NESTING MAPS OF GRASSMANNIANS

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ABSTRACT. Let F be a field and $\mathrm{Gr}(i, F^n)$ be the Grassmannian of i-dimensional linear subspaces of F^n. A map $f\colon \mathrm{Gr}(i, F^n) \longrightarrow \mathrm{Gr}(j, F^n)$ is called nesting if $l \subset f(l)$ for every $l \in \mathrm{Gr}(i, F^n)$. Glover, Homer and Stong showed that there are no continuous nesting maps $\mathrm{Gr}(i, \mathbb{C}^n) \longrightarrow \mathrm{Gr}(j, \mathbb{C}^n)$ except for a few obvious ones. We prove a similar result for algebraic nesting maps $\mathrm{Gr}(i, F^n) \longrightarrow \mathrm{Gr}(j, F^n)$, where F is an algebraically closed field of arbitrary characteristic. For $i = 1$ this yields a description of the algebraic subbundles of the tangent bundle to the projective space $\mathbb{P}_F^n$.

1. INTRODUCTION

Let F be a field. We shall denote the Grassmannian of i-dimensional linear subspaces of F^n by $\mathrm{Gr}(i, F^n)$. Suppose i, j and n are integers satisfying $1 \le i < j \le n - 1$. We shall say that a map of Grassmannians

$$(1.1) \qquad f\colon \mathrm{Gr}(i, F^n) \longrightarrow \mathrm{Gr}(j, F^n)$$

is *nesting*, if $l \subset f(l)$ for every $l \in \mathrm{Gr}(i, F^n)$. The starting point for this note is the following combinatorial result communicated to us by J. Buhler.

Theorem 1.1. *Let F be a finite field and $i < j$ be positive integers such that $n = i+j$. Then there exists a bijective nesting map (of sets) $f\colon \mathrm{Gr}(i, F^n) \longrightarrow \mathrm{Gr}(j, F^n)$.*

Theorem 1.1 can be deduced from a theorem of P. Hall about systems of distinct representatives; for details see Section 6.

It is natural to ask for what n, i and j there exist *algebraic* (and for $F = \mathbb{R}$, $F = \mathbb{C}$, *continuous*) nesting maps $f\colon \mathrm{Gr}(i, F^n) \longrightarrow \mathrm{Gr}(j, F^n)$.

Example 1.2. Let n be an even integer and ω be an alternating bilinear form on F^n, i.e., a non-degenerate bilinear form $\omega(v, w)$ such that $\omega(v, v) = 0$ for every $v \in F^n$. (If $\mathrm{char}\,(F) \ne 2$, then the last condition is equivalent to $\omega(v, w) = -\omega(w, v)$ for every $v, w \in F^n$, so "alternating" is the same as "symplectic".) Then $f\colon \mathrm{Gr}(1, F^n) \longrightarrow \mathrm{Gr}(n - 1, F^n)$, given by $f(l) = l^{\perp_\omega}$, is an algebraic nesting isomorphism. Here $l^{\perp_\omega}$ is the orthogonal complement with respect to ω. $\qquad\square$

Example 1.3. Suppose $n \geq 2$ is even, ω is an alternating form and H is a positive-definite Hermitian form on $\mathbb{C}^n$. Then we can define a *continuous* nesting map $g \colon \operatorname{Gr}(1, \mathbb{C}^n) = \mathbb{P}_{\mathbb{C}}^{n-1} \longrightarrow \operatorname{Gr}(2, \mathbb{C}^n)$ by $l \mapsto l \oplus (l^{\perp_\omega})^{\perp_H}$.

Note that if n is odd then there is no continuous map $g \colon \mathbb{P}_{\mathbb{C}}^{n-1} \longrightarrow \operatorname{Gr}(2, \mathbb{C}^n)$. This is a special case of Theorem 1.5(a) below, but it can also be seen directly as follows. Suppose g exists. Then $\alpha \colon l \mapsto l^{\perp_H} \cap g(l)$ is a continuous fixed point free map $\mathbb{P}_{\mathbb{C}}^{n-1} \longrightarrow \mathbb{P}_{\mathbb{C}}^{n-1}$, contradicting the Lefschetz fixed point theorem; see, e.g., [6, 30.13]. (Note that α is, indeed, well-defined, because $l \subset g(l)$ implies $g(l) \not\subset l^{\perp_H}$.) $\qquad\square$

The following results assert that, over an algebraically closed field F, these are essentially the only examples. First consider the case where $i \geq 2$.

