PROJECTIVELY SIMPLE RINGS

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ABSTRACT. We introduce the notion of a projectively simple ring, which is an infinite-dimensional $\mathbb{N}$-graded k-algebra A such that $\dim_k A/I < \infty$ for every nonzero 2-sided ideal $I \subset A$. We show that if a projectively simple ring A is strongly noetherian, is generated in degree 1, and has a point module, then A is equal in large degree to a twisted homogeneous coordinate ring $B = B(X, \mathcal{L}, \sigma)$. Here X is a smooth projective variety, σ is an automorphism of X with no proper σ-invariant subvariety (we call such automorphisms *wild*), and $\mathcal{L}$ is a σ-ample line bundle. We conjecture that if X admits a wild automorphism then every irreducible component of X is an abelian variety. We prove several results in support of this conjecture; in particular, we show that the conjecture is true if $\dim X \leq 2$. In the case where X is an abelian variety, we describe all wild automorphisms of X. Finally, we show that if A is projectively simple and admits a balanced dualizing complex, then proj A is Cohen-Macaulay and Gorenstein.

CONTENTS

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0. Introduction

Let k be a field. The present study grew out of the following question: What are the simple graded k-algebras $A = \bigoplus_{n=0}^{\infty} A_n$? Technically, such an algebra cannot be simple, since it always has ideals of the form $A_{\geq n}$ for each $n \geq 0$. Thus we are let to the following natural definition: we call A *projectively simple* if $\dim_k A = \infty$ and every nonzero graded ideal of A has finite codimension in A. If the reference to k is clear from the context, we will sometimes refer to A as a projectively simple graded ring.

Although it is easy to see that graded prime algebras of GK-dimension 1 which satisfy a polynomial identity are projectively simple, it is not immediately obvious that there are any examples more interesting than these. In fact, one interesting example has already appeared prominently in the literature: If S is the Sklyanin algebra of dimension 3 (as defined in [SV, Example 7.3]), then S has a central element g such that the factor ring $B = S/gS$ is a projectively simple domain of GK-dimension 2. In this case B may be constructed as a twisted homogeneous coordinate ring $B = B(E, \mathcal{L}, \sigma)$, where E is an elliptic curve and σ is a translation automorphism of E. Thus it is natural to look for other examples of projectively simple rings which also arise as twisted homogeneous coordinate rings; let us briefly review this construction.

Given X a projective scheme, $\mathcal{L}$ a line bundle on X, and σ an automorphism of X, set $\mathcal{L}_n = \mathcal{L} \otimes \sigma^*\mathcal{L} \otimes \cdots \otimes (\sigma^{n-1})^*\mathcal{L}$ for each $n \geq 0$. The *twisted homogeneous coordinate ring* associated to this data is the vector space $B(X, \mathcal{L}, \sigma) = \bigoplus_{n=0}^{\infty} \mathrm{H}^0(\mathcal{L}_n)$, which has a natural graded ring structure. As is true even in the commutative case where $\sigma = Id$, the rings obtained this way are typically well behaved only with additional assumptions on the sheaf $\mathcal{L}$. In particular, if $\mathcal{L}$ is σ*-ample* (see §2 below), then $B = B(X, \mathcal{L}, \sigma)$ is noetherian and its properties are closely related to the geometry of X. In this case one can describe the ideals of B geometrically and thus prove the following:

Proposition 0.1. *(Proposition 2.2) Let X be a projective scheme, σ an automorphism of X, and $\mathcal{L}$ a σ-ample line bundle. Then $B = B(X, \mathcal{L}, \sigma)$ is projectively simple if and only if σ satisfies the condition*

$$\sigma(Y) \neq Y \text{ for every nonempty reduced closed subscheme } Y \subsetneq X.$$

In the sequel we shall refer to σ satisfying this condition as a *wild* automorphism of X. If we seek explicit examples of projectively simple twisted homogeneous coordinate rings, then Proposition 0.1 inevitably leads us to the following purely geometric questions:

(i) Can we classify all pairs (X, σ), where X is a projective variety and σ is a wild automorphism of X?

(ii) for each (X, σ) as above, can we find σ-ample line bundles on X?

In the case where X is an abelian variety, we obtain the following complete answer to these questions. Recall that every automorphism σ of an abelian variety X can be written in the form $\sigma = T_b \cdot \alpha$, where $T_b : x \mapsto x + b$ is a translation by some $b \in X$ and α is an automorphism of X preserving the group structure; see, for example, [La1, Theorem 4, p.24].

Theorem 0.2. *(Theorem 7.2, Theorem 8.5) Let X be an abelian variety over an algebraically closed field k of characteristic zero.*

(a) *Suppose that $T_b : X \longrightarrow X$ is the translation automorphism by $b \in X$, and α is an automorphism of X preserving the group structure. Then $\sigma = T_b \cdot \alpha$ is a wild automorphism of X if and only if $\beta = \alpha - Id$ is nilpotent and b generates $X/\beta(X)$.*

(b) *If σ is a wild automorphism of X, then any ample invertible sheaf on X is σ-ample.*

Given an abelian variety X, we shall see in §7 below that it is easy to find many automorphisms σ satisfying the conditions in Theorem 0.2(a). Then as $\mathcal{L}$ varies over all ample sheaves on X, by Theorem 0.2(b) and Proposition 0.1 one gets many projectively simple twisted homogeneous coordinate rings $B(X, \mathcal{L}, \sigma)$.

We conjecture that there are no examples of wild automorphisms, other than those described in Theorem 0.2. More precisely,

Conjecture 0.3. *If an irreducible projective variety X admits a wild automorphism then X is an abelian variety.*

In Sections 3 – 6 we prove a number of results in support in Conjecture 0.3. In particular, in Theorem 6.5 below we show that this conjecture is true if $\dim(X) \leq 2$.

Having constructed some prototypical examples, we would like to say more about the structure of general projectively simple rings. Because twisted homogeneous coordinate rings are quite special among all graded rings, one might expect that the examples we have constructed so far are also very special. Rather surprisingly, the following theorem states that under certain hypotheses which are natural in the theory of noncommutative projective geometry, these are essentially the only possible examples.

Theorem 0.4. *(Theorem 2.4) Let k be an algebraically closed field and let A be a projectively simple noetherian k-algebra. Suppose that A is strongly noetherian, generated in degree 1 and has a point module. Then up to a finite dimensional vector space, A is isomorphic to a projectively simple twisted homogeneous coordinate ring $B(X, \mathcal{L}, \sigma)$ for a smooth projective variety X with σ-ample line bundle $\mathcal{L}$ and wild automorphism σ.*

It would be very interesting to know what kinds of projectively simple rings can appear if the various hypotheses of Theorem 0.4 are relaxed. In Example 2.5 below we show that there do exist projectively simple rings which are not strongly noetherian.

Our final class of results concerns the noncommutative projective scheme $\operatorname{Proj} A$ associated to a projectively simple ring A, and takes its inspiration from some known results about ungraded simple rings. It is shown in [YZ2] that a noetherian simple ring A has finite injective dimension provided that A admits a dualizing complex. This last condition is a natural one which holds for many important classes of rings such as the universal enveloping algebras of finite dimensional Lie algebras and factor rings of Artin-Schelter regular rings. Following the ideas in [YZ2], we show that some similar homological results are true in the graded setting.

Theorem 0.5. *(Proposition 10.1, Theorem 11.9) Let A be a projectively simple noetherian connected graded algebra with a balanced dualizing complex. Then*

(a) $\operatorname{Proj} A$ *is classically Cohen-Macaulay, namely, the dualizing complex for* $\operatorname{Proj} A$ *is isomorphic to $\omega[n]$ for a graded A-bimodule object ω in* $\operatorname{Proj} A$.
(b) $\operatorname{Proj} A$ *is Gorenstein, namely, the dualizing bimodule ω in part (a) is invertible. As a consequence, the structure sheaf $\mathcal{A}$ has finite injective dimension.*

If A satisfies the hypotheses of Theorem 0.5 and also $A = B(X, \mathcal{L}, \sigma)$ for some σ-ample $\mathcal{L}$, then the conclusions of Theorem 0.5 are immediate, since $\operatorname{QGr} A$ is equivalent to the category of quasi-coherent sheaves on X and X is smooth by Lemma 3.1(b) below. So Theorem 0.5 is aimed primarily at projectively simple rings that do not satisfy the conditions of Theorem 0.4.

To conclude this introduction, we would like to say a bit about our motivation for introducing and studying projectively simple rings. Let us describe a potential application to the theory of GK-dimension; see [KL] for an introduction to this subject. It is known that no algebra may have a non-integer value of GK-dimension between 0 and 2. Moreover, it is conjectured that there does not exist a noetherian connected graded domain with GK-dimension strictly between 2 and 3 [AS1, p. 2]. Note that if A is a Goldie prime ring and I is a nonzero ideal of A, then $\operatorname{GK}(A/I) \le \operatorname{GK}(A) - 1$ [KL, Proposition 3.15]. Thus if A is a connected graded noetherian prime ring with $2 < \operatorname{GK}(A) < 3$, then either A has a height one prime P with $\operatorname{GK}(A/P) = 1$, or else A is projectively simple. A proof that projectively simple rings have integer GK-dimension would eliminate one of these cases, thus making progress towards the proof of the full conjecture. The results of this paper do at least show that a projectively simple ring which satisfies the hypotheses of Theorem 0.4 must have integer GK-dimension, since this is true of twisted homogeneous coordinate rings.

Another application is to the classification of graded rings of low GK-dimension. Artin and Stafford have classified semiprime graded rings of GK-dimension 2 in terms of geometric data in [AS1], [AS2]. The classification of rings of GK-dimension 3, which correspond to noncommutative surfaces, is a subject of much current interest. Those rings of GK-dimension 3 which are also projectively simple represent a special subclass. If such a ring satisfies all of the hypotheses of Theorem 0.4 above, then it must be a twisted homogeneous coordinate ring in large degree. We classify all such twisted homogeneous coordinate rings in Proposition 9.2 below.

1. Elementary properties of projectively simple rings

Throughout this paper k is a commutative base field, and all rings will be k-algebras. In Section 2 we will assume that k is algebraically closed, and from Section 3 to Section 9 we will assume that k is algebraically closed and that $\operatorname{char} k = 0$. An algebra A is $\mathbb{N}$-*graded* (or *graded*) if $A = \bigoplus_{i \geq 0} A_i$ with $1 \in A_0$ and $A_i A_j \subseteq A_{i+j}$ for all $i, j \geq 0$. The graded algebra A is *locally finite* if $\dim_k A_i < \infty$ for all i. All algebras A in this article will be graded and locally finite, except when we consider localizations of A and in other obvious situations. If $A_0 = k$, then A is called *connected graded*. Let $\mathfrak{m}$ denote the graded ideal $A_{\geq 1}$.

We recall from the introduction the property which is the subject of this article:

Definition 1.1. A locally finite graded algebra A is called *projectively simple* if $\dim_k A = \infty$ and $\dim_k A/I < \infty$ for every nonzero graded ideal I of A.

It is useful to notice that since A is locally finite, the condition $\dim_k A/I < \infty$ for a graded ideal I is equivalent to the condition $I \supset A_{\geq n}$ for some n.

Our first result summarizes some easy observations about projectively simple rings. The proofs are straightforward, and we omit them.

Lemma 1.2. *Let A be a graded ring such that $\mathfrak{m} \neq 0$.*

- (a) *If A is projectively simple, then A is a finitely generated k-algebra.*
- (b) *If A is projectively simple, then A is prime. If A is also connected, then the only nonzero graded prime ideal of A is $\mathfrak{m}$.*
- (c) *If A is PI and finitely generated, then A is projectively simple if and only if A is prime of GK-dimension 1.*
- (d) *If A is projectively simple, then $\dim_k A/J < \infty$ for every (not necessarily graded) nonzero ideal J of A.*

Before pursuing further the abstract notion of a projectively simple ring, we should note that there do exist examples besides the trivial ones in Lemma 1.2(c) above. Let us mention here just one important example, which will turn out to be the model for all of our subsequent examples.

Example 1.3. Let E be an elliptic curve and let σ be an translation automorphism of E given by the rule $x \mapsto x + a$ where a is a point on E of infinite order. Let $\mathcal{L}$ be a very ample line bundle of E. Then the twisted homogeneous coordinate ring $B(E, \mathcal{L}, \sigma)$ (see §2 for the definition) is a projectively simple ring of GK-dimension 2 (see [AS1, 6.5(ii)] for a proof).

We can also give a non-constructive proof of the existence of projectively simple factor rings of quite general graded algebras.

Lemma 1.4. *Let A be finitely generated, connected graded and infinite dimensional over k. Then there is a graded ideal $J \subset A$ such that A/J is projectively simple.*

Proof. Let Φ be the set of all graded ideals $I \subset A$ such that $\dim_k A/I = \infty$. Suppose that $\{I_\alpha\}_{\alpha \in S}$ is a subset of Φ consisting of an ascending chain of (possibly uncountably many) ideals. We claim that the union

$$U = \bigcup_{\alpha \in S} I_\alpha$$

is in Φ. Suppose to the contrary that U is not in Φ. Then U contains $A_{\geq n}$ for some n. Let A be generated by elements of degree no more than d and let V be the finite dimensional vector space $\bigoplus_{i=0}^{d} A_{n+i}$. Then $V \subset U$ implies that $V \subset I_{\alpha_0}$ for some α_0 since $\{I_\alpha\}_{\alpha \in S}$ is an ascending chain. Hence I_{α_0} contains $A_{\geq n}$ since A is generated by elements of degree no more than d. This yields a contradiction, whence U is in Φ. By Zorn's lemma, the set Φ has a maximal object, say J. Then A/J is projectively simple. □

Next we will show that projective simplicity is preserved under base field extensions.

Lemma 1.5. *Let $k \subset L$ be an extension of fields, with k algebraically closed. Then A is a projectively simple k-algebra if and only if $A \otimes_k L$ is a projectively simple L-algebra.*

Proof. If R is any commutative k-algebra and V is any vector space over k, we write V_R for $V \otimes_k R$. Suppose that A is not projectively simple, but rather contains a nonzero ideal I with $\dim_k A/I = \infty$. Then I_L is a nonzero ideal of A_L and $\dim_L A_L/I_L = \infty$, so A_L is not projectively simple.

Now assume that A_L is not projectively simple, and let us show that A is not projectively simple. By assumption A_L has some nonzero ideal I such that A_L/I is not finite-dimensional. We may assume that $I = (g)$ is generated by one nonzero homogeneous element $g \in A_L$. Write $g = \sum_{i=0}^{q} b_i x_i$ where $0 \neq b_i \in L$ and $x_i \in A$. We may choose q as small as possible so that $\{x_0, \ldots, x_q\}$ are linearly independent over k. Replacing g by $b_0^{-1} g$ we may assume that $b_0 = 1$. Let $R = k[b_1, \ldots, b_q] \subset L$ be the commutative affine k-algebra generated by the $\{b_i\}$. Let J be the ideal of A_R generated by g. Then $(A_R/J) \otimes_R L \cong A_L/I$. Necessarily the degree n part $(A_R/J)_n$ is not R-torsion for all n in an infinite subset $\Phi \subset \mathbb{Z}$. Now let $\theta : R \to k$ be any k-algebra homomorphism (some such exists by the Nullstellensatz since k is algebraically closed), and extend it to a map $\theta : A_R \to A_k = A$. Then letting k be an R-module via θ, we have

$$A' = (A_R/J) \otimes_R k \cong A/(\theta(J)).$$

For each $n \in \Phi$, $A'_n \neq 0$ since $(A_R/J)_n$ is not R-torsion. Hence A' has infinite k-dimension. Now $\theta(g) = x_0 + \sum_{i=1}^{q} \theta(b_i) x_i$ is nonzero in $\theta(J)$ since $\{x_0, \cdots, x_q\}$ is linearly independent over k. Therefore A is not projectively simple. □

One might ask if Lemma 1.2(a) could be strengthened to say that a projectively simple ring is necessarily noetherian. Using Lemma 1.4 we can see that the answer is no.

Example 1.6. Let k be a countable field. By the Golod-Shafarevitch construction [He, Chapter 8], there is a finitely generated connected graded k-algebra A such that $\dim_k A = \infty$ and every element in $A_{\geq 1}$ is nilpotent. By Lemma 1.4, there is a graded ideal J such that $B := A/J$ is projectively simple, whence prime. Clearly $B_{\geq 1}$ is a nil ideal, in other words every element is nilpotent, but $B_{\geq 1}$ is not nilpotent since B is prime. By [MR, 2.3.7], B is neither left nor right noetherian.

Combining this with Lemma 1.5 one sees that if L is a field containing the algebraic closure of its prime field, then there exists a non-noetherian projectively simple algebra over L.

