$. This proves the claim.

Now observe that since $m = n^2 - 1$,

$$\dim((\mathrm{M}_n)^m /\!/ \mathrm{SL}_m) = mn^2 - (m^2 - 1) = n^2 = \dim(\mathrm{M}_n).$$

This means that the n^2 SL_m-invariant functions

$$\Delta(X_1, \ldots, X_m, e_{ji}) \colon (\mathrm{M}_n)^m \longrightarrow k$$

are algebraically independent over k. In other words, Y (viewed as a matrix in $T_{m,n} \subset \mathrm{M}_n(k[x_{ij}^{(h)}])$ has algebraically independent entries. Part (b) easily follows from this assertion; cf. [15, Lemma II.1.4].

Furthermore,

$$\begin{aligned} (T_{m,n})^{\mathrm{SL}_m} &= \mathrm{Morph}_{\mathrm{SL}_m \times \mathrm{PGL}_n}((\mathrm{M}_n)^m, \mathrm{M}_n) \\ &\simeq \mathrm{Morph}_{\mathrm{PGL}_n}((\mathrm{M}_n)^m /\!/ \mathrm{SL}_m, \mathrm{M}_n) \\ &\simeq \mathrm{Morph}_{\mathrm{PGL}_n}(\mathrm{M}_n, \mathrm{M}_n) = T_{1,n}, \end{aligned}$$

where $T_{1,n}$ is the trace ring of one generic $n \times n$-matrix. Here the last equality is a special case of Procesi's Theorem 2.2(a) (with $m = 1$). Since the chain of isomorphisms identifies Y with the identity map $\mathrm{M}_n \longrightarrow \mathrm{M}_n$, we conclude that

$$(T_{m,n})^{\mathrm{SL}_m} = k[c_1, \ldots, c_n, Y].$$

Since $Y^n + c_1 Y^{n-1} + \cdots + c_n = 0$, $k[c_1, \ldots, c_{n-1}, Y] = k[c_1, \ldots, c_n, Y]$. This proves the first assertion in (c).

To show that $c_1, \ldots, c_{n-1}, Y$ are algebraically independent over k, denote the eigenvalues of Y by $\lambda_1, \ldots, \lambda_n$. By part (b), $\lambda_1, \ldots, \lambda_n$ are algebraically independent over k. Since Y is algebraic over $k(c_1, \ldots, c_n)$, we have

$$\begin{aligned} \mathrm{trdeg}_k\, k(c_1, \ldots, c_{n-1}, Y) &= \mathrm{trdeg}_k\, k(c_1, \ldots, c_n, Y) \\ &= \mathrm{trdeg}_k\, k(c_1, \ldots, c_n) = \mathrm{trdeg}_k\, k(\lambda_1, \ldots, \lambda_n) = n. \end{aligned}$$

This shows that $c_1, \dots, c_{n-1}, Y$ are algebraically independent over k, thus completing the proof of (c).

(d) is an immediate consequence of (c) and Lemma 6.1(b). To prove (e), denote the central torus of GL_m by $\mathbb{G}_m$. Then

$$UD(m,n)^{\mathrm{GL}_m} = (\,UD(m,n)^{\mathrm{SL}_m})^{\mathbb{G}_m} = k(c_1, \dots, c_{n-1}, Y)^{\mathbb{G}_m}\,,$$

where $\mathbb{G}_m$ acts on the purely transcendental extension $k(c_1, \dots, c_{n-1}, Y)$ as follows: $t \cdot c_i \mapsto t^{im} c_i$ for $i = 1, \dots, n-1$, and and $t \cdot Y \mapsto t^m Y$. Part (e) easily follows from this description. □

8.6. **Remark.** Note that $c_1 = -\Delta(X_1, \dots, X_{n^2-1}, I_n)$, where I_n is the $n \times n$ identity matrix. By a theorem of Formanek, $(c_1)^2$ is an element of $\mathrm{Z}(G_{m,n})^{\mathrm{SL}_m}$ for $m = n^2 - 1$, see [10, Theorem 16].

9. Proof of Theorem 1.3

By Corollary 7.2, we may assume that k is algebraically closed. Set $A = (G_{m,n})^{\mathrm{SL}_m}$ and $B = (T_{m,n})^{\mathrm{SL}_m}$. By Theorem 1.2, A and B both have PI-degree n. Thus $\mathrm{Z}(A) = (\mathrm{Z}(G_{m,n}))^{\mathrm{SL}_m}$ and $\mathrm{Z}(B) = (\mathrm{Z}(T_{m,n}))^{\mathrm{SL}_m}$. Since SL_m is a reductive group, and since $T_{m,n}$ is a finitely generated k-algebra and a finite module over its center, B is a finite $\mathrm{Z}(B)$-module, and both B and $\mathrm{Z}(B)$ are finitely generated Noetherian k-algebras, cf. [26, Proposition 4.2]. Moreover, B is an FBN ring, cf. [12, 13.6.6].

