A NOTE ON RETRACTS AND LATTICES (AFTER D. J. SALTMAN)

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Abstract. This is an expository note based on the work of D. J. Saltman. We discuss the notions of retract rationality and retract equivalence and reprove some of the results from [Sa, Section 1] using geometric language.

1. Retracts

Throughout this note G will be a finite group and k will be a base field. We shall say that a dominant rational map $f\colon X \dashrightarrow Y$ is a model for a field extension $i\colon F \hookrightarrow E$ if $k(X) \simeq E$, $k(Y) \simeq F$, and i is induced by f (i.e., $i = f^*$).

Definition 1.1. Let $f\colon X \dashrightarrow Y$ be a dominant rational map of irreducible algebraic varieties. We say that Y is a *retract* of X via f if for every (not necessarily irreducible) subvariety $X_0 \neq X$ there exists a rational section $s\colon Y \dashrightarrow X$ of f such that $f(Y) \not\subset X_0$. If f is not specified, then the expression Y is a retract of X means "via some dominant rational map $X \dashrightarrow Y$".

Remark 1.2. If we view X as a variety over $k(Y)$ then the above definition can be restated as follows: Y is a retract of X if and only if $k(Y)$-points are dense in X.

Lemma 1.3. *(a) X is a retract of itself (via the identity map).*

(b) If Y is a retract of X via $f\colon X \dashrightarrow Y$ and Z is a retract of Y via $g\colon Y \dashrightarrow Z$ then Z is a retract of X via $gf\colon X \dashrightarrow Z$.

Proof. (a) Take s to be the identity map (for any X_0).

(b) Suppose $X_0 \neq X$ is a subvariety of X. Choose a section $s_1\colon Y \dashrightarrow X$ whose image is not contained in X_0. Now choose a section $s_2\colon Z \dashrightarrow Y$ whose image is not contained in the union of the indeterminacy locus of s_1 and $s_1^{-1}(X_0)$. Now $s = s_2 s_1\colon Z \dashrightarrow X$ is the section of gf we are looking for. □

The following lemma shows that the property of being a retract depends only on the field extension $k(X)/k(Y)$ and not on the specific model $f\colon X \dashrightarrow Y$ for this extension. Equivalently, this property does not change if we replace X and Y by birationally isomorphic varieties.

Lemma 1.4. *Consider the following commutative diagram of rational maps,*

$$\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[d, dashed, "\alpha"] & Y \arrow[d, dashed, "\beta"] \\ X' \arrow[r, dashed, "f'"] & Y' \end{tikzcd}$$

where f and f' are dominant and α and β are birational isomorphisms. If Y is a retract of X via f then Y' is a retract of X' via f'.

Proof. Given a subvariety $X_0' \neq X'$ of X', we want to construct a section $s' \colon Y' \dashrightarrow X'$ of f' whose image is not contained in X_0'. Let X_0 be the union of $\alpha^{-1}(X_0')$ and the indeterminacy locus of α and let $s \colon Y \dashrightarrow X$ be a section of f whose image is not contained in X_0. Then $s' = \alpha s \beta^{-1}$ has the desired properties: it is well defined and its image is not contained in X_0'. □

Example 1.5. (a) Suppose X is birationally isomorphic to $Y \times Z$, and k-points are dense in Z. (Note that latter condition is always satisfied if k is algebraically closed.) Then Y is a retract of X, via the projection $f \colon X \dashrightarrow Y$ to the first factor. Indeed, for every k-point $z \in Z$, consider the section $s_z \colon Y \longrightarrow Y \times Z$ of f given by $s_z(y) = (y, z)$. Since the union of the images of such sections is dense in X, we conclude that Y is a retract of X.

(b) If X is rational over Y (i.e. $k(X)$ is isomorphic to a purely transcendental extension of $k(Y)$) then Y is a retract of X. Indeed, in view of Lemma 1.4, we may assume that $X = Y \times \mathbb{A}_k^n$. Since k-points are dense in $\mathbb{A}_k^n$, the desired conclusion follows from part (a).