Now we discuss some further definitions which will be useful in the remainder of this section, and in §10-11. Let A be a noetherian graded ring, and let M and N be graded right A-modules. The group of module homomorphisms preserving degree is denoted $\operatorname{Hom}_A(M, N)$. Let $N(n)$ denote the nth degree shift of the module N. We write

$$\underline{\operatorname{Hom}}_A(M, N) = \bigoplus_{n \in \mathbb{Z}} \operatorname{Hom}_A(M, N(n)),$$

which is the group of all module homomorphisms from M to N in case M is finitely generated.

We say that a graded right module M is *torsion* if for every $x \in M$ there is an n such that $xA_{\geq n} = 0$. A noetherian module is torsion if and only if it is finite dimensional over k. We will also use a different kind of torsion property. A right A-module M is called *Goldie torsion* if for every element $x \in M$, the right annihilator $\operatorname{r.ann}(x)$ is an essential right ideal of A. Let $\operatorname{Kdim} M$ denote the Krull dimension of M; if A is prime then $\operatorname{Kdim} M < \operatorname{Kdim} A$ if and only if M is Goldie torsion. We call M *torsionfree* (respectively, *Goldie torsionfree*) if it does not contain a nonzero torsion (respectively, Goldie torsion) submodule.

Let A and B be graded rings. An (A, B)-bimodule is called *noetherian* if it is noetherian on both sides. Note that if M is a noetherian (A, B)-bimodule, then the largest torsion left submodule $\tau_A(M)$ of M and the largest torsion right submodule $\tau_B(M)$ of M coincide, both being equal to the largest finite-dimensional sub-bimodule of M. So we simply write $\tau(M)$ for this module and call it the torsion submodule of M. The bimodule $M/\tau(M)$ is torsionfree on both sides.

The following useful result shows how the projectively simple property gives strong information about the structure of graded bimodules. In particular, the two different notions of torsion defined above actually coincide for noetherian bimodules over projectively simple rings.

Lemma 1.7. *Let A be a noetherian projectively simple ring.*

(a) *Let B be any graded ring. If M is a noetherian (B, A)-bimodule such that M_A is Goldie torsion, then M is finite dimensional, whence torsion over A.*
(b) *Let M and N be noetherian graded (A, A)-bimodules such that M_A and N_A are not torsion. Then $\dim_k \underline{\mathrm{Hom}}_A(M_A, N_A) = \infty$.*

Proof. (a) Since ${}_BM$ is noetherian, $M = \sum_i Bx_i$ for a finite set of elements $\{x_i\} \subset M$. Thus

$$I := \mathrm{r.ann}(M_A) = \bigcap_i \mathrm{r.ann}(x_i) \neq 0,$$

where the final inequality follows from the fact that every $\mathrm{r.ann}(x_i)$ is essential. Thus M_A is finitely generated over a finite dimensional algebra A/I. Therefore M is finite dimensional.

(b) By part (a), M_A and N_A are not Goldie torsion. Since A is a prime Goldie ring, by replacing M_A by a factor module and N_A by a submodule, we may assume that both M and N are uniform Goldie torsionfree graded right A-modules, or homogeneous right ideals of A. Since A is prime, $N_{\geq n}M \neq 0$ for all n. Hence for each n there is a homogeneous element $x \in N_{\geq n}$ such that the left multiplication

$$l_x : m \to xm \quad \text{for all } m \in M$$

is a nonzero element in $\underline{\mathrm{Hom}}_A(M_A, N_A)$. Thus (b) follows. □

In the final result of this section, we note some circumstances under which the property of projective simplicity passes from one algebra to closely related algebras.

Lemma 1.8. *Let A be a noetherian graded prime ring.*

(a) *Suppose that B is a graded subring of A such that $\dim_k A/B < \infty$. Then A is projectively simple if and only if B is.*
(b) *Suppose that B is a projectively simple graded subring of a graded Goldie prime ring A such that A_B is finitely generated. Then A is projectively simple.*
(c) *Suppose that A is projectively simple, and let $B = A^{(n)}$ be the nth Veronese subring $\bigoplus_{i=0}^{\infty} A_{ni}$ for some $n \geq 2$. If B is prime and A_B (or ${}_BA$) is finitely generated, then B is projectively simple.*

Proof. (a) Suppose that A is not projectively simple. Let $0 \neq I$ be an ideal of A with $\dim_k A/I = \infty$. Then $J = I \cap B$ is a nonzero ideal of B with $\dim_k B/J = \infty$ and B is not projectively simple.

Conversely, suppose that A is projectively simple and let $0 \neq J$ be an ideal of B. Since A is projectively simple, $\dim_k A/(A_{\geq n}JA_{\geq n}) < \infty$. Also $A_{\geq n}JA_{\geq n} \subseteq BJB = J$ for some n, so $\dim_k B/J < \infty$ and B is projectively simple.

(b) Let I be a nonzero graded ideal of A. We want to show that $I \cap B$ is nonzero. Since A is Goldie prime, I contains a homogeneous regular element of A and thus

$$\operatorname{Kdim}(A/I)_B < \operatorname{Kdim} A_B = \operatorname{Kdim} B.$$

Hence the map $B \to A/I$ cannot be injective. Thus $B \cap I \neq 0$ and $B/(B \cap I)$ is finite dimensional because B is projectively simple. Now A/I is finitely generated over $B/(B \cap I)$, so it is also finite dimensional.

(c) We think of B as a subring of A which is zero except in degrees which are multiples of n. By [AZ1, 5.10(1)], B is noetherian. Let J be any nonzero right ideal of A. Then we claim that $J \cap B \neq 0$. Suppose this is not true; then if $0 \neq x \in J$ is any homogeneous element, then $x^n \in J \cap B = 0$. Thus J is a right nil ideal, and so J is a nilpotent ideal since A is noetherian [MR, 2.3.7]. Then since A is prime, $J = 0$, a contradiction.

Now let I be a nonzero graded ideal of B. Since B is prime, I contains a homogeneous regular element x. If $J = \operatorname{r.ann}_A x$, then $\operatorname{r.ann}_B x = J \cap B = 0$ since x is regular in B; then by the claim above, $J = 0$ and x is regular in A. Then A/IA is a noetherian (B, A)-bimodule which is Goldie torsion as a right A-module. By Lemma 1.7(a), $\dim_k A/IA < \infty$. Since $I = IA \cap B$, we conclude that $\dim_k B/I < \infty$. □

The result of Lemma 1.8(c) is false without the prime hypothesis on B, as is clear from the following example.

Example 1.9. Let $A = k\langle x, y\rangle/(x^2, y^2)$. Then A is a PI prime ring of GK-dimension one, so A is projectively simple by Lemma 1.2(c). The Veronese subring $A^{(2)}$ is isomorphic to $k[u, v]/(uv)$ where $u = xy$ and $v = yx$. Hence $A^{(2)}$ is semiprime, but not prime, so it cannot be projectively simple.

2. Twisted homogeneous coordinate rings

Starting now we work towards the goal of producing some interesting explicit examples of projectively simple rings. For this we take Example 1.3 as our model; we expect to find other projectively simple rings by looking at the class of twisted homogeneous coordinate rings, which we define and study in this section. We will also prove Theorem 0.4, which will show that under certain hypotheses, twisted homogeneous coordinate rings really are the only examples of projectively simple rings.

Assume throughout this section that k is an algebraically closed field. Let X be a commutative projective scheme, σ an automorphism of X and $\mathcal{L}$ an invertible sheaf on X. For any sheaf $\mathcal{F}$ on X, we use the notation $\mathcal{F}^\sigma$ for the pullback $\sigma^*(\mathcal{F})$. Now set

$$\mathcal{L}_n = \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}}$$

for all $n \geq 0$. The *twisted homogeneous coordinate ring* (or *twisted ring* for short) $B = B(X, \mathcal{L}, \sigma)$ is defined to be the graded vector space $\bigoplus_{n=0}^{\infty} \mathrm{H}^0(\mathcal{L}_n)$, with the multiplication rule

$$fg = f \otimes g^{\sigma^m} \quad \text{for} \quad f \in B_m,\ g \in B_n.$$

For more details about this construction see [AV] and [Ke].

The sheaf $\mathcal{L}$ is called *σ-ample* if for any coherent sheaf $\mathcal{F}$ on X, $\mathrm{H}^i(\mathcal{F} \otimes \mathcal{L}_n) = 0$ for all $i > 0$ and $n \gg 0$. This reduces to the usual notion of ampleness in the commutative case when σ is the identity. In case $\mathcal{L}$ is σ-ample, the ring $B = B(X, \mathcal{L}, \sigma)$ is noetherian and there are many nice relationships between the properties of B and the properties of X. For example, in this case there is an equivalence of categories $\mathrm{QGr}\, B \simeq \mathrm{Qch}\, X$, where $\mathrm{Qch}\, X$ is the category of quasi-coherent sheaves on X and $\mathrm{Proj}\, B = (\mathrm{QGr}\, B, \pi B)$ is the noncommutative projective scheme associated to B (see §10).

We now give in the next proposition a geometric characterization of which rings $B(X, \mathcal{L}, \sigma)$ are projectively simple, which proves Proposition 0.1 from the introduction. The answer involves the following geometric notion.

Definition 2.1. Let σ be an automorphism of a projective scheme X. Then we call σ *wild* if $\sigma(Y) \neq Y$ for every nonempty reduced closed subscheme $Y \subsetneq X$.

Proposition 2.2. *Let $B = B(X, \mathcal{L}, \sigma)$, where $\mathcal{L}$ is σ-ample. Then B is projectively simple if and only if σ is a wild automorphism of X.*

Proof. Call two right ideals J, J' of B equivalent if $J_{\geq n} = J'_{\geq n}$ for some $n \geq 0$. By [AS1, 4.4], the mapping $\mathcal{I} \mapsto \bigoplus_{n \geq 0} \mathrm{H}^0(\mathcal{I} \otimes \mathcal{L}_n)$ gives a bijective correspondence between ideal sheaves $\mathcal{I}$ of X which are σ-invariant (in other words $\mathcal{I}^\sigma \cong \mathcal{I}$), and equivalence classes of two-sided ideals of B.

Note that if $\mathcal{I}$ is a σ-invariant ideal sheaf which defines a closed subscheme Y of X, then the ideal sheaf $\mathcal{I}'$ defining the reduction Y_{red} of Y is also σ-invariant. Moreover, if Y is already reduced, then its defining ideal sheaf $\mathcal{I}$ is σ-invariant if and only if $\sigma(Y) = Y$. Thus B is projectively simple if and only if the only reduced subschemes Y of X with $\sigma(Y) = Y$ are $\emptyset$ and X. □

Next, we want to prove Theorem 0.4, which allows us to say that a projectively simple ring with certain other hypotheses must be a twisted homogeneous coordinate ring in large degree. Our proof will largely rely on results from the literature. For the reader's convenience, we will summarize the main ideas involved, but without full details.

Let A be a connected finitely $\mathbb{N}$-graded k-algebra. An *R-point module* for A, where R is a commutative k-algebra, is a cyclic graded right $A \otimes_k R$-module M, generated in degree 0, such that $M_0 \cong R$ and M_n is a locally free R-module of rank 1 for each $n \geq 0$. The algebra A is called *strongly noetherian* if for every commutative noetherian k-algebra C, the ring $A \otimes_k C$ is noetherian. If A is strongly noetherian, then by work of Artin and Zhang [AZ2] there exists a projective scheme X which parametrizes the point modules for A. More exactly, if we associate to each commutative k-algebra R the set of all isomorphism classes of R-point modules for A, then this rule defines a functor *Rings* $\rightarrow$ *Sets* and X *represents* this functor.

Now let A be strongly noetherian and generated in degree 1. For each base ring R one gets a map from the set of R-point modules to itself which is defined by the rule $M \mapsto M_{\geq 1}(1)$, and these maps induce an automorphism σ of the representing scheme X [KRS, Proposition 10.2]. Then using the same methods as in [ATV], there is a natural way to construct a ring homomorphism $\phi : A \to B(X, \mathcal{L}, \sigma)$ where $\mathcal{L}$ is a very ample invertible sheaf on X. In [RZ], the authors prove the following additional facts about such maps ϕ:

Proposition 2.3. [RZ] *Let A be a strongly noetherian connected graded ring generated in degree 1. Let X be the scheme parametrizing the point modules, and let σ be the automorphism of X induced by the map $M \mapsto M_{\geq 1}(1)$ on point modules. Then there is a graded ring homomorphism $\phi : A \to B(X, \mathcal{L}, \sigma)$ which is surjective in large degree, and such that $\mathcal{L}$ is σ-ample and very ample.*

Given the preceding proposition, it is now easy to show the following structure result for projectively simple rings, which proves Theorem 0.4 from the introduction.

Theorem 2.4. *Let A be a connected $\mathbb{N}$-graded k-algebra over an algebraically closed field k. Assume that A is generated in degree 1, strongly noetherian over k, and that some k-point module exists for A. Then if A is projectively simple, there is an injective ring map $\phi : A \to B = B(X, \mathcal{L}, \sigma)$, where $\mathcal{L}$ is σ-ample, and $\dim_k B/A < \infty$. The ring B is also projectively simple and σ is a wild automorphism of X.*

Proof. Let ϕ be the ring homomorphism of Proposition 2.3 above. By that proposition we have $\operatorname{im} \phi \supset B_{\geq m}$ for all $m \gg 0$. Since we assumed that A has a point module, the scheme X representing the point modules for A must be nonempty. Then since $\mathcal{L}$ is σ-ample and X is non-empty, the sheaf $\mathcal{L}_n$ has some nonzero global section for all $n \gg 0$ [Ke, Proposition 2.3], and so $B_n \neq 0$ for all $n \gg 0$. Then $\operatorname{im} \phi$ is not finite dimensional, so $\ker \phi$ is an ideal of A with $\dim_k A/\ker \phi = \infty$. Since A is projectively simple, $\ker \phi = 0$ and thus also $\dim_k B/A < \infty$. The ring A is also prime by Lemma 1.2(b), and B must be prime as well. Then B is projectively simple by Lemma 1.8(a). The automorphism σ is wild by Proposition 2.2. □

Because of the preceding result, we will focus the majority of our attention on twisted homogeneous coordinate rings in the remaining sections of the paper. We are very curious, though, what kinds of more general projectively simple rings may appear if the hypotheses of Theorem 2.4 are relaxed. We wonder how stringent the hypothesis that A has a point module is; do there exist, for example, any projectively simple rings which have no finitely generated modules of GK-dimension one?

We can offer an example of a projectively simple ring which is not strongly noetherian. The example is not too far away from the strongly noetherian case, since it has a closely related overring which is strongly noetherian and projectively simple. It also has a large supply of point modules. Again, it would be interesting to know if there are non-strongly noetherian examples of projectively simple rings which are significantly different from this one.

Example 2.5. Let k be algebraically closed of characteristic zero. Let $B = B(X, \mathcal{L}, \sigma)$ where X is integral (so B is a domain), and $\mathcal{L}$ is σ-ample and very ample. We construct special subrings of B which are "Naive noncommutative blowups" as studied in [KRS]. Let $c \in X$ be a smooth closed point, with corresponding ideal sheaf $\mathcal{I}$. Setting $\mathcal{L}_n = \mathcal{L} \otimes \mathcal{L}^{\sigma} \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}}$ and $\mathcal{I}_n = \mathcal{I}\mathcal{I}^{\sigma} \ldots \mathcal{I}^{\sigma^{n-1}}$ for each $n \geq 0$, we define the ring

$$R = R(X, \mathcal{L}, \sigma, c) = \bigoplus_{n \geq 0} \mathrm{H}^0(\mathcal{I}_n \otimes \mathcal{L}_n) \subset B(X, \mathcal{L}, \sigma).$$

Now assume in addition that X is an abelian variety, and that the point $c \in X$ *generates* X in the sense that the subgroup $\mathcal{C} = \{nc | n \in \mathbb{Z}\}$ of X is a Zariski dense set. Let σ be the translation automorphism $x \mapsto x + c$ of X. In Theorem 7.2 below we will prove that σ is wild, and so by Proposition 2.2, $B = B(X, \mathcal{L}, \sigma)$ is projectively simple. As in [KRS, Example 11.5], it follows that the set $\mathcal{C}$ is *critically dense*, which means that every infinite subset of it is dense. Then R is a noetherian domain by [KRS, Theorem 4.1].

Now let $0 \neq P$ be a homogeneous prime ideal of R. It follows from [KRS, Theorem 4.1 and Lemma 6.1] that the R-module (R/P) has a compatible B-structure in large degree. More exactly, there is some $n \geq 0$ and a left B-module M such that ${}_R M \cong {}_R(R/P)_{\geq n}$. Now either $P = R_{\geq 1}$, or else $P = \mathrm{ann}_R (R/P)_{\geq n} = (\mathrm{ann}_B M) \cap R$. In either case, P is the intersection with R of an ideal Q of B. Since B is projectively simple, $\dim_k B/Q < \infty$, and so $\dim_k R/P < \infty$. Thus R is projectively simple. However, R is not strongly noetherian [KRS, Theorem 9.2].

