ON A PROPERTY OF SPECIAL GROUPS

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ABSTRACT. We show that Conjectures I and II of Serre about the vanishing of H^1 cannot be strengthened.

1. INTRODUCTION

Let G be a linear algebraic group defined over an algebraically closed field k. We shall denote the radical of G by $\mathrm{Rad}(G)$ and $G/\mathrm{Rad}(G)$ by G_{ss}.

Recall that G is called *special* if $H^1(K,G) = \{1\}$ for every field extension K/k. Grothendieck [Gr, Section 5] showed that G is special if and only if $G_{ss} = S_1 \times \cdots \times S_r$, where each S_i is a connected and simply connected group of type SL or Sp; cf. also [PV, Section 2.6]. If G is not special, $H^1(K,G)$ may nevertheless vanish if K is a "small" field. This possibility is described by Conjectures I and II of Serre; see [Se$_1$] or [Se$_4$, Chapter III]. Let K be a perfect field (not necessarily containing k) of cohomological dimension d and let G be an algebraic group defined over K.

Conjecture I: If G is connected and $d \leq 1$ then $H^1(K,G) = \{1\}$.

Conjecture II: If G is semisimple, connected and simply connected and $d \leq 2$ then $H^1(K,G) = \{1\}$.

Conjecture I was proved by Steinberg [St$_1$]. Conjecture II remains open, though significant progress has been made in recent years; see [BP] and [Gi].

Our main result is the following partial converse to Conjectures I and II.

Theorem 1. *Let G be an algebraic group defined over an algebraically closed field k. Assume that $H^1(K,G) = \{1\}$ for some finitely generated field extension K of k of transcendence degree d.*

(a) If $d \geq 1$ then G is connected.

(b) If $d \geq 2$ then the universal covering map $\overline{G} \longrightarrow G_{ss}$ is purely inseparable. In particular, if $\mathrm{char}(k) = 0$ then G_{ss} is simply connected.

(c) If $d \geq 3$ and $\mathrm{Tors}(G) \neq \{\mathrm{char}(k)\}$ then G is special.

The symbol $\mathrm{Tors}(G)$ in part (c) denotes the set of torsion primes of G; for the definition and properties of torsion primes see [St$_2$] or [Se$_5$].

Informally speaking, the theorem may be interpreted as saying that Conjectures I and II cannot be extended or strengthened in a meaningful way, even for fields containing $\mathbb{C}$. (Indeed, if $\mathrm{char}(k) = 0$ then the cohomological dimension of K equals d; see [Se$_4$, Section II.4].) Our proof relies on results of Steinberg [St$_2$] about the existence of nontoral finite abelian subgroups in algebraic groups.

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2. Notations and preliminaries

In this note all algebraic varieties, morphisms, algebraic group actions, field extensions, etc., will be assumed to be defined over an algebraically closed base field k. By an algebraic variety we shall mean a reduced (but not necessarily irreducible) separated scheme of finite type over k. The algebra of regular (respectively, rational) functions on a variety X will be denoted by $k[X]$ (respectively, $k(X)$). As usual, $k(X)$ is a direct product of fields (one for each irreducible component of X).

By a G-variety X we shall mean an algebraic variety with a G-action; X is *generically free* if G acts freely on a dense open subset of X and *primitive* if $k(X)^G$ is a field. Equivalently, a G-variety X is primitive if G transitively permutes the irreducible components of X. If Y is an algebraic variety with $k(Y) = k(X)^G$ then the inclusion $k(Y) \hookrightarrow k(X)$ gives rise to a rational map $\pi \colon X \dashrightarrow Y$. This map, called the *rational quotient map*, has the property that $\pi^{-1}(y)$ is a single G-orbit for $y \in Y$ in general position; see [Ro_1, Theorem 2] and [Ro_2].

If A is a subgroup of G and Z is an A-variety then A acts on $G \times Z$ by

$$a \cdot (g, z) \longrightarrow (ga^{-1}, az)\,. \tag{2.1}$$

The rational quotient $(G \times Z)/A$ for this action is called the *homogeneous fiber product* of G and X and is denoted by $G *_A X$. Since the G-action on $G \times Z$ (by left multiplication in the first factor) commutes with the above-defined A-action on $G \times Z$, it descends to $G *_A Z$, thus making $G *_A Z$ into a G-variety. (A priori G only acts rationally on $G *_A Z$ but by [Ro_1, Theorem 1] one can find a model with a regular action.) In the sequel we will only use this construction in the case where A is a finite abelian group, and Z is an affine A-variety. In this setting we have a natural birational model for $(G \times Z)/A$, namely the geometric quotient $\mathrm{Spec}(k[G \times Z]^A)$ (see [MFK, Corollary 1.2 and Amplification 1.3]). The symbol $G *_A Z$ will always refer to this model.