Theorem 1.4. *Suppose $2 \leq i < j \leq n - 1$. Then*

(a) *there does not exist a continuous nesting map $\operatorname{Gr}(i, \mathbb{C}^n) \longrightarrow \operatorname{Gr}(j, \mathbb{C}^n)$,*

(b) *there does not exist an algebraic nesting map $\operatorname{Gr}(i, F^n) \longrightarrow \operatorname{Gr}(j, F^n)$ for any algebraically closed field F.*

For $i = 1$ the continuous and the algebraic cases diverge. We shall write $\mathbb{P}_F^{n-1}$ in place of $\operatorname{Gr}(1, F^n)$.

Theorem 1.5. *Suppose $2 \leq j \leq n - 1$. Then*

(a) *a continuous nesting map $\mathbb{P}_{\mathbb{C}}^{n-1} \longrightarrow \operatorname{Gr}(j, \mathbb{C}^n)$ exists if and only if n is even and $j = 2$ or $j = n - 1$.*

(b) *Let F be an algebraically closed field and $f \colon \mathbb{P}_F^{n-1} \longrightarrow \operatorname{Gr}(j, F^n)$ be an algebraic nesting map. Then n is even, $j = n-1$, and f is as in Example 1.2.*

Nesting maps $\mathbb{P}_F^{n-1} \longrightarrow \operatorname{Gr}(j, F^n)$ are easily seen to be in a natural 1-1-correspondence with rank $j - 1$ subbundles of $T(\mathbb{P}_F^{n-1})$; cf. Lemma 5.1. Using this correspondence, we can rephrase Theorem 1.5 as follows:

Corollary 1.6. *Let F be an algebraically closed field and $T(\mathbb{P}_F^{n-1})$ be the tangent bundle to the projective space $\mathbb{P}_F^{n-1}$.*

(a) *$T(\mathbb{P}_{\mathbb{C}}^{n-1})$ has a topological (complex) subbundle of rank $1 \leq r \leq n - 2$ if and only if n is even and $r = 1$ or $n - 2$.*

(b) *If n is odd then $T(\mathbb{P}_F^{n-1})$ has no nontrivial algebraic subbundles. If n is even then the only non-trivial algebraic subbundle of $T(\mathbb{P}_F^{n-1})$ (up to an automorphism of $\mathbb{P}_F^{n-1}$) is the null correlation bundle of rank $n - 2$.*

Here the null-correlation bundle on $\mathbb{P}_F^{n-1}$ (defined, e.g., in [10, Section 4.2] or [2, p. 128]) is the rank $n - 2$ bundle associated to the nesting map of Example 1.2. (Note that different choices of the alternating form in Example 1.2 give rise to bundles that differ by an automorphism of $\mathbb{P}_F^{n-1}$.)

Theorem 1.4(a) is due to Stong [13, Theorem 2]. For a field F of characteristic zero, part (b) follows from part (a) by the Lefschetz priniciple. Theorem 1.5(a) and Corollary 1.6(a) are due to Glover, Homer and Stong [5, Theorem 2(ii) and Theorem 1.1(ii)]. (A special case of Corollary 1.6(a) was considered earlier by Bott [2, Corollary 1.5].) For fields F of characteristic

zero, part (b) can be deduced from these results and the work of Roan [11, Main Theorem].

The purpose of this note is to give uniform characteristic-free proofs of Theorem 1.4, Theorem 1.5 and Corollary 1.6. We also include a proof of Theorem 1.1 in a short appendix. An application of Theorem 1.5 can be found in a recent paper of Bergman [1].

Acknowledgments

2. Proof of Theorem 1.4

We begin by recalling some well-known facts about the cohomology of the Grassmannian; for details we refer the reader to [4, Chapter 14] (see also [3] or [8]).

Let σ_i be the ith elementary symmetric polynomial in the independent variables $x_1, \ldots, x_n$. The cohomology ring $H^*(\mathrm{Gr}(i, F^n), \mathbb{Z})$ is isomorphic to the quotient of $\mathbb{Z}[\sigma_1, \ldots \sigma_i]$, by the ideal generated by the Schur functions S_λ, where λ ranges over the partitions of n whose Young tableaux are not contained in a rectangle with i rows and $n-i$ columns. Here, by convention, σ_i is the Schur function for a column with i rows.