3. Wild automorphisms and algebraic group actions

As we saw in the preceding section, to study projectively simple twisted homogeneous coordinate rings we need to better understand wild automorphisms. This will be the subject of sections 3 – 7. We begin with several simple observations.

Lemma 3.1. *Let X be a projective scheme.*

(a) *If σ is a wild automorphism of X, then X is reduced.*
(b) *If σ is a wild automorphism, then X is smooth.*
(c) *Assume that X is reduced with irreducible components $X_1, X_2, \ldots X_m$. Then σ is a wild automorphism of X if and only if the permutation of the X_i induced by σ is a single m-cycle, and σ^m restricts to a wild automorphism of each X_i. In this case, X is a disjoint union of $X_1, \cdots, X_m$.*
(d) *If X is integral, then an automorphism σ of X is wild if and only if σ^n is wild for all $n \geq 1$.*

Proof. (a) X_{red} is a non-trivial subscheme of X preserved by σ; hence, $X = X_{\text{red}}$.

(b) By part (a), Y is reduced. Let Y be the singular locus of X. Then Y is closed, σ-invariant and $Y \neq X$. Since σ is wild, Y is empty.

(c) The orbit of each component is preserved by σ; hence, there can only be one such orbit. Since the subscheme

$$Y = \bigcup_{i \neq j} X_i \cap X_j \subsetneq X$$

is σ-invariant, it has to be empty, i.e., $X_1, \ldots, X_m$ are disjoint. If $\sigma^m(Z) = Z$ for some subscheme $Z \subsetneq X_i$, then $\sigma(Z') = Z'$ where $Z' = \bigcup_{j=0}^{m-1} \sigma^j(Z) \subsetneq X$; thus $Z = \emptyset$ and so $\sigma^m|_{X_i}$ is wild.

(d) If σ preserves a subscheme $Y \subset X$, then so does σ^n. Conversely, if σ^n preserves $Y \subsetneq X$ then σ preserves $\bigcup_{j=0}^{n-1} \sigma^j(Y) \subsetneq X$. □

By part (d) of the Lemma, it is clear that we lose nothing essential in our understanding of wild automorphisms by restricting to the case where X is integral, and we will usually do so.

thus

The rest of this section will be devoted to proving Conjecture 0.3 under the additional assumption that σ can be extended to a finite-dimensional family of automorphisms (or even of birational isomorphisms) of X. Before proceeding, let us make a few remarks about our terminology. In order to avoid arithmetic complications (in what is already a difficult geometric problem), we shall assume throughout that k is an algebraically closed field of characteristic zero. For convenience, we often use the language of varieties rather than schemes; for us, a *variety* is a (not necessarily irreducible) reduced separated scheme of finite type over k. For a variety X the notation $p \in X$ always means that p is a closed point of X. By an *algebraic group* we mean a variety with a compatible group structure. Note that we do not assume that algebraic groups are linear, but by definition they must have finitely many components. If G is an algebraic group, then for each $h \in X$ one has a *translation automorphism* T_h defined by $g \mapsto hg$. An *abelian variety* is a complete irreducible variety X over k which is also an algebraic group.

Proposition 3.2. *Let X be an irreducible projective variety. Suppose that an algebraic group $G \subset \operatorname{Aut}(X)$ acts regularly on X such that $\sigma \in G$ acts by a wild automorphism. Then X is an abelian variety and some power of σ is a translation automorphism after a group structure in X is chosen properly.*

Proof. First we make some reductions. Let H be the closure in G of the subgroup $\{\sigma^i\}_{i \in \mathbb{Z}}$. Then H is an algebraic group. Since H is a closure of an abelian group, H is abelian. Without loss of generality, we may assume from now on that $G = H$, so in particular G is abelian. Since G has finitely many components, σ^n must lie in the connected component G_e containing the identity element of G, for some $n \geq 1$. Since X is irreducible, σ^n is also wild (Lemma 3.1(d)). Thus we may replace σ by σ^n and G by G_e, and so we may assume that G is irreducible.

Choose any $x \in X$, and let Y be the closure of Gx in X. Since $\sigma \in G$, $\sigma(Gx) \subset Gx$ and hence $\sigma(Y) \subset Y$. Since σ is wild, $Y = X$. Now the rule $g \mapsto gx$ defines a morphism $f : G \to X$. By Chevalley's theorem [Hart, Exercise II.3.19], the image Gx of the map f must be constructible, and so it must contain an open subset U of its closure $Y = X$. Hence $Gx = GU = \bigcup_{g \in G} gU$ is dense and open in X. Since Gx is σ-stable, $Z := X - Gx$ is a σ-stable closed subvariety. Since σ is wild, we conclude that Z is empty and thus $X = Gx$.

Next, let G_0 be the stabilizer of x. Since G is abelian, G_0 is the stabilizer of every point in $Gx = X$. Since G is a subgroup of $\mathrm{Aut}(X)$ and X is a variety over an algebraically closed field, automorphisms of X are determined by their actions on closed points and so G_0 is trivial. Hence the morphism $f : G \to X$ is bijective.

Since our standing assumption is that the base field k has characteristic 0, it now follows from [Hum, Theorem 4.6] that f is birational. Let V be the largest open set of X such that $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is an isomorphism. Then V is σ-invariant, and since σ is wild, we see that $V = X$ and f is an isomorphism. Therefore the group structure of G may be transferred to X via f, G must be projective, and both X and G are abelian varieties. The isomorphism f transforms the translation automorphism T_σ of G to the translation automorphism $T_{f(\sigma)} = \sigma$ of X. Since we replaced σ by σ^n during the proof, we see that some power of the original σ is a translation automorphism of X as required. □

The previous theorem can be generalized to the case of birational actions. Let $\mathrm{Bir}(X)$ be the group of all birational maps over k from X to itself.

Corollary 3.3. *Let X be an irreducible projective variety admitting a wild automorphism σ. Suppose that $P \subset \mathrm{Bir}(X)$ is an algebraic group that acts birationally on X such that $\sigma \in P$. Then X is an abelian variety and some power of σ is a translation after a group structure in X is chosen properly.*

Proof. Let $G = \{\tau \mid \sigma\tau = \tau\sigma\}$ be the centralizer of σ in P. Since P is an algebraic group, so is G. Clearly $\sigma \in G$. We want to show that $G \subset \mathrm{Aut}(X)$. For any $\tau \in G$, let Y_τ be the indeterminacy locus of the rational map $\tau \colon X \dashrightarrow X$. Since $\sigma\tau = \tau\sigma$, Y_τ is σ-stable. Since σ is wild, Y_τ is empty, whence τ is regular. Similarly, τ^{-1} is regular. Therefore $G \subset \mathrm{Aut}(X)$. The assertion now follows from Theorem 3.2. □

In Remark 8.3 below, we will see that the conclusions of Theorem 3.2 and Corollary 3.3 may be strengthened to say that σ is, in fact, itself a translation automorphism.

4. Wild automorphisms and numerical invariants

In this section we will show that the existence of a wild automorphism imposes strong restrictions on two important numerical invariants of a projective variety, the Kodaira dimension and the Euler characteristic. We will continue to assume that the base field k is algebraically closed and of characteristic zero.

We begin by considering the Kodaira dimension. Suppose that X is a smooth projective variety, and let ω_X be the canonical sheaf. For a sheaf $\mathcal{F}$ on X we write $\Gamma(X, \mathcal{F})$ for the global sections of $\mathcal{F}$. The *Kodaira dimension* (or *canonical dimension*) of X, denoted by $\kappa(X)$, is defined to be the transcendence degree of the canonical ring $\bigoplus_{i \geq 0} \Gamma(X, \omega_X^{\otimes i})$ minus 1. Similarly, the *anti-canonical dimension* $\bar{\kappa}(X)$ of X is the transcendence degree of the anti-canonical ring $\bigoplus_{i \geq 0} \Gamma(X, \omega_X^{\otimes -i})$ minus 1. We will see next that the existence of a wild automorphism limits the possible values of κ and $\bar{\kappa}$.

Let $\mathcal{L}$ be an invertible sheaf on a projective variety X and let σ be an automorphism of X. A σ-linearization of $\mathcal{L}$ is an isomorphism

$$p\colon \sigma^*(\mathcal{L}) \xrightarrow{\simeq} \mathcal{L}\,.$$

Informally speaking, we can think of p as a way to lift σ from X to $\mathcal{L}$. If p is chosen, we shall say that $\mathcal{L}$ is σ-linearized and identify $\sigma^*(\mathcal{L})$ and $\mathcal{L}$ via p.

Proposition 4.1. *Suppose an irreducible projective variety X admits a wild automorphism σ. If $\mathcal{L}$ is a σ-linearized invertible sheaf on X then either*

(a) *$\mathcal{L}$ is isomorphic to $\mathcal{O}_X$, or*
(b) $\Gamma(X, \mathcal{L}) = 0 = \Gamma(X, \mathcal{L}^{-1})$.

Proof. Suppose that (b) fails, in other words that either $\mathcal{L}$ or $\mathcal{L}^{-1}$ has a nonzero global section. Note that a σ-linearization of $\mathcal{L}$ induces a σ-linearization of $\mathcal{L}^{-1}$. Thus after switching the roles of $\mathcal{L}$ and $\mathcal{L}^{-1}$ if necessary, we may assume that $V := \Gamma(X, \mathcal{L}) \neq (0)$.

Since $\mathcal{L}$ is linearized, pullback by σ induces an automorphism $\widetilde{\sigma} = p \circ \sigma^*$ of V. Since k is algebraically closed, V must have an eigenvector f for the action of $\widetilde{\sigma}$. The vanishing set $Z(f) \subsetneq X$ of the global section f is then fixed by σ. Since σ is wild, $Z(f) = \emptyset$. This implies that $\mathcal{L}$ is generated by the global section f, i.e., the map $i : \mathcal{O}_X \to \mathcal{L}$ sending 1 to f is an isomorphism. □

Corollary 4.2. *Suppose an irreducible projective variety X of dimension $d \geq 1$ admits a wild automorphism. Then either (i) $\kappa(X) = \bar{\kappa}(X) = -1$ or (ii) $\omega_X^{\otimes n} \cong \mathcal{O}_X$ for some $n \geq 1$. (In the latter case $\kappa(X) = \bar{\kappa}(X) = 0$.) In particular, X cannot be a Fano variety or a variety of general type.*

Proof. Apply Proposition 4.1 with $\mathcal{L} = \omega_X^{\otimes n}$. If (a) holds for some $n \geq 1$ then $\omega_X^{\otimes n} \cong \mathcal{O}_X$; if instead (b) holds for all $n \geq 1$ then $\kappa(X) = \bar{\kappa}(X) = -1$.

The last assertion is an immediate consequence of the first: if X is Fano then $\bar{\kappa}(X) = d \geq 1$ and if X is of general type then $\kappa(X) = d \geq 1$. □

Corollary 4.3. *Let X be an abelian variety, and suppose that $Y \subseteq X$ is an irreducible subvariety of X such that Y admits a wild automorphism. Then Y is a translate of an abelian subvariety of X.*

Proof. By Corollary 4.2, since Y has a wild automorphism it follows that $\kappa(Y) \leq 0$. Then a theorem of Ueno, which is proved for the case $k = \mathbb{C}$ in [U, 10.1 and 10.3], and for a general algebraically closed field in [Mo, 3.7], states that Y must be a translate of an abelian subvariety of X. □

In the rest of this section we work out what the existence of a wild automorphism tells us about the Euler characteristic (or the arithmetic genus) of a variety. As a consequence, we will see that rationally connected varieties have no wild automorphisms.

Given a regular map $f\colon X \longrightarrow X$, we define the *algebraic Lefschetz number* $L(f, X)$ by the formula

$$L(f, X) = \sum_{q \geq 0} (-1)^q \mathrm{Trace}\left(f^*\colon \mathrm{H}^q(X, \mathcal{O}_X) \longrightarrow \mathrm{H}^q(X, \mathcal{O}_X)\right).$$

Note that the sum on the right is well-defined because $\mathrm{H}^q(X, \mathcal{O}_X)$ is a finite-dimensional k-vector space for every $q \geq 0$ [Hart, III.5.2] and $\mathrm{H}^q(X, \mathcal{O}_X) = (0)$ for $q > \dim(X)$ [Hart, III.2.7]. We also define the *algebraic Euler characteristic* $\chi(\mathcal{O}_X)$ of X as

$$\chi(\mathcal{O}_X) = L(\mathrm{id}, X) = \sum_{q \geq 0} (-1)^q \dim \mathrm{H}^q(X, \mathcal{O}_X).$$

Proposition 4.4. *Let X be a smooth irreducible projective variety and let σ be an automorphism of X.*

(a) If $\chi(\mathcal{O}_X) \neq 0$, then some power of σ has a fixed point.

(b) If $\dim(X) \geq 1$ and σ is wild then the algebraic Euler characteristic $\chi(\mathcal{O}_X) = 0$. Equivalently, the arithmetic genus $p_a(X) = (-1)^{\dim(X)+1}$.

Proof. (a) By the "holomorphic Lefschetz fixed point theorem", if σ has no fixed points then $L(\sigma, X) = 0$. A proof of this result over $k = \mathbb{C}$ can be found in [GH, Section III.4, p. 426] and over an arbitrary algebraically closed field k of characteristic zero in [TT]. (Note that the sum in the right hand side of the formula is empty if there are no fixed points.)

Now assume that σ^n has no fixed point for any $n \geq 1$. Set $W = \bigoplus_{q \geq 0} \mathrm{H}^q(X, \mathcal{O}_X)$ and $d = \dim_k(W)$. Then the automorphism σ^* of W induced by σ satisfies an equation of the form

$$(\sigma^*)^d + c_1(\sigma^*)^{d-1} + \cdots + c_d \,\mathrm{id}_W = 0$$

for some $c_i \in k$. Since σ^* is an automorphism of W, $c_d \neq 0$. By the linearity of the trace, we have

$$L(\sigma^d, X) + c_1 L(\sigma^{d-1}, X) + \cdots + c_{d-1} L(\sigma, X) + c_d L(\mathrm{id}_X, X) = 0.$$

Then since $L(\sigma^n, X) = 0$ for all $n \geq 1$, $L(\mathrm{id}_X, X) = \chi(\mathcal{O}_X) = 0$.

(b) By Lemma 3.1(d) no power of σ can have a fixed point. Hence, part (a) tells us that $\chi(\mathcal{O}_X) = 0$. The second assertion is simply a restatement of the first, since by definition, $p_a(X) = (-1)^{\dim(X)}(\chi(\mathcal{O}_X) - 1)$ [Hart, p. 230]. □

Remark 4.5. If X is an irreducible variety defined over $k = \mathbb{C}$ which has a wild automorphism, then the same arguments as in Proposition 4.4 show that $e(X) = 0$, where $e(X)$ is the usual (topological) Euler characteristic.

Proposition 4.6. *Suppose an irreducible projective variety X of dimension $d \geq 1$ has a wild automorphism. Then*

(a) X carries a non-vanishing regular differential m-form ω for some odd $m \geq 1$. Moreover, ω can be chosen so that

$$\sigma^*\omega = c\,\omega \text{ for some } c \in k^*. \tag{4.7}$$

(b) X is not rationally connected.

(c) X is not unirational.

Recall that an algebraic variety X is called *rationally connected* if there is a family of irreducible rational curves in X such that two points in general position can be connected by a curve from this family; see [Ko, Section IV.3]. X is called *unirational* if it admits a dominant rational map $\mathbb{P}^n \dashrightarrow X$ for some $n \geq 1$.

Proof. (a) Let $h^{n,m} = h^{n,m}(X) = \dim \mathrm{H}^n(X, \bigwedge^m \Omega_X)$ be the Hodge numbers of X. Here Ω_X is the sheaf of differential 1-forms on X and $\bigwedge^m \Omega_X$ is the sheaf of differential m-forms.

We claim that $h^{0,m} > 0$ for some odd $m \geq 1$. Since $h^{n,m} = h^{m,n}$ (see, for example, [D, pp. 54-55]), we have

$$p_a(X) = \sum_{i=0}^{d-1}(-1)^i h^{d-i,0} = \sum_{m=1}^{d}(-1)^{d-m}h^{0,m}\,.$$

By Proposition 4.4, $p_a(X) = (-1)^{d+1}$, i.e.,

$$\sum_{m=1}^{d}(-1)^{m+1}h^{0,m} = 1\,. \tag{4.8}$$

If $h^{0,m} = 0$ for every odd $m \geq 1$, then every term in the left hand side of (4.8) is non-positive, a contradiction.

Thus for some odd m we have $V := \mathrm{H}^0(X, \bigwedge^m \Omega_X) \neq (0)$. Now we can choose ω to be an eigenvector for the linear automorphism σ^* of the finite-dimensional k-vector space V. In other words, (4.7) holds. Finally, the vanishing locus Y of ω is a closed σ-invariant subvariety of X. Since σ is wild, $Y = \emptyset$, i.e., ω is a non-vanishing differential form, as claimed.

