Math 152: Linear Systems – Winter 2004

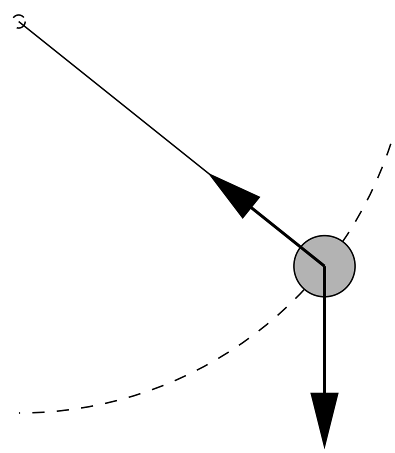
1: Vectors and Geometry

Vectors

Vectors are used to describe quantities that have both a magnitude and a direction. You are probably familiar with vector quantities in two and three dimensions, such as forces and velocities.

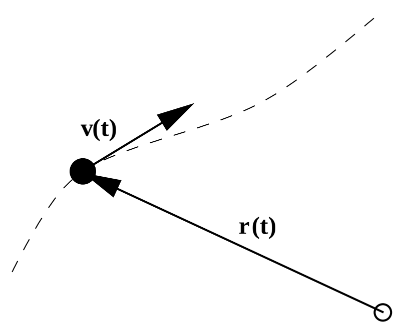
Later in this course we will see that vectors can also describe the configuration of a mechanical system of weights and springs, or the collections of voltages and currents in an electrical circuit. These more abstract vector quantities are not so easily visualized since they take values in higher dimensional spaces.

We begin this course by discussing the geometry of vectors in two and three dimensions. In two and three dimensions, vectors can be visualized as arrows. Before we can draw a vector, we have to decide where to place the tail of the vector. If we are drawing forces, we usually put the tail of the vector at the place where the force is applied. For example, in the diagram the forces acting on a pendulum bob are gravity and the restraining force along the shaft.



Forces acting on a pendulum

If we are drawing the velocity of a particle at a given time, we would place the tail of the velocity vector $\mathbf{v}(t)$ at the position of the particle at that time.



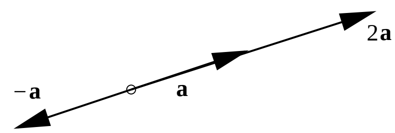
Position and velocity of a particle

Once we have chosen a starting point for the tails of our vectors (i.e., an origin for space), every point in space corresponds to exactly one vector, namely the vector whose tail is at the origin and whose head is at the given point. For example, in the diagram above, we have chosen an arbitrary point as the origin (marked with a circle) and identified position of the particle with the vector $\mathbf{r}(t)$.

Multiplication by a number and vector addition

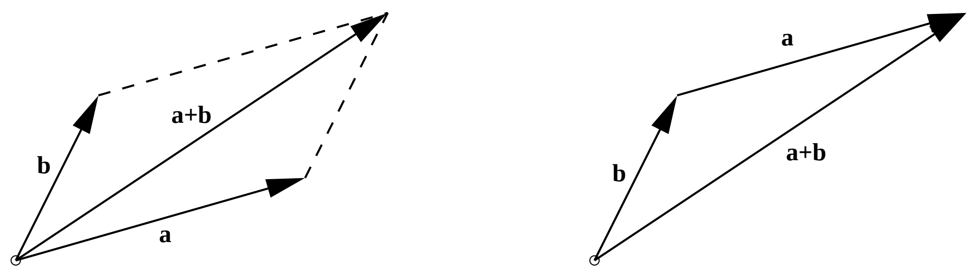
There are two basic operations defined for vectors. One is multiplication of a vector by a number (also called scalar multiplication). The other is addition of two vectors.

A vector $\mathbf{a}$ can be multiplied by a number (or scalar) s to produce a new vector $s\mathbf{a}$. If s is positive then $s\mathbf{a}$ points in the same direction as $\mathbf{a}$ and has length s times the length of $\mathbf{a}$. If s is negative then $s\mathbf{a}$ points in the direction opposite to $\mathbf{a}$ and has length $|s|$ times the length of $\mathbf{a}$.



