

# TRACE FORMS OF GALOIS EXTENSIONS IN THE PRESENCE OF A FOURTH ROOT OF UNITY

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ABSTRACT. We study quadratic forms that can occur as trace forms  $q_{L/K}$  of Galois field extensions  $L/K$ , under the assumption that  $K$  contains a primitive 4th root of unity. M. Epkenhans conjectured that  $q_{L/K}$  is always a scaled Pfister form. We prove this conjecture and classify the finite groups  $G$  which admit a  $G$ -Galois extension  $L/K$  with a non-hyperbolic trace form. We also give several applications of these results.

## 1. INTRODUCTION

The trace form of a finite separable field extension (or, more generally of an étale algebra)  $L/K$  is the non-degenerate quadratic form  $q_{L/K}: x \mapsto \text{tr}_{L/K}(x^2)$  defined over  $K$ . In this paper we shall address the following problem: Given a finite group  $G$ , which quadratic forms over  $K$  are trace forms of  $G$ -Galois extensions  $L/K$ ? This question has been extensively studied; see, e.g. [5] and the references there. In [9] D.-S. Kang and the second author obtained the following partial answer:

**Theorem 1.1.** *Let  $L/K$  be a  $G$ -Galois extension and let  $S$  be a Sylow 2-subgroup of  $G$ . Assume*

- (a)  *$S$  is not abelian, and*
- (b)  *$K$  contains a primitive  $e$ th root of unity, where*

$$e = \min\{\exp(H) \mid H \text{ is a non-abelian subgroup of } S\}.$$

*Then the trace form  $q_{L/K}$  is hyperbolic over  $K$ .*

In this paper we will study trace forms of  $G$ -Galois extensions  $L/K$ , assuming only that  $K$  contains a primitive 4th root of unity. M. Epkenhans has conjectured that in this situation  $q_{L/K}$  is always a scaled Pfister form. Our first main result is a proof of this conjecture. Before giving the precise statement, we introduce some notations.

If  $G$  is a group and  $i \geq 1$  is an integer, we set  $G^i = \langle g^i \mid g \in G \rangle \triangleleft G$ . If  $S$  is a finite 2-group, then  $S^2 = \text{Fr}(S)$  is the Frattini subgroup of  $S$ . The Frattini rank  $r$  of  $S$  is the rank of the elementary abelian group  $S/S^2 \simeq (\mathbb{Z}/2)^r$ . Note

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that the Frattini rank of  $S$  equals the cardinality of any minimal generating set of  $S$ ; see, e.g., [18, 7.3].

**Theorem 1.2.** *Suppose  $K$  is a field containing a primitive 4th root of unity,  $L/K$  is  $G$ -Galois extension,  $S$  is a Sylow 2-subgroup of  $G$ , and  $r$  is the Frattini rank of  $S$ . Then the trace form  $q_{L/K}$  is Witt-equivalent to the scaled Pfister form  $\langle |S| \rangle \otimes \langle\langle a_1, \dots, a_r \rangle\rangle$ , for some  $a_1, \dots, a_r \in K^*$ .*

Several remarks are in order, regarding Theorem 1.2. First of all, both Theorem 1.1 and 1.2 remain true for Galois  $K$ -algebras  $L$  that are not necessarily fields. The reason is that both are enough to check for a single “versal”  $G$ -Galois algebra, which is a field; cf. e.g., [9, Proposition 2.5].

Secondly, Theorem 1.2 was previously known for  $|S| \leq 16$ ; see [5, Corollary 6, p. 227].

Thirdly, the “scaling factor” of  $\langle |S| \rangle$  presents only a minor inconvenience in working with the trace form  $q_{L/K}$ . It can be dropped if  $|S|$  is a square in  $K$  (and, in particular, if  $K$  contains a primitive 8th root of unity; cf. Remark 9.1) and replaced by  $\langle 2 \rangle$  in all other cases.

Finally, the requirement that  $K$  should contain a primitive 4th root of unity is essential. Indeed, let  $K = \mathbb{Q}$  and  $L = \mathbb{Q}(\sqrt{2 + \sqrt{2}})$ . By [5, Proposition 8] (with  $q = a = b = 1$  and  $D = 2$ ), the field extension  $L/K$  is Galois, with  $\text{Gal}(L/K) = \mathbb{Z}/4$  and the trace form  $q_{L/K} = \langle 1, 2, 1, 1 \rangle$ . This form is positive-definite and thus anisotropic. Consequently,  $q_{L/K}$  cannot be Witt-equivalent to a 2-dimensional form. This shows that Theorem 1.2 fails for this extension.

Our second main result is a complete description of those finite groups  $G$  which admit a  $G$ -Galois extension  $L/K$  with a non-hyperbolic trace form. (Here we assume that  $K$  contains a primitive root of unity of degree  $2^m$  for a fixed  $m \geq 2$ .) It turns out that these groups belong to a rather small but interesting family that was previously studied for entirely different reasons.

**Theorem 1.3.** *Let  $G$  be a finite group,  $S$  be a Sylow 2-subgroup of  $G$  and  $m \geq 2$  be an integer. Then the following conditions are equivalent:*

- (a) *there exists a  $S$ -Galois extension  $E/F$  such that  $F$  contains a primitive root of unity of degree  $2^m$  and the trace form  $q_{E/F}$  is not hyperbolic,*
- (b) *there exists a  $G$ -Galois extension  $L/K$  such that  $K$  contains a primitive root of unity of degree  $2^m$  and the trace form  $q_{L/K}$  is not hyperbolic,*
- (c)  *$T/T^{2^m}$  is abelian for every subgroup  $T$  of  $S$ ,*
- (d) *there exist an integer  $s \geq m$ , an abelian subgroup  $A \triangleleft S$ , and an element  $t \in S$  such that  $S = \langle A, t \rangle$  and  $tat^{-1} = a^{1+2^s}$  for every  $a \in A$ .*

A simple argument based on Sylow’s theorem shows that condition (c) is equivalent to  $H/H^{2^m}$  being abelian for every subgroup  $H$  of  $G$  (see Remark 5.2). Note also that the  $G$ -Galois extension  $L/K$  in part (b) can be chosen so that  $\text{char}(K) = 0$  (see Remark 7.3) and  $K$  does not contain a primitive root of unity of degree  $2^{m+1}$  (see Remark 5.1).

The 2-groups  $T$  appearing in condition (c) are *powerful* in the sense of Lubotzky and Mann [13]. Their results on the structure of powerful groups will be used in the proof of Theorem 1.3, along with theorems of Iwasawa [8] and Engler-Koenigsmann [6].

Theorems 1.2 and 1.3 have a natural cohomological interpretation. Let  $G$  be a finite group,  $S$  be a Sylow 2-subgroup of  $G$ ,  $r$  be the Frattini rank of  $S$  and  $K$  be a field containing a primitive root of unity of degree  $2^m$  for some integer  $m \geq 2$ . Then to every  $G$ -Galois field extension  $L/K$  (and, more generally, to a  $G$ -Galois  $K$ -algebra  $L$ ) we can associate the well-defined cohomology class  $\phi(L) = (a_1) \cdot (a_2) \cdots (a_r)$  in  $H^r(K, \mathbb{Z}/2\mathbb{Z})$ , where  $a_1, \dots, a_r$  are as in Theorem 1.2. In other words,  $\phi(L)$  is the Arason invariant of the Pfister form  $\langle |S| \rangle \otimes q_{L/K}$ ; cf. [1, Section 1]. The map  $\phi$  so defined is easily seen to be a cohomological invariant

$$\phi: H^1(*, G) \longrightarrow H^r(*, \mathbb{Z}/2\mathbb{Z}),$$

where  $*$  ranges over the category of fields containing a primitive  $2^m$ th root of unity. (Recall that the non-abelian cohomology set  $H^1(K, G)$  parametrizes  $G$ -Galois algebras over  $K$ .) Theorem 1.3 gives equivalent conditions for this cohomological invariant to be non-trivial.

The rest of this paper is structured as follows. Theorem 1.2 is proved in Sections 2 and 3. Theorem 1.3 is proved in Sections 4 - 7. In Section 8 we discuss a number of applications of these results. In particular, we show that the trace form of a  $G$ -Galois field extension  $L/K$  is hyperbolic if the field  $K$  is “sufficiently small” in a suitable sense (see Proposition 8.1) or if  $G$  is a simple group whose Sylow 2-subgroups are non-abelian (see Proposition 8.2). In the last section we give a description of quadratic forms that can occur as trace forms of  $M(2^n)$ -Galois extensions, where

$$M(2^n) = \langle \sigma, \tau | \sigma^{2^{n-1}} = 1 = \tau^2, \tau\sigma\tau = \sigma^{1+2^{n-2}} \rangle.$$

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## 2. ORTHOGONAL 2-GROUPS

Most of our subsequent results will be based on the following lemma, communicated to us by J.-P. Serre.