Two remarks are now in order. First of all, the cohomology class corresponding to σ_i lies in $H^{2i}(\mathrm{Gr}(i, F^n), \mathbb{Z})$; however, for notational convenience, we will continue to use the natural grading induced from $\mathbb{Z}[x_1, \ldots, x_n]$, so that $\deg(\sigma_i) = i$. Secondly, since $\mathbb{Z}[\sigma_1, \ldots \sigma_i]$ is generated over $\mathbb{Z}$ by the Schur functions S_λ, where λ ranges over partitions with at most i rows, we identify $H^*(\mathrm{Gr}(i, F^n), \mathbb{Z})$ with $\mathbb{Z}[\sigma_1, \ldots \sigma_i]/I$, where I is a homogeneous ideal generated by elements of degree $> n - i$.

We will prove both parts of Theorem 1.4 by the same computation. (If $\mathrm{char}(F) > 0$ in part (b), we replace the cohomology ring $H^*(\mathrm{Gr}(i, F^n), \mathbb{Z})$ by the Chow ring; our computation will remain valid there; cf. [4, Chapter 14].)

Let V be the trivial bundle of rank n on $\mathrm{Gr}(i, F^n)$ and let $\mathcal{T}_i$ be the tautological bundle of rank i. (The fiber of $\mathcal{T}_i$ over $l \in \mathrm{Gr}(i, F^n)$ consists of the vectors in l.) Since f is a nesting map,

$$\mathcal{T}_i \subset f^*(\mathcal{T}_j) \text{ and } f^*(\mathcal{T}_j) \subset V,$$

where $\subset$ means "subbundle of". In other words,

(2.1) $$\mathcal{B} = f^*(\mathcal{T}_j)/\mathcal{T}_i \text{ is a subbundle of } \mathcal{A} = V/\mathcal{T}_i,$$

where $\mathrm{rank}(\mathcal{A}) = n - i$ and $\mathrm{rank}(\mathcal{B}) = j - i$.

Now recall that the rth Chern class of the tautological bundle is given by $c_r(\mathcal{T}_i) = (-1)^r \sigma_r$ (cf., e.g., [8, p. 111]), so that the total Chern class is

$$c_{\text{tot}}(\mathcal{T}_i) = (1 - \sigma_1 + \sigma_2 - \cdots + (-1)^i \sigma_i).$$

Since $\mathcal{A} := V/\mathcal{T}_i$, we have

$$(2.2) \qquad c_{\text{tot}}(\mathcal{A})(1 - \sigma_1 + \sigma_2 - \cdots + (-1)^i \sigma_i) = 1,$$

where $c_{\text{tot}}(\mathcal{A})$ is the total Chern class of $\mathcal{A}$, and 1 represents the total Chern class of the trivial bundle V on $\text{Gr}(i, F^n)$.

Since $\mathcal{A}$ has a subbundle $\mathcal{B}$ or rank $n - j$, $c_{\text{tot}}(\mathcal{A})$ factors as a product of elements of degree $n - j$ and $j - i$ in $H^*(\text{Gr}(i, F^n), \mathbb{Z})$. We will show that this is impossible. Consider the degree-preserving homomorphism

$$\phi \colon \mathbb{Z}[x_1, \ldots, x_i, \ldots, x_n] \longrightarrow \mathbb{Z}[x_1, \ldots, x_i]$$

given by $\phi(x_r) = x_r$ if $1 \leq r \leq i$ and $\phi(x_r) = 0$ if $i + 1 \leq r \leq n$. Under this homomorphism $\mathbb{Z}[\sigma_1, \ldots, \sigma_i]$ maps isomorphically to $\mathbb{Z}[\phi(\sigma_1), \ldots, \phi(\sigma_i)]$, where $\phi(\sigma_r)$ is the rth elementary symmetric function in $x_1, \ldots, x_i$. Thus it is enough to prove that $\phi(c_{\text{tot}}(\mathcal{A}))$ is irreducible in $\mathbb{Z}[\phi(\sigma_1), \ldots, \phi(\sigma_i)]$. We will, in fact, prove that $\phi(c_{\text{tot}}(\mathcal{A}))$ is irreducible in $\mathbb{Z}[x_1, \ldots, x_i]$ and even in $\mathbb{C}[x_1, \ldots, x_i]$. Applying ϕ to both sides of (2.2), we see that

$$1 = \phi(c_{\text{tot}}(\mathcal{A}))(1 - \phi(\sigma_1) + \phi(\sigma_2) - \cdots + (-1)^i \phi(\sigma_i)) =$$
$$= \phi(c_{\text{tot}}(\mathcal{A}))(1 - x_1)(1 - x_2) \ldots (1 - x_i).$$