Scalar multiplication.

To add two vectors $\mathbf{a}$ and $\mathbf{b}$ and we draw the parallelogram that has $\mathbf{a}$ and $\mathbf{b}$ as two of its sides. The vector $\mathbf{a}+\mathbf{b}$ is has its tail at the origin and its head at the vertex of the parallelogram opposite the origin. Alternatively we can imagine sliding (or translating) one of the vectors, without changing its direction, so that its tail sits on the head of the other vector. (In the diagram we translated the vector $\mathbf{a}$.) The sum $\mathbf{a}+\mathbf{b}$ is then the vector whose tail is at the origin and whose head coincides with the vector we moved.



Vector addition

Problem 1.1: Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be a fixed non-zero vectors. Describe and sketch the following sets of points in two and three dimensions:

(i) $\{s\mathbf{a} : s \in \mathbb{R}\}$ (i.e., the set of all scalar multiples of $\mathbf{a}$)

(ii) $\{s\mathbf{a} : s > 0\}$ (i.e., the set of all positive scalar multiples of $\mathbf{a}$)

(iii) $\{\mathbf{b} + s\mathbf{a} : s \in \mathbb{R}\}$

(iv) $\{s\mathbf{a} + t\mathbf{b} : s, t \in \mathbb{R}\}$

(v) $\{\mathbf{c} + s\mathbf{a} + t\mathbf{b} : s, t \in \mathbb{R}\}$

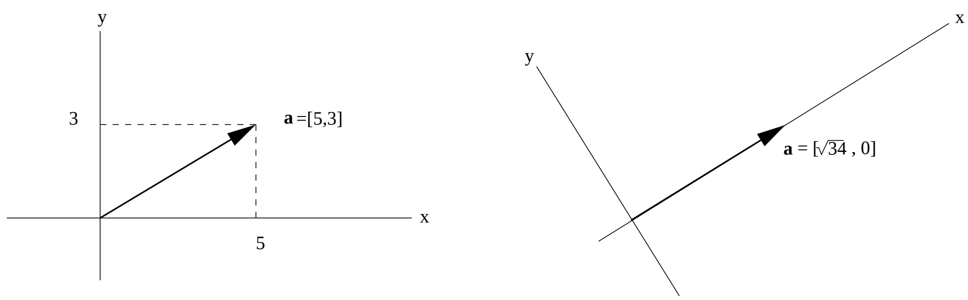
Problem 1.2: Describe the vectors $\mathbf{a} - \mathbf{b}$ and $\mathbf{b} - \mathbf{a}$.

Problem 1.3: Find an expression for the midpoint between $\mathbf{a}$ and $\mathbf{b}$. Find an expression for a point one third of the way between $\mathbf{a}$ and $\mathbf{b}$.

Problem 1.4: Find an expression for the line segment joining $\mathbf{a}$ and $\mathbf{b}$.

Co-ordinates

In order to do calculations with vectors, we have to introduce co-ordinate axes. Once we have chosen in what directions the co-ordinate axes will lie, we can specify a vector by giving its components in the co-ordinate directions.