(b) follows from (a). Indeed, if X is rationally connected then $h^{0,m}(X) = 0$ for all $m \geq 0$; see [Ko, Corollary IV.3.8].

(c) follows from (b) because a unirational variety is rationally connected; see [Ko, Example IV.3.2.6.2] or [LBR, Lemma 3.4.1]. □

5. Wild automorphisms and the Albanese map

Let X be an irreducible variety. Then associated to X is the *Albanese variety* $\mathrm{Alb}(X)$, which is an abelian variety, and the *Albanese map* $f : X \to \mathrm{Alb}(X)$. These constructions have the following properties: $\mathrm{Alb}(X)$ is generated as an algebraic group by the image of f (in the sense of Definition 7.1 below), and given any other regular map $g : X \to Y$ where Y is an abelian variety, there is a regular map $h : \mathrm{Alb}(X) \to Y$ such that $hf = g$. The dimension of $\mathrm{Alb}(X)$ is called the *irregularity* of X and is denoted by $q(X)$ (or just q if the reference to X is clear from the context). Equivalently, $q(X) = \dim_k \mathrm{H}^1(X, \mathcal{O}_X)$; see for example [BoHu, Section I.6] or [U, Lemma 9.22].

We shall now see that if an irreducible variety has a wild automorphism then its Albanese map must have various special properties.

Proposition 5.1. *Let X be an irreducible projective variety of dimension d and irregularity q and let $\pi\colon X \longrightarrow \mathrm{Alb}(X)$ be the Albanese map. Suppose that σ is a wild automorphism of X. By the universal property of the Albanese map, σ induces an automorphism of $\mathrm{Alb}(X)$, which we will denote by $\overline{\sigma}$.*

(a) *π is surjective.*
(b) *$\overline{\sigma}$ is a wild automorphism of $\mathrm{Alb}(X)$.*
(c) *π is smooth.*
(d) *The fiber $X_t = \pi^{-1}(t)$ is a smooth irreducible variety of dimension $d - q$ for every $t \in \mathrm{Alb}(X)$.*
(e) *$q \leq d$ and if $q = d$ then X is an abelian variety.*

Note that by [Mo, Theorem 8.1], part (a) holds for every variety X with $\kappa(X) = 0$. On the other hand, if $\kappa(X) = -1$ and we do not assume that X has a wild automorphism, then part (a) may fail; for example, the Albanese map for the ruled surface $X \longrightarrow C$ is surjective if and only if C is a curve of genus ≤ 1; see [GH, p. 554].

Proof. (a) Let $\overline{X} = \pi(X)$. We claim that

$$\overline{\sigma} \text{ restricts to a wild automorphism of } \overline{X}. \tag{5.2}$$

Indeed, assume the contrary, say $\overline{\sigma}(\overline{Y}) \subset \overline{Y}$ for $\emptyset \subsetneq \overline{Y} \subsetneq \overline{X}$. Then setting $Y = \pi^{-1}(\overline{Y})$, we see that $\emptyset \subsetneq \sigma(Y) \subseteq Y$, a contradiction. Now by Corollary 4.3, $\overline{X}$ is a translate of an abelian subvariety in $\mathrm{Alb}(X)$. By the definition of the Albanese map, this implies that $\overline{X} = \mathrm{Alb}(X)$; see [La1, II.3].

(b) This follows from part (a) and (5.2).

(c) Since $\mathrm{char}\, k = 0$ and we know that X must be nonsingular (see Lemma 3.1(b)), by generic smoothness there exists a Zariski open subset

$U \subseteq \mathrm{Alb}(X)$ such that π is smooth over U [Hart, Corollary III.10.7]. Then clearly π is smooth over the $\overline{\sigma}$-invariant open subset

$$W = \bigcup_{i \in \mathbb{Z}} \overline{\sigma}^i(U)$$

of $\mathrm{Alb}(X)$. Since $\overline{\sigma}$ is wild, $W = \mathrm{Alb}(X)$.

(d) The fact that each X_t is smooth of dimension $d-q$ is immediate from part (c). To show that X_t is irreducible, consider the Stein decomposition

$$\pi\colon X \xrightarrow{\alpha} X' \xrightarrow{\beta} \mathrm{Alb}(X)\,,$$

where α has connected fibers and β is finite. The variety X' may be defined explicitly as **Spec** $\pi_*(\mathcal{O}_X)$ [Hart, III.1.5]; in particular, X' is irreducible, since it is covered by the spectra of domains. Our goal is to show that β is an isomorphism. If we can prove this, then each X_t is connected and nonsingular, so must be irreducible.

Since the automorphism $\overline{\sigma}$ of $\mathrm{Alb}(X)$ acts also on the sheaf of graded algebras $\pi_*(\mathcal{O}_X)$ over $\mathrm{Alb}(X)$, we get an induced automorphism $\widetilde{\sigma}$ of X'. Since β is a finite surjective map and $\overline{\sigma}$ is a wild automorphism of $\mathrm{Alb}(X)$, it follows that $\widetilde{\sigma}$ is a wild automorphism of X'. Thus X' is nonsingular by Lemma 3.1(b). Then as in the proof of part (c), since β is smooth over an open set and $\overline{\sigma}$ is wild, β must be a smooth morphism of relative dimension 0, in other words an étale map. Now by a theorem of Serre and Lang [Mu, Section IV.18], X' has a structure of an abelian variety such that β is a regular homomorphism. By the universal property of the Albanese map, α factors through π. In other words, β has an inverse, so β is an isomorphism as desired.

(e) The inequality $q \leq d$ is an immediate consequence of part (a). If $q = d$, then by (c) π is a smooth morphism of relative dimension 0, so étale. By [Mu, Section IV.18] again, X has the structure of an abelian variety. □

6. Wild automorphisms of algebraic surfaces

In this section we will prove Conjecture 0.3 in the case where $\dim(X) \leq 2$. Our proof relies on the classification of algebraic surfaces. We will start with the most difficult cases, where X is assumed to be a ruled surface or a hyperelliptic surface, and defer the rest of the proof until the end of this section.

Recall that an algebraic surface X is called *ruled* if it admits a surjective morphism

$$\pi\colon X \to C \tag{6.1}$$

to a smooth curve C, and such that the fiber of π over each point of C is isomorphic to $\mathbb{P}^1$. Such a morphism always has a section $s\colon C \longrightarrow X$; for example, see [Ko, Corollary IV.6.6.2].

Lemma 6.2. *A (smooth projective) ruled surface cannot have a wild automorphism.*

Proof. Assume the contrary: σ is a wild automorphism of a ruled surface X. Let $\pi : X \longrightarrow C$ be as in (6.1) and let g be the genus of C. Since $p_a(X) = -g$ (see [Hart, Corollary V.2.5]), Proposition 4.4 tells us that $g = 1$, so that C is an elliptic curve.

By [Hart, Corollary V.2.5] the irregularity $q(X) = g = 1$, so the Albanese variety $\mathrm{Alb}(X)$ is an elliptic curve. We claim that C is, in fact, the Albanese variety and π is the Albanese map for X. Indeed, let $\alpha\colon X \longrightarrow \mathrm{Alb}(X)$ be the Albanese map. By the universal property of the Albanese map, π factors through α, in other words

$$\pi\colon X \xrightarrow{\alpha} \mathrm{Alb}(X) \xrightarrow{\beta} C\,,$$

where β is a covering map of elliptic curves. Since the fibers of π are connected (each fiber is isomorphic to $\mathbb{P}^1$), we conclude that β is one-to-one, i.e., is a bijective morphism between smooth curves $\mathrm{Alb}(X)$ and C. We conclude that β is an isomorphism; this proves the claim. (For a different proof of this claim over $k = \mathbb{C}$, see [GH, p. 554].)

Now let $s\colon C \longrightarrow X$ be a section of π, with the property that $C_0 = s(C) \subset X$ has the minimal possible self-intersection number. Following [Hart, Section V.2], we denote this number by $-e$. The group $\mathrm{Num}(X)$ of divisors in X up to numerical equivalence is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, and is generated by C_0 and F, where F is a fiber of π; the intersection form on $\mathrm{Num}(X)$ is given by $C_0 \cdot F = 1$, $F^2 = 0$ and $C_0^2 = -e$ [Hart, Proposition V.2.3]. As we saw in Proposition 5.1, σ acts on the fibers of π; since these fibers are all algebraically equivalent, we have $\sigma(F) \equiv F$ in $\mathrm{Num}(X)$. We claim that C_0 and $\sigma(C_0)$ are numerically equivalent, that is,

$$\sigma(C_0) \equiv C_0 \text{ in } \mathrm{Num}(X). \tag{6.3}$$

Indeed, suppose that $\sigma(C_0) \equiv aC_0 + bF$ for some $a, b \in \mathbb{Z}$. Then $a = \sigma(C_0) \cdot F = 1$, and since $C_0^2 = \sigma(C_0)^2$, we see that $b = 0$, thus proving (6.3). We now consider three cases.

1. $e > 0$. In view of (6.3), $C_0 \cdot \sigma(C_0) = C_0^2 < 0$; consequently, $\sigma(C_0) = C_0$, contradicting the fact that σ is wild.

2. $e = 0$. Since σ is wild (and hence, so is σ^2), we may assume that C_0, $\sigma(C_0)$ and $\sigma^2(C_0)$ are three distinct curves in X. Since $C_0^2 = 0$, formula (6.3) tells us that C_0, $\sigma(C_0)$ and $\sigma^2(C_0)$ are mutually disjoint in X. We now appeal to the general fact that any ruled surface $X \longrightarrow C$ with three mutually disjoint sections C_1, C_2 and C_3 is isomorphic to $\mathbb{P}^1 \times C$ (over C). Indeed, for such X there is a (unique) isomorphism $\mathbb{P}^1 \times C \longrightarrow X$ (over C), sending $\{0\} \times C$, $\{1\} \times C$ and $\{\infty\} \times C$ to C_0, C_1 and C_2. Thus we may assume that $X = \mathbb{P}^1 \times C$, where C is an elliptic curve. In this case the canonical divisor of X is $K_X = -2(\{\mathrm{pt}\} \times C)$ [Hart, Lemma V.2.10], so that $\bar{\kappa}(X) > 0$, contradicting Corollary 4.2.

3. $e < 0$. By [Hart, Theorems V.2.12 and V.2.15] there is only one surface X in this category, with $e = -1$. (This surface is often denoted

by $\mathbf{P_1}$.) By [Mar, Theorem 3(4)], the automorphism group $\mathrm{Aut}(X)$ has a normal subgroup

$$\Delta = \{\mathrm{id}, \delta_1, \delta_2, \delta_3\} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

such that $\overline{\delta_1} = \overline{\delta_2} = \overline{\delta_3} = \mathrm{id}_C$ for $i = 1, 2, 3$. Here $\overline{\delta}$ is the automorphism of C induced by δ (as in Proposition 5.1(b)). Let

$$Y = X^{\delta_1} \cup X^{\delta_2} \cup X^{\delta_3} ,$$

where X^{δ} denotes the fixed point set of δ. Since $\sigma(X^{\delta}) = X^{\sigma\delta\sigma^{-1}}$, we see that $\sigma(Y) = Y$. Clearly, $Y \neq X$; it thus remains to show that $X^{\delta} \neq \emptyset$ for $\delta = \delta_1, \delta_2, \delta_3$. Indeed, for every $p \in C$ we have an automorphism

$$\delta_{|\,\pi^{-1}(p)} \colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1,$$

where $\mathbb{P}^1$ stands for $\pi^{-1}(p)$. Since every automorphism of $\mathbb{P}^1$ has at least one fixed point, we see that $X^{\delta} \neq \emptyset$, as claimed.

This completes the proof of Lemma 6.2. □

Next we consider hyperelliptic surfaces. Recall that a hyperelliptic surface X is a surface of the form $(E \times F)/G$, where E and F are elliptic curves, and G is a finite subgroup of E acting on $E \times F$ via

$$g \cdot (x, y) \mapsto (x + e, \alpha(g) \cdot y)$$

for some $e \in X$ and injective homomorphism $\alpha \colon G \hookrightarrow \mathrm{Aut}(F)$. There are only seven possibilities for E, F, G, and α; see [BeMi, Table 1.1], [GH, p. 590] or [BoMu, p. 37].

Lemma 6.4. *A hyperelliptic surface cannot have a wild automorphism.*

Proof. Suppose that X is a hyperelliptic surface and σ is an automorphism of X. By [BeMi, Lemmas 1.2 and 2.1], σ can be lifted to an automorphism $\widetilde{\sigma} \in \mathrm{Aut}(E) \times \mathrm{Aut}(F)$ which normalizes G and such that the diagram

$$\begin{array}{ccc} E \times F & \xrightarrow{\widetilde{\sigma}} & E \times F \\ \downarrow & & \downarrow \\ X = (E \times F)/G & \xrightarrow{\sigma} & X = (E \times F)/G \end{array}$$

commutes. (In fact, by [BeMi, Lemma 3.1] $\widetilde{\sigma}$ centralizes G, but we shall not use this.) Consequently, σ descends to an automorphism σ_0 of $F/\alpha(G) \simeq \mathbb{P}^1$ (cf. [BoMu, Theorem 4, pp. 35-36]) such that the extended diagram

$$\begin{array}{ccc} E \times F & \xrightarrow{\widetilde{\sigma}} & E \times F \\ \downarrow & & \downarrow \\ X = (E \times F)/G & \xrightarrow{\sigma} & X = (E \times F)/G \\ \downarrow & & \downarrow \\ \mathbb{P}^1 \simeq F/\alpha(G) & \xrightarrow{\sigma_0} & \mathbb{P}^1 \simeq F/\alpha(G) \end{array}$$

commutes; here the vertical arrows are the natural projections. Now recall that every automorphism of $\mathbb{P}^1$ has a fixed point. If $x \in F/\alpha(G) \simeq \mathbb{P}^1$ is a fixed point of σ_0 then the preimage of x in $X = (E \times F)/G$ is a σ-invariant curve. This shows that σ cannot be wild.

The above argument relied on the results of [BeMi] which are stated under the assumption that $k = \mathbb{C}$. However, we observe that the the proofs of Lemmas 1.2 and 2.1 in [BeMi] remain valid over any algebraically closed field k of characteristic 0. □

Theorem 6.5. *Let X be an irreducible projective variety of dimension ≤ 2 over an algebraically closed field k of characteristic 0. If X admits a wild automorphism then X is an abelian variety.*

Proof. By Lemma 3.1(b), X is smooth. The case $\dim X = 0$ is trivial. Suppose $\dim(X) = 1$. Recall that the arithmetic genus $p_a(X)$ and the geometric genus $p_g(X)$ of a smooth curve X coincide; see [Hart, Proposition IV.1.1]. Hence, by Proposition 4.4, $p_g(X) = 1$, i.e., X is an elliptic curve or equivalently, a 1-dimensional abelian variety.

From now on we will assume that X is a smooth surface. By Proposition 4.4, $p_a(X) = -1$ and thus the irregularity is

$$q(X) = p_g(X) - p_a(X) = p_g(X) + 1\,;$$

see [Hart, Remark 7.12.3]. Here the geometric genus $p_g(X) = \dim \mathrm{H}^0(X, \omega)$ is either 0 or 1; see Proposition 4.1. If $p_g(X) = 1$ then $q(X) = 2$, and X is an abelian variety; see Proposition 5.1(e).

Now suppose $p_g(X) = 0$ and $q(X) = 1$. To complete the proof, it is enough to show that this case cannot occur. Let $\pi \colon X \longrightarrow C$ the Albanese map. Since $q(X) = 1$, C is an elliptic curve. We claim that X is minimal, i.e., X contains no smooth rational curves D with $D^2 = -1$. Indeed, if such a curve existed on X, we would have $\pi(D) = \{p\}$, where p is a point of C. In other words, D would be contained in a fiber of π. By Proposition 5.1, π is a smooth map with irreducible fibers. In particular, $D = \pi^{-1}(p)$. But then $D^2 = 0$ (because the fibers of π are disjoint and algebraically equivalent). This contradicts our assumption that $D^2 = -1$, thus proving the claim.

We now appeal to the Castelnuovo-Enriques classification of algebraic surfaces. By Corollary 4.2, the Kodaira dimension $\kappa(X) = -1$ or 0. If $\kappa(X) = -1$ then X is either rational or ruled [Hart, Theorem V.6.1]. By Proposition 4.6, X cannot be rational and by Lemma 6.2, X cannot be ruled. Thus we may assume without loss of generality that $\kappa(X) = 0$. Here there are four possibilities: (1) a $K3$ surface, (2) an Enriques surface, (3) an abelian surface, or (4) a hyperelliptic surface; see [BoHu, p. 373] or [Hart, Theorem V.6.3]. Of these four types, only (4) has $p_g(X) = 0$ and $q = 1$. On the the hand, by Lemma 6.4, X cannot be hyperelliptic. This shows that $p_g(X) = 0$ and $q(X) = 1$ is impossible, and the proof of the theorem is complete. □

7. Wild automorphisms of abelian varieties

In this section we will classify wild automorphisms of abelian varieties. We will assume throughout that k is algebraically closed of characteristic zero. We first review some basic definitions and facts from the theory of abelian varieties; see [Mu] and [La1] for detailed treatments of this material. As the name suggests, the group law of an abelian variety must be commutative [La1, Theorem II.1.1], and so we will use additive notation.

