

# Birational isomorphisms between twisted group actions

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**Abstract.** Let  $X$  be an algebraic variety with a generically free action of a connected algebraic group  $G$ . Given an automorphism  $\phi: G \rightarrow G$ , we will denote by  $X^\phi$  the same variety  $X$  with the  $G$ -action given by  $g: x \rightarrow \phi(g) \cdot x$ .

We construct examples of  $G$ -varieties  $X$  such that  $X$  and  $X^\phi$  are not  $G$ -equivariantly isomorphic. The problem of whether or not such examples can exist in the case where  $X$  is a vector space with a generically free linear action, remains open. On the other hand, we prove that  $X$  and  $X^\phi$  are always stably birationally isomorphic, i.e.,  $X \times \mathbb{A}^m$  and  $X^\phi \times \mathbb{A}^m$  are  $G$ -equivariantly birationally isomorphic for a suitable  $m \geq 0$ .

## 1. Introduction

Throughout this note all algebraic varieties, algebraic groups, group actions and maps between them will be defined over a fixed base field  $k$ . By a  $G$ -variety  $X$  we shall mean an algebraic variety with a (regular) action of a linear algebraic group  $G$ . A morphism (respectively, rational map, birational isomorphism) of  $G$ -varieties is a  $G$ -equivariant morphism (respectively, rational map, birational isomorphism).

Given an automorphism  $\phi$  of  $G$ , we can “twist” a group action  $\alpha: G \times X \rightarrow X$  by  $\phi$  to obtain a new  $G$ -action  $\alpha^\phi$  on  $X$  as follows:

$$\alpha^\phi: G \times X \xrightarrow{(\phi, id)} G \times X \xrightarrow{\alpha} X.$$

Note that the new action has the same orbits as the old one. If  $X$  is a  $G$ -variety (via  $\alpha$ ) then we will denote the “twisted”  $G$ -variety (i.e.,  $X$  with the  $G$ -action via  $\alpha^\phi$ ) by  $X^\phi$ .

Now suppose that the action  $\alpha$  of  $G$  on  $X$  is generically free, i.e., that there exists a  $G$ -invariant open dense subset  $U$  of  $X$  such that the stabilizer of every geometric point of  $U$  is trivial. In this paper we will address the following question:

**Problem 1.1.** Are  $X$  and  $X^\phi$  birationally isomorphic as  $G$ -varieties over  $k$ ?

<sup>1</sup>Z. Reichstein was partially supported by an NSERC research grant.

<sup>2</sup>A. Vistoli was partially supported by the University of Bologna, funds for selected research topics.

*Mathematics Subject Classification.* 14L30, 14E07, 16K20.

*Key words and phrases.* Group action, algebraic group, no-name lemma, birational isomorphism, central simple algebra, Galois cohomology.

Katsylo [6] conjectured that when  $k = \mathbb{C}$  generically free linear  $G$ -representations  $V$  and  $W$  are ( $G$ -equivariantly) birationally isomorphic if and only if  $\dim(V) = \dim(W)$ . This conjecture is known to be false for some finite groups  $G$ ; in particular, there are counterexamples where  $W = V^\phi$  for an automorphism  $\phi$  of  $G$ , see [16, Section 7]. V. L. Popov has suggested that counterexamples of this form should also exist in the case where  $G$  is connected. (In this case Katsylo's conjecture is still open.) The present paper was largely motivated by this suggestion.

Before proceeding to state our main results, we will make two simple observations. First, the answer to Problem 1.1 depends only on the class of  $\phi$  in the group of outer automorphisms of  $G$ . Indeed, suppose  $\phi' = \phi \circ \text{inn}_h$ , where  $\text{inn}_h: G \rightarrow G$  is conjugation by  $h \in G$ , i.e.,  $\phi(g) = hgh^{-1}$ . Then  $X^\phi$  and  $X^{\phi'}$  are isomorphic via  $x \mapsto h \cdot x$ . In particular,  $X$  and  $X^\phi$  are always isomorphic if  $G$  has no outer automorphisms, e.g., if  $G$  is the full symmetric group  $S_n$  ( $n \neq 6$ ) or if  $G$  is a semisimple algebraic group whose Dynkin diagram has no non-trivial automorphisms; cf. [10, Theorem 25.16]. The latter class of groups includes every split (almost) simple algebraic group, other than those of type  $A_n$ ,  $D_n$  and  $E_6$ ; cf. [10, Sections 24A and 25B].

Secondly, the  $G$ -invariant rational functions for  $X$  and  $X^\phi$  are exactly the same, i.e.,

$$k(X)^G = k(X^\phi)^G \subset k(X). \quad (1)$$

Recall that the inclusion  $k(X) \subset k(X)^G$  induces a dominant rational map  $\pi: X \dashrightarrow X/G$ , which is called *the rational quotient map*. The  $k$ -variety  $X/G$  is defined (up to birational isomorphism) by the condition  $k(X/G) = k(X)^G$ ; cf. [13, Section 2.4]. Thus (1) can be rephrased by saying that a rational quotient map  $\pi$  for  $X$  is also a rational quotient map for  $X^\phi$ .

The main results of this note are Theorems 1.2 and 1.3 below.