By Corollary 3.3(b) and Remark 6.3, the transcendence degrees of both B and $\mathrm{Z}(B)$ are $t = (m-1)n^2 - m^2 + 2$. For notational simplicity, set

$$\mu(S) = \limsup_{d \to \infty} \frac{\dim_k S[d]}{d^{t-1}}$$

for any graded k-algebra $S = \oplus_{d \geq 0} S[d]$. By [25, Lemma 6.1] (cf. also [11, §12.6]), $f(d) = \dim_k B[d]$ is eventually periodically polynomial, i.e., there are polynomials $f_1, \dots, f_s$ with rational coefficients such that $f(d) = f_i(d)$ whenever d is large enough and congruent to i modulo s; moreover, the maximum of the degrees of the f_i is $t-1$. Consequently $\mu(B)$ exists and is equal to the largest among the leading coefficients of those f_i of degree $t-1$. A similar argument shows that $\mu(Z(B))$ exists and is a nonzero number.

Consider the multilinear central polynomial g_n for $n \times n$ matrices used in the proof of Proposition 6.2. Since it is multilinear and nonzero on A, we can find a nonzero evaluation c of g_n at homogeneous elements of A; this c is homogeneous. Since c is also an evaluation of g_n on $G_{m,n}$, $cT_{m,n} \subset G_{m,n}$, so that $cB \subset A$ and $c\,\mathrm{Z}(B) \subset \mathrm{Z}(A)$. Then for all integers $d \geq j$, $cB[d-j] \subseteq A[d] \subseteq B[d]$, where $j = \deg c$. Replacing c by c^s if necessary, we may assume that s divides j. Consequently, whenever d is large enough and congruent to i modulo s,

$$f_i(d-j) \leq \dim_k A[d] \leq f_i(d)\,.$$

It follows easily that $\mu(A)$ exists and is equal to the largest among the leading coefficients of those f_i of degree $t-1$. A similar argument shows that $\mu(Z(A))$ exists and is a nonzero number. □

9.1. **Remark.** The above proof shows that $\mu((G_{m,n})^{\mathrm{SL}_m}) = \mu((T_{m,n})^{\mathrm{SL}_m})$, and that $\mu(\mathrm{Z}((G_{m,n})^{\mathrm{SL}_m})) = \mu(\mathrm{Z}((T_{m,n})^{\mathrm{SL}_m}))$.

9.2. **Remark.** Consider the GL_m-representation on R, where $R = G_{m,n}$, $T_{m,n}$, $\mathrm{Z}(G_{m,n})$ or $\mathrm{Z}(T_{m,n})$. Recall that irreducible polynomial representations of GL_m are indexed by partitions $\lambda = (\lambda_1, \dots, \lambda_s)$ with $s \le m$ parts; cf. [9, Section 2]. Denote the multiplicity of the irreducible representation corresponding to λ in R by $\mathrm{mult}_\lambda(R)$. Writing (r^m) for the partition $\lambda = (r, ..., r)$ (m times), we have

(a) $\dim_k R^{\mathrm{SL}_m}[d] = 0$ if d is not a multiple of m, and

(b) $\dim_k R^{\mathrm{SL}_m}[rm] = \mathrm{mult}_{(r^m)}(R)$ for any integer $r \ge 1$.

Proof. (a) Assume $R^{\mathrm{SL}_m}[d]$ is nonzero. Then it is a direct sum of one-dimensional representations of GL_m of the form $M = \mathrm{Span}_k(v)$. Moreover, any such representation is given by $g(v) = \det(g)^r v$ for some integer r; cf., e.g., [9, Theorem 3(a)]. On the other hand, substituting $g = tI_m$, where $t \in k$ and I_m is the $m \times m$ identity matrix, we obtain, $g(v) = t^d v$. Since $\det(tI_m) = t^m$, we see that $d = rm$, as claimed.