in $\mathbb{Z}[x_1, \ldots, x_i]/\phi(I)$. Since $\phi(c_{\text{tot}}(\mathcal{A}))$ has degree $\leq n - i$, and I (and thus $\phi(I)$) is generated by homogeneous elements of degree $> n - i$,

$$(2.3) \qquad \phi(c_{\text{tot}}(\mathcal{A})) = \left((1 - x_1)^{-1}(1 - x_2)^{-1} \ldots (1 - x_i)^{-1}\right)_{|n-i} =$$
$$\left((1 + x_1 + x_1^2 + \ldots)(1 + x_2 + x_2^2 + \ldots) \ldots (1 + x_i + x_i^2 + \ldots)\right)_{|n-i} =$$
$$1 + \theta_1(x_1, \ldots, x_i) + \theta_2(x_1, \ldots, x_i) + \cdots + \theta_{n-i}(x_1, \ldots, x_i),$$

where $_{|n-i}$ means that we cut the series at the terms of degree at most $n - i$ and $\theta_d(x_1, \ldots, x_i)$ denotes the sum of all monomials of degree d in $x_1, \ldots, x_i$. It remains to show that $1 + \theta_1(x_1, \ldots, x_i) + \cdots + \theta_{n-i}(x_1, \ldots, x_i)$ is irreducible in $\mathbb{C}[x_1, \ldots, x_i]$. Homogenizing this polynomial with respect to an additional variable x_0, we obtain $\theta_{n-i}(x_0, x_1, \ldots, x_i)$. Thus Theorem 1.4 is a consequence of the following:

Lemma 2.1. $\theta_d(x_0, x_1, \ldots, x_i)$ *is irreducible in* $\mathbb{C}[x_0, \ldots, x_i]$ *for any* $i \geq 2$ *and* $d \geq 1$.

Proof. It suffices to consider the case $i = 2$. For notational convenience, we will write x, y and z for x_0, x_1 and x_2. In other words, we want to prove that the projective curve $X_d \subset \mathbb{P}^2$ given by $\theta_d(x, y, z) = 0$ is irreducible. It suffices to show that X_d is non-singular. We proceed by induction on d.

The base case, $d = 1$, is trivial, since $\theta_1(x, y, z) = x + y + z$ cuts out a line in $\mathbb{P}^2$. The inductive step will rely on the formulas

$$(2.4) \qquad \theta_d(x, y, z) = x\theta_{d-1}(x, y, z) + \theta_d(y, z)$$

and

(2.5) $\qquad (\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z})\theta_d(x, y, z) = (d + 2)\theta_{d-1}(x, y, z)\,.$

Suppose $p = (x_0 : y_0 : z_0)$ is a singular point of X_d. By (2.5),

(2.6) $\qquad\qquad\qquad \theta_{d-1}(p) = 0,$ i.e., $p \in X_{d-1}.$

We claim that $x_0 y_0 z_0 \neq 0$. Indeed, assume the contrary, say, $p = (0 : 1 : z_0)$. Then formulas (2.4), and (2.6) say that $\theta_d(1, z_0) = 0$, i.e., $z_0 \neq 1$ is a $(d+1)$th root of unity. On the other hand, by (2.6),

$$0 = \theta_{d-1}(p) = \theta_{d-1}(0, 1, z_0) = \theta_{d-1}(1, z_0)\,,$$

so that $z_0 \neq 1$ is a dth root of unity, a contradiction. This proves the claim.

Differentiating (2.4) with respect to x, we obtain

$$\frac{\partial}{\partial x}\theta_d(x, y, z) = \theta_{d-1}(x, y, z) + x\frac{\partial}{\partial x}\theta_{d-1}(x, y, z)\,,$$

and similarly for y and z. Combining this with (2.4) and (2.6), and keeping in mind that $x_0 y_0 z_0 \neq 0$, we obtain

$$\theta_{d-1}(p) = \frac{\partial}{\partial x}\theta_{d-1}(p) = \frac{\partial}{\partial y}\theta_{d-1}(p) = \frac{\partial}{\partial z}\theta_{d-1}(p) = 0\,.$$

Thus p is a singular point of X_{d-1}, contradicting our induction assumption. This completes the proof of Lemma 2.1 and of Theorem 1.4. $\qquad\square$

3. The Schwarzenberger conditions

Our proof of Theorem 1.5 in the next section will rely on the following result about vector bundles on projective spaces.