**Theorem 1.2.** *Let  $X$  be a generically free  $G$ -variety and  $\phi: G \rightarrow G$  be an automorphism of  $G$ . Then the  $G$ -varieties  $X$  and  $X^\phi$  are stably birationally isomorphic. More precisely there exists an integer  $m \geq 0$  and a  $G$ -equivariant birational isomorphism*

$$f: X \times \mathbb{A}^m \dashrightarrow X^\phi \times \mathbb{A}^m$$

*such that the diagram*

$$\begin{array}{ccc} X \times \mathbb{A}^m & \xrightarrow{f} & X^\phi \times \mathbb{A}^m \\ \downarrow & & \downarrow \\ X/G & \xlongequal{\quad} & X^\phi/G, \end{array}$$

*commutes.*

Here  $\mathbb{A}^m$  is the  $m$ -dimensional affine space with trivial  $G$ -action, and the vertical map on the left is the composition of the projection  $X \times \mathbb{A}^m \rightarrow X$  with the rational quotient map  $X \dashrightarrow X/G$  (similarly for the vertical map on the right).

**Theorem 1.3.** *Let  $n \geq 3$  and let  $\phi$  be the (outer) automorphism of  $G = \text{PGL}_n$  given by  $g \mapsto (g^{-1})^{\text{transpose}}$ . Assume the base field  $k$  contains a primitive  $n$ th root of unity. Then there exists a generically free  $\text{PGL}_n$ -variety  $X$  such that  $X$  and  $X^\phi$  are not birationally isomorphic over  $k$ .*

This gives a negative answer to Problem 1.1. The question of whether or not similar examples can exist in the case where  $X$  is a vector space with a generically free linear action of a connected linear algebraic group  $G$ , remains open.

## 2. The no-name lemma

Recall that a  $G$ -bundle  $\pi: E \rightarrow X$  is an algebraic vector bundle with a  $G$ -action on  $E$  and  $X$  such that  $\pi$  is  $G$ -equivariant and the action of every  $g \in G$  restricts to a linear map  $\pi^{-1}(x) \rightarrow \pi^{-1}(gx)$  for every  $x \in X$ .

Our proof of Theorem 1.2 in the next section will heavily rely the following result.

**Lemma 2.1.** (No-name Lemma) *Let  $\pi: E \rightarrow X$  be a  $G$ -bundle of rank  $r$ . Assume that the  $G$ -action on  $X$  is generically free. Then there exists a birational isomorphism  $\pi: E \xrightarrow{\sim} X \times \mathbb{A}^r$  of  $G$ -varieties such that the following diagram commutes*

$$\begin{array}{ccc} E & \xrightarrow{\phi} & X \times \mathbb{A}^r \\ \downarrow \pi & \swarrow \text{pr}_1 & \\ X & & \end{array} \quad (2)$$

Here  $G$  is assumed to act trivially on  $\mathbb{A}^r$ , and  $\text{pr}_1$  denotes the projection to the first factor.

The term “no-name lemma” is due to Dolgachev [4]. In the case where  $G$  is a finite group, it is a variant of Hilbert’s Theorem 90 classically known as Speiser’s Lemma [21]; for a modern treatment, see [5, Proposition 1.1], [9, Proposition 1.3] or [20, Appendix 3]. In the case where the base field  $k$  is algebraically closed,  $\text{char}(k) = 0$ , and  $G$  is an arbitrary linear algebraic group, to the best of our knowledge the no-name lemma first appeared in [1], [6]. For a detailed proof in this setting, see [2, Section 4].

In the sequel we would like to use Lemma 2.1 in the case where  $k$  is not necessarily algebraically closed. With this in mind, we will prove a more general variant of this result (Proposition 2.3 below). For the rest of this section we will work over an arbitrary base field  $k$ .

**Remark 2.2.** Suppose  $G$  is a group scheme of finite type over  $k$ ,  $X$  is an arbitrary quasi-separated scheme (or algebraic space) over  $k$  on which  $G$  acts (quasi-separated means that the diagonal embedding  $X \hookrightarrow X \times_{\text{Spec } k} X$  is quasi-compact; this is automatically satisfied when  $X$  is of finite type over  $k$ ). Assume further that the  $G$ -action on  $X$  is *free*, i.e., that the stabilizers of all geometric points of  $X$  are trivial (as group schemes). Equivalently, the morphism  $G \times_{\text{Spec } k} X \rightarrow X \times_{\text{Spec } k} X$  defined in functorial terms by  $(g, x) \mapsto (gx, x)$  is categorically injective (i.e., it is injective on geometric points and unramified). Then, by a result of Artin [8, Corollaire 10.4], the quotient sheaf  $X/G$  in the fppf topology is an algebraic space, and  $G \times_{\text{Spec } k} X = X \times_{X/G} X$ . There is a Zariski open dense subspace  $V \subseteq X/G$  that is a scheme ([7, Proposition 6.7]); if  $U$  is the inverse image of  $V$  in  $X$ , then the restriction  $U \rightarrow U/G = V$  is a  $G$ -torsor (i.e. a principal  $G$ -bundle in the fppf topology, cf. [3]). In the case where  $X$  is a  $k$ -variety, this is a the rational quotient map, i.e.,  $k(U/G) = k(X)^G$ ; cf. [13, Lemma 2.1].

**Proposition 2.3.** *Assume that  $G$  is a group scheme of finite type over  $k$ , acting on a quasi-separated  $k$ -scheme  $X$ , with a non-empty invariant open subscheme on which the action is free. Let  $\mathcal{E}$  be a  $G$ -equivariant locally free sheaf of rank  $r$  on  $X$ . Then there exists a non-empty open  $G$ -invariant subscheme  $U$  of  $X$ , such that the restriction  $\mathcal{E}|_U$  is isomorphic to the trivial  $G$ -equivariant sheaf  $\mathcal{O}_U^r$ .*

To see that Lemma 2.1 (over an arbitrary base field  $k$ ) follows from Proposition 2.3, recall the well-known equivalence between the category of  $G$ -equivariant vector bundles on  $X$  and the category of  $G$ -equivariant locally free sheaves on  $X$ . One passes from a  $G$ -bundle  $V \rightarrow X$  to the  $G$ -equivariant locally free sheaf of sections of  $V$ ; conversely, to each  $G$ -equivariant locally free sheaf  $\mathcal{E}$  on  $X$  one associates the spectrum of the sheaf of symmetric algebras of the dual  $\mathcal{E}^\vee$  over  $X$ .