**Lemma 2.1.** *Let  $K$  be a field containing a primitive 4th root of unity,  $(V, q)$  be a non-degenerate finite-dimensional quadratic space over  $K$  and  $G$  be a finite 2-subgroup, acting orthogonally on  $V$ . Then  $V$  can be decomposed as*

an orthogonal sum  $V = V^{\text{Fr}(G)} \oplus V_0$ , such that the restriction of  $q$  to  $V_0$  is hyperbolic.

Here, as usual,  $V^{\text{Fr}(G)} = \{v \in V \mid h(v) = v \text{ for every } h \in \text{Fr}(G)\}$ , and we allow the trivial hyperbolic quadratic space  $V_0 = \{0\}$ .

*Proof.* We argue by induction on  $\dim(V) + |G|$ . Assume, to the contrary, that the lemma fails for some  $V$ ,  $q$  and  $G$ ; choose a counterexample with  $\dim(V) + |G|$  as small as possible. Then  $G$  acts faithfully on  $V$ ; otherwise we could obtain a counterexample with a smaller value of  $\dim(V) + |G|$  by keeping the same  $V$  and replacing  $G$  by  $G/N$ , where  $N$  is the kernel of this action.

We claim that every index 2 subgroup of  $G$  is elementary abelian. Indeed, assume the contrary:  $\text{Fr}(H) \neq \{1\}$  for some index 2 subgroup  $H$ . Equivalently,  $V^{\text{Fr}(H)} \neq V$ . Since  $|H| + \dim V < |G| + \dim V$ , our induction hypothesis applies and we can write  $V$  as an orthogonal sum

$$V = V^{\text{Fr}(H)} \oplus V_1,$$

where the restriction of  $q$  to  $V_1$  is hyperbolic. In particular,  $(V^{\text{Fr}(H)}, q|_{V^{\text{Fr}(H)}})$  is a regular quadratic space; see [11, p. 11, Corollary 2.6]. Since  $\text{Fr}(H)$  is a normal subgroup of  $G$ , the action of  $G$  restricts to  $V^{\text{Fr}(H)}$ . This restricted action is once again orthogonal, and since  $\dim V^{\text{Fr}(H)} < \dim V$ , we can apply our induction assumption to write  $V^{\text{Fr}(H)}$  as an orthogonal sum

$$V^{\text{Fr}(H)} = V^{\text{Fr}(G)} \oplus V_2,$$

where the restriction of  $q$  to  $V_2$  is hyperbolic. To sum up,

$$V = V^{\text{Fr}(H)} \oplus V_1 = V^{\text{Fr}(G)} \oplus V_0,$$

where the restriction of  $q$  to  $V_0 = V_1 \oplus V_2$  is hyperbolic, contradicting our choice of  $V$  and  $G$ . This contradiction proves the claim.

If every element of  $G$  has order  $\leq 2$  then  $G$  is itself elementary abelian. In this case the lemma is trivial, because  $\text{Fr}(G) = \{1\}$ . Thus we may assume  $G$  has an element  $g$  of order 4. By the claim we just proved,  $g$  is not contained in any subgroup of  $G$  of index 2. In other words,  $\langle g \rangle$  is not contained in any proper subgroup of  $G$ , i.e.,  $G = \langle g \rangle \simeq \mathbb{Z}/4$ . We shall thus concentrate on this case for the rest of the proof. Note that  $\text{Fr}(G) = \langle g^2 \rangle$ . We now proceed with an explicit description of  $V_0$ .

Now recall that  $K$  is assumed to contain a primitive 4th root of unity; we will denote it by  $\zeta$ . Since  $g^4 = 1$ , we can decompose  $V$  as a direct sum of the four eigenspaces for  $g$ :

$$(2.1) \quad V = V_1 \oplus V_{-1} \oplus V_\zeta \oplus V_{-\zeta},$$

where  $V_\alpha = \{v \in V \mid g(v) = \alpha v\}$ . Note that if  $x \in V_\alpha$  and  $y \in V_\beta$  then

$$B(x, y) = B(g(x), g(y)) = \alpha\beta B(x, y)$$

and thus

$$(2.2) \quad B(x, y) = 0 \text{ whenever } \alpha\beta \neq 1.$$

Here  $B$  denotes the bilinear form associated with the quadratic form  $q$ .

In particular  $V^{\text{Fr}(G)} = V_1 \oplus V_{-1}$  is orthogonal to  $V_\zeta \oplus V_{-\zeta}$ , and thus we can take  $V_0 = V_\zeta \oplus V_{-\zeta}$ . By (2.2) both  $V_\zeta$  and  $V_{-\zeta}$  are totally isotropic. Thus  $V_0$  contains a totally isotropic space of dimension at least half the dimension of  $V_0$ . Observe also that from (2.2), and from our assumption that  $q$  is non-degenerate on  $V$ , it follows that  $q$  is non-degenerate on  $V_0$ . Thus we see that  $V_0$  is hyperbolic; see [11, Chapter 1, Theorem 3.4(i)]. To sum up,

$$V = (V_1 \oplus V_{-1}) \oplus (V_\zeta \oplus V_{-\zeta}) = V^{\text{Fr}(G)} \oplus V_0,$$

where the restriction of  $q$  to  $V_0$  is hyperbolic. This contradicts our choice of  $G$  and  $V$ , thus completing the proof of Lemma 2.1.  $\square$

**Corollary 2.2.** *Let  $G$  be a finite 2-group and  $L/K$  be a  $G$ -Galois extension. Assume  $K$  contains a primitive 4th root of unity. Then*

- (a)  $q_{L/K} \simeq \langle |\text{Fr}(G)| \rangle \otimes q_{L^{\text{Fr}(G)}/K}$ .
- (b) More generally, for any normal subgroup  $H \subset \text{Fr}(G)$ ,

$$q_{L^H/K} \simeq \langle [\text{Fr}(G) : H] \rangle \otimes q_{L^{\text{Fr}(G)}/K}.$$

Here  $\simeq$  denotes Witt equivalence.

*Proof.* (a) The 2-group  $G$  acts orthogonally on the quadratic space  $(V = L, q_{L/K})$  over  $K$ . By Lemma 2.1,  $q_{L/K}$  is Witt-equivalent to its restriction to  $L^{\text{Fr}(G)}$ . Finally, for every  $x \in L^{\text{Fr}(G)}$ , we have

$$q_{L/K}(x) = |\text{Fr}(G)| q_{L^{\text{Fr}(G)}}(x),$$

and part (a) follows.

(b) Apply part (a) to the  $G/H$ -Galois extension  $L^H/K$ , remembering that  $\text{Fr}(G/H) = \text{Fr}(G)/H$ .  $\square$

### 3. CONCLUSION OF THE PROOF OF THEOREM 1.2

As usual, given  $a_1, a_2, \dots, a_n \in K^*$ ,  $\ll a_1, \dots, a_n \gg = \otimes_{i=1}^n \langle 1, -a_i \rangle$  will denote an  $n$ -fold Pfister form. Note that since we always assume  $K$  contains a primitive 4th root of unity,

$$\ll a_1, \dots, a_n \gg \simeq \otimes_{i=1}^n \langle 1, a_i \rangle.$$

We now begin the proof of Theorem 1.2 by reducing to the case where  $G = S$  is a 2-group.

**Lemma 3.1.** *Let  $G$  be a finite group,  $K$  be a field containing a primitive 4th root of unity,  $L/K$  be a  $G$ -Galois extension,  $S$  be the Sylow 2-subgroup of  $G$ ,  $K_1 = L^S$  and  $\phi: W(K) \rightarrow W(K_1)$  be the natural (extension of scalars) homomorphism of Witt rings.*

- (a) (cf. [2, 6.1.1])  $q_{L/K_1} = \phi(q_{L/K})$  in  $W(K_1)$ .
- (b)  $q_{L/K}$  is hyperbolic if and only if  $q_{L/K_1}$  is hyperbolic.