A primitive generically free G-variety X gives rise to an element of $H^1(k(X)^G, G)$, which we shall denote by $\mathrm{cl}(X)$. If $\mathrm{cl}(X) = 1$ then X is birationally isomorphic to a G-variety of the form $X_0 \times G$, where G acts on $X_0 \times G$ by left translations on the second factor; see e.g., [Po, Section 1.3]. If A is a subgroup of G and if Z is a primitive generically free A-variety then $\mathrm{cl}(G *_A Z)$ is the image of $\mathrm{cl}(Z)$ under the natural map $H^1(K, A) \longrightarrow H^1(K, G)$, where $K = k(G *_A Z)^G = k(Z)^A$.

Our proof of Theorem 1 will rely on the following characteristic-free version of [RY_2, Lemma 4.3].

Lemma 2. *Let G be an algebraic group and let X be a generically free primitive G-variety. Suppose a finite abelian subgroup A of G is not contained in any Borel subgroup. If X has a smooth A-fixed point then* $\mathrm{cl}(X) \neq 1$ *in* $H^1(k(X)^G, G)$.

Proof. Assume the contrary: $\mathrm{cl}(X) = 1$ and thus X is birationally isomorphic to $X_0 \times G$, for some irreducible variety X_0, where G acts by left translations on the second factor. Let B be a Borel subgroup of G. Consider the rational G-equivariant map

$$X \simeq X_0 \times G \dashrightarrow G/B\,.$$

Since G/B is a complete variety and X has a smooth A-fixed point, the Going Down Theorem [RY_1, Proposition A.1] tells us that G/B also has an A-fixed point;

denote this point by y. Thus $A \subset \mathrm{Stab}(y)$, and $\mathrm{Stab}(y)$, being a conjugate of B, is a Borel subgroup of G, a contradiction. $\square$

Remark 3. If $|A|$ is prime to $\mathrm{char}(k)$ (and, in particular, if $\mathrm{char}(k) = 0$), then A does not lie in any Borel subgroup of G if and only if A is nontoral (i.e., A does not lie in any subtorus of G); see [Bor$_2$, Theorem 10.6(5)].

3. Smooth fixed points

Our strategy for proving Theorem 1, i.e., for showing that $H^1(K, G) \neq \{1\}$ for suitable G and K, will be as follows:

(i) find a finite abelian subgroup A of G that does not lie in any Borel subgroup,

(ii) construct a generically free G-variety X with a smooth A-fixed point such that $k(X)^G = K$, and

(iii) apply Lemma 2.

In this section we shall concentrate on (ii). Our starting point is the following well-known lemma; for lack of a suitable reference, we supply an elementary proof below.

Lemma 4. *Let C be a cyclic group. Then there exists a C-Galois cover $Y \longrightarrow \mathbb{P}^1$ of irreducible smooth projective curves totally ramified at 0.*

Proof. Case I: If $n = |C|$ is prime to $p = \mathrm{char}(k)$ then the cover $\mathbb{P}^1 \longrightarrow \mathbb{P}^1$ given by $t \mapsto t^n$ has the desired properties.

Case II: Suppose $n = p^m$. By [L, Exercise VIII.12.5] (or [FJ, Lemma 24.43]) there exists an irreducible $\mathbb{Z}/p^m\mathbb{Z}$-cover $Y \longrightarrow \mathbb{P}^1$, where Y is a smooth curve. We claim that such a cover is totally ramified at some point $x \in \mathbb{P}^1$ (which can be taken to be 0). Indeed, since the subgroups of $\mathbb{Z}/p^m\mathbb{Z}$ are totally ordered by inclusion, there exists $y_0 \in Y$ whose inertia group $C_0 = \mathrm{Stab}(y_0)$ contains $\mathrm{Stab}(y)$ for every $y \in \mathbb{P}^1$. Then the subcover $Y/C_0 \longrightarrow \mathbb{P}^1$ is an unramified Galois C/C_0-cover of $\mathbb{P}^1$. This is only possible if $C/C_0 = \{1\}$, i.e., $Y \longrightarrow \mathbb{P}^1$ is totally ramified at 0.