Note also that in the course of proving Lemma 2.1, we may assume without loss of generality that  $X$  is *primitive*, i.e.,  $G$  transitively permutes the irreducible components of  $X$  (equivalently,  $k(X)^G$  is a field). Indeed, an arbitrary  $G$ -variety  $X$  is easily seen to be birationally isomorphic to a disjoint union of primitive  $G$ -varieties  $X_1, \dots, X_r$ , and it suffices to prove Lemma 2.1 for each  $X_i$ . On the other hand, if  $X$  is primitive, then every non-empty  $G$ -invariant open subset is dense in  $X$ . This shows that Lemma 2.1 follows from Proposition 2.3, as claimed.

*Proof of Proposition 2.3.* After replacing  $X$  by a non-empty open subscheme we may assume that the action of  $G$  on  $X$  is free. By passing to a dense invariant subscheme of  $X$ , we may assume that  $X/G$  is a scheme, and  $X \rightarrow X/G$  is a  $G$ -torsor; see Remark 2.2. By descent theory, the  $G$ -equivariant sheaf  $\mathcal{E}$  comes from a locally free sheaf  $\mathcal{F}$  on  $X/G$ ; see, for example, [22, Theorem 4.46]. By restricting to a non-empty open subscheme of  $X/G$  once again, we may assume that  $\mathcal{F}$  is isomorphic to  $\mathcal{O}_{X/G}^r$ . Then  $\mathcal{E}$  is  $G$ -equivariantly isomorphic to  $\mathcal{O}_X^r$ , as claimed. ■

**Remark 2.4.** The same argument goes through if the base field  $k$  (or, equivalently, the base scheme  $\text{Spec}(k)$ ) is replaced by an algebraic space  $B$ , so that  $X$  is defined over  $B$ , and the group scheme  $G$  is assumed to be flat and finitely presented over  $B$ .

### 3. Proof of Theorem 1.2

We will prove Theorem 1.2 in two steps: first in the case where  $X = V$  is a generically free linear representation of  $G$ , then for arbitrary  $X$ .

**Step 1:** Suppose  $X = V$  is a generically free linear representation of  $G$ .

Let  $m = \dim(V)$ . By the no-name lemma, there exist  $G$ -equivariant birational isomorphisms  $\alpha$  and  $\beta$  such that the diagram

$$\begin{array}{ccc}
 V \times \mathbb{A}^m \xrightarrow{\alpha} V \times V^\phi & & V \times V^\phi \xrightarrow{\beta} V^\phi \times \mathbb{A}^m \\
 \downarrow pr_1 & \downarrow pr_1 & \downarrow pr_2 & \downarrow pr_1 \\
 V & \xlongequal{\quad} & V & & V^\phi & \xlongequal{\quad} & V^\phi \\
 \downarrow \pi_V & \downarrow \pi_V & \downarrow \pi_{V^\phi} & \downarrow \pi_{V^\phi} \\
 V/G & \xlongequal{\quad} & V/G & \xlongequal{\quad} & V^\phi/G & \xlongequal{\quad} & V^\phi/G
 \end{array}$$

commutes. Now we can take  $f = \beta \circ \alpha: V \times \mathbb{A}^m \dashrightarrow V^\phi \times \mathbb{A}^m$ .

**Step 2:** Suppose  $X$  is an arbitrary generically free  $G$ -variety.

Let  $V$  be a generically free linear representation of  $G$  and  $p: X \times V \rightarrow V$  be the projection onto the second factor. By the no-name lemma,  $X \times V$  is birationally isomorphic to  $X \times \mathbb{A}^m$ ; this yields a dominant rational map of  $G$ -varieties  $X \times \mathbb{A}^m \dashrightarrow V$ , which we will continue to denote by  $p$ . After replacing  $X$  by  $X \times \mathbb{A}^m$ , we may assume that there exists a dominant rational map  $p: X \dashrightarrow V$ . We now consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{p} & V \\ \downarrow & & \downarrow \\ X/G & \xrightarrow{p/G} & V/G, \end{array}$$

where the vertical arrows are rational quotient maps. We claim that  $X$  is birationally isomorphic to the fiber product  $X/G \times_{V/G} V$ , where the  $G$ -action on this fiber product is induced from the  $G$ -action on  $V$  (in other words,  $G$  acts trivially on  $X/G$  and on  $V/G$ ). In the case where  $k$  is an algebraically closed field of characteristic zero, this is proved in [14, Lemma 2.16]. For general  $k$ , choose an open invariant subscheme  $U$  of  $V$  such that  $G$  acts freely over  $U$ , the quotient  $U/G$  is a scheme, and the projection  $U \rightarrow U/G$  is a  $G$ -torsor; see Remark 2.2. By restricting  $X$ , we may assume that the morphism  $X \rightarrow X/G$  is also a  $G$ -torsor, and that  $X$  maps into  $U$ . Then we get a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{p} & U \\ \downarrow & & \downarrow \\ X/G & \xrightarrow{p/G} & U/G \end{array}$$

where the columns are  $G$ -torsors and the top row is  $G$ -equivariant. Any such diagram is well known to be cartesian; this proves our claim.