(c) Let  $a \in K^*$ . Then  $q_{L/K} = \langle a \rangle \otimes \ll a_1, \dots, a_r \gg$  in  $W(K)$ , for some  $a_1, \dots, a_r \in K^*$ , if and only if  $q_{L/K_1} = \langle a \rangle \otimes \ll b_1, \dots, b_r \gg$  in  $W(K_1)$  for some  $b_1, \dots, b_r \in K_1^*$ .

*Proof.* (a)  $\phi(q_{L/K})$  is clearly the trace form of the  $K_1$ -algebra  $L_1 = L \otimes_K K_1$  and  $L_1$  is isomorphic, as a  $K_1$ -algebra, to

$$(3.1) \quad L \oplus \dots \oplus L \text{ (} m \text{ times),}$$

where  $m = [G : S]$  is odd. Moreover, (3.1) is an orthogonal direct sum with respect to the trace form. Thus

$$\phi(q_{L/K}) = q_{L/K_1} \oplus \dots \oplus q_{L/K_1} \text{ (} m \text{ times);}$$

cf. [3, Theorem I.5.1]. Since we are assuming  $K$  (and thus  $K_1$ ) contains a primitive 4th root of unity,  $2W(K_1) = \{0\}$ , and part (a) follows.

By Springer's theorem,  $\phi$  is injective; see, e.g., [11, Theorem 7.2.3]. Part (b) now follows from (a).

(c) By Rost's theorem on the descent of Pfister forms [16, Section 3] (see also [2, 4.4.1]),  $\langle a \rangle \otimes q_{L/K}$  is Witt-equivalent to a Pfister form over  $K$  if and only if  $\langle a \rangle \otimes q_{L/K_1}$  is Witt-equivalent to a Pfister form over  $K_1$ .  $\square$

We now continue with the proof of Theorem 1.2. By Lemma 3.1(c) we may assume that  $G$  is a 2-group. By Corollary 2.2

$$q_{L/K} \simeq \langle | \text{Fr}(G) | \rangle \otimes q_{L^{\text{Fr}(G)}/K}.$$

Note that  $L^{\text{Fr}(G)}/K$  is a  $G/\text{Fr}(G)$ -Galois extension, where  $G/\text{Fr}(G) \simeq (\mathbb{Z}/2)^r$ . Thus it is enough to prove Theorem 1.2 in the case where  $\text{Gal}(L/K)$  is an elementary abelian 2-group; indeed, if we know that

$$q_{L^{\text{Fr}(G)}/K} \simeq \langle |G/\text{Fr}(G)| \rangle \otimes (r\text{-fold Pfister form}).$$

then by Corollary 2.2(a)

$$q_{L/K} \simeq \langle | \text{Fr}(G) | \rangle \otimes q_{L^{\text{Fr}(G)}/K} \simeq \langle |G| \rangle \otimes (r\text{-fold Pfister form}),$$

as claimed.

Now assume  $G = (\mathbb{Z}/2)^r$ . Here any  $G$ -Galois extension  $L/K$  has the form  $L = K(\sqrt{a_1}, \dots, \sqrt{a_r})$ , for some  $a_1, \dots, a_r \in K^*$ , and an easy computation in the basis  $\{a_1^{\frac{\epsilon_1}{2}} \dots a_r^{\frac{\epsilon_r}{2}}\}$ , with  $\epsilon_1, \dots, \epsilon_r = 0, 1$ , shows that

$$(3.2) \quad q_{L/K} \simeq \langle 2^r \rangle \otimes \ll a_1, \dots, a_r \gg;$$

cf. [2, 6.2.1] or [9, Lemmas 2.1(b) and 2.2]. This completes the proof of Theorem 1.2.  $\square$

## 4. IWASAWA STRUCTURES

An *Iwasawa structure of level  $s \geq 1$*  on a 2-group  $G$  is a normal abelian subgroup  $A$  and an element  $t$  such that  $G = \langle A, t \rangle$  and

$$tat^{-1} = a^{1+2^s} \text{ for every } a \in A.$$

Informally speaking, the higher the level is, the closer  $G$  is to an abelian group. In particular, if  $\exp(A) = 2^e$  and  $s \geq e$  then  $G$  is abelian. Conversely, any finite abelian 2-group  $G$  of exponent  $\leq 2^s$  admits an Iwasawa structure of level  $s$ , with  $A = G$  and  $t = \{1\}$ .

If a 2-group  $G$  admits an Iwasawa structure of level  $\geq 2$ , we will call  $G$  an Iwasawa group. Note that the level of an Iwasawa group  $G$  is not well-defined in general, since  $G$  may admit Iwasawa structures of different levels (see Example 4.2 below).

For any 2-group  $G$  we define the *strength* of  $G$  by

$$\text{str}(G) = \max \{m \mid G/G^{2^m} \text{ is abelian}\}.$$

In particular,  $\text{str}(G) = \infty$  iff  $G$  is abelian and  $\text{str}(G) \geq 2$  iff  $G$  is *powerful* in the sense of Lubotzky and Mann; cf. [13, Definition, p. 499].

**Lemma 4.1.** *Suppose that  $G$  is a finite 2-group which admits an Iwasawa structure  $(A, t)$  of level  $s$ . Then*

- (a)  $[G, G] = A^{2^s}$ ,
- (b)  $\text{str}(G) \geq s$ ,
- (c) If  $s \geq 2$  then  $G^{2^m} = \langle A^{2^m}, t^{2^m} \rangle$  for every  $m \in \mathbb{N}$ .

*Proof.* (a) From the definition of an Iwasawa structure of level  $s$ , we see that  $A^{2^s} \subset [G, G]$  and  $G/A^{2^s}$  is abelian. Hence,  $[G, G] = A^{2^s}$ .

(b) By part (a)  $G/A^{2^s}$  is commutative. Hence, so is  $G/G^{2^s}$ , and part (b) follows.

(c) By part (b),  $\text{str}(G) \geq 2$ . Thus  $[G, G] \subset G^4$ , i.e.,  $G$  is a powerful 2-group. The desired conclusion now follows from [4, Theorem 2.7].  $\square$

We remark that part (c) remains true even if  $s = 1$ . This stronger assertion will not be used in the sequel; we leave it as an exercise for the reader.

**Example 4.2.** The inequality  $\text{str}(G) \geq s$  may be strict, even if  $G$  is non-abelian. Indeed, let

$$G = \langle a, t \mid a^{2^5} = 1, a^{2^2} = t^{2^3}, tat^{-1} = a^{1+8} \rangle.$$

One checks readily that  $G$  is a metacyclic group of order  $2^8$  and that  $G$  admits an Iwasawa structure  $(A, t)$  of level 3, where  $A = \langle a \rangle$ . We claim that  $\text{str}(G) = 4$ . By Lemma 4.1,  $[G, G] = \langle a^8 \rangle$ . Since  $a^8 = t^{16}$ , we see that  $[G, G]$  is contained in  $G^{16}$  but not in  $G^{32} = \langle a^{32}, t^{32} \rangle = \langle a^{16} \rangle$ . Thus  $\text{str}(G) = 4$ , as claimed.

On the other hand, observe that  $G$  admits another Iwasawa structure  $(\tilde{A}, \tilde{t})$  of level 4, where  $\tilde{A} = \langle t \rangle$  and  $\tilde{t} = a^{-1}$ . Indeed, have  $\tilde{t}\tilde{t}\tilde{t}^{-1} =$

$a^{-1}ta = t^{1+2^4}$ . Thus we see that by switching the role of  $t$  and  $a^{-1}$ , we are able to find another Iwasawa structure whose level equals the strength of  $G$ . In the next lemma we shall show that such a switch is always possible.

**Lemma 4.3.** *Suppose  $G$  be a non-abelian Iwasawa 2-group. Then*

$$\text{str}(G) = \max\{\text{level}(A, t)\},$$

where the maximum is taken over all Iwasawa structures  $(A, t)$  on  $G$ .

*Proof.* Let  $m = \text{str}(G)$  and  $(A, t)$  is an Iwasawa structure on  $G$  of level  $s$ . By Lemma 4.1,  $s \leq m$ . If  $s = m$  we are done. Thus we may assume  $s < m$ . Our goal is to construct another Iwasawa structure on  $G$  of level  $m$ .

Since  $G$  is an Iwasawa 2-group,  $m \geq 2$ . Thus  $[G, G] \subset G^4$ , so that  $G$  is a powerful group. By Lemma 4.1,

$$A^{2^s} = [G, G] \subset G^{2^m} = \langle A^{2^m}, t^{2^m} \rangle.$$

We now see that the group  $G^{2^m}/A^{2^m}$  is cyclic, and hence, so is its subgroup  $A^{2^s}/A^{2^m}$ . Since  $s < m$  this implies that  $A^{2^s}$  is itself cyclic.