Definition 1.6. Let $i \colon F \hookrightarrow E$ be a field extension, where E and F are finitely generated over k. We shall say that F is a retract of E (via i) if Y is a retract of X (via f) for some (and thus for every) model $f \colon X \dashrightarrow Y$ of E/F.

Equivalently, F is a retract of E if for some (and thus any) algebraic F-variety X such that $F(X) = E$, the F-points are dense in X.

If the field extension i is not specified then the expression "F is a retract of E" means "via some field extension E/F.

2. Generators of tori

In this section we make some simple observations about algebraic tori. This material will be used in the proof of Theorem 3.1 in the next section (but not anywhere else in this paper).

Let $T = (k^*)^r$ be an algebric torus. We will say that $t \in T$ *generates* T, if the cyclic subgroup $<t> = \{t^n \mid n \in \mathbb{Z}\}$ is Zariski dense in T. We will say that $t_1, \dots, t_r \in k^*$ are *multiplicatively independent*, if $t_1^{a_1} \dots, t_r^{a_r} = 1$ implies $a_1 = \cdots = a_r = 0$. (Here $a_1, \dots, a_r$ are integers.)

Lemma 2.1. *Let $t = (t_1, \dots, t_r) \in T$ for some $t_1, \dots, t_r \in k^*$. Then the following conditions are equivalent:*

(a) $\{t^n \mid n \in \mathbb{Z} - \{0\}\}$ *is dense in* T*,*

(b) t *is a generator of* T*,*

(c) t *does not lie in any proper closed subgroup of* T*,*

(d) $\chi(t) \neq 1$ *for any non-trivial character of* T*,*

(e) $t_1, \dots, t_r$ *are multiplicatively independent.* □

Proof. (a) ⇒ (b): Obvious.

(b) ⇔ (a): The set $C = \{t^n \mid n \in \mathbb{Z} - \{0\}\}$ differs from $<t>$ in at most one point, i.e.,

$$<t> \setminus \{id\} \subset C \subset <t> .$$

Hence, C is dense in T if and only if so is $<t>$.

(b) ⇒ (c): If t lies in a closed subgroup S then the closure of $<t>$ is contained in S.

(c) ⇒ (b): The Zariski closure of $<t>$ is a closed subgroup of T.

(c) ⇔ (d): Immediate from [Sp, Corollary 2.5.3].

(d) ⇔ (e): Every character χ of $T = (k^*)^n$ has the form $\chi(t_1, \dots, t_r) = t_1^{a_1} \dots t_r^{a_r}$ for some integers $a_1, \dots, a_r$; see, e.g. [Sp, Exercise 2.5.12]. □

Lemma 2.2. *If $t \in T$ is a generator of T then t^n is also a generator for any integer $n \neq 0$.*

Proof. Assume the contrary: t^n is not a generator. Then by Lemma 2.1, there is a non-trivial character $\chi \colon T \longrightarrow k^*$ such that $\chi(t^n) = 1$. Since χ^n is also a non-trivial character of T, and $\chi^n(t) = \chi(t^n) = 1$, we conclude that t is not a generator of T, contradicting our assumption. □

Lemma 2.3. *Let T_{gen} be the set of generators of T. Assume either* $\operatorname{char}(k) = 0$ *or* $\operatorname{char}(k) = p$ *but k is not algebraic over its prime field $\mathbb{F}_p$. Then T_{gen} is Zariski dense in T.*

Proof. First we claim that $T_{gen} \neq \emptyset$. If $\operatorname{char}(k) = 0$ then k contains the field of rational numbers. By Lemma 2.1 any point of the form $t = (p_1, \dots, p_r)$, where $p_1, \dots, p_r$ are distinct prime integers, is a generator of T (distinct primes are clearly multiplicatively independent). If $\operatorname{char}(k) = p$, then, by our assumption, k contains a copy of the polynomial ring $\mathbb{F}_p[x]$, and we use the same argument as above, with $\mathbb{Z}$ and $\mathbb{Q}$ replaced by $\mathbb{F}_p[x]$ and $\mathbb{F}_p(x)$, respectively. That is, if $f_1(x), \dots, f_r(x)$ are distince irreducible polynomials then $t = (f_1, \dots, f_r)$ is a generator of T. This proves the claim.