Similarly,  $X^\phi \simeq X^\phi/G \times_{V^\phi/G} V^\phi$ . By Step 1 there is a  $G$ -equivariant birational isomorphism  $f: V \times \mathbb{A}^m \dashrightarrow V^\phi \times \mathbb{A}^m$  which makes the diagram

$$\begin{array}{ccccc} & & V \times \mathbb{A}^m & & \\ & & \downarrow \pi_V & \searrow f & \\ X/G & \xrightarrow{p/G} & V/G & & V^\phi \times \mathbb{A}^m \\ & \searrow id & & \searrow id & \downarrow \pi_{V^\phi} \\ & & X^\phi/G & \xrightarrow{p/G} & V^\phi/G \end{array}$$

commute. Consequently,  $f$  induces a  $G$ -equivariant birational isomorphism between the fiber products  $X/G \times_{V/G} (V \times \mathbb{A}^m)$  and  $X^\phi/G \times_{V^\phi/G} (V^\phi \times \mathbb{A}^m)$  i.e., between  $X \times \mathbb{A}^m$  and  $X^\phi \times \mathbb{A}^m$ . This completes the proof of Theorem 1.2.

**Remark 3.1.** Our proof shows that the integer  $m$  in the statement of Theorem 1.2 can be taken to be the minimal value of  $2 \dim(W)$ , as  $W$  ranges over the generically free linear representations of  $G$ .

#### 4. Examples where $X$ and $X^\phi$ are birationally isomorphic

The question of whether or not  $X$  and  $X^\phi$  are birationally isomorphic over  $k$  is delicate in general. Birational isomorphism over  $K = k(X)^G$  is more accessible because it has a natural interpretation in terms of Galois cohomology. In this section we will show that in many cases  $X$  and  $X^\phi$  are, indeed, birationally isomorphic over  $K$  (and thus over  $k$ ).

Let  $X$  be a primitive  $G$ -variety. Recall that  $X$  is called primitive if  $G$  transitively permutes the irreducible components of  $X$ ; see Section 2. That is,  $X$  is primitive if  $K = k(X)^G$  is a field or equivalently, if the rational quotient variety  $X/G$  is irreducible. As we saw in Remark 2.2, after replacing  $X$  by an open  $G$ -invariant subvariety we may assume that the rational quotient map  $\pi: X \dashrightarrow X/G$  is regular and is a  $G$ -torsor. Hence, the  $G$ -action on  $X$  gives rise to a Galois cohomology class in  $H^1(K, G)$ , which we shall denote by  $[X]$ ; cf., [18], [12].

An automorphism  $\phi$  of  $G$  induces an automorphism  $\phi_*$  of the (pointed) cohomology set  $H^1(K, G)$ , where  $[X^\phi] = \phi_*([X])$ . The  $G$ -varieties  $X$  and  $X^\phi$  are birationally isomorphic over  $K$  if and only if  $\phi_*([X]) = [X]$  in  $H^1(K, G)$ .

**Example 4.1.** If  $[X] = 1$  then  $\phi_*([X]) = 1$ ; hence,  $X$  and  $X^\phi$  are birationally isomorphic. Explicitly, in this case  $X$  is birationally isomorphic to the “split”  $G$ -variety  $Y \times G$ , with  $G$  acting on the second component by left translations, and a birational isomorphism between  $Y \times G$  and  $(Y \times G)^\phi$  is given by  $(y, g) \mapsto (y, \phi(g))$ . In particular, if  $G$  is a special group, i.e.,  $H^1(K, G) = \{1\}$  for every  $K/k$ , then  $X$  and  $X^\phi$  are birationally isomorphic for every generically free  $G$ -variety  $X$ . Examples of special groups are  $G = \mathrm{GL}_n$ ,  $\mathrm{SL}_n$ ,  $\mathrm{Sp}_{2n}$ ; see [19, Chapter X].

The following lemma extends this simple argument a bit further.

**Lemma 4.2.** *Let  $X$  be a primitive generically free  $G$ -variety and  $K = k(X)^G$ . Suppose  $[X]$  lies in the image of the natural map  $H^1(K, G_0) \rightarrow H^1(K, G)$ , where  $G_0$  is a closed subgroup of  $G$  such that  $\phi|_{G_0} = \mathrm{id}: G_0 \rightarrow G_0$ . Then  $X$  and  $X^\phi$  are birationally isomorphic as  $K$ -varieties.*

**Proof.** Let  $i: G_0 \hookrightarrow G$  be the inclusion map. The commutative diagram

$$\begin{array}{ccc} G_0 & \xrightarrow{i} & G \\ \downarrow \mathrm{id} & & \downarrow \phi \\ G_0 & \xrightarrow{i} & G \end{array}$$

of groups induces a commutative diagram of cohomology sets

$$\begin{array}{ccc} H^1(K, G_0) & \xrightarrow{i_*} & H^1(K, G) \\ \downarrow \mathrm{id} & & \downarrow \phi_* \\ H^1(K, G_0) & \xrightarrow{i_*} & H^1(K, G). \end{array}$$

Since  $[X]$  is in the image of  $i_*$ , this diagram shows that  $\phi_*([X]) = [X]$ . ■

Note that if  $[X] = 1$  then  $[X]$  is the image of the trivial element of  $H^1(K, G_0)$ , where  $G_0 = \{1\}$ . Example 4.1 is thus a special case of Lemma 4.2. We now turn to a more sophisticated application of Lemma 4.2, with non-trivial  $G_0$ .