Let  $a^{2^s} = t^{2^m}$  be a generator of  $A^{2^s}$  with  $a \in A$ . Since the order of  $a$  is equal to the exponent of  $A$ , we see that there exists a subgroup  $B$  of  $A$  such that  $A = \langle a \rangle \oplus B$ . Moreover, since  $A^{2^s}/A^{2^m}$  is cyclic, we see that  $B^{2^s} = \{1\}$ . Therefore,  $tbt^{-1} = b^{1+2^s} = b$  for each  $b \in B$ , and  $B$  is a subgroup of the center  $Z(G)$  of  $G$ .

Set  $\tilde{A} = \langle t, B \rangle$  and  $\tilde{t} = a^{-1}$ . We claim that  $(\tilde{A}, \tilde{t})$  is an Iwasawa structure on  $G$  of level  $m$ . First we have

$$\langle \tilde{A}, \tilde{t} \rangle = \langle t, B, a^{-1} \rangle = \langle t, A \rangle = G.$$

Also  $\tilde{A}$  is an abelian subgroup of  $G$  as  $B \subset Z(G)$ . Further  $\tilde{t}\tilde{t}\tilde{t}^{-1} = a^{-1}ta = a^{-1}a^{1+2^s}t = a^{2^s}t = t^{1+2^m}$ , as  $a^{2^s} = t^{2^m}$ . Because  $\tilde{A} = \langle B, t \rangle$  and  $B \subset Z(G)$  we see that  $\tilde{t}\tilde{a}\tilde{t}^{-1} = \tilde{a}^{1+2^m}$  for each  $\tilde{a} \in \tilde{A}$ . Hence  $(\tilde{A}, \tilde{t})$  is the Iwasawa structure of level  $m$ .  $\square$

**Remark 4.4.** In view of Lemma 4.3, a 2-group  $S$  satisfies condition (d) of Theorem 1.3 if and only if it is an Iwasawa group of strength  $\geq m$ .

## 5. PROOF OF THEOREM 1.3 (A) $\implies$ (B) $\implies$ (C) $\implies$ (D)

(a)  $\implies$  (b): Let  $k$  be the subfield of  $F$  generated by the prime field and the primitive  $2^m$ th root of unity and let  $V$  be a faithful linear representation of  $G$  over  $k$  (e.g., we can take  $V$  to be the group algebra  $k[G]$ ). Denote the field of rational functions on  $V$  by  $k(V)$ . Since the trace form of the  $S$ -Galois extension  $E/F$  is not hyperbolic [9, Proposition 2.5] tells us that the trace form of  $k(V)/k(V)^S$  is not hyperbolic. Now by Lemma 3.1(b),  $k(V)/k(V)^G$  is not hyperbolic either. Thus we can take  $L = k(V)$  and  $K = k(V)^G$ .

(b)  $\implies$  (c): Let  $L/K$  be a  $G$ -Galois field extension with a non-hyperbolic trace form, as in (b). Assume, to the contrary, that  $T/T^{2^m}$  is non-abelian for some subgroup  $T$  of  $S$ . Then the trace form of  $L/L^T$  is still non-hyperbolic;



see [9, Lemma 2.1(c)]. Thus, replacing  $G$  by  $T$  and  $K$  by  $L^T$ , we may assume  $G = T$ .

Now let  $H = G^{2^m}$ . Then  $L^H/K$  is a Galois extension with Galois group  $G/H$ , which by our assumption, is non-abelian of exponent  $\leq 2^m$ . Thus, by Theorem 1.1,  $q_{L^H/K}$  is hyperbolic. Now, since  $H \subset G^2 = \text{Fr}(G)$ , Corollary 2.2 tells us that  $q_{L/K}$  is hyperbolic as well, contradicting our assumption.

(c)  $\implies$  (d): By our assumption every subgroup  $T$  of  $S$  satisfies  $[T, T] \subset T^4$ , i.e.,  $T$  is powerful. By [13, Theorem 4.3.1] this implies that  $S$  is modular but not Hamiltonian. On the other hand, by a theorem of Iwasawa [8] modular non-Hamiltonian 2-groups are precisely the 2-groups that admit an Iwasawa structure of level  $\geq 2$ .<sup>1</sup>

It remains to show that  $S$  admits an Iwasawa structure of level  $s \geq m$ . First suppose  $S$  is abelian. Then, as we pointed out in Section 4, we can take  $A = S$ ,  $t = 1$ , and  $s = \max\{m, e\}$ , where  $e$  is the exponent of  $S$ . Now assume  $S$  is not abelian. Then by our assumption (c),  $\text{str}(S) \geq m$ . The desired conclusion now follows from Lemma 4.3.  $\square$

**Remark 5.1.** *If  $\text{char}(K) = 0$  then the  $G$ -Galois extension  $L/K$  in part (b) can be chosen so that  $K$  does not contain a primitive root of unity of degree  $2^{m+1}$ .*

*Proof.* Let  $k = \mathbb{Q}(\zeta_{2^m})$  be the subfield of  $K$  generated by its prime subfield and a primitive  $2^m$ th root of unity. Let  $V = k^n$  be a faithful  $G$ -representation (over  $k$ ), as in the proof of the implication (a)  $\implies$  (b). Since the trace form of the  $S$ -Galois extension  $E/F$  is not hyperbolic, [9, Proposition 2.5] tells us that the trace form of  $k(V)/k(V)^S$  is not hyperbolic. Thus we can replace  $E$  by  $E' = k(V)$  and  $F$  by  $F' = k(V)^G$ . Since  $k$  is algebraically closed in  $E'$ ,  $E'$  (and hence,  $F'$ ) does not contain a primitive root of unity of degree  $2^{m+1}$ .  $\square$

The same argument goes through in characteristic  $p$ , provided that  $k = \mathbb{F}_p(\zeta_{2^m})$  does not contain  $\zeta_{2^{m+1}}$ .

**Remark 5.2.** *Condition (c) of Theorem 1.3 is equivalent to*

*(c')  $H/H^{2^m}$  is abelian for every subgroup  $H$  of  $G$ .*

*Proof.* Clearly, (c')  $\implies$  (c). To prove the converse, let  $T$  be a Sylow 2-subgroup of  $H$ . After replacing  $S$  by a conjugate Sylow subgroup in  $G$ , we may assume  $T \subset S$ . Let  $\bar{T}$  be the image of  $T$  in  $H/H^{2^m}$ . We claim that  $\bar{T} = H/H^{2^m}$ . Indeed, on the one hand, the exponent of  $H/H^{2^m}$  divides  $2^m$ , so that  $H/H^{2^m}$  is a 2-group. On the other hand, since  $T$  is a Sylow 2-subgroup of  $H$ , the index  $[H : T]$  is odd. The index of  $\bar{T}$  in  $H/H^{2^m}$  is thus both odd and a power of 2; hence,  $\bar{T} = H/H^{2^m}$ , as claimed.

<sup>1</sup>The proofs of Iwasawa's theorem in [8] and [20, Theorem 14] had some gaps that were later pointed out and closed by Napolitani [14]. For a detailed exposition of Iwasawa's theorem and related group-theoretic results, we refer the reader to [17].

Consequently,

$$T/T^{2^m} \xrightarrow{\text{onto}} T/(T \cap H^{2^m}) \simeq H/H^{2^m}.$$

If  $T/T^{2^m}$  is abelian, then so is  $H/H^{2^m}$ . This shows that (c)  $\implies$  (c').  $\square$

**Remark 5.3.** Let  $G$  be a finite group. If  $A$  and  $B$  are subgroups of  $G$ , we shall denote the set of intermediate subgroups  $A \subset X \subset B$  by  $[A, B]$ . This set is naturally a lattice, where  $X \wedge Y = X \cap Y$  and  $X \vee Y =$  subgroup generated by  $X$  and  $Y$ .