Now suppose $t \in T_{gen}$. Lemma 2.2 tells us $\{t^n \mid n \in \mathbb{Z} - \{0\}\}$ lies in T_{gen}. By Lemma 2.1, this set is dense in T. Hence, T_{gen} is also dense in T, as claimed. □

3. Lattices and retracts

As usual, by a G-lattice L we shall mean $\mathbb{Z}^n$, with a linear G-action, i.e., an integral representation $G \longrightarrow \mathrm{GL}_n(\mathbb{Z})$. We will say that L is faithful if this representation is faithful, i.e., if every non-trivial element of G acts non-trivially on L. We will say that L is *permutation*, if it has a $\mathbb{Z}$-basis permuted by G.

Given a G-lattice L, we will write $k[L]$ for its group ring, $k(L)$ for the fraction field of $k[L]$, and X_L for the spectrum of $k[L]$. The G-action on L induces G-actions on $k[L]$, $k(L)$ and X_L. Ignoring these G-actions for a moment, we see that since $L \simeq \mathbb{Z}^n$, $k[L]$ is isomorphic to the Laurent polynomial ring $k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, $k(L)$ is isomorphic to the purely transcendental field extension $k(x_1, \dots, x_n)$ of k, and X_L is isomorphic to the n-dimensional torus $(k^*)^n$.

For the rest of this paper we will assume that $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) = p$ and k is not algebraic over the prime field $\mathbb{F}_p$, so that we can use Lemma 2.3.

Theorem 3.1. *Let G be a finite group and $L \subset M$ be G-lattices. Then $k(M)^G$ is a retract of $k(L \oplus M)^G$.*

Proof. The inclusion $i \colon L \hookrightarrow M$ gives rise to a dominant morphism $i_* \colon X_M \dashrightarrow X_L$. Consider the maps

$$X_M \xrightarrow{\alpha} X_L \times X_M \xrightarrow{\beta} X_M$$

of G-varieties, where $\alpha = (i_*, \mathrm{id})$ and β is the projection to the second component. Passing to rational G-quotients, we obtain

$$X_M/G \overset{\alpha/G}{\dashrightarrow} (X_L \times X_M)/G \overset{\beta/G}{\dashrightarrow} X_M/G\,.$$

The function fields of X_M/G and $(X_L \times X_M)/G$ are, by definition, $k(M)^G$ and $k(L \oplus M)^G$. Hence, it suffices to show that $(X_L \times X_M)/G$ is a retract of X_M/G via β/G.

First observe that since $\beta \circ \alpha = \mathrm{id}_{X_M}$ (and hence, $\beta/G \circ \alpha/G = \mathrm{id}_{X_M/G}$), α/G is a section of β/G. Note also that β (and thus β/G) does not depend on i_*, but α (and thus α/G) does. Thus if we modify $i_* \colon X_M \longrightarrow X_L$ by composing it with a G-equivariant dominant map $\sigma \colon X_M \longrightarrow X_M$, we will create another section of β/G. In the sequel I will take $\sigma = \sigma_n$ to be the dominant map $X_M \longrightarrow X_M$ induced by $\mathrm{mult}_n \colon M \longrightarrow M$, where mult_n is multiplication by a fixed integer $n \neq 0$. (Note that X_M is, by definition, a torus, and $\sigma_n(t) = t^n$ for any $t \in X_M$.) It remains to show that the union of the images of the sections obtained in this way is dense in $(X_L \times X_M)/G$. In fact, it suffices to show that the union Z of the images of

$$\alpha_n = (i_* \sigma_n, \mathrm{id}) \colon X_M \longrightarrow X_L \times X_M$$

is dense in $X_L \times X_M$, as n ranges over non-zero integers. More concretely, the k-points of Z are of the form $(i_*(x^n), x)$, where n ranges over the non-zero integers and x ranges over X_M (and x^n denotes the nth power of x in the torus X_M).