Let X and Y be arbitrary abelian varieties throughout the following discussion. We write $\mathrm{Hom}(X, Y)$ for the *homomorphisms* from X to Y, which are by definition the regular morphisms preserving the group structure. We also write $\mathrm{End}(X)$ for $\mathrm{Hom}(X, X)$, the *endomorphisms* of X. The words *automorphism* and *morphism* are used for regular maps which are not necessarily assumed to be homomorphisms of groups; for example, for any $a \in X$ one has the translation automorphism $T_a : x \mapsto x + a$. In fact, every regular map $\sigma : X \to Y$ between abelian varieties is of the form $\sigma = T_b \cdot \alpha$, where $\alpha \in \mathrm{Hom}(X, Y)$ and $b \in Y$; see [La1, Theorem 4, p.24]. We say that $\psi \in \mathrm{Hom}(X, Y)$ is an *isogeny* if ψ is surjective and has finite kernel. For example, for any abelian variety X and $p \in \mathbb{Z}$ the map $p \cdot Id_X : X \to X$ defined by $x \mapsto px$ is an isogeny [La1, Corollary IV.2.1]. If there exists an isogeny from X to Y, then there also must exist an isogeny from Y to X [La1, p. 29], and X and Y are said to be *isogeneous.*

A fundamental result is the *complete reducibility* theorem of Poincaré: if Z is an abelian subvariety of X, then there exists another abelian subvariety $Z' \subseteq X$ such that $Z + Z' = X$ and $Z \cap Z'$ is a finite group [La1, Theorem II.1.6]; in this case clearly X is isogeneous to $Z \times Z'$. An abelian variety X is *simple* if 0 and X are the only closed irreducible subgroups of X. Abelian varieties satisfy unique decomposition into simples in the following sense: every abelian variety X is isogeneous to a product $X_1 \times X_2 \times \cdots \times X_n$ where each X_i is a simple abelian variety, and the simples appearing in such a decomposition are unique up to isogeny and order of the factors [La1, page 30]. If X' is a closed subgroup of an abelian variety X, then the factor group X/X' has a natural structure of an abelian variety [La1, p. 3].

Next, we recall some results about the structure of the endomorphism ring $\mathrm{End}(X)$. Let X be isogeneous to the product $\prod_{i=1}^{h} X_i^{n_i}$ where the X_i are simple abelian varieties, mutually non-isogeneous. By [La1, Corollary VII.1.2], $\mathrm{End}(X)$ is a torsionfree $\mathbb{Z}$-module, and so there is an embedding of rings $\mathrm{End}(X) \subset \mathrm{End}_{\mathbb{Q}}(X) := \mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then by [La1, Theorem 7, p.30], each $D_i = \mathrm{End}_{\mathbb{Q}}(X_i)$ is a finite-dimensional division algebra over $\mathbb{Q}$, and

$$\mathrm{End}_{\mathbb{Q}}(X) \cong \prod_{i=1}^{h} M_{n_i}(D_i).$$

In particular, if X and Y are isogeneous then $\mathrm{End}_{\mathbb{Q}}(X) \cong \mathrm{End}_{\mathbb{Q}}(Y)$. If R is a ring with identity, we call an element $r \in R$ *unipotent* if $r - 1$ is nilpotent and *quasi-unipotent* if some power of r is unipotent. We shall apply these terms in particular to endomorphisms $\alpha \in \mathrm{End}(X)$. (The reader should not confuse these definitions with the use of the same terms in the theory of twisted homogeneous coordinate rings in [Ke]; see the discussion in §8 below.) Finally,

Definition 7.1. The algebraic subgroup of an abelian variety X *generated* by $S \subset X$ is the Zariski closure of the (abstract) subgroup $<S>$ of X generated by S. In particular, we say that S *generates* X if S is not contained in any proper closed subgroup of X. If $S = \{s_1, s_2, \dots\}$, then we will sometimes say that $s_1, s_2, \dots$ generate X.

Now we present the main theorem of this section, which gives a characterization of wild automorphisms of abelian varieties, and proves Theorem 0.2(a).

Theorem 7.2. *Let $\sigma = T_b \cdot \alpha$ be an automorphism of X, where $\alpha \in \mathrm{End}(X)$ is an automorphism and $b \in X$. Let $\beta = \alpha - Id$, and set $S = \{b, \beta(b), \beta^2(b), \dots\} \subset X$. Then the following are equivalent:*

(a) *σ is wild.*
(b) *α is unipotent and S generates X.*
(c) *α is unipotent and the image $\overline{b}$ of b generates $X/\beta(X)$.*

Proof. Let Z be the algebraic subgroup of X generated by S.

(b) $\Leftrightarrow$ (c). If S generates X then clearly $\overline{b}$ generates $X/\beta(X)$. For the converse, assume that $\overline{b}$ generates $X/\beta(X)$. Then $Z + \beta(X) = X$. Applying β on both sides, we obtain $\beta(Z) + \beta^2(X) = \beta(X)$. Now

$$X = Z + \beta(X) = Z + \beta(Z) + \beta^2(X) = Z + \beta^2(X);$$

the last equality follows from the fact that $\beta(Z) \subseteq Z$. Continuing in this manner, we obtain

$$X = Z + \beta^i(X) \text{ for } i = 1, 2, \dots$$

Since β is nilpotent, we conclude that $Z = X$, as desired.

(a) $\Rightarrow$ (b): Suppose that $\alpha - Id$ is not nilpotent. Let Y_s be the connected component containing 0 of the closed subgroup $\ker(\alpha - Id)^s$ of X. Then $Y_s \subset Y_{s+1}$ for all s, and this sequence of irreducible closed sets must stabilize, so we may set $Y = Y_s$ for $s \gg 0$. Clearly $\alpha(Y) \subset Y$. Hence α induces an endomorphism of the abelian variety X/Y, denoted by $\overline{\alpha}$. By the definition of Y, $\overline{\alpha} - Id$ is an isogeny of X/Y. Let π be the quotient map $X \to X/Y$ and let $\overline{\sigma} = T_{\pi(b)} \cdot \overline{\alpha}$, which is an automorphism of X/Y. Since $\overline{\alpha} - Id$ is an isogeny, it is surjective, and so there exists some $\overline{x}$ such that $(\overline{\alpha} - Id)(\overline{x}) = -\pi(b)$. Then $\overline{x} \in X/Y$ is a fixed point of the automorphism $\overline{\sigma}$. Set $W = \pi^{-1}(\overline{x}) \subset X$. Then $\sigma(W) = W$, and $W \neq X$ since $Y \neq X$. Hence σ is not wild, a contradiction.

Thus we have proved that α is unipotent. Since $\sigma(Z) \subset Z$, and σ is wild, we conclude also that $Z = X$.

(b, c) $\Rightarrow$ (a): First we observe that for the purpose of this proof we may replace σ by σ^n, where $n \geq 1$; Indeed, σ is wild if and only if σ^n is wild; see Lemma 3.1(d). Moreover, $\sigma^n = T_{b'} \cdot \alpha^n$, where

$$b' = \big(\sum_{i=0}^{n-1} \alpha^i\big)(b) = \big(\sum_{i=1}^{n} \binom{n}{i} \beta^{i-1}\big)(b).$$

If $\beta = \alpha - Id$ is nilpotent, then so is $\alpha^n - Id$. If $\overline{b}$ is the the image of b in $X' = X/\beta(X)$, then the image of b' in X' is $\overline{b'} = n\overline{b}$, so that $\overline{b'}$ generates $X/\beta(X)$ if and only if $\overline{b}$ does.

Let $W \subset X$ be a nonempty subvariety of minimal possible dimension such that $\sigma(W) \subset W$. Our goal is to show that $W = X$; this will imply that σ is wild. Replacing σ by σ^n if necessary, we may assume also that W is irreducible. Note that by the minimality of $\dim(W)$, σ restricts to a wild automorphism of W. By Corollary 4.3, W is a translate $H + x = T_x(H)$ of some abelian subvariety H of X. Now set

$$\sigma' = T_{-x} \sigma T_x = T_{b+\beta(x)} \cdot \alpha.$$

Then $\sigma'(H) = H$, so that $c = \sigma'(0) = b + \beta(x) \in H$. Thus $\alpha(H) \subset H$, from which it follows that $\beta(H) \subset H$. In particular, H contains $S' = \{c, \beta(c), \beta^2(c), \dots\}$. Since the image of c in $X/\beta(X)$ coincides with $\overline{b}$, the implication (c) $\Rightarrow$ (b) proved above shows that S' generates X. Since $S' \subset H$, we conclude that $H = X$ and hence $W = X$, as we wished to show. □

For the remainder of this section we will study the conditions for wildness given by Theorem 7.2 a bit further. Following part (c) of that theorem, we see that we would like to understand when a single element of an abelian variety will generate that variety. We analyze this situation next. The first two parts of the proposition below show that if the base field k is uncountable, then "most" points $a \in X$ will generate X. Since an arbitrary X is isogeneous to some product of simple abelian varieties, the remaining parts may be combined to give more precise information about which points $a \in X$ generate X.

Proposition 7.3. *Let X be an abelian variety and $a \in X$ some point.*

(a) *The element a generates X if and only if $f(a) \neq 0$ for every $0 \neq f \in$* End(X).
(b) *There is a countable set of closed subgroups $\{G_\alpha\}$ of X such that a generates X if and only if $a \notin \bigcup G_\alpha$.*
(c) *If $f : X \to Y$ is an isogeny, then a generates X if and only if $f(a)$ generates Y.*
(d) *If X is simple, then a generates X if and only if a is a point of infinite order on X.*
(e) *Let $X = X_1 \times X_2 \times \cdots \times X_n$ where the X_i are abelian varieties such that* Hom$(X_i, X_j) = 0$ *for all $i \neq j$. Then $a = (a_1, a_2, \dots, a_n)$ generates X if and only if a_i generates X_i for every $i = 1, \dots, n$.*

(f) *Let $Y = X^{\times n}$ and $a = (a_1, \dots, a_n) \in X^{\times n}$. Then a generates Y if and only if the following condition holds: given endomorphisms $\theta_i \in \mathrm{End}(X)$ with $\sum_{i=1}^n \theta_i(a_i) = 0$, one must have $\theta_i = 0$ for $i = 1, \dots, n$.*

Proof. (a) If $f(a) = 0$ for some $0 \neq f \in \mathrm{End}(X)$ then the algebraic subgroup Z generated by a is contained in $\mathrm{Ker}(f) \neq X$, so that a does not generate X.

Conversely, suppose that a does not generate X, so that a is contained in a closed subgroup $Z \subsetneq X$. Let Z_0 be the connected component of Z containing the identity element 0; then $pa \in Z_0$ for some integer $p \geq 1$. By complete reducibility, there is a non-trivial abelian subvariety $W \subset X$ such that $X = Z_0 + W$ and $Z_0 \cap W$ is finite. Then we may choose some isogeny $\theta : X/Z_0 \to W$, and thus we have a nonzero endomorphism

$$f: \quad X \overset{p \cdot Id_X}{\longrightarrow} X \longrightarrow X/Z_0 \overset{\theta}{\longrightarrow} W \hookrightarrow X$$

such that $f(a) = 0$.

(b) For every $0 \neq \alpha \in \mathrm{End}(X)$, let $G_\alpha = \ker \alpha$. By [Mu, Theorem IV.19.3], $\mathrm{End}(X)$ is countable, so part (a) implies the result.

(c) Suppose that a generates a closed subgroup A in X and that $f(a)$ generates B in Y. Then clearly $f(A) = B$. Now it is easy to see that

$$A = X \iff \dim(A) = \dim(X) \iff \dim(B) = \dim(Y) \iff B = Y.$$

(d) Every nonzero endomorphism of a simple abelian variety is an isogeny, which has finite kernel. Conversely, if a is a point on X of finite order and $G = \{na\}_{n \in \mathbb{Z}}$, then a is in the kernel of the isogeny $X \to X/G$. Now apply part (a).

(e) This is an easy consequence of the fact that every endomorphism of X maps X_i to X_i for all i.

(f) Suppose that a does not generate Y. Then by part (a), there exists an $0 \neq \alpha \in \mathrm{End}(Y)$ such that $\alpha(a) = 0$. If $Z \subset \mathrm{im}\, \alpha$ is a simple abelian subvariety, then Z is isogeneous to some simple abelian variety appearing in the decomposition of X into simples. It follows that $\mathrm{Hom}(\mathrm{im}\, \alpha, X) \neq 0$. Then picking a nonzero homomorphism $f : \mathrm{im}\, \alpha \to X$, we see that $\theta := f\alpha : Y \to X$ is a nonzero homomorphism such that $\theta(a) = 0$. But $\theta(a) = \sum_{i=1}^n \theta_i(a_i)$ for some $\theta_i \in \mathrm{End}(X)$, at least one of which is nonzero. The converse is proved by reversing this argument. □

Let us also make some remarks concerning unipotent automorphisms of abelian varieties. Of course, any abelian variety X has at least the unipotent automorphism $\sigma = Id$. In the next result we will identify those X for which there exist non-identity unipotent automorphisms, and show how to construct some of them. Suppose that $X = Y^{\times n}$ where Y is an abelian variety. Then for any integer matrix $M \in M_n(\mathbb{Z})$ we may define an endomorphism $\alpha_M \in \mathrm{End}(X)$, as follows. Write an arbitrary point in X as a column vector $\overline{x}$ with entries from Y. Then let α_M be defined by the formula $\alpha_M(\overline{x}) = M\overline{x}$, where the right hand side is "matrix multiplication" performed using the $\mathbb{Z}$-module structure of Y.

Proposition 7.4. (a) *Let $X = Y^{\times n}$ for some abelian variety Y. Then for $M \in \mathrm{GL}_n(\mathbb{Z})$, the automorphism $\alpha_M \in \mathrm{End}(X)$ is (quasi)-unipotent if and only if M is a (quasi)-unipotent matrix in $\mathrm{GL}_n(\mathbb{Z})$.*

(b) *Let X be an abelian variety which is isogeneous to a product $\prod_i X_i$, where the X_i are simple abelian varieties. Then X has a unipotent automorphism $Id \neq \alpha \in \mathrm{End}(X)$ if and only if X_i and X_j are isogeneous for some $i \neq j$.*

Proof. (a) Since the mapping $\psi : M \to \alpha_M$ is a ring homomorphism from $M_n(\mathbb{Z})$ to $\mathrm{End}(X)$, it is clear that M unipotent implies α_M unipotent. Conversely, suppose that M is not unipotent, so $M - Id$ is not nilpotent. One may easily check that $\ker \psi = 0$, and thus $\alpha_{M-Id} = \alpha_M - Id$ is not nilpotent in $\mathrm{End}(X)$, so α_M is not unipotent. Thus M is unipotent if and only if α_M is unipotent; the same statement for quasi-unipotence follows immediately.

(b) Suppose first that the X_i are pairwise non-isogeneous. Then $\mathrm{End}(X) \subset \mathrm{End}_{\mathbb{Q}}(X)$, where $\mathrm{End}_{\mathbb{Q}}(X) \cong \prod \mathrm{End}_{\mathbb{Q}}(X_i)$ is a product of division rings, and hence every nilpotent element in this ring is zero. Now if $\alpha \in \mathrm{End}(X)$ is a unipotent automorphism, then $\alpha - Id = 0$ and so $\alpha = Id$.

Conversely, if X_i and X_j are isogeneous for some $i \neq j$, then the ring $R := \mathrm{End}_{\mathbb{Q}}(X_i \times X_i) \cong \mathrm{End}_{\mathbb{Q}}(X_i \times X_j)$ embeds in $\mathrm{End}_{\mathbb{Q}}(X)$. Here $R \cong M_2(D)$, where $D = \mathrm{End}_{\mathbb{Q}}(X_i)$ is a division ring finite over $\mathbb{Q}$. By part (a), R contains many unipotent automorphisms of the form α_M for non-identity unipotent matrices $M \in \mathrm{GL}_2(\mathbb{Z})$. □

8. Unipotency on the Neron-Severi Group

As usual, we assume in this section that our base field k is algebraically closed of characteristic zero.