*Let  $S$  be a Sylow 2-subgroup of  $G$ . Suppose for some subgroups  $A$  and  $B$  of  $S$ , the map  $\varphi_{A,B} : [A, A \vee B] \longrightarrow [A \wedge B, B]$ , defined by  $\varphi_{A,B}(X) = A \wedge X$ , is not a lattice isomorphism. Then the trace form  $q_{L/K}$  is hyperbolic for every  $G$ -Galois extension  $L/K$  such that  $K$  contains a primitive 4th root of unity.*

*Proof.* If  $\varphi_{A,B}$  is not a lattice isomorphism for some  $A$  and  $B$  then the lattice  $[\{1\}, S]$  is not modular; see [17, Theorem 2.1.5]. Then, by Iwasawa's theorem (the easy direction),  $S$  does not satisfy condition (d) of Theorem 1.3. The desired conclusion follows from the implication (b)  $\implies$  (d).  $\square$

## 6. PROOF OF THEOREM 1.3 (D) $\implies$ (A): PRELIMINARY REDUCTIONS

We begin by observing that for the purpose of proving the implication (d)  $\implies$  (a), we may assume that  $G = S$  is a 2-group and that  $m = s$ . We shall say that  $S$  admits a non-hyperbolic trace form if it satisfies condition (a) of Theorem 1.3.

It is easy to see that every abelian 2-group admits a non-hyperbolic trace form; see, e.g., [9, Remark 3.2]. Thus we will assume from now on that  $S$  is non-abelian. Recall that by our assumption (d),  $S = \langle A, t \rangle$ , where  $A$  is abelian and

$$(6.1) \quad tat^{-1} = a^{1+2^s} \text{ for every } a \in A.$$

Our proof of the implication (d)  $\implies$  (a) of Theorem 1.3 will consist of two parts. In this section we will reduce the problem to the case where  $A = (\mathbb{Z}/2^e\mathbb{Z})^r$  and  $S$  is a semidirect product of  $A$  and  $\langle t \rangle$ ; in the next section we will show that every  $S$  of this form admits a non-hyperbolic trace form. (Note that here  $r$  is the Frattini rank of  $A$ ; the Frattini rank of  $S$  is  $r + 1$ .)

In order to facilitate working with Iwasawa groups, we will write them in terms of generators and relations. Decompose the abelian 2-group

$$A = \langle a_1 \rangle \times \cdots \times \langle a_r \rangle \simeq \mathbb{Z}/2^{e_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/2^{e_r}\mathbb{Z},$$

as a product of cyclic subgroups, where  $a_i$  has order  $2^{e_i}$ . Then  $\exp(A) = 2^e$ , where  $e = \max\{e_1, \dots, e_r\}$ . Since  $S$  is non-abelian,

$$(6.2) \quad s < e.$$

Denote the order of the image of  $t$  in  $G/A$  by  $2^q$  and let  $a_0 = t^{2^q} \in A$ . Note that the order of  $a_0$  in  $A$  is  $2^{-q}|\langle t \rangle|$  and, since  $a_0$  commutes with  $t$ ,  $a_0^{2^s} = 1$  in  $A$ .

**Lemma 6.1.** (a) *The group  $X = A * \langle t \rangle / \langle tat^{-1} = a^{1+2^s} \mid a \in A \rangle$  is isomorphic to  $A \rtimes \langle t \rangle$ , with the action of  $t$  on  $A$  given by (6.1). Here  $A * \langle t \rangle$  denotes the free product of the subgroups  $A$  and  $\langle t \rangle$  of  $G$ .*

(b) *Let  $c \in A$  be an element of order  $2^{-q}|\langle t \rangle|$ , satisfying  $c^{2^s} = 1$  and  $Y = A * \langle t \rangle / \langle t^{2^q} = c, tat^{-1} = a^{1+2^s} \mid a \in A \rangle$ . Then every element of  $Y$  can be uniquely written in the form  $at^i$  for some  $a \in A$  and  $0 \leq i < 2^q$ .*

(c)  *$S$  is isomorphic to  $Z = A * \langle t \rangle / \langle t^{2^q} = a_0, tat^{-1} = a^{1+2^s} \mid a \in A \rangle$ .*

*Proof.* (a) Consider the natural surjective homomorphism  $X \rightarrow A \rtimes \langle t \rangle$ , taking  $a$  to  $a$  and  $t$  to  $t$ . Since  $X$  has at most  $|A| \times |\langle t \rangle|$  elements (every element of  $X$  can be written in the form  $at^i$  for some  $a \in A$  and  $0 \leq i < |\langle t \rangle|$ ), this homomorphism is an isomorphism.

(b) The defining relations of  $Y$  tell us that every element of  $Y$  can be written as  $at^i$ , with  $a \in A$  and  $i \in \{0, 1, \dots, 2^q - 1\}$ . To prove uniqueness, it is enough to show that  $|Y| = 2^q \cdot |A|$ . Note that  $Y$  is the quotient of  $X = A \rtimes \langle t \rangle$  by the central cyclic subgroup  $C = \langle ct^{-2^q} \rangle$ . (This subgroup is central in  $X$  because  $c^{2^s} = 1$  in  $A$ .) Since  $c$  has order  $2^{-q}|\langle t \rangle|$  in  $A$  and  $t^{2^q}$  has order  $2^{-q}|\langle t \rangle|$  in  $\langle t \rangle$ , we have  $|C| = 2^{-q}|\langle t \rangle|$

$$|Y| = \frac{|X|}{|C|} = \frac{|A| \cdot |\langle t \rangle|}{|C|} = 2^q |A|,$$

as desired.

(c) Every element of  $S$  can be uniquely written in the form  $at^i$ , for some  $a \in A$  and  $0 \leq i < 2^q$ . Thus the natural surjective homomorphism  $Z \rightarrow S \simeq \langle A, t \rangle$  is an isomorphism.  $\square$

We are now ready to prove the main result of this section. We will continue to use the notations of Lemma 6.1.

**Reduction 6.2.** In the proof of the implication (d)  $\implies$  (a) of Theorem 1.3 we may assume without loss of generality that

- (1)  $e_1 = \dots = e_r$  and
- (2)  $S$  is a semidirect product of  $A$  and  $\langle t \rangle$ .

*Proof.* We will use the following two simple “moves” to go from an arbitrary Iwasawa group to one satisfying (1) and (2):

(i) If  $H$  is a subgroup of  $G$  and  $G$  admits a non-hyperbolic trace form then so does  $H$ .

(ii) Suppose  $T$  is a 2-group and  $N$  be a normal subgroup of  $T$  contained in  $T^2 = \text{Fr}(T)$ . If  $T$  admits a non-hyperbolic trace form then so does  $T/N$ .

(ii) is immediate from Corollary 2.2. To prove (i), note that if the trace form of a  $G$ -Galois extension  $L/K$  is not hyperbolic then neither is the trace form of  $L/L^H$ ; see, e.g., [9, Lemma 2.1(c)]

(1) Let  $e = \max\{e_1, \dots, e_r\}$  and embed  $A$  in the abelian group

$$B = \langle b_1 \rangle \times \dots \times \langle b_r \rangle \simeq \mathbb{Z}/2^e \times \dots \times \mathbb{Z}/2^e,$$

where each  $b_i$  has order  $2^e$  and  $a_i = b_i^{2^{e-e_i}}$  for all  $i = 1, 2, \dots, r$ . Let

$$S_1 = B * \langle t \rangle / \langle t^{2^q} = a_0, tbt^{-1} = b^{1+2^s} \mid b \in B \rangle.$$

Then there is a natural homomorphism  $S \simeq Z \longrightarrow S_1$ , which sends  $t$  to  $t$  and  $a$  to  $a$  for every  $a \in A \subset B$ . By Lemma 6.1(b), this homomorphism is injective. Thus by (i) we may replace  $S$  by  $S_1$ . This completes the proof of (1).

From now on, we will assume that  $e_1 = \dots = e_r = e$ .

(2) Let  $X$  and  $Z$  be as in Lemma 6.1. Consider the natural homomorphism  $f: X \longrightarrow Z \simeq S$  which sends  $t$  to  $t$  and  $a$  to  $a$  for every  $a \in A$ . By Lemma 6.1(a)  $X \simeq A \rtimes \langle t \rangle$ . It now suffices to show that  $\text{Ker}(f) \subset \text{Fr}(X) = X^2$ ; part (2) will then follow from (ii), with  $T = X$ . For notational convenience, we will denote the image  $t$  in  $S$  by  $\bar{t}$ .