Let $(X_M)_{gen}$ be the set of generators of X_M (viewed as a torus $(k^*)^{\mathrm{rank}(M)}$). By Lemma 2.3 $(X_M)_{gen}$ is Zariski dense in X_M, and by Lemma 2.1

$$D_x = \{x^n \,|n \in \mathbb{Z} - \{0\}\}$$

is Zariski dense in X_M for any $x \in (X_M)_{gen}$. Since $i_* \colon X_M \longrightarrow X_L$ is dominant, $i_*(D_x)$ is Zariski dense in X_L. To sum up, Z contains $i_*(D_x) \times \{x\}$ for every $x \in (X_M)_{gen}$. We thus conclude that the closure $\overline{Z}$ of Z in $X_L \times X_M$ contains $X_L \times (X_M)_{gen}$. Since $(X_M)_{gen}$ is dense in X_M, this implies $\overline{Z} = X_L \times X_M$, as claimed. □

4. Stable retracts

Definition 4.1. Let X and Y be irreducible varieties defined over k.

(a) We will say that Y is a *stable retract* of X (and write $Y \preceq X$) if Y is a retract of $X \times \mathbb{A}_k^m$. Similarly, we will say that a field F is a stable retract of E (and write $F \preceq E$) if F is a retract of a purely transcendental extension $E(t_1, \dots, t_m)$ of F.

(b) We will say X and Y are *stably retract equivalent* (and write $X \sim Y$) if X is a stable retract of Y and Y is a stable retract of X. Similarly, we will say that finitely generated field extensions F and E of k are stably retract equivalent (and write $F \sim E$) if F is a stable retract of E and E is a stable retract of F.

(c) X is called *retract rational* if it is stably retract equivalent to a point (here by a point I mean $\mathrm{Spec}(k)$). A finitely generated field extension F of k is retract rational if it is stably retract equivalent to k.

Example 4.2. Clearly if F/k is stably rational then it is stably retract rational.

Lemma 4.3. *(a) $\sim$ is an equivalence relation on the set of varieties/k (or of finitely generated field extensions of k).*

(b) $\preceq$ induces a partial order on the set of stable retract equivalence classes of varieties (or finitely generated field extensions F/k),

(c) If X and Y (or F and E) are stably isomorphic over k, then they are stably retract equivalent.

Proof. (a) $\sim$ is symmetric by definition. It is reflexive and transitive by Lemma 1.3.

(b) Suppose $\preceq$ is a reflexive and transitive relation on a set S. Then the relation $s \sim t$, defined by ($s \sim t$ if $s \preceq t$ and $t \preceq s$), is easily seen to be an equivalence relation, and $\preceq$ induces a partial order on the set $S/\sim$ of equivalence classes in S. This is precisely the construction we carried out, with $S =$ set of irreducible varieties (or finitely generated field extension of k).

(c) Suppose Z is rational over both X and Y. Then $X \sim Z$ and $Y \sim Z$ (see Example 4.2), and since $\sim$ is an equivalence relation, we conclude that $X \sim Y$. □

Theorem 4.4. *Let M be faithful G-lattice. Then $k(M)^G$ is a stable retract of $k(L \oplus M)^G$ for any G-lattice L.*

This theorem is a variant of Theorem 3.1. The conclusion of Theorem 4.4 is a little weaker ("retract" has been replaced by "stable retract"), and M is assumed to be faithful. On the other hand, L is no longer required to be a sublattice of M.

Proof. Recall that L is contained in some permutation lattice P; see [Sa, Lemma 1.2]. Let $M' = M \oplus P$. Then L is contained in M' and thus $k(M')^G$ is a retract of $k(L \oplus M')^G$ by Theorem 3.1. By the no-name lemma, $k(M')^G$ is stably equivalent to $k(M)^G$, and $k(L \oplus M')^G = k(L \oplus M \oplus P)^G$ is stably equivalent to $k(L \oplus M)^G$. This shows that $k(M)^G$ is a stable retract of $k(L \oplus M)^G$, as claimed. □

Corollary 4.5. *(cf.* [Sa, Lemma 1.5]*) Let G be a finite group, L be a faithful G-lattice and $G \hookrightarrow \mathrm{GL}(V)$ be a faithful linear representation of G. Then $k(V)^G$ is a stable retract of $k(L)^G$.*

Proof. Use Theorem 4.4, with M = faithful permutation G-lattice. By the no-name lemma, $k(M)^G$ is stably equivalent to $k(V)^G$, $k(L \oplus M)^G$ is stably equivalent to $k(L)^G$ (here we use the fact that L is faithful!), and the corollary follows. □

Corollary 4.6. *Let $G \hookrightarrow \mathrm{GL}(V)$ be a faithful linear representation of G and let N be a G-lattice such that $k(N)^G$ is stably retract equivalent to $k(V)^G$.*

Then $k(M)^G$ is stably retract equivalent to $k(V)^G$ for any faithful direct summand M of N.