Proposition 3.1. *Suppose E is a continuous vector bundle of rank $1 \leq r \leq m-2$ on $\mathbb{P}_{\mathbb{C}}^m$ or an algebraic vector bundle on $\mathbb{P}_F^m$, where F is an algebraically closed field. Let $p(t) = 1 + c_1 t + \cdots + c_r t^r$ be the Chern polynomial of E.*

Suppose the (complex) roots $w_1, \ldots, w_r$ of $p(t)$ are distinct and lie on the unit circle. Then either $r = 1$ and $p(t) = 1 \pm t$ or $r = 2$ and $p(t) = 1 - t^2$.

Note that by a theorem of Kronecker, $w_1, \ldots, w_r$ are necessarily roots of unity; however, we shall not use this in the proof.

Proof. Since $w_1, \ldots, w_r$ lie on the unit circle each $w_i^{-1} = \overline{w_i}$ is also a root of $p(t)$, so that

$$p(t) = (1 - w_1 t) \ldots (1 - w_r t)\,.$$

Let $B_{k,l} = \sum_{i=1}^r \binom{k - w_i}{l}$. Our argument is based on the Schwarzenberger conditions, which require that $B_{s,m}$ should be an integer for every $s \in \mathbb{Z}$; see [9, Theorem 22.4.1].

Lemma 3.2. *(a) $B_{s,m} = 0$ for every $s = 1, \ldots, m - 2$.*

(b) $B_{1,k} = 0$ for $k = 3, \ldots, m$.

Proof. (a) Since $B_{s,m}$ is an integer, it is enough to show that $|B_{s,m}| < 1$. Indeed, for each $i = 1, \ldots, r$,

$$\left| \binom{s - w_i}{m} \right| \leq \frac{1}{m!} (|s - w_i| \ldots |2 - w_i|) \cdot (|1 - w_i| \cdot | - w_i| \cdot | - 1 - w_i|) \cdot$$

$$(| - 2 - w_i| \ldots |s - m + 1 - w_i|) \leq$$

$$\frac{1}{m!} ((s + 1) \cdot s \cdot \ldots \cdot 3) \cdot |1 - w^2| \cdot (3 \cdot 4 \cdot \ldots \cdot (m - s)) \leq$$

$$\frac{(s + 1)!(m - s)!}{2m!} = \frac{(m + 1)}{2 \binom{m + 1}{s + 1}} \leq \frac{(m + 1)}{2 \binom{m + 1}{2}} = \frac{1}{m}.$$

Note that we have used the inequality $\binom{m + 1}{s + 1} \geq \binom{m + 1}{2}$, which is valid for any $s = 1, \ldots, m - 2$. Now

$$|B_{s,m}| \leq \sum_{i=1}^{r} \left| \binom{s - w_i}{m} \right| \leq r \frac{1}{m} < 1,$$

and part (a) follows.

(b) Combining part (a) with the identity $B_{s,m} - B_{s-1,m} = B_{s-1,m-1}$, we conclude that $B_{i,m-1} = 0$ for $i = 1, \ldots, m - 3$. Repeating this argument, we see that for every $j = 0, \ldots, m - 3$, and every $i = 1, \ldots, m - 2 - j$, we have $B_{i,m-j} = 0$. Now set $i = 1$ and $k = m - j$, and part (b) follows. $\square$

We now return to the proof of Proposition 3.1. By the Chinese Remainder theorem, the semisimple $\mathbb{Q}$-algebra $F = \mathbb{Q}[x]/(p(x))$ is isomorphic to $\mathbb{Q}(w_1) \oplus \cdots \oplus \mathbb{Q}(w_d)$ via $p(x) \mapsto (p(w_1), \ldots, p(w_d))$. Thus for every $f(x) \in F$,

$$\mathrm{tr}_{F/\mathbb{Q}} \, f(x) = \sum_{i=1}^{r} f(w_i).$$

In particular, setting $a = -(1 - x)x(-1 - x) \in F$, and $b_0 = 1$, $b_1 = -2 - x$, $b_2 = (-2 - x)(-3 - x)$, $\ldots$, $b_{m-3} = (-2 - x)(-3 - x) \ldots (2 - m - x)$ in F, and applying Lemma 3.2(b), we see that

(3.1) $\qquad \mathrm{tr}_{F/\mathbb{Q}}(ab_i) = (i + 3)! B_{1,i+3} = 0$ for every $i = 0, 1, \ldots, m - 3$.