Suppose  $a\bar{t}^i \in \text{Ker}(f)$  for some  $a \in A$  and  $0 \leq i < |\langle t \rangle|$ ; in other words,  $a\bar{t}^i = 1$  in  $S$ . Then, since the order of  $\bar{t}A$  in  $S/A$  is  $2^q$ , we conclude that  $i$  is a multiple of  $2^q$ . In particular, since  $S$  is not abelian, we have  $q \geq 1$  and thus  $t^i \in X^2$ . It remains to show that  $a \in X^2$ . Indeed, since  $a = \bar{t}^{-i}$  in  $S$ ,  $a$  and  $\bar{t}$  commute in  $S$ , i.e.,  $a^{2^s} = 1$  in  $S$ . Since we are assuming that  $A \simeq (\mathbb{Z}/2^e\mathbb{Z})^r$  and  $s < e$ , cf. part (1) and (6.2), we conclude that  $a \in A^2$  in  $A$ , and consequently  $a \in X^2$  in  $X$ , as claimed.  $\square$

## 7. CONCLUSION OF THE PROOF OF THEOREM 1.3 (D) $\implies$ (A)

In view of Reduction 6.2, it remains to prove the following

**Proposition 7.1.** *Let  $S = A \rtimes \langle t \rangle$ , where  $\langle t \rangle$  is a finite cyclic 2-group, acting on  $A = (\mathbb{Z}/2^e\mathbb{Z})^r$  by  $tat^{-1} = a^{1+2^s}$ , and  $2 \leq s < e$ . Then there exists a  $S$ -Galois extension  $E/F$  such that  $F$  contains a primitive root of unity  $\zeta_{2^s}$  of degree  $2^s$  and the trace form  $q_{E/F}$  is non-hyperbolic.*

Our proof of Proposition 7.1 below relies on valuation theory; our primary background references are [6], [15] and [22]. We shall denote the finite field of order  $q$  by  $\mathbb{F}_q$ .

**Lemma 7.2.** *For every integer  $s \geq 2$ , there exists a field  $F$  with a 2-henselian valuation  $v$  with value group  $\Gamma_v$ , and residue field  $\mathcal{K}$ , such that*

(i)  $\text{char } \mathcal{K} \neq 2$ ,

(ii)  $F$  contains a primitive root of unity  $\zeta_{2^s}$  of degree  $2^s$  but does not contain the primitive root of unity  $\zeta_{2^{s+1}}$  of degree  $2^{s+1}$ ,

(iii)  $\dim_{\mathbb{F}_2} \Gamma_v / 2\Gamma_v \geq r$ .

(iv)  $\mathcal{K}(2) = \mathcal{K}(\zeta_{2^\infty})$ , where  $\mathcal{K}(\zeta_{2^\infty})$  is the extension of  $\mathcal{K}$  obtained by adjoining all  $2^n$ th roots of unity to  $\mathcal{K}$ , for  $n = 1, 2, \dots$  and  $\mathcal{K}(2)$  is the maximal 2-extension of  $\mathcal{K}$  in some algebraic closure of  $\mathcal{K}$ .

Moreover, we can choose  $F$  so that  $\text{char}(F) = 0$ .

*Proof.* We shall give two constructions: a simple one in prime characteristic and a slightly more complicated one in characteristic zero.

Construction 1: Observe that  $5^{2^{s-2}} - 1$  is divisible by  $2^s$  but not by  $2^{s+1}$  for any integer  $s \geq 2$ ; see, e.g., [18, 5.3.17]. Therefore if  $q = 5^{2^{s-2}}$  then  $\zeta_{2^s} \in \mathbb{F}_q$  but  $\zeta_{2^{s+1}} \notin \mathbb{F}_q$ . Let  $F = \mathbb{F}_q((X_1))((X_2)) \dots ((X_r))$  be the field of the iterated power series in variables  $X_1, \dots, X_r$  over  $\mathbb{F}_q$  and  $v$  be the natural 2-henselian valuation  $v : F \longrightarrow \mathbb{Z} \times \dots \times \mathbb{Z}$  ( $r$ -times), where  $\mathbb{Z} \times \dots \times \mathbb{Z}$  is lexicographically ordered. One also has  $\mathcal{K}(v) = \mathbb{F}_q$ , so that properties (i)-(iv) hold.

Construction 2: Alternatively consider the field

$$F = \mathbb{Q}_p((x_1))((x_2)) \dots ((x_r))$$

of characteristic 0 and the natural 2-henselian valuation

$$v : F \longrightarrow \mathbb{Z} \times \dots \times \mathbb{Z} \text{ (} r \text{ times)}.$$

This valuation composed with the  $p$ -adic valuation on  $\mathbb{Q}_p$  (see e.g., [15, p. 63]) yields a new 2-henselian valuation  $v' : F \longrightarrow \mathbb{Z} \times \dots \times \mathbb{Z}$  ( $(r+1)$ -times) with a residue field  $\mathcal{K}(v') = \mathbb{F}_p$ . (The fact that  $v'$  is again 2-henselian follows from [15, Proposition 10, page 211]; see also [10, p. 4].) Thus  $v'$  satisfies conditions (i), (iii) and (iv).

It remains to show that we can choose the prime  $p$  so that condition (ii) holds. We claim that for each  $s \in \mathbb{N}$  there is a prime  $p$  such that  $\zeta_{2^s} \in \mathbb{Q}_p$  but  $\zeta_{2^{s+1}} \notin \mathbb{Q}_p$ . By Hensel's Lemma it is enough to show that for each  $s \in \mathbb{N}$  there exists a prime  $p$  such that  $p - 1$  is divisible by  $2^s$  but not by  $2^{s+1}$ . To construct  $p$ , note that by Dirichlet's theorem there exists  $n \in \mathbb{N}$  such that  $p = (1 + 2^s) + 2^{s+1}n$  is a prime number; this prime  $p$  has the desired properties.  $\square$

For the rest of this section, we shall assume that  $F$ ,  $v$ ,  $\Gamma_v$  and  $\mathcal{K}$  are as in Lemma 7.2,  $\mathbb{Z}_2$  is the additive group of 2-adic integers and furthermore,

- $F(2)$  is the maximal 2-extension of  $F$  in some algebraic closure,
- $G_F(2) := \text{Gal}(F(2)/F)$  is the Galois group of  $F(2)/F$ ,
- $T_v \simeq \mathbb{Z}_2 \times \dots \times \mathbb{Z}_2$  ( $d$ -times), where  $d = \dim_{F_2} \Gamma_v/2\Gamma_v$ . Here  $T_v$  denotes the inertia subgroup of  $G_F(2)$  associated with  $v$ ,
- $w$  is the unique valuation of  $F(2)$  which extends  $v$  on  $F$ .

By a result of Engler and Koenigsmann [6, Proposition 1.1b],

$$G_F(2) \simeq (T_v \times G_{\mathcal{K}(\zeta_{2^\infty})}(2)) \rtimes \mathbb{Z}_2,$$

where  $\mathbb{Z}_2 = \langle \sigma \rangle$  and the action of  $\sigma$  on  $T_v$  is  $\sigma^{-1}\tau\sigma = \tau^{2^s+1}$  for every  $\tau \in T_v$ .

It is also worthwhile to recall that  $T_v/T_v^2$  is the Pontrjagin dual of  $\Gamma_v/2\Gamma_v$ , and this duality is induced by the Kummer pairing

$$\langle \cdot, \cdot \rangle : T_v/T_v^2 \times \Gamma_v/2\Gamma_v \longrightarrow \{\pm 1\},$$

where  $\langle [\theta], [f] \rangle = \theta(\sqrt{f})/\sqrt{f}$  for each  $\theta \in T_v$  and  $f \in F^*$ . Here  $[\theta] \in T_v/T_v^2$  and  $[f] \in \Gamma_v/2\Gamma_v$  denote the images in  $\theta$  and  $f$  in the factor groups  $T_v/T_v^2$  and  $\Gamma_v/2\Gamma_v$ , respectively.