In the last section we saw how to construct many examples of wild automorphisms σ of an abelian variety X. Then by Proposition 2.2, we see that we obtain lots of examples of projectively simple twisted homogeneous coordinate rings $B(X, \mathcal{L}, \sigma)$, as long as we can find σ-ample invertible sheaves $\mathcal{L}$ on X. We shall show in this section that it will suffice to take $\mathcal{L}$ to be any ample invertible sheaf on X. To do this, we apply criteria of Keeler for σ-ampleness, which will require us to consider the induced action of automorphisms of X on the group of divisors modulo numerical equivalence.

Let us begin with a review of Keeler's results from [Ke]. Let X be any projective scheme, and let $\mathrm{Pic}\, X$ be the Picard group of all invertible sheaves on X. Two invertible sheaves $\mathcal{L}$ and $\mathcal{L}'$ on X are *numerically equivalent* if $(\mathcal{L}.C) = (\mathcal{L}'.C)$ for all integral curves $C \subset X$. Define the group $\mathrm{Num}(X)$ to be $\mathrm{Pic}\, X$ modulo numerical equivalence; it is isomorphic to $\mathbb{Z}^m$ for some m. When X is an abelian variety, then $\mathrm{Num}(X)$ is isomorphic to the Neron-Severi group $NS(X)$ [Hart, p.367]. Any morphism $\sigma : X \to X$ naturally induces via pullback an endomorphism of the group $\mathrm{Num}(X)$, or equivalently a matrix $P_\sigma \in M_m(\mathbb{Z})$.

Recall that we defined the notions of unipotence and quasi-unipotence for elements of a ring in the last section. Now let X be an projective scheme and $\sigma : X \to X$ any automorphism. Then we will call σ *Num-unipotent* if the corresponding matrix $P_\sigma \in M_m(\mathbb{Z})$ is unipotent and *Num-quasi-unipotent* if $P_\sigma \in M_m(\mathbb{Z})$ is quasi-unipotent. (Note that in [Ke] σ is called simply unipotent or quasi-unipotent, respectively.) Keeler proved that if σ is Num-quasi-unipotent then every ample invertible sheaf $\mathcal{L}$ on X is σ-ample, and if σ is not Num-quasi-unipotent then no σ-ample invertible sheaf exists [Ke, Theorem 1.2]. Thus to construct σ-ample sheaves on abelian varieties, we just need to understand which automorphisms of abelian varieties are Num-quasi-unipotent.

Let X be an abelian variety. For every invertible sheaf $\mathcal{L} \in \operatorname{Pic} X$ and every $a \in X$, let $\phi_{\mathcal{L}}(a) = (T_a)^*(\mathcal{L}) \otimes \mathcal{L}^{-1} \in \operatorname{Pic} X$. The *Theorem of the Square* [Mu, Corollary 4, p.59] states that

$$(T_{a+b})^*(\mathcal{L}) \otimes \mathcal{L} \cong (T_a)^*(\mathcal{L}) \otimes (T_b)^*(\mathcal{L}) \quad \text{for all } a, b \in X.$$

It follows that $\phi_{\mathcal{L}}$ is a group homomorphism from X to $\mathrm{Pic}(X)$. Now we can easily handle the case of translation automorphisms.

Lemma 8.1. *Let X be an abelian variety with $a \in X$ and let T_a be the corresponding translation automorphism. Then the induced automorphism P_{T_a} of $\mathrm{Num}(X)$ is the identity. In particular, T_a is Num-unipotent.*

Proof. The *Picard variety* of X, denoted by $\mathrm{Pic}^0(X)$, is the subgroup of $\mathrm{Pic}(X)$ consisting of all $\mathcal{L}$ such that $(T_a)^*(\mathcal{L}) \cong \mathcal{L}$ for all $a \in X$, in other words those $\mathcal{L}$ such that $\phi_{\mathcal{L}}$ is identically 0. Then another application of the Theorem of the Square shows that for every $\mathcal{L} \in \mathrm{Pic}(X)$ and $a \in X$, $\phi_{\mathcal{L}}(a) = (T_a)^*(\mathcal{L}) \otimes \mathcal{L}^{-1} \in \mathrm{Pic}^0(X)$ (see [Mu, p. 74]). Since any divisor in $\mathrm{Pic}^0(X)$ is numerically equivalent to 0 [La1, Proposition IV.3.4], $\mathcal{L}$ and $(T_a)^*(\mathcal{L})$ induce the same element of $\mathrm{Num}(X)$, and so it follows that $P_{T_a} = Id$ as required. □

Since an arbitrary automorphism $\sigma \in \mathrm{Aut}(X)$ has the form $\sigma = T_b \cdot \alpha$ for $b \in X$ and $\alpha \in \mathrm{End}(X)$, we see from the preceding result that σ and α induce the same map on $\mathrm{Num}(X)$. Thus from now on we may restrict our attention to automorphisms in $\mathrm{End}(X)$. We would like to say that if $\alpha \in \mathrm{End}(X)$ is a quasi-unipotent automorphism, then P_α is a quasi-unipotent automorphism of $\mathrm{Num}(X)$. This is not obvious, because the map $P : \mathrm{End}(X) \to \mathrm{End}(\mathrm{Num}(X))$ defined by $\alpha \mapsto P_\alpha$ is only a homomorphism of multiplicative semigroups, not a ring map. Nevertheless, with a little more work we will succeed in proving that P preserves quasi-unipotency in Proposition 8.5 below. The main step is to express unipotency purely in terms of the multiplicative structure, which will require the following technical bit of algebra.

Lemma 8.2. *Let $D_1, D_2, \ldots, D_h$ be division algebras over $\mathbb{Q}$. Fix some integers $n_i > 0$ and let $M \in \prod_{i=1}^{h} M_{n_i}(D_i)$.*

(a) *If M is unipotent, then M^p is conjugate to M for all $p > 0$. If M is quasi-unipotent, then M^p is conjugate to M^q for some $0 < p < q$.*

(b) *Assume that each D_i is a (commutative) field. Then the converses of both statements in part (a) hold.*

(c) *Let $g : \prod_{i=1}^{h} M_{n_i}(D_i) \to M_m(F)$ be a homomorphism of multiplicative semigroups, where F is a field. Then g preserves quasi-unipotency (respectively, unipotency).*

Proof. In both parts it is easy to reduce to the case where $h = 1$, and we do so, writing $D = D_1$ and $n = n_1$.

(a) Let $M \in M_n(D)$ be unipotent. It is an exercise in linear algebra that in this case M has a Jordan canonical form (this is of course not necessarily true for arbitrary matrices in $M_n(D)$). Also, M and M^p must have the same Jordan form for all $p \geq 0$, so they are conjugate.

Suppose instead that M is quasi-unipotent. Then M^s is unipotent for some $s > 0$, and so M^s and M^{sp} are conjugate for all $p \geq 0$.

(b) Assume now that D is commutative, and let $M \in M_n(D)$. Let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of M in some algebraic closure of D. Then M is quasi-unipotent if and only if all of the λ_i are roots of unity.

Suppose that M^p is conjugate to M^q for some $0 < p < q$. Then the ordered set $\{\lambda_1^p, \cdots, \lambda_n^p\}$ is a permutation of the ordered set $\{\lambda_1^q, \cdots, \lambda_n^q\}$. Let $\tau \in S_n$ be that permutation, namely, $\lambda_i^p = \lambda_{\tau(i)}^q$ for all i. If $w = n!$, then τ^w is the identity. Hence we have

$$\lambda_i^{p^w} = \lambda_{\tau(i)}^{p^{w-1}q} = \cdots = \lambda_{\tau^t(i)}^{p^{w-t}q^t} = \cdots = \lambda_{\tau^w(i)}^{q^w} = \lambda_i^{q^w}.$$

Thus $\lambda_i^{q^w - p^w} = 1$ for all i and M is quasi-unipotent.

If M is conjugate to M^p for all $p > 0$, then M is quasi-unipotent by the last paragraph. Hence M^p is unipotent for some p. Since M is conjugate to M^p, M is unipotent.

(c) This is an immediate consequence of parts (a) and (b). □

Remark 8.3. Lemma 8.2(a) also allows us to improve Theorem 3.2 and Corollary 3.3 slightly. Suppose that X is an abelian variety, that $\sigma \in \mathrm{Aut}(X)$ is wild, and that some power of σ is a translation. Then $\sigma = T_b \cdot \alpha$ with $\alpha \in \mathrm{End}(X)$ and $\alpha^n = Id$ for some $n \geq 0$. By Theorem 7.2, α is unipotent. Then by Lemma 8.2(a), α is conjugate to $\alpha^n = Id$, so $\alpha = Id$. Thus the conclusion of Theorem 3.2 and Corollary 3.3 may be improved to the statement that σ is a translation.

Now we return to our abelian variety X, and show how the above lemma may be applied to the map $P : \mathrm{End}(X) \to \mathrm{End}(\mathrm{Num}(X))$. First, we want to know the action of P on the "scalars".

Lemma 8.4. *Let X be an abelian variety. Let $\alpha = n \cdot Id_X \in \mathrm{End}(X)$ where $n \in \mathbb{Z}$. Then $P_\alpha = n^2 \cdot Id_{\mathrm{Num}(X)}$.*

Proof. This is immediate from [La1, Proposition 2, p. 92]. □

Let $\mathrm{End}_{\mathbb{Q}}(X) = \mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathrm{Num}_{\mathbb{Q}}(X) = \mathrm{Num}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. By Lemma 8.4, if we define $P_{n^{-1}Id_X} = n^{-2} Id_{\mathrm{Num}(X)}$ for $n \geq 1$ then P naturally extends to a semigroup homomorphism $P : \mathrm{End}_{\mathbb{Q}}(X) \to \mathrm{End}_{\mathbb{Q}}(\mathrm{Num}(X))$.

Let X be isogeneous to a product $\prod_{i=1}^h X_i^{n_i}$ where the X_i are simple abelian varieties, mutually non-isogeneous. As we stated in §7, each $D_i = \mathrm{End}_{\mathbb{Q}}(X_i)$ is a finite dimensional division algebra over $\mathbb{Q}$, and $\mathrm{End}_{\mathbb{Q}}(X) \cong \prod_{i=1}^h M_{n_i}(D_i)$. Also, letting m be the rank of $\mathrm{Num}(X)$ we have $\mathrm{End}_{\mathbb{Q}}(\mathrm{Num}(X)) \cong M_m(\mathbb{Q})$. We may now prove our main result on Num-quasi-unipotence for automorphisms of abelian varieties. This settles Theorem 0.2(b) from the introduction.

Theorem 8.5. *Let X be an abelian variety.*

(a) *The map $P : \mathrm{End}_{\mathbb{Q}}(X) \to \mathrm{End}_{\mathbb{Q}}(\mathrm{Num}(X))$ preserves (quasi)-unipotency.*
(b) *If $\alpha \in \mathrm{Aut}(X)$ is (quasi)-unipotent and $b \in X$ is any point, then $\sigma = T_b \cdot \alpha \in \mathrm{Aut}(X)$ is Num-(quasi)-unipotent.*
(c) *If σ is a wild automorphism of X, then σ is Num-unipotent and any ample invertible sheaf $\mathcal{L}$ on X is σ-ample.*

Proof. (a) As we saw in the comments before the theorem, the map P is a semigroup homomorphism from $\prod_{i=1}^h M_{n_i}(D_i)$ to $M_m(\mathbb{Q})$. Thus the result is immediate from Lemma 8.2(c).

(b) Since P is a semigroup homomorphism, this follows from part (a) and Lemma 8.1.

(c) This follows from part (b), Theorem 7.2(b), and [Ke, Theorem 1.2]. □

We have finally shown all of the necessary pieces to demonstrate the existence of a large supply of projectively simple twisted homogeneous coordinate rings.

Corollary 8.6. *Let X be an abelian variety with automorphism $\sigma = T_b \cdot \alpha$, where $\alpha \in \mathrm{End}(X)$ is unipotent and the image of b generates $X/(\alpha - Id)(X)$. Then if $\mathcal{L}$ is any ample invertible sheaf on X, then the ring $B(X, \mathcal{L}, \sigma)$ is projectively simple.*

Proof. By Theorem 7.2, σ is a wild automorphism. By Theorem 8.5, the sheaf $\mathcal{L}$ is σ-ample. Then by Proposition 2.2, $B(X, \mathcal{L}, \sigma)$ is a projectively simple ring. □

In [AV, Example 5.18], Artin and Van den Bergh gave an example of automorphism σ of a surface Y, defined by "translation along a pencil of elliptic curves", which has the property that P_σ is unipotent, but not the identity. There seem to be few other explicit examples in the literature of automorphisms of projective schemes which are Num-quasi-unipotent, but whose action on $\mathrm{Num}(X)$ is non-trivial. In part (c) of the next result we show that we can produce numerous such examples in the setting of abelian varieties.

Lemma 8.7. *Let X be an abelian variety.*

(a) *If $f \in \operatorname{End}(X)$ is an isogeny, then P_f is nonzero.*
(b) *If $\beta \in \operatorname{End}_{\mathbb{Q}}(X)$ is nonzero, then P_β is nonzero.*
(c) *If $\alpha \in \operatorname{End}(X)$ is unipotent and $\alpha \neq Id$, then P_α is unipotent and $P_\alpha \neq Id$.*

Proof. (a) This follows from Lemma 8.4 and the fact that there is a $g \in \operatorname{End}(X)$ such that $fg = n \cdot Id_X$ [La1, p. 29].

(b) Replacing β by $n\beta$ for some $n \geq 1$ we may assume that $\beta \in \operatorname{End}(X)$. If Y is the image of β, then there are endomorphisms $f : Y \to X$ and $g : X \to Y$ such that $g\beta f \in \operatorname{End}(Y)$ is an isogeny. Now the assertion follows from (a).

(c) By Theorem 8.5, P_α is unipotent, so we just need to show that P_α is not the identity. Let $\beta = \alpha - Id$, so that β is nilpotent. Let p be the maximal integer such that $\beta^p \neq 0$. Then

$$\alpha^n = \sum_{i=0}^{p} \binom{n}{i} \beta^i$$

for all $n \geq 0$. By [La1, Proposition 2, p. 92],

$$P_{\alpha^n} = \frac{1}{2} \sum_{i=0}^{p} \sum_{j=0}^{p} \binom{n}{i} \binom{n}{j} D(\beta^i, \beta^j) \tag{8.8}$$

for all $n \geq 0$, where $D(f, g) = P_{f+g} - P_f - P_g$ for all $f, g \in \operatorname{End}(X)$. For all n the right hand side of (8.8) is equal to $Q(n)$, where Q is some fixed polynomial

$$Q(z) = \sum_{i=0}^{2p} h_i z^i \in \operatorname{End}(\operatorname{Num}(X))[z] = M_m(\mathbb{Z})[z].$$

Now suppose that P_α is the identity. Then P_{α^n} is the identity for all $n \geq 1$, and thus $Q(n) = Id$ for all $n \geq 1$. Let $|M|$ be notation for the norm of an arbitrary matrix $M \in M_m(\mathbb{Z})$ in the Euclidean topology on $\mathbb{R}^{m^2}$. Now if $h_i \neq 0$ for some $i \geq 1$, then clearly $\lim_{n\to\infty} |Q(n)| = \infty$, contradicting the fact that $Q(n) = Id$ for all $n \geq 1$. Thus $h_i = 0$ for all $i \geq 1$.

In particular, the leading coefficient h_{2p} of Q is $(p!)^{-2} D(\beta^p, \beta^p) = 0$. But using Lemma 8.4,

$$D(\beta^p, \beta^p) = P_{2\beta^p} - 2P_{\beta^p} = 4P_{\beta^p} - 2P_{\beta^p} = 2P_{\beta^p},$$

so $P_{\beta^p} = 0$ (recall our standing assumption that $\operatorname{char} k = 0$). This contradicts part (b). Thus $P_\alpha \neq Id$. □

In order to illustrate the results of this section, we conclude with a simple example where the action of automorphisms on $\operatorname{Num}(X)$ may be calculated explicitly.

Proposition 8.9. *Let E be an elliptic curve without complex multiplication (in other words* $\mathrm{End}(E) \cong \mathbb{Z}$*), and let $X = E \times E$. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in$* $\mathrm{GL}_2(\mathbb{Z})$ *be arbitrary and let $\alpha = \alpha_M \in$* $\mathrm{End}(X)$ *be the corresponding automorphism; explicitly, $\alpha(x, y) = (ax + by, cx + dy)$.*

(a) *The divisors $C^1 = 0 \times E$, $C^2 = E \times 0$, and $C^3 = \{(x, x) \mid x \in E\}$ form a basis for* $\mathrm{Num}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^3$.

(b) *With respect to the basis given in (a), $P_\alpha \in M_3(\mathbb{Z})$ is equal to*

$$\begin{pmatrix} a^2 - ab & c^2 - cd & (a+c)^2 - (a+c)(b+d) \\ b^2 - ab & d^2 - cd & (b+d)^2 - (a+c)(b+d) \\ ab & cd & (a+c)(b+d) \end{pmatrix}.$$

(c) $\det P_\alpha = (\det M)^3$.