(b) If $d = rm$ and $0 \ne v \in R^{\mathrm{SL}_m}[rm]$ then the partition associated to the 1-dimensional irreducible GL_m-module $M = \mathrm{Span}(v)$ is (r^m); cf. , e.g., [9, Theorem 2]. Now consider the direct sum decomposition $R = \oplus R_\lambda$, where R_λ is the sum of all irreducible GL_m-submodules of R with associated partition λ. The argument of part (a) shows that $R_{(r^m)} = R^{\mathrm{SL}_m}[rm]$. Moreover, since $\dim(M) = 1$, we have

$$\dim_k R^{\mathrm{SL}_m}[rm] = \dim_k R_{(r^m)} = \mathrm{mult}_{(r^m)}(R)\,,$$

as claimed. □

10. Standard polynomials

Let $G_{m,n}$ be the ring of m generic $n \times n$-matrices. By Theorem 1.2, $(G_{m,n})^{\mathrm{SL}_m}$ is a PI domain of degree n, whenever $2 \le m \le n^2 - 2$. We will now describe one particular element of this ring. Let

$$F_m(x_1, \dots, x_m) = \sum_{\sigma \in \mathrm{S}_n} (-1)^\sigma x_{\sigma(1)} \dots x_{\sigma(n)} \in k\{x_1, \dots, x_m\}$$

be the standard polynomial. Since F_m is multilinear and alternating, one checks easily that for $g \in \mathrm{GL}_m$,

$$g(F_m) = \det(g) \cdot F_m; \qquad (10.1)$$

see, e.g., [23, 1.4.12]. Substituting m generic $n\times n$-matrices $X_1,\dots,X_m$ into F_m, we obtain

$$f_{m,n} = F_m(X_1,\dots,X_m) = \sum_{\sigma\in \mathrm{S}_n} (-1)^\sigma X_{\sigma(1)}\dots X_{\sigma(n)} \in G_{m,n}\,.$$

From (10.1), we see that $f_{m,n}\in (G_{m,n})^{\mathrm{SL}_m}$. By the Amitsur-Levitzki Theorem, $f_{m,n}=0$ iff $m\geq 2n$.

Fix $m,n\geq 2$ and let K be the center of $UD(m,n)$.

10.2. **Proposition.** *For $2\leq m<2n$, $K(f_{m,n})$ generates a GL_m-stable maximal subfield of $UD(m,n)$.*

The proof is algebraic in nature and works in characteristic $\neq 2$.

Proof. The fact that $K(f_{m,n})$ is a GL_m-stable subfield follows from (10.1). In order to prove that this subfield is maximal, it suffices to verify that $f_{m,n}$ has an eigenvalue of multiplicity 1. (Indeed, if, say, $n=d\cdot[K(f_{m,n}):K]$, then the characteristic polynomial $p(t)$ of $f_{m,n}$ in $UD(m,n)$ has the form $p(t)=q(t)^d$, where $q(t)$ is the minimal polynomial of $f_{m,n}$ over K. This shows that the multiplicity of each eigenvalue of $f_{m,n}$ is divisible by d.)

Since the multiplicity of eigenvalues cannot decrease when evaluating $f_{m,n}$ in M_n, it suffices now to show that $f_{m,n}$ (or equivalently, F_m) has some evaluation in M_n with an eigenvalue of multiplicity one. We now proceed to construct such an evaluation. Since

$$F_m(1,x_2,\dots,x_m)=F_{m-1}(x_2,\dots,x_m)$$

for m odd (cf. [23, Exercise 1.2.3]), we may assume that m is even, say $m=2r-2$, with $1<r\leq n$. In M_n, consider the sequence of m matrix units

$$e_{1,2}, e_{2,2}, e_{2,3}, e_{3,3},\dots, e_{r-2,r-1}, e_{r-1,r-1}, e_{r-1,r}, e_{r,1}\,.$$

When permuting these matrix units cyclically, their product is nonzero; for any other permutation, their product is zero. Since an m-cycle is odd, it follows that F_m evaluated at these matrix units is

$$e_{1,1}-e_{2,2}+e_{2,2}-+\dots-e_{r-1,r-1}+e_{r-1,r-1}-e_{r,r}=e_{1,1}-e_{r,r}\,,$$

which has 1 as an eigenvalue of multiplicity one (since $\mathrm{char}(k)\neq 2$). ☐

We do not know an explicit expression for any non-constant element of $(G_{m,n})^{\mathrm{SL}_m}$ (as a polynomial in the generic $n\times n$-matrices $X_1,\dots,X_m$) in the case where $2n\leq m\leq n^2-2$; we leave this as an open question. Note that for $m=n^2$ and $m=n^2-1$, such elements are exhibited in Remarks 8.2 and 8.6.

Acknowledgments

We are grateful to A. Berele and E. Formanek for helpful comments.

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