**Lemma 4.3.** *Let  $k$  be a field of characteristic  $\neq 2$ ,  $n$  be a positive integer,  $a_1, \dots, a_n \in k^*$ ,*

$$q(x_1, \dots, x_n) = a_1 x_1^2 + \dots + a_n x_n^2$$

*be a non-degenerate quadratic form and  $\phi$  be the automorphism of the special orthogonal group  $\mathrm{SO}(q)$  given by*

$$\phi(g) = hgh^{-1}, \text{ where } h = \mathrm{diag}(-1, 1, \dots, 1) \in \mathrm{O}(q). \quad (3)$$

*Then  $X$  and  $X^\phi$  are birationally isomorphic over  $K = k(X)^{\mathrm{SO}(q)}$  (and hence, over  $k$ ) for any irreducible generically free  $\mathrm{SO}(q)$ -variety  $X$ .*

We remark that if  $n$  is odd then  $\phi$  is an inner automorphism; indeed,  $\phi(g) = (-h)g(-h)^{-1}$  and  $-h \in \mathrm{SO}(q)$ . In this case a  $K$ -isomorphism between  $X$  and  $X^\phi$  is given by  $x \mapsto (-h) \cdot (x)$ . Thus the above lemma is only of interest in the case where  $n$  is even; however, the argument below is valid for every  $n \geq 1$ .

**Proof.** Let  $D_0 \simeq (\mathbb{Z}/2\mathbb{Z})^{n-1}$ ,  $D \simeq (\mathbb{Z}/2\mathbb{Z})^n$  be the subgroups of diagonal matrices in  $\mathrm{SO}(q)$ ,  $\mathrm{O}(q)$  respectively, and  $i: D_0 \hookrightarrow \mathrm{SO}(q)$ ,  $j: D \hookrightarrow \mathrm{O}(q)$  be the natural inclusion maps. Note that  $\phi$  restricts to a trivial automorphism of  $D_0$ . Thus, in view of Lemma 4.2 it suffices to show that  $i_*: H^1(K, D_0) \rightarrow H^1(K, \mathrm{SO}(q))$  is surjective.

Consider the commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & D_0 & \longrightarrow & D & \xrightarrow{\det} & \mathbb{Z}/2\mathbb{Z} \longrightarrow 1 \\ & & \downarrow i & & \downarrow j & & \\ & & \mathrm{SO}(q) & \longrightarrow & \mathrm{O}(q) & & \end{array}$$

of algebraic groups and the induced commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & H^1(K, D_0) & \longrightarrow & (K^*/(K^*)^2)^n & \xrightarrow{p} & K^*/(K^*)^2 \longrightarrow 1 \\ & & \downarrow i_* & & \downarrow j_* & & \\ & & H^1(K, \mathrm{SO}(q)) & \longrightarrow & H^1(K, \mathrm{O}(q)) & & \end{array}$$

in cohomology. The top row is an exact sequence of abelian groups;  $H^1(K, D_0)$  can thus be identified with the kernel of the product map  $p: (K^*/(K^*)^2)^n \rightarrow K^*/(K^*)^2$ , where  $p(b_1, \dots, b_n) = b_1 \dots b_n \pmod{(K^*)^2}$ .

Now recall that  $H^1(K, \mathrm{O}(q))$  is in a natural 1-1 correspondence with isometry classes of  $n$ -dimensional quadratic forms  $q'$  and that  $j_*$  takes  $(b_1, \dots, b_n) \in (K^*/K^{*2})^n$  to the quadratic form  $q' = a_1 b_1 x_1^2 + \dots + a_n b_n x_n^2$ . Similarly,  $H^1(K, \mathrm{SO}(q))$  is in a natural 1-1 correspondence with isometry classes of  $n$ -dimensional quadratic forms  $q'$  such that  $q'$  has the same discriminant as  $q$ , and  $i_*$  takes  $(b_1, \dots, b_n) \in (K^*/K^{*2})^n$ , with  $b_1 \dots b_n = 1$  in  $K^*/K^{*2}$ , to  $q' = a_1 b_1 x_1^2 + \dots + a_n b_n x_n^2$ ; cf., [10, Section 29.E] or [18, III, Appendix 2, §2]. It is clear from this description that both  $i_*$  and  $j_*$  are surjective.  $\blacksquare$

We will now assume that the base field  $k$  is algebraically closed (and still of characteristic  $\neq 2$ ). In this case, up to isomorphism, there is only one non-degenerate quadratic form  $q$  of each dimension  $n \geq 1$ , so we will set  $q(x_1, \dots, x_n) = x_1^2 + \dots + x_n^2$  and write  $\mathrm{SO}_n$  in place of  $\mathrm{SO}(q)$ .

**Proposition 4.4.** *Let  $k$  be an algebraically closed base field of characteristic  $\neq 2$  and let  $X$  be a generically free  $\mathrm{SO}_n$ -variety defined over  $k$ . Then  $X$  and  $X^\phi$  are birationally isomorphic over  $K = k(X)^{\mathrm{SO}_n}$  for every automorphism  $\phi$  of  $\mathrm{SO}_n$ .*

**Proof.** If  $n = 1$  then the group  $\mathrm{SO}_n$  is trivial, and there is nothing to prove. If  $n = 2$  then  $\mathrm{SO}_n$  is isomorphic to the 1-dimensional torus  $\mathbb{G}_m$ . Since this group is special (see e.g., [18, Proposition X.2]), in this case the proposition follows from Example 4.1.

We may thus assume that  $n \geq 3$ . As we remarked in the introduction, we are allowed to replace  $\phi$  by  $\phi' = \phi \circ \mathrm{inn}_h$ , where  $\mathrm{inn}_h$  denotes conjugation by  $h \in \mathrm{SO}(q)$ ; indeed,  $X^\phi$  and  $X^{\phi'}$  are isomorphic (over  $K$ ) via  $x \mapsto h \cdot x$ . For this reason we will be interested in the group  $\mathrm{Out}(\mathrm{SO}_n)$  of outer automorphisms of  $\mathrm{SO}_n$ .