Proof. By Corollary 4.5, we have $k(V)^G \preceq k(M)^G$, where $\preceq$ stands for "is a stable retract of", as before. On the other hand, by Theorem 4.4 $k(M)^G \preceq k(N)^G \sim k(V)^G$. This shows that $k(M)^G$ and $k(V)^G$ are retract equivalent. □

Example 4.7. If N is a quasi-permutation lattice then $k(N)^G$ and $k(V)^G$ are stably isomorphic; see [LL, Proposition 1.4]. Consequently, Corollary 4.6 tells us that $k(M)^G$ and $k(V)^G$ are stably retract equivalent for any faithful direct summand M of a quasi-permutation lattice.

Remark 4.8. Saltman has shown that there exist faithful G-lattices M such that $k(M)^G$ and $k(V)^G$ are not stably retract equivalent; see [Sa, Theorem 2.8].

Corollary 4.9. *(cf.* [Sa, Corollary 1.6]*) If $k(L)^G$ is stably retract rational for some faithful G-lattice L then $k(V)^G$ is stably retract rational for every faithful linear representation $G \hookrightarrow \mathrm{GL}(V)$.*

Proof. Our goal is to show that $k \sim k(V)^G$, where $\sim$ denotes stable retract equivalence, as before.

Since $k(V)^G \preceq k(L)^G$ (by Corollary 4.5) and $k(L)^G \sim k$ (given), we conclude that $k(V)^G \preceq k$. The opposite inequality, $k \preceq k(V)^G$ follows from Remark 1.2, since $k(V)^G$ is unirational over k (and hence, k-points are dense on any model of $k(V)^G/k$). $\square$

5. Appendix: Smooth retracts

Saltman's terminology is a bit different from ours; see [Sa, p. 223]. Given a field extension E/F, he says that F is *a retraction* of E if there is a F-variety X, such that $F(X) = E$ and X has a F-point. If F-points are dense in X (i.e., F is a retract of E in the sense of our Definition 1.6), Saltman calls F is a *dense retraction* of E. He remarks that the relationship between these notions, beyond the obvious implication (dense retraction) $\Rightarrow$ (retraction), is obscure.

The problem, as I see it, is that the notion of retraction defined above is difficult to work with, because the existence of a F-point is, in general, not a birational invariant of X. The following "intermediate" notion seems more natural to me (even though I have no applications for it at this point).

Definition 5.1. Suppose $\mathrm{char}(F) = 0$. Given a field extension E/F, we will say that F is *a smooth retract* of E, if the following equivalent conditions hold.

(a) There exists a F-variety X such that $F(X) = E$ and X has a smooth F-point.

(b) For every smooth proper F-variety X' such that $F(X') = E$, X' has a F-point.

The implication (a) $\Rightarrow$ (b), is a consequence of Nishimura's Lemma [RY, Proposition A6] (see also [N]) to the birational isomorphism $X \dashrightarrow X'$. To prove the implication (b) $\Rightarrow$ (a), we only need to show that there exists a smooth proper F-variety X' such that $F(X') = E$ (then we can take $X = X'$). To construct such X', embed some model of E/F in a projective space $\mathbb{P}^n_F$, take the closure and resolve its singularities (this requires the assumption $\mathrm{char}(F) = 0$!).

Using Nishimura's lemma one can reprove the results of this note (and in particular, Lemmas 1.3 and 4.3) for smooth retracts. I opted to use "dense retracts" instead, because this leads to stronger versions of Theorems 3.1, 4.4 and their corollaries.

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