Note since $1, x, \ldots, x^{r-1}$ are $\mathbb{Q}$-linearly independent in F, so are $b_0, \ldots, b_{r-1}$. Since we are assuming $r \leq m - 2$, this implies that $b_0, b_1, \ldots, b_{m-3}$ span F. But the trace form $<\mathbf{x}, \mathbf{y}> = \mathrm{tr}_{F/\mathbb{Q}}(xy)$ is non-singular on F (viewed as an r-dimensional $\mathbb{Q}$-vector space); thus (3.1) is only possible if $a = 0$ in F. In other words, $a(w_i) = (1 - w_i)(-w_i)(-1 - w_i) = 0$ or, equivalently, $w_i = \pm 1$ for every $i = 1, \ldots, r$. Since $w_1, \ldots, w_r$ are assumed to be distinct, we conclude that either $r = 1$ and $w_1 = \pm 1$ or $r = 2$ and $\{w_1, w_2\} = \{-1, 1\}$. This completes the proof of Proposition 3.1 $\square$

4. Proof of Theorem 1.5

From now on we will assume that $i = 1$. That is, we will be interested in nesting maps $\mathbb{P}_F^{n-1} \longrightarrow \mathrm{Gr}(j, F^n)$, where $2 \leq j \leq n - 1$.

Lemma 4.1. *Suppose there exists a continuous nesting map $f \colon \mathbb{P}_{\mathbb{C}}^{n-1} \longrightarrow \mathrm{Gr}(j, \mathbb{C}^n)$ (respectively, an algebraic nesting map $f \colon \mathbb{P}_F^{n-1} \longrightarrow \mathrm{Gr}(j, F^n)$), where $2 \leq j \leq n - 1$ and F is an algebraically closed field.*

(a) There exists topological (respectively, algebraic) vector bundles $\mathcal{B}$ and $\mathcal{C}$ on $\mathbb{P}_{\mathbb{C}}^{n-1}$ (respectively, $\mathbb{P}_F^{n-1}$) of ranks $n - j$ and $j - 1$ with Chern polynomials $p(t)$ and $q(t) \in \mathbb{Z}[t]$ such that $p(t)q(t) = 1 + t + \cdots + t^{n-1}$. In fact, we can take $\mathcal{C} = \mathcal{A}/\mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are defined in (2.1).

(b) n is even and either $j = n - 1$ and $p(t) = t + 1$ or $j = 2$ and $q(t) = t + 1$. □

Proof. (a) We specialize the argument of Section 2 to the case where $i = 1$. In this case the cohomology ring $H^*(\mathbb{P}^{n-1}, F^n) = H^*(\mathrm{Gr}(i, F^n), \mathbb{Z})$ reduces to $\mathbb{Z}[h]/(h^n = 0)$, where h is the class of a hyperplane section in $\mathbb{P}^{n-1}$. (In Section 2 we denoted h by $\phi(\sigma_1)$.) Defining $\mathcal{A}$ and $\mathcal{B}$ as in (2.1) (with $i = 1$), setting $\mathcal{C} = \mathcal{A}/\mathcal{B}$, and writing t for x_1, we see that

$$p(t)q(t) = \phi(c_{\mathrm{tot}}(\mathcal{A})) = 1 + \theta_1(t) + \cdots + \theta_{n-1}(t) = 1 + t + \cdots + t^{n-1};$$

cf. (2.3).

(b) If $n = 3$, the polynomial $1 + t + t^2$ is irreducible, contradicting part (a). Thus we may assume $n \geq 4$. In this case, $1 \leq n - j \leq n - 3$ or $1 \leq j - 1 \leq n - 3$, and thus Proposition 3.1 (with $m = n - 1$) applies to the bundle $\mathcal{B}$ and to the bundle $\mathcal{C}$ of part (a). Since $p(t)$ and $q(t)$ have no multiple factors, their roots are nth roots of unity, and $p(1), q(1) \neq 0$, we see that the only possibilities for the Chern polynomials are the ones listed in part (b). □

We now turn to the proof of Theorem 1.5. Part (a) is an immediate consequence of the above lemma. To prove part (b) of Theorem 1.5, assume that there exists an algebraic nesting map

$$f \colon \mathbb{P}_F^{n-1} \longrightarrow \mathrm{Gr}(j, F^n).$$

By Lemma 4.1(b), n is even and $j = 2$ or $n - 1$. Our goal is to show that (i) $j = 2$ is impossible, and (ii) if $j = n - 1$ then $f(l) = l^{\perp_\omega}$ for some alternating form ω on F^n, as in Example 1.2. (For $F = \mathbb{C}$, (ii) was proved in by Roan [11]; we will give a short characteristic-free proof below.)