(d) *α is quasi-unipotent if and only if P_α is.*

Proof. (a, b) These parts are the statements of [Ko, Exercise II.4.16.2] and [Ko, Exercise II.4.16.6]), respectively, and we omit the proofs.

(c) This is an easy computation using part (b).

(d) By part (b), all entries of P_α are polynomial functions of a, b, c, d. Hence the semigroup homomorphism $P : GL_2(\mathbb{Z}) \to GL_3(\mathbb{Z})$ can be extended to a group homomorphism $P : GL_2(\mathbb{C}) \to GL_3(\mathbb{C})$ with the same formula given in (b). Since P is a group homomorphism, the desired assertion holds if and only if it holds after conjugation.

By conjugation we may assume that α has one of the two forms $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ or $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$. Then one may check using the formula in part (d) that the eigenvalues of P_α are either $\{a^2, d^2, ad\}$ or $\{a^2, a^2, a^2\}$, respectively. Hence P_α is quasi-unipotent if and only if M is a quasi-unipotent matrix, if and only if α_M is a quasi-unipotent, by Proposition 7.4(a). □

9. Classification of low-dimensional cases

In this section we tie together our previous results to classify projectively simple twisted homogeneous coordinate rings of small GK-dimension. Again we assume that k is an algebraically closed field of characteristic zero.

We will use the following result of Keeler about the GK-dimension of twisted homogeneous coordinate rings [Ke, Theorem 6.1]. Let $B = B(X, \mathcal{L}, \sigma)$ where $\mathcal{L}$ is σ-ample. Then $\mathrm{GKdim}\, B$ is an integer satisfying the inequalities

$$j + \dim X + 1 \leq \mathrm{GKdim}\, B \leq j(\dim X - 1) + \dim X + 1, \tag{9.1}$$

where $j + 1$ is the size of the largest Jordan block of the matrix $P_\sigma \in \mathrm{End}(\mathrm{Num}(X))$ giving the induced action of σ on $\mathrm{Num}\, X$. It is a fact that j is always even.

Proposition 9.2. *Let $B = B(X, \mathcal{L}, \sigma)$, where $\mathcal{L}$ is σ-ample. Then B is a projectively simple ring with* $\mathrm{GKdim}(B) = 2$ *or* 3 *if and only if*

(a) *X is an abelian variety of dimension* $\mathrm{GKdim}(B) - 1$*,*
(b) *σ is the the translation T_b, and*
(c) *b generates X.*

Proof. Suppose first that B is projectively simple. Since $\mathrm{GKdim}\, B \leq 3$, by (9.1) we must have $\dim X \leq 2$, and since j in that equation is even, the only possibility is $j = 0$, which forces $\mathrm{GKdim}(B) = \dim X + 1$. Also, σ is wild by Proposition 2.2. By Theorem 6.5, X must be an abelian variety. Let $\sigma = T_b \cdot \alpha$ for $\alpha \in \mathrm{End}(X)$ and $b \in X$. Note that $P_\sigma = P_\alpha$ by Lemma 8.1. By Theorem 7.2, α is unipotent. If α is not the identity, then by Lemma 8.7(c), $P_\sigma = P_\alpha$ is unipotent and not the identity. In this case P_σ has a non-trivial Jordan block, and so $j \neq 0$ in (9.1), a contradiction. We conclude that $\alpha = Id$ and $\sigma = T_b$ is a translation. By Theorem 7.2 again, b generates X.

Conversely, suppose that X is an abelian variety with $\dim X = 1$ or 2 and that $\sigma = T_b$ where b generates X. By Theorem 7.2, σ is wild. Then B is projectively simple by Proposition 2.2. Moreover, $P_\sigma = Id$ by Lemma 8.1, so $j = 0$ in (9.1) and $\mathrm{GKdim}(B) = \dim X + 1$. □

One can prove analogous results for other small values of dimension. We omit the similar proof of the following.

Proposition 9.3. *Let $B = B(X, \mathcal{L}, \sigma)$ where X is an abelian variety and $\mathcal{L}$ is σ-ample.*

(a) *Suppose that* $\mathrm{GKdim}\, B = 4$*. Then B is projectively simple if and only if* $\dim X = 3$*, $\sigma = T_b$, and b generates X.*
(b) *Suppose that* $\mathrm{GKdim}\, B = 5$*. Then B is projectively simple if and only if either*
    (1) $\dim X = 4$*, $\sigma = T_b$, and b generates X, or*
    (2) $\dim X = 2$*, $\sigma = T_b, \cdot\alpha$ for some non-identity unipotent $\alpha \in$* $\mathrm{End}(X)$*, and the image $\overline{b}$ of b generates $X/(\alpha - Id)(X)$.*

10. The Cohen-Macaulay property

Our goal in the these last two sections is to study some of the homological properties of projectively simple rings and their associated noncommutative schemes. In particular we will prove Theorem 0.5 from the introduction. In order to get interesting results, we take as our basic hypothesis in these two sections that all algebras have a balanced dualizing complex. This condition holds for many important classes of algebras.

The objects of interest and the methods we employ here are rather different from those of the preceding sections, and we must begin with a review of the definitions of noncommutative projective schemes [AZ1], Serre duality [YZ1], and dualizing complexes [Ye, VdB]. We assume that the reader is familiar with the basics of derived categories and follow the notations used in [YZ1]. For example, the nth complex shift of a complex X is denoted by $X[n]$.

Let A be a right noetherian graded k-algebra. Let $\operatorname{Gr} A$ be the category of graded right A-modules and let $\operatorname{gr} A$ be the full subcategory of $\operatorname{Gr} A$ consisting of the noetherian objects. Recall from §1 that a graded A-module M is called *torsion* if for every $m \in M$ there is some $n \geq 0$ such that $mA_{\geq n} = 0$. Let $\operatorname{Tors} A$ denote the full subcategory of $\operatorname{Gr} A$ consisting of all torsion objects and let $\operatorname{tors} A$ be the full subcategory of noetherian torsion objects, or equivalently modules of finite k-dimension. The *noncommutative projective spectrum* of A is defined to be $\operatorname{Proj} A := (\operatorname{QGr} A, \mathcal{A}, s)$ where $\operatorname{QGr} A$ is the quotient category $\operatorname{Gr} A/\operatorname{Tors} A$, the object $\mathcal{A}$ is the image of A in $\operatorname{QGr} A$, and s is the auto-equivalence of $\operatorname{QGr} A$ induced by the degree shift $M \to M(1)$ for $M \in \operatorname{Gr} A$. If we want to work with noetherian objects only, then $\operatorname{proj} A := (\operatorname{qgr} A, \mathcal{A}, s)$ is also called the projective spectrum of A. The canonical functor from $\operatorname{Gr} A \to \operatorname{QGr} A$ (and from $\operatorname{gr} A$ to $\operatorname{qgr} A$) is denoted by π. If $M \in \operatorname{Gr} A$, we will use the corresponding calligraphic letter $\mathcal{M}$ for $\pi(M)$ if there is no chance of confusion. For example, $\mathcal{A} = \pi(A)$.

Let $X = \operatorname{Proj} A$. For $\mathcal{N} \in \operatorname{QGr} A$, the i-th cohomology group of $\mathcal{N}$ is defined to be

$$\mathrm{H}^i(X, \mathcal{N}) = \operatorname{Ext}^i_{\operatorname{QGr} A}(\mathcal{A}, \mathcal{N})$$

for all $i \geq 0$. Then the (right) cohomological dimension of X is defined to be

$$\operatorname{cd}(X) = \max\{i \mid \mathrm{H}^i(X, \mathcal{N}) \neq 0 \text{ for some } \mathcal{N} \in \operatorname{QGr} A\}.$$

Suppose that $d = \operatorname{cd}(X)$ is finite. An object $\omega \in \operatorname{qgr} A$ is called a *dualizing sheaf* for X if there is a natural isomorphism

$$\theta^0 : \mathrm{H}^d(X, \mathcal{N})^* \to \operatorname{Hom}_{\operatorname{qgr} A}(\mathcal{N}, \omega)$$

for all $\mathcal{N} \in \operatorname{qgr} A$. Here $(-)^*$ means the k-vector space dual. We say that X satisfies *Serre duality* if a dualizing sheaf ω exists. In this case we say that X is *classically Cohen-Macaulay* if θ^0 can be extended to a sequence of natural isomorphisms

$$\theta^i : \mathrm{H}^{d-i}(X, \mathcal{N})^* \to \operatorname{Ext}^i_{\operatorname{qgr} A}(\mathcal{N}, \omega)$$

for all $\mathcal{N}$ and $0 \leq i \leq d$. If A is commutative, then the concepts of dualizing sheaf and the Cohen-Macaulay property which we have defined here agree with the usual commutative notions.

The notion of a balanced dualizing complex for a noncommutative graded algebra was introduced by Yekutieli in [Ye]. Let $\mathrm{D}(\operatorname{Gr} A)$ denote the derived category of graded right A-modules. Given a complex Y, we use the notation $\mathrm{h}^i(Y)$ for the ith cohomology of Y. Let A° be the opposite algebra of A, and let $A^e = A \otimes_k A^\circ$. A *dualizing complex* for a noetherian graded ring A is a bounded complex $R \in \mathrm{D}(\operatorname{Gr} A^e)$ such that

(a) R has finite injective dimension over A and A°,
(b) The A-bimodule $\mathrm{h}^i(R)$ is noetherian on both sides for all $i \in \mathbb{Z}$, and
(c) The natural maps $A \to \operatorname{RHom}_A(R, R)$ and $A \to \operatorname{RHom}_{A^\circ}(R, R)$ are isomorphisms in the derived category $\mathrm{D}(\operatorname{Gr} A^e)$.

Let $\Gamma_{\mathfrak{m}} : \operatorname{Gr} A^e \to \operatorname{Gr} A^e$ be the *torsion functor* $\lim_{n\to\infty} \underline{\operatorname{Hom}}_A(A/\mathfrak{m}^n, -)$. Writing $\mathfrak{m}^\circ = (A^\circ)_{\geq 1}$, the functor $\Gamma_{\mathfrak{m}^\circ}$ is defined similarly. If A has a dualizing complex R, then R is called *balanced* if there are isomorphisms $\mathrm{R}\,\Gamma_{\mathfrak{m}}(R) \cong A^*$ and $\mathrm{R}\,\Gamma_{\mathfrak{m}^\circ}(R) \cong A^*$ in $\mathrm{D}(\operatorname{Gr} A^e)$.

Yekutieli proved that if A is noetherian with a balanced dualizing complex R, then $\omega = \mathrm{h}^{-(d+1)}(\mathcal{R})$ is a dualizing sheaf for $\operatorname{Proj} A$, where $\mathcal{R} = \pi(R)$ [YZ1, 4.2(4)]. In addition, X is classically Cohen-Macaulay if and only if $\omega[d+1]$ is isomorphic to $\mathcal{R}$ in the derived category $\mathrm{D}(\operatorname{QGr} A)$ [YZ1, 4.2(5)], or equivalently if and only if $\mathcal{R}$ has nonzero cohomology in only one term.

A powerful criterion of Van den Bergh [VdB, 6.3] says that A admits a balanced dualizing complex if and only if A satisfies the left and right χ condition and $X = \operatorname{Proj} A$ has finite left and right cohomological dimension. Here, A is said to satisfy the (right) χ condition if $\dim_k \underline{\operatorname{Ext}}^i_A(A/\mathfrak{m}, M) < \infty$ for all $M \in \operatorname{gr} A$ and all $i \geq 0$, and similarly on the left. It follows from Van den Bergh's criterion that many important classes of rings admit a balanced dualizing complex, although there are examples of noetherian rings which do not because they fail the χ condition [SZ], [KRS].

We may now prove Theorem 0.5(a).

Proposition 10.1. *Let A be a connected graded, noetherian, projectively simple ring admitting a balanced dualizing complex. Then the associated projective scheme* $\operatorname{Proj} A$ *is classically Cohen-Macaulay.*

Proof. By [VdB, 6.3], A satisfies the χ condition and has finite cohomological dimension on both sides, and moreover the balanced dualizing complex over A is given by

$$R = \mathrm{R}\,\Gamma_{\mathfrak{m}}(A)^*.$$

Let $X = \operatorname{Proj} A$. Note that $\mathrm{R}^i\,\Gamma_{\mathfrak{m}}(N) \cong \mathrm{H}^{i-1}(X, \mathcal{N})$ for all $N \in \operatorname{Gr} A$ and all $i \geq 2$ [AZ1, 7.2(2)]. Thus if $\operatorname{cd}(\operatorname{Proj} A) = d$, then $\mathrm{R}^n\,\Gamma_{\mathfrak{m}} = 0$ for $n > d+1$, and so $\mathrm{h}^n(R) = 0$ for all $n < -d-1$. Let $\mathcal{R} = \pi(R)$. Since the dualizing sheaf $\omega = \mathrm{h}^{-(d+1)}(\mathcal{R})$ is necessarily nonzero, $M := \mathrm{h}^{-(d+1)}(R)$ is not finite dimensional over k. Let $j_0 = -d-1$ and put $j = \max\{i \mid \mathrm{h}^i(\mathcal{R}) \neq 0\}$. We want to show that $j_0 = j$. If this is the case, then $\omega \cong \mathcal{R}[-d-1]$ and we are done by [YZ1, 4.2(5)].

By the definition of a dualizing complex, $\mathrm{h}^n(R)$ is noetherian on both sides for every n. Thus for every $n > j$, $\mathrm{h}^n(R)$ is finite dimensional over k because $\mathrm{h}^n(\mathcal{R}) = 0$, but $N := \mathrm{h}^j(R)$ is not finite dimensional since $\mathrm{h}^j(\mathcal{R}) \neq 0$.

Let Y be the truncation $\tau^{\leq j}R$ and Z the truncation $\tau^{\geq (j+1)}R$. Then we have a distinguished triangle in the derived category $\mathrm{D}(\operatorname{Gr} A)$

$$Y \to R \to Z \to Y[1]$$

which induces a long exact sequence

$$\cdots \to \operatorname{Ext}^i_A(Z, R) \to \operatorname{Ext}^i_A(R, R) \to \operatorname{Ext}^i_A(Y, R) \to \cdots.$$

Since $\mathrm{h}^n(Z)$ is finite dimensional and $\mathrm{h}^n(R)$ is noetherian for all n, it follows from the χ condition and induction on the lengths of the bounded complexes Z and R that $\mathrm{Ext}^i_A(Z, R)$ is finite dimensional for all i. If we show that $\mathrm{Ext}^i_A(Y, R)$ is infinite dimensional for some i, then it will follow that $\mathrm{Ext}^i_A(R, R)$ is also infinite dimensional. Since Y is bounded above at j with $\mathrm{h}^j(Y) = \mathrm{h}^j(R) = N$ and R is bounded below at j_0 with $\mathrm{h}^{j_0}(R) = M$, every nonzero map from N to M induces a nonzero element in $\mathrm{Ext}^{j_0-j}_A(Y, R)$. Thus there is an injection of vector spaces

$$\underline{\mathrm{Hom}}_A(N, M) \to \mathrm{Ext}^{j_0-j}_A(Y, R).$$

Now since N and M are not torsion, by Lemma 1.7(b) $\underline{\mathrm{Hom}}_A(N, M)$ is infinite dimensional. Hence $\mathrm{Ext}^{j_0-j}_A(Y, R)$ is infinite dimensional, whence $\mathrm{Ext}^{j_0-j}_A(R, R)$ is infinite dimensional. But since R is a dualizing complex, by definition we we have $\mathrm{Ext}^i_A(R, R) = 0$ for all $i \neq 0$. Thus $j = j_0$. □

11. The Gorenstein property

In this section, we will complete the proof of Theorem 0.5 by showing that if A is a projectively simple ring with a balanced dualizing complex, then the dualizing sheaf ω must be invertible. This is equivalent to $X = \mathrm{Proj}\, A$ being Gorenstein in the commutative case. In preparation for this result, we will first study graded bimodules over two rings.

Throughout this section, let A and B be noetherian finitely graded prime k-algebras, with graded quotient rings Q and T respectively. Given two noetherian (A, B)-bimodules M and N, we say M and N are *p.isomorphic* (short for projectively isomorphic), and write $M \cong_p N$, if there exists an isomorphism of bimodules $f : M_{\geq n} \to N_{\geq n}$ for some $n \gg 0$. The map f induces an isomorphism $\pi(M) \to \pi(N)$ in $\mathrm{QGr}\, B$ (and similarly in $\mathrm{QGr}\, A^\circ$). A map $f : M \to N$ is called a *p.isomorphism* if $f_{\geq n} : M_{\geq n} \to N_{\geq n}$ is an isomorphism for some $n \gg 0$.