Case III: Suppose $n = p^m q$, and $(p, q) = 1$. Write $C = C_1 \times C_2$, where C_1 and C_2 are cyclic groups of orders q and p^m respectively. Let $Y_1 \longrightarrow \mathbb{P}^1$ (with $Y_1 = \mathbb{P}^1$) be the C_1-cover constructed in Case I, $Y_2 \longrightarrow \mathbb{P}^1$ be the C_2-cover constructed in Case II, and Y be the normalization of $Y_1 \times_{\mathbb{P}^1} Y_2$. Since both Y_1 and Y_2 are totally ramified at 0, the number of points of Y above 0 divides both p^m and q. Hence, this number is equal to 1, so that $Y \longrightarrow \mathbb{P}^1$ is totally ramified at 0. $\square$

If A be a finite abelian group, we shall refer to the minimal number of generators of A as $\mathrm{rank}(A)$.

Lemma 5. *Let A be an finite abelian group of rank r and let K be a finitely generated field extension of k of transcendence degree $d \geq r$. Then there exists an A-variety Y such that (i) $k(Y)^A = K$ and (ii) Y has a smooth A-fixed point.*

Proof. First assume $K = k(t_1, \dots, t_d)$ is a purely transcendental extension of k. Write $A = C_1 \times \cdots \times C_r$ as a product of r cyclic groups. By Lemma 4 for each $i = 1, \dots, r$ there exists a C_i-Galois cover $\psi_i \colon V_i \longrightarrow \mathbb{A}^1$, where V_i is smooth and ψ is totally ramified at 0. Now let $V = V_1 \times \cdots \times V_r \times \mathbb{A}^{d-r}$, where C_i acts on the ith component. Then V is smooth and

$$\psi = \psi_1 \times \cdots \times \psi_r \times id \colon V \longrightarrow \mathbb{A}^d$$

is a Galois A-cover (so that $k(V)^A = k(\mathbb{A}^d) = k(t_1, \dots, t_d)$), totally ramified at $0 \in \mathbb{A}^d$. Thus V has the desired properties.

For general K we will obtain the desired A-variety Y as the fiber product $Y = Y_0 \times_{\mathbb{A}^d} V$ via a diagram of the form

$$\begin{array}{ccc} & & V \\ & & \downarrow \psi \\ Y_0 & \xrightarrow{\phi} & \mathbb{A}^d \end{array}$$

where Y_0 is an affine variety such that $k(Y_0) = K$. To construct ϕ choose a transcendence basis $s_1, \dots, s_d$ for K/k so that K is a finite separable extension of $k(s_1, \dots, s_d)$ (this can always be done by [L, Corollary X.6.3]). The inclusion $k(s_1, \dots, s_d) \subset K = k(Y_0)$ induces a rational map $\phi \colon Y_0 \dashrightarrow \mathbb{A}^d = \mathbb{A}^d$; after replacing Y_0 by an affine dense open subset we may assume this map is regular. By our choice of s_i, this map is etale at some point $y_0 \in Y$; after composing ϕ with a translation in $\mathbb{A}^d$, we may assume $\phi(y_0) = 0 \in \mathbb{A}^d$.

It remains to check that for ϕ so defined, the A-variety $Y = Y_0 \times_{\mathbb{A}^d} V$ (with A acting trivially on Y_0) satisfies conditions (i) and (ii) of the lemma. Indeed, (i) follows from $[k(Y) : k(Y_0)] = |A|$ and $k(Y_0) \subset k(Y)^A$. To prove (ii), note that $y = (y_0, 0)$ is an A-fixed point of Y. To see that Y is smooth at this point, recall that the morphism $Y \longrightarrow V$ is obtained from ϕ by a base change, and hence, is étale at y. Since V is smooth, we conclude that Y is smooth at y, as claimed. □

Remark 6. After passing to an affine open A-invariant neighborhood of the A-fixed point, we may assume the variety Y in Lemma 5 is affine.