Proof of (i): Suppose $j = 2$. Consider the exact sequences

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{n-1}}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^{n-1}}^{\oplus n} \longrightarrow \mathcal{A} \longrightarrow 0$$
$$0 \longrightarrow \mathcal{B} \longrightarrow \mathcal{A} \longrightarrow \mathcal{C} \longrightarrow 0$$

of algebraic vector bundles on $\mathbb{P}_F^{n-1}$. From the first sequence, we see that $H^0(\mathbb{P}_F^{n-1}, \mathcal{A}) = F^n$. By Lemma 4.1(b), $\mathcal{B}$ is a line bundle with Chern polynomial $1 + t$, i.e., $\mathcal{B} = \mathcal{O}(1)$ on $\mathbb{P}_F^{n-1}$. Thus $H^0(\mathbb{P}_F^{n-1}, \mathcal{B}) = F^n$, and the second sequence yields

$$(0) \longrightarrow F^n \longrightarrow F^n \longrightarrow H^0(\mathbb{P}_F^{n-1}, \mathcal{C}) \longrightarrow (0) \,,$$

which means that $\mathcal{C}$ has no global sections. On the other hand, $\mathcal{C}$, being a quotient of $\mathcal{A}$, is generated by global sections. This contradiction shows that f cannot exist for $j = 2$.

Proof of (ii): Assume $j = n - 1$. Consider the exact sequence

$$0 \longrightarrow \mathcal{T}_{n-1} \longrightarrow \mathcal{O}_{\check{\mathbb{P}}^{n-1}}^{\oplus n} \longrightarrow L \longrightarrow 0 \,,$$

where $\mathcal{T}_{n-1}$ is the tautological bundle on $\mathrm{Gr}(n - 1, F^n) = \check{\mathbb{P}}^{n-1}$. Pulling back this sequence to $\mathbb{P}^{n-1}$ via f, we obtain an exact sequence

$$0 \longrightarrow f^*(\mathcal{T}_{n-1}) \longrightarrow \mathcal{O}_{\mathbb{P}^{n-1}}^{\oplus n} \longrightarrow f^*(L) \longrightarrow 0$$

of vector bundles on $\mathbb{P}^{n-1}$. Since L is a line bundle generated by global sections, $L = \mathcal{O}_{\check{\mathbb{P}}^{n-1}}(d)$ for some $d > 0$. Since f is nesting, $\mathcal{T}_1 = \mathcal{O}_{\mathbb{P}^{n-1}}(-1) \subset f^*(\mathcal{T}_{n-1})$. Recall that the bundles $\mathcal{B}$ and $\mathcal{C}$ in the statement of Lemma 4.1 are defined by $\mathcal{B} = f^*(\mathcal{T}_{n-1})/\mathcal{T}_1$ and $\mathcal{C} = \mathcal{A}/\mathcal{B}$, where $\mathcal{A} = \mathcal{O}_{\mathbb{P}^{n-1}}^{\oplus n}/\mathcal{T}_1$; cf. (2.1). Thus the line bundle $\mathcal{C}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^{n-1}}/f^*(\mathcal{T}_{n-1}) = f^*(L)$. By Lemma 4.1(b), the Chern polynomial of $\mathcal{C}$ is $1 + t$; in other words, $\mathcal{C} = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$. On the other hand, $\mathcal{C} = f^*(L) = f^*(\mathcal{O}_{\check{\mathbb{P}}^{n-1}}(d))$. This is only possible if $d = 1$ and f is induced by a non-singular linear map

$$f^* \colon H^0(\check{\mathbb{P}}^{n-1}, \mathcal{O}_{\check{\mathbb{P}}^{n-1}}(1)) \longrightarrow H^0(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$$

In other words, f is induced by a non-singular linear map $f^* \colon F^n \longrightarrow (F^n)^*$, where $(F^n)^*$ is the dual vector space to F^n, $\mathbb{P}^{n-1} = \mathbb{P}(F^n)$ and $\check{\mathbb{P}}^{n-1} = \mathbb{P}((F^n)^*)$. Now $\omega(x, y) = f^*(x)(y)$ is a non-singular bilinear form on F^n. Since f is nesting, $\omega(x, x) = 0$ for every $x \in F^n$, i.e., ω is alternating. This shows that f is obtained by the construction in Example 1.2, as claimed. The proof of Theorem 1.5 is now complete. $\qquad\square$

5. Proof of Corollary 1.6

Corollary 1.6 is an immediate consequence of Theorem 1.5 and the lemma below.