We are now ready to finish the proof of Proposition 7.1. Suppose  $G_{\mathcal{K}(\zeta_{2^\infty})}(2) = \{1\}$ , i.e.,  $\mathcal{K}(2) = \mathcal{K}(\zeta_{2^\infty})$ . Then we have

$$G_F(2) \simeq T_v \rtimes \mathbb{Z}_2.$$

Since  $d = \dim_{F_2} \Gamma_v/2\Gamma_v \geq r$  we deduce that

$$T_v = \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{r \text{ times}} \times S$$

for some suitable subgroup  $S$  of  $T_v$ . Therefore there exists a surjective homomorphism  $\tilde{\varphi} : T_v \rightarrow A$  which projects the first factor on  $A$  and is trivial on  $S$ . Because the action of  $\sigma$  on  $T_v$  is given by  $\sigma^{-1}\tau\sigma = \tau^{1+2^s}$  for each  $\tau \in T_v$ , we see that  $\tilde{\varphi}$  extends uniquely to a surjective homomorphism

$$\varphi : G_F(2) \rightarrow S \text{ such that } \varphi(\sigma) = t^{-1}.$$

Let  $R$  be the kernel of  $\varphi$  and  $E$  the fixed field of  $R$ . Then  $E/F$  is Galois and  $\text{Gal}(E/F) \simeq S$ . From the fact that  $T_v \simeq \text{Hom}(\Gamma_w/\Gamma_v, \zeta_{2^\infty})$  (see [6, page 2474]) and the fact that the outer factor  $\mathbb{Z}_2$  in the semidirect decomposition of  $G_F(2)$  as  $T_v \rtimes \mathbb{Z}_2$  is  $\text{Gal}(F(\zeta_{2^\infty})/F)$ , we see that the maximal Galois subextension  $E'/F$  of  $E/F$  with a Galois group of exponent 2 has the form

$$E' = F(\sqrt{a_1}, \dots, \sqrt{a_r}, \zeta_{2^{s+1}}),$$

where  $a_1, a_2, \dots, a_r \in F^*$  such that their values  $v(a_1), \dots, v(a_r) \in \Gamma_v$  are linearly independent in  $\Gamma_v/2\Gamma_v$  over  $\mathbb{F}_2$ .

From [22, Proposition 4.7] we see that the Pfister form

$$\ll a_1, \dots, a_r, \zeta_{2^s} \gg$$

is non-hyperbolic. By Corollary 2.2 the trace form of  $E/F$  is Witt equivalent to a scalar multiple of  $\ll a_1, \dots, a_r, \zeta_{2^s} \gg$ , which is also non-hyperbolic. This completes the proof of Proposition 7.1 and thus of Theorem 1.3.  $\square$

**Remark 7.3.** Our proof shows that if the equivalent conditions (a) - (d) of Theorem 1.3 hold then the fields  $F$  and  $K$  in parts (a) and (b) can be chosen to be of characteristic zero.

## 8. APPLICATIONS

### Trace forms over “small” fields.

**Proposition 8.1.** *Let  $G$  be a finite group,  $S$  be a Sylow 2-subgroup of  $G$ ,  $K$  be a field containing a primitive 4th root of unity and  $L/K$  be a  $G$ -Galois extension. Denote the Frattini rank of  $S$  by  $r$ .*

- (a) *If  $K$  is a  $C_{r-1}$ -field then the trace form  $q_{L/K}$  is hyperbolic.*
- (b) *If  $\text{cd}_2(K) \leq r - 1$  then the trace form  $q_{L/K}$  is hyperbolic.*
- (c) *If  $K$  is a number field and  $r \geq 3$  (i.e.,  $S$  cannot be generated by two elements) then the trace form  $q_{L/K}$  is hyperbolic.*

Here  $\text{cd}_2(K)$  refers to the 2-cohomological dimension of  $K$ . For the definition of cohomological dimension and of the  $C_i$  property for fields, see [19, II.4].

*Proof.* By Theorem 1.2 it is enough to show that under the assumptions of the corollary every  $r$ -fold Pfister form  $q$  over  $K$  is hyperbolic.

In part (a)  $q$  is necessarily isotropic and, hence, hyperbolic; see, e.g., [11, Corollary 10.1.6]. In part (b), by Milnor's conjecture (recently proved by Voevodsky [21])  $q$  lies in  $I^{r+1}$ , where  $I$  is the fundamental ideal in the Witt ring  $W(K)$  and by the Arason-Pfister theorem this is only possible if  $q$  is hyperbolic; see [11, Corollary 10.3.4].

Part (c) is a special case of (b), since a totally imaginary number field has cohomological dimension 2; see [19, II.4.4]. However, a much more elementary argument, based on the Hasse-Minkowski principle, is available in this case. Indeed, every quadratic form of dimension  $\geq 5$  over  $K$  is isotropic; see [11, Corollary 3.5, p. 169]. In particular, for  $r \geq 3$ , every  $r$ -fold Pfister form is isotropic and hence hyperbolic over  $K$ .  $\square$

### Simple groups.

**Proposition 8.2.** *Let  $G$  be a finite simple group and let  $S$  be the Sylow 2-subgroup of  $G$ . Then the following are equivalent.*

- (a)  $S$  is abelian, and
- (b) *There exists a  $G$ -Galois field extension  $L/K$  such that  $K$  contains a primitive 4th root of unity and the trace form  $q_{L/K}$  is not hyperbolic.*

*Proof.* By Theorem 1.3 it is sufficient to prove that  $S$  cannot be a non-abelian Iwasawa group. Equivalently (via Iwasawa's theorem [8])  $S$  cannot be a non-abelian modular non-Hamiltonian 2-group. The last assertion is an immediate consequence of [24, Proposition 4.2]. (It can also be deduced from [17, page 197, Exercise 1].)  $\square$

For the sake of completeness we remark if a finite simple group  $G$  has an abelian 2-Sylow subgroup  $S$  then  $S$  is necessarily elementary abelian (see [7, Theorem 4.2.3]); moreover, Walter [23] classified all finite simple groups  $G$  with this property.

**The extension problem.** Let  $G$  be a finite group and  $N$  be a normal subgroup of  $G$  and  $K \subset L$  be a  $G/N$ -Galois field extension. Recall that the *extension problem* for this data is the question of existence of a tower  $K \subset L \subset M$ , such that  $M/K$  is a  $G$ -Galois field extension, and the natural quotient map  $\text{Gal}(M/K) \longrightarrow \text{Gal}(L/K)$  coincides with  $G \longrightarrow G/N$ .

Now assume that  $G$  is a nonabelian 2-group of Frattini rank  $r$ ,  $N = \text{Fr}(G) = G^2$ , and  $L = K(\sqrt{a_1}, \dots, \sqrt{a_r})$  is a multiquadratic extension of  $K$  of degree  $2^r$  such that  $\text{Gal}(L/K) \cong G/\text{Fr}(G) = (\mathbb{Z}/2\mathbb{Z})^r$ . Assume also that  $K$  contains a primitive  $e$ th root of unity, where

$$e = \min\{\exp(H) \mid H \text{ is a non-abelian subgroup of } G\}.$$

**Proposition 8.3.** *If the extension problem for  $G$ ,  $N$ , and  $L/K$  defined above has a solution, then the  $r$ -fold Pfister form  $\ll a_1, \dots, a_r \gg$  is a hyperbolic over  $K$ .*

*Proof.* Suppose  $L/K$  is the required  $G$ -Galois field extension. Then from Theorem 1.1 we see that the trace form  $q_{L/K}$  is hyperbolic. But from Corollary 2.2(a) we see that  $q_{L/K}$  is Witt equivalent to a scalar multiple of  $\ll a_1, \dots, a_r \gg$ . Hence  $\ll a_1, \dots, a_r \gg$  is hyperbolic as required.  $\square$

## 9. WHICH QUADRATIC FORMS ARE TRACE FORMS?

We now return to the question we posed at the beginning of the Introduction. Let  $G$  be a finite group and  $K$  be a field containing  $\sqrt{-1}$ . Which quadratic forms  $q$  over  $K$  can occur as trace forms of  $G$ -Galois field extension  $L/K$ ? In view of Theorem 1.3 we may assume that the Sylow 2-subgroup  $S$  of  $G$  is an Iwasawa 2-group; otherwise every trace form will be hyperbolic. By Theorem 1.2

$$q \simeq |S| \otimes (r\text{-fold Pfister form})$$

but, in general, we do not know which  $r$ -fold Pfister forms can occur, even if  $G = S$  is a 2-group. In this section we will describe the trace forms for one particular family of groups.

Recall that the modular group  $M(2^n)$  of order  $2^n$  is defined as

$$M(2^n) = \langle \sigma, \tau \mid \sigma^{2^{n-1}} = 1 = \tau^2, \tau\sigma\tau = \sigma^{1+2^{n-2}} \rangle.$$

In the sequel

$$(9.1) \quad \text{we will always assume that } n \geq 4.$$

It is easy to see that  $M(2^n)$  is an Iwasawa group of order  $2^n$ , exponent  $2^{n-1}$  and strength  $n - 2$ . Setting  $A = \langle \sigma \rangle$ , we see that  $(A, \tau)$  is an Iwasawa structure on  $M(2^n)$  of level  $n - 2$ . Note also that the Frattini subgroup of  $M(2^n)$  is  $\text{Fr}(M(2^n)) = \langle \sigma^2 \rangle$ .