Proposition 7. *Suppose an algebraic group G has a finite abelian subgroup A such that A does not lie in any Borel subgroup of G. Then $H^1(K, G) \neq \{1\}$ for any finitely generated field extension K/k of transcendence degree $d \geq \operatorname{rank}(A)$.*

Proof. Choose an A-variety Y and a smooth A-fixed point $y \in Y$, as in Lemma 5. Set $X = G *_A Y$. (Note that by Remark 6 we can choose Y to be affine.) We claim that $\operatorname{cl}(X) \neq 1$ in $H^1(K, G)$. Recall that X is defined as the geometric quotient for the A-action on $G \times Y$ given by (2.1). Let $\pi \colon G \times Y \longrightarrow X$ be the geometric quotient map (π is dual to the inclusion $k[G \times Y]^A \hookrightarrow k[G \times Y]$ of k-algebras).

In view of Lemma 2, it is enough to show that $x = \pi(1_G, y)$ is a smooth A-fixed point of X. Since y is fixed by A, we have

$$a \cdot x = \pi(a \cdot (1_G, y)) = \pi(a, y) = \pi(1_G, a^{-1}y) = \pi(1_G, y)$$

for every $a \in A$, i.e., x is fixed by A. To see that X is smooth at x, note that A acts on $G \times Y$ with trivial stabilizers and thus π is etale at x. Since, $G \times Y$ is smooth at $(1_G, y)$, we conclude that X is smooth at $x = \pi(1_G, y)$, as claimed (cf. [PV, Proposition 4.22]). □

4. Proof of Theorem 1

We begin by reducing the proof to the case of a semisimple group. We claim that if Theorem 1 holds for G_{ss} then it holds for G. Indeed, suppose $H^1(K, G) \neq \{1\}$. The short exact sequence $1 \longrightarrow \operatorname{Rad}(G) \longrightarrow G \longrightarrow G_{ss} \longrightarrow 1$ gives rise to the exact sequence

$$H^1(K, \operatorname{Rad}(G)) \longrightarrow H^1(K, G) \longrightarrow H^1(K, G_{ss})$$

of pointed sets; see [Se_2, p. 125]. Since the radical $\mathrm{Rad}(G)$ is special (as is every connected solvable group), $H^1(K, \mathrm{Rad}(G)) = \{1\}$; thus $H^1(K, G) \neq \{1\}$ implies $H^1(K, G_{ss}) \neq \{1\}$. Applying Theorem 1 to G_{ss}, we conclude that G_{ss} is connected in part (a), the universal covering map for $(G_{ss})_{ss}$ is purely inseparable in part (b), and G_{ss} is special in part (c). But this immediately implies that G is connected in part (a), the universal covering map for G_{ss} is purely inseparable in part (b), and G is special in part (c), as desired.

From now on we shall assume that G is semisimple.

Proof of part (a): Assume the contrary: G is not connected. By [BS, Lemme 5.11] we can write G in the form $G = FG^0$, where F is a finite group (cf. also [V, Proposition 7]). Let a be an element of F that does not lie in G^0. Applying Proposition 7 to $A = \langle a \rangle$, we conclude that $H^1(K, G) \neq 1$, a contradiction.

Proof of part (b): In view of part (a) we may assume that $G = G_{ss}$ is connected. Let $\phi\colon \overline{G} \longrightarrow G$ be the universal covering map. Assume to the contrary, that ϕ is not inseparable. Then the order of the fundamental group of G_{ss} is divisible by a prime $p \neq \mathrm{char}(k)$; see [St_2, 2.4]. By [St_2, Theorem 2.27], G has a finite nontoral elementary abelian p-subgroup A of rank ≤ 2. This subgroup does not lie in any Borel subgroup of G; see Remark 3. Now Proposition 7 tells us that $H^1(K, G) \neq \{1\}$, a contradiction.

Proof of part (c): Assume the contrary: G is not special. Then G has a torsion prime p; see [Se_5, 1.3] or [Se_3, Théorème 1(c)]. Since $\mathrm{Tors}(G) \neq \{\mathrm{char}(k)\}$, we may assume $p \neq \mathrm{char}(k)$. Now by [St_2, Theorem 2.28] G has a finite nontoral elementary abelian p-subgroup A of rank ≤ 3. Applying Proposition 7 to this subgroup (cf. Remark 3), we conclude, that $H^1(K, G) \neq \{1\}$, a contradiction. $\square$

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