Since  $\mathrm{SO}_n$  is semisimple for every  $n \geq 3$  (in fact, it is even simple, unless  $n = 4$ ), we have

$$|\mathrm{Out}(\mathrm{SO}_n)| \leq |\mathrm{Aut}(\mathrm{Dyn}(\mathrm{SO}_n))|,$$

where  $\mathrm{Dyn}(\mathrm{SO}_n)$  is the Dynkin diagram of  $\mathrm{SO}_n$ ; see [10, Theorem 25.16].

If  $n = 2m + 1 \geq 3$  is odd, then  $\mathrm{SO}_n$  is a simple group of type  $B_m$ . In this case  $\mathrm{Dyn}(\mathrm{SO}_n)$  has no non-trivial automorphisms and thus  $\mathrm{Out}(\mathrm{SO}_n) = \{1\}$ . In other words,  $\phi$  is an inner automorphism, and the proposition follows.

Now suppose  $n = 2m \geq 4$  is even. Recall that  $\mathrm{SO}_4$  is a semisimple group of type  $A_1 + A_1$ , and  $\mathrm{SO}_n$  is a simple group of type  $D_m$  for any  $n \geq 6$ . In both cases  $|\mathrm{Aut}(\mathrm{Dyn}(\mathrm{SO}_n))| = 2$  and thus  $|\mathrm{Out}(\mathrm{SO}_n)| \leq 2$ . On the other hand, it is easy to see that the automorphism  $\phi$  defined in (3) is not inner. Thus  $|\mathrm{Out}(\mathrm{SO}_n)| = 2$ , and after composing  $\phi$  with an inner automorphism, we may assume that either  $\phi = \mathrm{id}$  or  $\phi$  is as in (3). In the former case the assertion of the proposition is trivial, in the latter it follows from Lemma 4.3.  $\blacksquare$

## 5. Proof of Theorem 1.3

Recall that elements of  $H^1(K, \mathrm{PGL}_n)$  are in a natural 1-1 correspondence with

(i) generically free  $\mathrm{PGL}_n$ -varieties  $X$ , with  $k(X)^{\mathrm{PGL}_n} = K$ , up to birational isomorphism over  $K$ , or alternatively, with

(ii) central simple algebras  $A/K$  of degree  $n$ , up to  $K$ -isomorphism;

see [18], [12], [15]. We will denote the central simple algebra corresponding to an irreducible generically free  $\mathrm{PGL}_n$ -variety  $X$  (respectively, to an element  $\alpha \in H^1(K, \mathrm{PGL}_n)$ ) by  $A_X$  (respectively, by  $A_\alpha$ ). If  $\phi: \mathrm{PGL}_n \rightarrow \mathrm{PGL}_n$  is the automorphism given by  $g \mapsto (g^{-1})^{\mathrm{transpose}}$  then  $A_{\phi_*(\alpha)}$  is the opposite algebra  $A_\alpha^{\mathrm{op}}$ ; cf. e.g., [19, pp. 152-153]. In other words,  $A_{X^\phi} = A_X^{\mathrm{op}}$ . The following lemma gives a necessary and sufficient conditions for  $X$  and  $X^\phi$  to be birationally isomorphic over  $K$ .

**Lemma 5.1.** *Let  $X$  be an irreducible generically free  $\mathrm{PGL}_n$ -variety. Then the following conditions are equivalent.*

- (a)  $X$  and  $X^\phi$  are birationally isomorphic over  $K = k(X)^{\mathrm{PGL}_n}$ ,
- (b)  $A_X$  is  $K$ -isomorphic to  $A_X^{op}$ ,
- (c)  $A_X$  has order 1 or 2 in the Brauer group  $\mathrm{Br}(K)$ .

**Proof.** (a)  $\iff$  (b): Let  $\alpha = [X] \in H^1(K, \mathrm{PGL}_n)$ . Then as we observed in Section 4,  $[X^\phi] = \phi_*(\alpha)$ . Thus  $A_{X^\phi} = A_{\phi_*(\alpha)} = A_\alpha^{op} = A_X^{op}$ , so that  $X$  and  $X^\phi$  are birationally isomorphic over  $K$  if and only if  $A_X$  is  $K$ -isomorphic to  $A_X^{op}$ .

The equivalence of (b) and (c) is obvious, since  $A^{op}$  is the inverse of  $A$  in  $\mathrm{Br}(K)$ . ■

Lemma 5.1 does not directly address Problem 1.1, i.e., it does not tell us under what conditions  $X$  and  $X^\phi$  are birationally isomorphic over the base field  $k$ . Note however, that a birational isomorphism  $\alpha: X \rightarrow X^\phi$  defined over  $k$ , restricts to a  $k$ -automorphism of the field of invariants  $K = k(X)^{\mathrm{PGL}_n} = k(X^\phi)^{\mathrm{PGL}_n}$ . Our proof of Theorem 1.3 is based on the observation that if  $\mathrm{Aut}_k(K) = \{1\}$ , then

- $X$  and  $X^\phi$  are isomorphic over  $k \iff$
- $X$  and  $X^\phi$  are isomorphic over  $K \iff$
- $A_X$  has order 1 or 2 in the Brauer group of  $K$ .

Thus in order to prove Theorem 1.3 it suffices to construct (i) a finitely generated field extension  $K/k$  such that  $\mathrm{Aut}_k(K) = \{1\}$  and (ii) a central simple algebra  $A/K$  of degree  $n$  and exponent  $n$ . These constructions are carried out in Lemmas 5.2 and 5.3 below.