Lemma 5.1. *The following are in natural (i.e., PGL_n-equivariant) correspondence:*

(a) algebraic nesting maps $f \colon \mathbb{P}_F^{n-1} \longrightarrow \mathrm{Gr}(j, F^n)$,

(b) algebraic subbundles of rank $j - 1$ of the bundle $\mathcal{A} = \mathcal{O}^{n+1}/\mathcal{O}(-1)$ on $\mathbb{P}_F^{n-1}$,

(c) algebraic subbundles of rank $j - 1$ of the tangent bundle $T(\mathbb{P}_F^{n-1})$,

For $F = \mathbb{R}$ or $\mathbb{C}$ the lemma remains true if "algebraic" is replaced by "continuous".

Proof. Let $\mathcal{T}_j$ be the tautological bundle on $\mathrm{Gr}(j, F^n)$ (as in Section 2); in particular, $\mathcal{T}_1 = \mathcal{O}(-1)$ is the tautological line bundle on $\mathbb{P}_F^{n-1}$. Given a nesting map $f \colon \mathbb{P}_F^{n-1} \longrightarrow \mathrm{Gr}(j, F^n)$, we associate to it the subbundle $\mathcal{B} = f^*(\mathcal{T}_j)/\mathcal{O}(-1)$ of $\mathcal{A}$ of rank $j - 1$; cf. (2.1).

Conversely, a subbundle $\mathcal{B}$ of $\mathcal{A}$ of rank $j - 1$ lifts to a subbundle $\mathcal{B}'$ of $\mathcal{O}^n$ of rank j containing $\mathcal{O}(-1)$, and we can define a nesting map

$$f \colon \mathbb{P}_F^{n-1} \longrightarrow \mathrm{Gr}(i, F^n)$$

by $p \mapsto \mathcal{B}'(p)$. This establishes a natural bijective correspondence between (a) and (b).

To show that (b) and (c) are in a natural bijective correspondence, note that $\mathcal{A} = T(\mathbb{P}_F^{n-1})(-1)$; see, e.g., [10, p. 6] or [7, p. 409]. The same argument works in the continuous case. $\qquad\square$

6. Appendix: Proof of Theorem 1.1

For every $l \in \mathrm{Gr}(i, F^n)$, let

$$X_l = \{L \in \mathrm{Gr}(j, F^n) \mid l \subset L\}.$$

Since the sets $\mathrm{Gr}(i, F^n)$ and $\mathrm{Gr}(j, F^n)$ have the same cardinality, a bijective nesting map $f \colon \mathrm{Gr}(i, F^n) \longrightarrow \mathrm{Gr}(j, F^n)$ may be viewed as a system of distinct representatives for the collection of subsets $\{X_l \mid l \in \mathrm{Gr}(i, F^n)\}$ of $\mathrm{Gr}(j, F^n)$. Indeed, given a system of distinct representatives $\{x_l\}$, with $x_l \in X_l$, the map defined by $f(l) = x_l$ is nesting and bijective. Conversely, given f, as in Theorem 1.1, the elements $x_l = f(l)$ form a system of distinct representatives for $\{X_l\}$.

Thus, by a theorem of P. Hall (see, e.g., [12, Theorem 5.1.1]), we only need to check that

$$(6.1) \qquad\qquad |X_{l_1} \cup \cdots \cup X_{l_k}| \geq k$$

for every choice of distinct elements $l_1, \ldots, l_k \in \mathrm{Gr}(i, F^n)$.

Let N be the number of j-planes in F^n containing a given i-plane. Since $i + j = n$,

$$N = |\mathrm{Gr}(j - i, F^{n-i})| = |\mathrm{Gr}(n - j, F^{n-i})| = |\mathrm{Gr}(i, F^j)|$$

is also the number of i-planes in F^n contained in a given j-plane. To prove (6.1), we will count the number of elements in the set

$$W = \{(l, L) \mid \text{where } l \subset L \text{ and } l = l_1, \ldots, l_k\}$$

in two ways. On the one hand, projecting to the first component, we see that $|W| = kN$. On the other hand, projecting to the second component, we obtain $|W| \leq |X_{l_1} \cup \cdots \cup X_{l_k}|N$. Thus

$$|W| = kN \leq |X_{l_1} \cup \cdots \cup X_{l_k}|N$$

and (6.1) follows. $\qquad\square$

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