If M is a graded right B-module and N is a graded left B-module, the graded tensor product of M and N over B is denoted by $M \underline{\otimes}_B N$. As usual, if M is a graded (A, B)-bimodule then M is called *invertible* if there is a graded (B, A)-bimodule N with bimodule isomorphisms $M \underline{\otimes}_B N \cong A$ and $N \underline{\otimes}_A M \cong B$. Analogously, M is called *p.invertible* if there exists such a graded (B, A)-bimodule N with $M \underline{\otimes}_B N \cong_p A$ and $N \underline{\otimes}_A M \cong_p B$, and N is called the *p.inverse*.

Next, we want to define a different but related notion of invertibility. We call a noetherian (A, B)-bimodule M *generically invertible* if

(a) M is Goldie torsionfree on both sides,
(b) M is *evenly localizable* in the sense that

$$Q \underline{\otimes}_A M \cong Q \underline{\otimes}_A M \underline{\otimes}_B T \cong M \underline{\otimes}_B T, \quad \text{and}$$

(c) $Q \underline{\otimes}_A M = M \underline{\otimes}_B T$ is an invertible (Q, T)-bimodule.

Although it is useful to state all three conditions in the above definition, we should note that in general condition (b) is actually a consequence of condition (a), as follows.

Lemma 11.1. *If M is a noetherian (A,B)-bimodule which is Goldie torsionfree on both sides, then M is evenly localizable.*

Proof. First we claim that the (A,B)-bimodule $M' = Q \underline{\otimes}_A M$ is Goldie torsionfree on the right. This is because every finitely generated graded right B-submodule of $Q \underline{\otimes}_A M$ is contained in $qA \underline{\otimes}_A M \cong M$ for some homogeneous $q \in Q$.

Thus right multiplication by any element $b \in B$ must induce a left Q-module injection $\psi_b : M' \to M'$. Since ${}_QM'$ is of finite rank over the graded-semisimple ring Q, the map ψ_b must be a bijection for all b and so $Q \underline{\otimes}_A M \cong Q \underline{\otimes}_A M \underline{\otimes}_B T$. A symmetric argument gives $M \underline{\otimes}_B T \cong Q \underline{\otimes}_A M \underline{\otimes}_B T$. □

The reader may check that an invertible bimodule is generically invertible. The following key proposition is a kind of converse statement for projectively simple rings. The proof will be given below after several more lemmas.

Proposition 11.2. *Let A and B be connected graded, noetherian, projectively simple rings admitting balanced dualizing complexes. Then every generically invertible noetherian (A,B)-bimodule is p.invertible.*

Let M be an (A,B)-bimodule. Then we may define the (B,A)-bimodules ${}^\vee M = \underline{\mathrm{Hom}}_B(M_B, B_B)$ and $M^\vee = \underline{\mathrm{Hom}}_{A^\circ}({}_AM, {}_AA)$. Induced by evaluation, we get a B-bimodule homomorphism

$$\phi_M : {}^\vee M \underline{\otimes}_A M \to B \tag{11.3}$$

and an A-bimodule homomorphism

$$\xi_M : M \underline{\otimes}_B M^\vee \to A, \tag{11.4}$$

both of which are graded morphisms of degree 0. The bimodules $M^\vee$ and ${}^\vee M$ will be our candidates for the p.inverse of M.

We shall need the following technical result, which says loosely that "Ext commutes with localization". The proof is an easy generalization to the graded setting of ([BL, Proposition 1.6]).

Lemma 11.5. *Let B be a graded noetherian prime ring. Let T be the graded quotient ring of B. Let M_B be a finitely generated graded B-module and let N be a graded B-bimodule such that $T \underline{\otimes}_B N \cong N \underline{\otimes}_B T$. Then*

$$T \underline{\otimes}_B \underline{\mathrm{Ext}}^i_B(M_B, {}_BN_B) \cong \underline{\mathrm{Ext}}^i_T(M \underline{\otimes}_B T, N \underline{\otimes}_B T)$$

for all i.

We next show that to prove Proposition 11.2 it is enough to know that $M^\vee$ and ${}^\vee M$ are noetherian.

Lemma 11.6. *Let A and B be two prime noetherian projectively simple rings. Let M be a generically invertible noetherian (A, B)-bimodule. If $M^\vee$ and ${}^\vee M$ are noetherian on both sides, then M is p.invertible with p.inverse $M^\vee \cong_p {}^\vee M$.*

Proof. We will show that the evaluation map

$$\phi_M : {}^\vee M \underline{\otimes}_A M \to B$$

of (11.3) is a p.isomorphism. By the definition of generic invertibility, the (Q, T)-bimodule $L = M \underline{\otimes}_B T \cong Q \underline{\otimes}_A M$ is invertible. By Morita theory, the inverse of L must be given by the (T, Q)-bimodule $L^{-1} = \underline{\mathrm{Hom}}_T(L, T_T)$. By Lemma 11.5, we have

$$T \underline{\otimes}_B {}^\vee M \cong \underline{\mathrm{Hom}}_T(L, T_T) = L^{-1}.$$

Hence ${}^\vee M$ is nonzero and the the map ϕ_M is nonzero. Since B is projectively simple, the map ϕ_M must be surjective in large degree.

Now consider the map $T \underline{\otimes}_B \phi_M$. This map must be an isomorphism, since it may be identified with the chain of isomorphisms

$$T \underline{\otimes}_B {}^\vee M \underline{\otimes}_A M \cong L^{-1} \underline{\otimes}_A M \cong L^{-1} \underline{\otimes}_Q (Q \underline{\otimes}_A M) \cong T.$$

It follows that the left B-module $\ker \phi_M$ is Goldie torsion. Since by assumption ${}^\vee M$ is a noetherian bimodule, ${}^\vee M \underline{\otimes}_A M$ and thus $\ker \phi_M$ are also noetherian bimodules. Hence $\ker \phi_M$ is finite dimensional by Lemma 1.7, and ϕ_M is a p.isomorphism as we wished. The proof that the map ξ_M of (11.4) is a p.isomorphism is analogous. Finally, that ${}^\vee M \cong_p M^\vee$ is a formal consequence of the fact that ϕ_M and ξ_M are both p.isomorphisms. □

Although it is easy to see that ${}^\vee M$ is left noetherian over B, in general there is no reason that ${}^\vee M$ should be right noetherian over A. The dualizing complex provides the extra information needed to prove such a fact.

Lemma 11.7. *Let A and B be two graded noetherian rings admitting balanced dualizing complexes R_A and R_B respectively. Let M be a noetherian graded (A, B)-bimodule. If N is a graded noetherian right B-module, then $\underline{\mathrm{Ext}}^i_B(M, N)$ is a noetherian right A-module for all i.*

Proof. First, note that by [VdB, 5.1 and 4.8],

(11.8)

$$\mathrm{RHom}_{A^\circ}(M, R_A) \cong \mathrm{R}\,\Gamma_{\mathfrak{m}_{A^\circ}}(M)^* \cong \mathrm{R}\,\Gamma_{\mathfrak{m}_B}(M)^* \cong \mathrm{RHom}_B(M, R_B).$$

The functor $D(-) = \mathrm{RHom}_B(-, R_B)$ gives a duality from $\mathrm{D}(\mathrm{Gr}\, B)$ to $\mathrm{D}(\mathrm{Gr}\, B^\circ)$ which restricts to a duality between the subcategories of complexes with finitely generated cohomology groups [Ye, 3.4]. Thus $\mathrm{RHom}_B(M, R_B)$ is a complex of graded left B-modules with noetherian cohomologies. Applying the same reasoning to the duality functor $\mathrm{RHom}_{A^\circ}(-, R_A)$ and using (11.8), we see that the cohomology groups of $\mathrm{RHom}_B(M, R_B)$ are also noetherian right A-modules.

Now since $D(-)$ is a duality we have

$$\mathrm{RHom}_B(M,N) \cong \mathrm{RHom}_{B^\circ}(D(N),D(M)).$$

As we showed above, $D(M)$ has noetherian cohomologies on the right, and so $\mathrm{Ext}^i_{B^\circ}(D(N),D(M))$ also has noetherian right cohomologies. Thus

$$\mathrm{Ext}^i_B(M,N) \cong \mathrm{Ext}^i_{B^\circ}(D(N),D(M))$$

is noetherian as a right A-module. □

Proof of Proposition 11.2. By Lemma 11.6 it suffices to show that $M^\vee$ and ${}^\vee M$ are noetherian bimodules. We only show this for ${}^\vee M$; the proof for $M^\vee$ is symmetric. It is clear that ${}^\vee M = \mathrm{Hom}_B(M_B, B_B)$ is a noetherian left B-module. By Lemma 11.7, $\mathrm{Hom}_B(M,B)$ is also noetherian as a right A-module. □

Let A be a graded ring with balanced dualizing complex R, and suppose that $X = \mathrm{Proj}\,A$ is classically Cohen-Macaulay. Set $M = \mathrm{h}^{-(d+1)}(R)/\tau$ where $d = \mathrm{cd}(\mathrm{Proj}\,A)$ and τ is the torsion submodule of $\mathrm{h}^{-(d+1)}(R)$. When we say that the dualizing sheaf $\omega = \pi(M)$ is *invertible*, we mean that M is an p.invertible (A,A)-bimodule in the sense defined earlier. Now we are ready to prove Theorem 0.5(b).

Theorem 11.9. *Let A be a connected graded, noetherian, projectively simple ring with a balanced dualizing complex R. Let ω be the dualizing bimodule of* $\mathrm{Proj}\,A$*. Then ω is invertible and $\mathcal{A} = \pi(A)$ has finite injective dimension in the category* $\mathrm{QGr}\,A$*.*

Proof. Let $M = \mathrm{h}^{-(d+1)}(R)/\tau$ as in the comments before the proof. Then M is torsionfree on both sides, so Goldie torsionfree on both sides by Lemma 1.7(a).

We need to show that M is p.invertible. We already know that M is Goldie torsionfree on both sides. Let Q be the graded fraction ring of A. Then M is evenly localizable to Q by Lemma 11.1. Since we have already proven that $\mathrm{Proj}\,A$ is classically Cohen-Macaulay in Proposition 10.1, we know that for every $i \neq -d-1$, $\mathrm{h}^i(R)$ is finite dimensional. By [YZ3, 6.2(1)], the complex $Q \otimes_A R \otimes_A Q$ is a graded dualizing complex for the ring Q, and since M is evenly localizable this complex is just a shift of $M \otimes_A Q$. Since Q is graded semisimple artinian with graded global dimension zero, a dualizing bimodule for Q is a progenerator on both sides. Thus $M \otimes_A Q$ is invertible over Q and hence by definition M is generically invertible. Then by Proposition 11.2, M is p.invertible.

By the definition of a dualizing complex R has finite injective dimension. This implies that $\pi(R)$ and hence ω has finite injective dimension in $\mathrm{QGr}\,A$. Since M is a p.invertible bimodule, the functor $-\otimes_A M^\vee$ induces an auto-equivalence of $\mathrm{QGr}\,A$ and maps ω to $\mathcal{A}$. Therefore $\mathcal{A}$ has finite injective dimension in $\mathrm{QGr}\,A$. □

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References

[AS1] M. Artin and J.T. Stafford, *Noncommutative graded domains with quadratic growth*, Invent. Math. **122** (1995), no. 2, 231–276.

[AS2] M. Artin and J.T. Stafford, *Semiprime graded algebras of dimension two*, J. Algebra **227** (2000), no. 1, 68–123.

[ATV] M. Artin, J. Tate and M. Van den Bergh, *Some algebras associated to automorphisms of elliptic curves*, The Grothendieck Festschrift, Vol. I, 33–85, Progr. Math., **86**, Birkhäuser Boston, Boston, MA, 1990.

[AV] M. Artin and M. Van den Bergh, *Twisted homogeneous coordinate rings*, J. Algebra **133** (1990), no. 2, 249–271.

[AZ1] M. Artin and J.J. Zhang, *Noncommutative projective schemes*, Adv. Math. **109** (1994), no. 2, 228–287.

[AZ2] M. Artin and J.J. Zhang, *Abstract Hilbert schemes*, Algebr. Represent. Theory **4** (2001), no. 4, 305–394.

[BeMi] C. Bennett and R. Miranda, *The automorphism groups of the hyperelliptic surfaces*, Rocky Mountain J. Math. **20** (1990), no. 1, 31-37.

[BoHu] E. Bombieri and D. Husemoller, *Classification and embeddings of surfaces*, Algebraic Geometry (Proc. Sympos. Pure Math., Vol **29**, Humboldt State Univ., Arcata, Calif., 1974), 329-420.

[BoMu] E. Bombieri and D. Mumford, *Enriques' classification of surfaces in char. p. II*, Complex analysis and algebraic geometry, pp. 23–42. Iwanami Shoten, Tokyo, 1977.

[BL] K.A. Brown and T. Levasseur, *Cohomology of bimodules over enveloping algebras*, Math. Z. **189** (1985), no. 3, 393–413.

[D] V. I. Danilov, *Cohomology of algebraic varieties*, in Algebraic geometry, II, Encyclopaedia Math. Sci., **35**, Springer, Berlin, 1996, 1–125.

[GH] P. Griffiths and J. Harris, Principles of Algebraic Geometry, Wiley-Interscience, New York, 1978.

[Han] M. Hanamura, *On the birational automorphism groups of algebraic varieties*, Compositio Math. **63** (1987), no. 1, 123–142.

[Hart] R. Hartshorne, Algebraic geometry. GTM 52. Springer-Verlag, New York-Heidelberg, 1977.

[He] I.N. Herstein, Noncommutative rings, Carus Mathematical Monographs, 15. Mathematical Association of America, Washington, DC, 1994.

[Hum] J.E. Humphreys, Linear algebraic groups, GTM 21, Springer-Verlag, New York-Heidelberg, 1975.

[Ke] D.S. Keeler, *Criteria for σ-ampleness*, J. Amer. Math. Soc. **13** (2000), no. 3, 517–532 (electronic).

[KRS] D.S. Keeler, D. Rogalski and J.T. Stafford, *Naive noncommutative blowing up*, preprint. Available at www.arxiv.org/abs/math.RA/0306244.

[Ko] J. Kollár, *Rational curves on algebraic varieties*, Springer-Verlag, Berlin, 1996.

[KL] G. R. Krause and T. H. Lenagan, *Growth of algebras and Gelfand-Kirillov dimension*, revised ed., American Mathematical Society, Providence, RI, 2000.

[La1] S. Lang, *Abelian varieties*, Springer-Verlag, New York, 1983, Reprint of the 1959 original.

[La2] H. Lange, *Produkte elliptischer Kurven*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II 1975, no. 8, 95–108.

[LBR] L. Le Bruyn, Z. Reichstein, *Smoothness in algebraic geography*, Proc. of the London Math. Society, **79**, no. 1 (1999), 158–190.

[Mar] M. Maruyama, *On automorphism groups of ruled surfaces*, J. Math. Kyoto Univ. **11**, (1971) 89-112.

[MR] J.C. McConnell and J.C. Robson, “Noncommutative Noetherian Rings,” Wiley, Chichester, 1987.

[Mo] S. Mori, *Classification of higher-dimensional varieties*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), 269–331, Proc. Sympos. Pure Math., 46, Part 1, Amer. Math. Soc., Providence, RI, 1987.

[Mu] D. Mumford, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, No. 5, Bombay, 1970.

[RZ] D. Rogalski and J.J. Zhang, *Graded rings with commutative function field*, work in progress.

[SZ] J.T. Stafford and J.J. Zhang, *Examples in non-commutative projective geometry*, Math. Proc. Cambridge Philos. Soc. **116** (1994), no. 3, 415–433.

[SV] J.T. Stafford and M. Van den Bergh, *Noncommutative curves and noncommutative surfaces*, Bull. Amer. Math. Soc. (N.S.) **38** (2001), 171–216.

[TT] D. Toledo and Y.L.L. Tong, *The holomorphic Lefschetz formula*, Bull. Amer. Math. Soc. **81** (1975), no. 6, 1133–1135.

[U] K. Ueno, Classification theory of algebraic varieties and compact complex spaces. Notes written in collaboration with P. Cherenack. Lecture Notes in Mathematics, Vol. 439. Springer-Verlag, Berlin-New York, 1975.

[VdB] M. Van den Bergh, *Existence theorems for dualizing complexes over non-commutative graded and filtered rings*, J. Algebra **195** (1997), no. 2, 662–679.

[Ye] A. Yekutieli, *Dualizing complexes over noncommutative graded algebras*, J. Algebra **153** (1992), no. 1, 41–84.

[YZ1] A. Yekutieli and J.J. Zhang, *Serre duality for noncommutative projective schemes*, Proc. Amer. Math. Soc. **125** (1997), no. 3, 697–707.

[YZ2] A. Yekutieli and J.J. Zhang, *Dualizing Complexes and Tilting Complexes over Simple Rings*, J. Algebra, **256** (2002), no. 2, 556–567.

[YZ3] A. Yekutieli and J.J. Zhang, *Differential Algebras of Finite Type*, preprint.

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