**Lemma 5.2.** *For any field  $k$  there exists a smooth projective geometrically connected algebraic surface  $S/k$  such that  $S(k) \neq \emptyset$  which admits no non-trivial birational automorphisms. In other words,  $\mathrm{Aut}_k k(S) = \{1\}$ .*

**Proof.** According to [11], there exist smooth projective geometrically connected curves  $C/k$ , of arbitrary genus  $g \geq 3$ , such that the automorphism group of the extension  $C_{\bar{k}}$  to the algebraic closure of  $k$  is trivial. It is easy to see that all the examples constructed in [11] have rational points over  $k$  (for example, each has a rational point over the origin in  $\mathbb{P}_k^1$ , and another one at infinity). Choose two such curves  $C_1$  and  $C_2$ , of genus  $g_1$  and  $g_2$  respectively, with  $g_1 < g_2$ , and set  $S = C_1 \times C_2$ . By the Hurwitz formula every map from  $C_1$  to  $C_2$  is constant and every non-constant separable map from  $C_i$  to itself is the identity.

We claim that every birational automorphism  $f: S \dashrightarrow S$  is trivial. To prove this, we may extend the base field to  $\bar{k}$ , and assume that  $k$  is algebraically closed. Note that there are no rational curves in  $S$ , since there are no non-constant maps from  $\mathbb{P}^1$  to  $C_1$  or to  $C_2$ . By the standard result on the resolution of indeterminacies of rational maps by sequences of blow-ups, we see that every rational map  $S \dashrightarrow S$  is in fact regular. Hence a birational isomorphism of  $S$  is a biregular isomorphism.

Since every morphism  $C_1 \rightarrow C_2$  is constant, we see that the composite  $S \xrightarrow{f} S \xrightarrow{\mathrm{pr}_2} C_2$  factors through the second projection  $\mathrm{pr}_2: S \rightarrow C_2$  to give a

commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{f} & S \\ \downarrow \text{pr}_2 & & \downarrow \text{pr}_2 \\ C_2 & \xrightarrow{g} & C_2. \end{array}$$

Since  $\text{pr}_2 \circ f$  is smooth,  $g$  must be separable, so it is the identity. Then  $f$  sends each fiber  $\text{pr}_2^{-1}(y) \simeq C_1$  isomorphically to itself; and each of these restrictions must be the identity. This implies that  $f$  is the identity, as claimed. ■

**Lemma 5.3.** *Let  $X$  be an algebraic variety over a field  $k$  of dimension  $d \geq 2$  with a smooth rational  $k$ -point,  $K$  its fraction field,  $n$  a positive integer not divisible by  $\text{char}(k)$  such that  $k$  contains all  $n$ th roots of 1. Then there exists a division algebra  $D/K$  of degree  $n$  and exponent  $n$ .*

**Proof.** Let  $x \in X(k)$  be a smooth rational point; consider a system of local parameters  $t_1, \dots, t_d$  in the local ring  $\mathcal{O}_x(X)_{X,x}$ ; here  $d = \dim(X) = \text{trdeg}_k(K) \geq 2$ . We claim that the symbol algebra  $D = (t_1, t_2)_n$ , i.e., the  $K$ -algebra given by generators  $x_1, x_2$  and relations  $x_1^n = t_1$ ,  $x_2^n = t_2$ ,  $x_1 x_2 = \zeta x_2 x_1$  has exponent  $n$  in  $\text{Br}(K)$ . (Here  $\zeta_n$  is a primitive  $n$ th root of unity in  $k$ .)

To prove this, consider the completion  $\widehat{\mathcal{O}}_x(X) = k[[t_1, \dots, t_d]]$  of the local ring  $\mathcal{O}_x(X)$ , where  $k[[t_1, \dots, t_d]]$  denotes the ring of formal power series in the variables  $t_1, \dots, t_d$ . Note that  $\mathcal{O}_x(X) \subset \widehat{\mathcal{O}}_x(X)$  and thus, after passing to the fields of fractions,  $K \subset k((t_1, \dots, t_d))$ . The image of  $D$  under the restriction map  $\text{Br}(K) \rightarrow \text{Br}(k((t_1, \dots, t_d)))$  is the symbol algebra  $D' = (t_1, t_2)_n$  over  $k((t_1, \dots, t_d))$ . A simple valuation-theoretic argument shows that  $D'$  has exponent  $n$ ; cf. [17, Proposition 3.3.26]. Hence, so does  $D$ . This completes the proof of Lemma 5.3 and thus of Theorem 1.3. ■

**Remark 5.4.** Theorem 1.3 remains valid if the condition that  $k$  contains a primitive  $n$ th root of unity is replaced by the (weaker) condition that  $k$  contains a primitive  $m$ th root of unity for some divisor  $m$  of  $n$  such that  $m \geq 3$ . The proof is the same, except that instead of the symbol algebra  $D = (t_1, t_2)_n$  of degree  $n$  and exponent  $n$ , we use the algebra  $M_{n/m}(E)$  of degree  $n$  and exponent  $m$ , where  $E = (t_1, t_2)_m$ .

**Remark 5.5.** Theorem 1.3 fails for  $n = 2$ ; indeed,  $A$  and  $A^{\text{op}}$  are isomorphic over  $K$  for any central simple algebra  $A/K$  of degree 2. Alternatively,  $\text{PGL}_2 \simeq \text{SO}_3$ , so if Theorem 1.3 were true for  $n = 2$ , it would contradict Proposition 4.4.

## 6. Further examples

In this section we will assume that  $G$  is a finite group and  $k$  is an algebraically closed field of characteristic zero.

**Proposition 6.1.** (a) For every finitely generated field extension  $K/k$  with  $K \neq k$  and every finite group  $G$  there exists a  $G$ -Galois extension  $L/K$ . Equivalently, there exists an irreducible  $G$ -variety  $X$  such that  $k(X)^G = K$ .