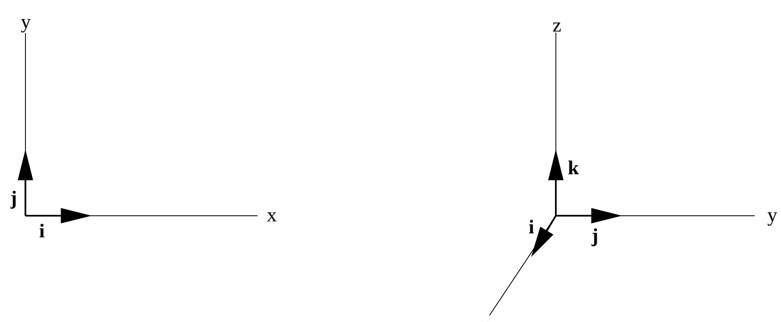


Two choices of co-ordinate axes

In the diagram we see two choices of x and y axes. For the first choice of axes, the vector $\mathbf{a}$ has co-ordinates $[5,3]$ and for the second choice of axes the co-ordinates are $[\sqrt{34},0]$. In a given problem, it makes sense to choose the axes so that at least some of the vectors have a simple representation. For example, in analyzing the forces acting on a pendulum, we would either choose the y axis either to be vertical, or to lie along the shaft of the pendulum.

We can choose to write the co-ordinates of a vector in a row, as above, or in a column, like $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$. Later on, we will almost always write vectors as columns. But in this chapter we will write vectors as rows.

A convenient way to choose the co-ordinate axes is to specify unit vectors (that is, vectors of length one) that lie along each of the axes. These vectors are called standard basis vectors and are denoted $\mathbf{i}$ and $\mathbf{j}$ (in two dimensions) and $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ (in three dimensions). (Sometimes they are also denoted $\mathbf{e}_1$ and $\mathbf{e}_2$ (in two dimensions) and $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$(in three dimensions).)



Unit vectors

The unit vectors have co-ordinates

$$\begin{aligned}\mathbf{i} &= \mathbf{e}_1 = [1,0]\\ \mathbf{j} &= \mathbf{e}_2 = [0,1]\end{aligned}$$

in two dimensions, and

$$\begin{aligned}\mathbf{i} &= \mathbf{e}_1 = [1,0,0]\\ \mathbf{j} &= \mathbf{e}_2 = [0,1,0]\\ \mathbf{k} &= \mathbf{e}_3 = [0,0,1]\end{aligned}$$

in three dimensions.

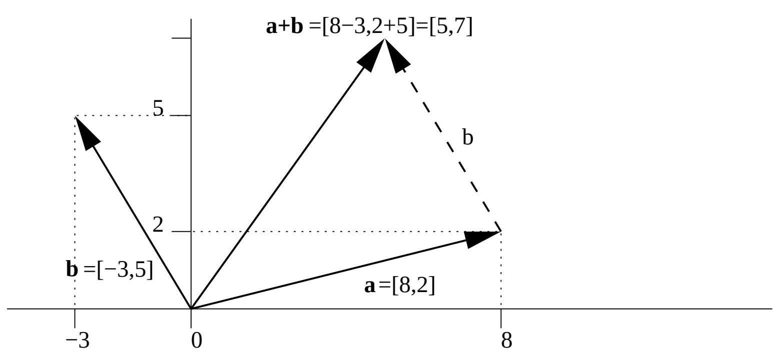
Often, we make no distinction between a vector and its co-ordinate representation. In other words, we regard the co-ordinate axes as being fixed once and for all. Then a vector in two dimensions is simply a list of two numbers (the components) $[a_1, a_2]$, and a vector in three dimensions is a list of three numbers $[a_1, a_2, a_3]$. Vectors in higher dimensions are now easy to define. A vector in n dimensions is a list of n numbers $[a_1, a_2, \ldots, a_n]$.

When a vector is multiplied by a number, each component is scaled by the same amount. Thus if $\mathbf{a} = [a_1, a_2]$, then

$$\begin{aligned}s\mathbf{a} &= s[a_1, a_2]\\ &= [sa_1, sa_2]\end{aligned}$$

Similarly, when two vectors are added, their co-ordinates are added componentwise. So if $\mathbf{a} = [a_1, a_2]$ and $\mathbf{b} = [b_1, b_2]$, then

$$\begin{aligned}\mathbf{a} + \mathbf{b} &= [a_1, a_2] + [b_1, b_2]\\ &= [a_1 + b_1, a_2 + b_2]\end{aligned}$$