For future reference we record the following elementary observation. As usual, we shall denote the class of  $a \in K^*$  in  $K^*/(K^*)^2$  by  $[a]$ .

**Remark 9.1.** Let  $K$  be a field containing a primitive 4th root of unity  $\zeta_4$ . Then  $2\zeta_4 = (1 + \zeta_4)^2$  and thus

$$(9.2) \quad [2] = [\zeta_4] \text{ in } K^*/(K^*)^2.$$

In particular,

- (i) if  $K$  contains a primitive 8th root of unity then 2 is a square in  $K$  and
- (ii) if  $K$  contains a primitive root of unity  $\zeta_{2^{n-2}}$  then  $2^n$  is a square in  $K$ .

Indeed, (i) is immediate from (9.2). To prove (ii), consider two cases:  $n = 4$  and  $n \geq 5$ ; see (9.1). If  $n = 4$  then  $2^4 = 4^2$  is certainly a square. For  $n \geq 5$  (cf. (9.1)), (ii) follows from (i).

We now proceed with the main result of this section. As usual,  $\zeta_i$  will denote a primitive  $i$ th root of unity.



**Proposition 9.2.** *Let  $n \geq 4$  be an integer,  $K$  be a field such that  $\zeta_{2^{n-2}} \in K$  but  $\zeta_{2^{n-1}} \notin K$  and  $q$  be a non-degenerate  $2^n$ -dimensional quadratic form over  $K$ . Then the following are equivalent:*

(a)  *$q$  is Witt equivalent to the trace form of some  $M(2^n)$ -Galois field extension  $L/K$ .*

(b)  *$q$  is Witt equivalent to  $\ll \zeta_{2^{n-2}}, a \gg$  for some  $a \in K^*$ , where  $[a] \neq [1]$ ,  $[\zeta_{2^{n-2}}]$  in  $K^*/(K^*)^2$ .*

Our assumption that  $\zeta_{2^{n-1}} \notin K$  is harmless, since otherwise Theorem 1.1 tells us that the trace form of every  $M(2^n)$ -Galois extension is hyperbolic. On the other hand, the assumption that  $\zeta_{2^{n-2}} \in K$  is essential.

*Proof.* Set  $K' = K(\zeta_{2^{n-1}})$ , where  $\zeta_{2^{n-1}}$  is a primitive root of unity of degree  $2^{n-1}$ . By our assumption on  $K$ ,  $[K' : K] = 2$ .

(b)  $\implies$  (a): Suppose  $q \simeq \ll \zeta_{2^{n-2}}, a \gg$ , where  $a \neq [1]$ ,  $[\zeta_{2^{n-2}}]$  in  $K^*/(K^*)^2$ . We will construct an  $M(2^n)$ -Galois extension  $L/K$  whose trace form is Witt equivalent to  $q$  by modifying [9, Example 6.1], due to Serre.

Let  $L = K'(\sqrt[2^{n-1}]{a})$ . By our assumption on  $[a]$ ,  $a$  is not a square in  $K'$ . Thus  $[L : K'] = 2^{n-1}$  (see, e.g., [12, Theorem VIII.9.16]) and consequently,  $[L : K] = 2^n$ . Now the computations in [9, Example 6.1] show that  $L/K$  is an  $M(2^n)$ -Galois extension whose trace form  $q_{L/K}$  is Witt equivalent to  $\langle 2^n \rangle \otimes \ll \zeta_{2^{n-2}}, a \gg$ . Finally by Remark 9.1(ii),  $2^n$  is a square in  $K$  and thus the factor of  $\langle 2^n \rangle$  can be removed. In other words,  $q$  is Witt equivalent to  $\ll \zeta_{2^{n-2}}, a \gg$ , as claimed.

(a)  $\implies$  (b): Assume that  $q = q_{L/K}$  for some  $M(2^n)$ -Galois extension  $L/K$ . Then  $q \otimes_K K'$  is the trace form of the  $M(2^n)$ -Galois  $K'$ -algebra  $L \otimes_K K'$ . By Theorem 1.1, we know that  $q \otimes_K K'$  is hyperbolic. (Recall that Theorem 1.1 applies to Galois algebras as well as field extensions; see the first remark after the statement of Theorem 1.2 in Section 1.) On the other hand, combining Theorem 1.2 and Remark 9.1, we see that  $q$  is Witt equivalent to a 2-fold Pfister form. The basic theory of Pfister forms (see, e.g., [1, p. 465]) now tells us that  $q$  is Witt equivalent to  $\ll \zeta_{2^{n-2}}, a \gg$  for some  $a \in K^*$ .

It remains to show that  $a$  can always be chosen so that  $[a] \neq [1]$ ,  $[\zeta_{2^{n-2}}]$  in  $K^*/(K^*)^2$ . Note that if  $[a] = [1]$  or  $[\zeta_{2^{n-2}}]$  then  $\ll \zeta_{2^{n-2}}, a \gg$  is a hyperbolic trace form. Thus in order to finish the proof of the proposition, it suffices to establish assertions (i) and (ii) below. Recall that a field  $K$  containing a primitive 4th root of unity  $\zeta_4$  is called *rigid* if and only if for every  $k \notin (K^*)^2$ , the form  $\langle 1, k \rangle$  represents only the classes  $[1]$  and  $[k]$  in  $K^*/(K^*)^2$ ; cf. [25, Section 3].

(i) If  $K$  is rigid then no  $M(2^n)$ -Galois field extension  $L/K$  has a hyperbolic trace form.

(ii) If  $K$  is not rigid then  $\ll \zeta_{2^{n-2}}, b \gg$  is hyperbolic for some  $b \in K^*$  such that  $[b] \neq [1]$ ,  $[\zeta_{2^{n-2}}]$  in  $K^*/(K^*)^2$ .

In other words, if  $K$  is rigid then the case where  $[a] = [1]$  or  $[\zeta_{2^{n-2}}]$  can never occur. If  $K$  is not rigid then, after possibly replacing  $a$  by  $b$ , we can always assume that  $[a] \neq [1]$ ,  $[\zeta_{2^{n-2}}]$  in  $K^*/(K^*)^2$ .

To prove (i), note that if  $L/K$  is an  $M(2^n)$ -Galois extension then  $L^{\text{Fr}(M(2^n))}$  is a  $\mathbb{Z}/2 \times \mathbb{Z}/2$ -Galois extension of  $K$ . Hence,  $L^{\text{Fr}(M(2^n))}$  has the form  $K(\sqrt{a}, \sqrt{b})$  for some  $a, b \in K^*$ , where  $a$  and  $b$  are  $\mathbb{F}_2$ -linearly independent in  $K^*/(K^*)^2$ . By Corollary 2.2(a),

$$q \simeq \langle |\text{Fr}(M(2^n))| \rangle \otimes q_{K(\sqrt{a}, \sqrt{b})/K}.$$

Here  $|\text{Fr}(M(2^n))| = 2^{n-2}$  because  $\text{Fr}(M(2^n))$  is the cyclic subgroup of  $M(2^n)$  generated by  $\sigma^2$ . Combining this with formula (3.2) for  $q_{K(\sqrt{a}, \sqrt{b})/K}$ , we obtain

$$q \simeq \langle 2^n \rangle \otimes \ll a, b \gg \simeq \ll a, b \gg,$$

where the factor of  $\langle 2^n \rangle$  can be removed in view of Remark 9.1(ii). Over a rigid field such a form cannot be isotropic, since otherwise  $\langle 1, a \rangle$  would take on the same value as  $\langle b \rangle \otimes \langle 1, a \rangle$ , thus making  $[a]$  and  $[b]$  linearly dependent over  $\mathbb{F}_2$ . This proves (i).

To prove (ii), we appeal to [25, Theorem 2.16(2)], which tells us that over a non-rigid field  $K$  the form  $\langle 1, \zeta_{2^{n-2}} \rangle$  assumes a value  $b$  such that  $[b] \neq [1]$ ,  $[\zeta_{2^{n-2}}]$  in  $K^*/(K^*)^2$ . Then  $\ll \zeta_{2^{n-2}}, b \gg$  is hyperbolic, as claimed.  $\square$

**Remark 9.3.** Suppose  $n = 4$ . Then by Remark 9.1, we can replace the form  $\ll \zeta_4, a \gg$  in the statement of Proposition 9.2 by  $\ll 2, a \gg$ . This way we recover [5, Corollary 6(b)] for  $G = M(16)$ .

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