(b) Suppose  $\text{Aut}_k(K) = \{1\}$  and  $\phi: G \rightarrow G$  is an outer automorphism of a finite group  $G$ . Then for every  $X$ , as in (a), the  $G$ -varieties  $X$  and  $X^\phi$  are not birationally isomorphic.

**Proof.** (a) By the Riemann existence theorem there exists a  $G$ -Galois extension  $L_0/k(t)$ , where  $t$  is an independent variable. Hence, there exists a  $G$ -Galois extension  $L_1/K(t)$ , where  $L_1 = L_0 \otimes_{k(t)} K(t)$ . The Hilbert irreducibility theorem now allows us to construct a  $G$ -Galois extension  $L/K$  by suitably specializing  $t$  in  $K$ .

(b) Irreducible  $G$ -varieties  $X$  (up to birational isomorphism) such that  $k(X)^G = K$ , are in 1-1 correspondence with  $G$ -Galois field extensions  $L/K$ . A birational isomorphism  $\alpha: X \dashrightarrow X^\phi$  of  $G$ -varieties induces an isomorphism

$$\begin{array}{ccc} L = k(X) & \xrightarrow{\alpha} & k(X^\phi) = L \\ | & & | \\ K & \xlongequal{\quad} & K. \end{array}$$

Then  $\alpha \in \text{Gal}(L/K) = G$ , and since the above diagram commutes, we have  $\alpha g(l) = \phi(g)\alpha(l)$  for every  $g \in G$  and  $l \in L$ . In other words,  $\phi(g) = \alpha g \alpha^{-1}$ , contradicting our assumption that  $\phi$  is an outer automorphism. ■

**Acknowledgment.** The authors would like to thank P. Brosnan and V. L. Popov for informative discussions. We are also grateful to F. Knop and V. L. Popov for helpful comments on earlier versions of this paper.

## References

- [1] F. Bogomolov, P. Katsylo, *Rationality of certain quotient varieties*, Mat. Sbornik, **126**(168) (1985), no. 4, 584–589.
- [2] V. Chernousov, Ph. Gille, Z. Reichstein, *Resolving  $G$ -torsors by abelian base extensions*, J. Algebra **296** (2006), no. 2, 561–581.
- [3] M. Demazure, P. Gabriel, “Introduction to algebraic geometry and algebraic groups”, North-Holland Mathematics Studies, **39**, North-Holland Publishing Co., Amsterdam-New York, 1980, xiv+357 pp.
- [4] I. V. Dolgachev, *Rationality of fields of invariants*, Algebraic Geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Amer. Math. Soc., Providence, RI, 1987, pp. 3–16.
- [5] S. Endo, T. Miyata, *Invariants of finite abelian groups*, J. Math. Soc. Japan **25** (1973), 7–26.
- [6] P. Katsylo, *On the birational classification of linear representations*, Max Planck Institute für Mathematik preprint MPI-92-1 (1992).
- [7] D. Knutson, “Algebraic spaces”, Lecture Notes in Mathematics, Vol. 203. Springer-Verlag, Berlin-New York, 1971. vi+261 pp.

- [8] G. Laumon, L. Moret-Bailly, “Champs Algébriques”, *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*, **39**, Springer-Verlag, Berlin, 2000. xii+208 pp.
- [9] H. W. Lenstra, Jr., *Rational functions invariant under a finite abelian group*, *Invent. Math.* **25** (1974), 299–325.
- [10] M.-A. Knus, A. Merkurjev, M. Rost, J.-P. Tignol, *The book of involutions*, *Colloquium Publications* **44**, American Mathematical Society, Providence, RI, 1998, xxii+593 pp.
- [11] B. Poonen, *Varieties without extra automorphisms. I. Curves*, *Math. Res. Lett.* **7** (2000), no. 1, 67–76.
- [12] V. L. Popov, *Sections in invariant theory*, *The Sophus Lie Memorial Conference (Oslo, 1992)*, *Scand. Univ. Press*, Oslo, 1994, 315–361.
- [13] V. L. Popov and E. B. Vinberg, “Invariant Theory”, *Algebraic Geometry IV*, *Encyclopedia of Mathematical Sciences* **55**, Springer, 1994, 123–284.
- [14] Z. Reichstein, *On the notion of essential dimension for algebraic groups*, *Transform. Groups* **5** (2000), no. 3, 265–304.
- [15] Z. Reichstein, N. Vonessen, *Polynomial identity rings as rings of functions*, arXiv: math.RA/0407152, *J. Algebra*, to appear.
- [16] Z. Reichstein, B. Youssin, *A birational invariant for algebraic group actions*, *Pacific J. Math.* **204** (2002), no. 1, 223–246.
- [17] L. H. Rowen, “Polynomial identities in ring theory”, *Pure and Applied Mathematics* **84**, Academic Press, New York-London, 1980, xx+365 pp.
- [18] J.-P. Serre, “Galois Cohomology”, Springer-Verlag, Berlin, 1997. x+210 pp.
- [19] J.-P. Serre, “Local Fields”, *Graduate Texts in Mathematics*, **67**, Springer-Verlag, New York-Berlin, 1979. viii+241 pp.
- [20] I. R. Shafarevich, “Basic Algebraic Geometry. 1. Varieties in projective space”, second edition, Springer-Verlag, Berlin, 1994. xx+303 pp.
- [21] A. Speiser, *Zahlentheoretische Sätze aus der Gruppentheorie*, *Math. Zeitschrift* **5** (1919), 1–6.
- [22] A. Vistoli, *Notes on Grothendieck topologies, fibered categories and descent theory*, available at [www.dm.unibo.it/~vistoli/descent.pdf](http://www.dm.unibo.it/~vistoli/descent.pdf).

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Received  
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