Adding vector co-ordinates

The analogous formulas hold in three (and higher dimensions). If $\mathbf{a} = [a_1, a_2, \ldots, a_n]$ and $\mathbf{b} = [b_1, b_2, \ldots, b_n]$, then

$$\begin{aligned}s\mathbf{a} &= s[a_1, a_2, \ldots, a_n]\\ &= [sa_1, sa_2, \ldots, sa_n]\\ \mathbf{a} + \mathbf{b} &= [a_1, a_2, \ldots, a_n] + [b_1, b_2, \ldots, b_n]\\ &= [a_1 + b_2, a_2 + b_2, \ldots, a_n + b_n]\end{aligned}$$

Properties of vector addition and scalar multiplication

Let $\mathbf{0}$ denote the zero vector. This is the vector all of whose components are zero. The following properties are intuitive and easy to verify.

1. $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$	2. $\mathbf{a}+(\mathbf{b}+\mathbf{c})=(\mathbf{a}+\mathbf{b})+\mathbf{c}$
3. $\mathbf{a}+\mathbf{0}=\mathbf{a}$	4. $\mathbf{a}+(-\mathbf{a})=\mathbf{0}$
5. $s(\mathbf{a}+\mathbf{b})=(s\mathbf{a}+s\mathbf{b})$	6. $(s+t)\mathbf{a}=s\mathbf{a}+t\mathbf{a}$
7. $(st)\mathbf{a}=s(t\mathbf{a})$	8. $1\mathbf{a}=\mathbf{a}$

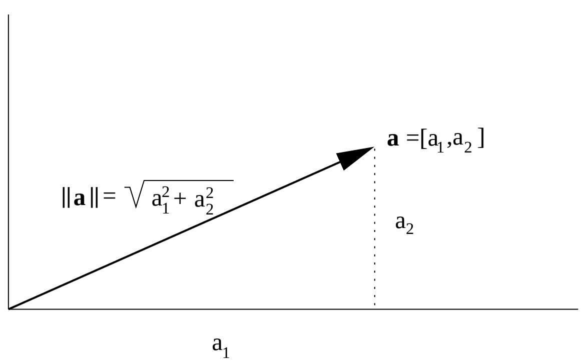
The follow from similar properties which hold for numbers. For example, for numbers a_1 and b_1 we know that $a_1+b_1=b_1+a_1$. Thus

$$\begin{aligned}\mathbf{a}+\mathbf{b} &= [a_1,a_2]+[b_1,b_2]\\ &= [a_1+b_1,a_2.b_2]=[b_1+a_1,b_2+a_2]\\ &= [b_1,b_2]+[a_1,a_2]=\mathbf{b}+\mathbf{a},\end{aligned}$$

so property 1 holds. Convince yourself that the rest of these properties are true. (What is the vector $-\mathbf{a}$?). It might seem like a waste of time fussing over obvious properties such as these. However, we will see when we come to the cross product and matrix product, that sometimes such “obvious” properties turn out to be false!

Length of a vector

It follows from the Pythagorean formula that the length $\|\mathbf{a}\|$ of $\mathbf{a}=[a_1,a_2]$ satisfies $\|\mathbf{a}\|^2=a_1^2+a_2^2$.



Pythagorean formula

Thus

$$\|\mathbf{a}\|=\sqrt{a_1^2+a_2^2}.$$

Similarly, for a vector $\mathbf{a}=[a_1,a_2,a_3]$ in three dimensions,

$$\|\mathbf{a}\|=\sqrt{a_1^2+a_2^2+a_3^2}.$$

The distance between two vectors $\mathbf{a}$ and $\mathbf{b}$ is the length of the difference $\mathbf{b}-\mathbf{a}$.

Problem 1.5: Find the equation of a sphere centred at $\mathbf{a}=[a_1,a_2,a_3]$ with radius r. (Hint: the sphere is the set of points $\mathbf{x}=[x_1,x_2,x_3]$ whose distance from $\mathbf{a}$ is r

Problem 1.6: Find the equation of a sphere if one of its diameters has endpoints $[2,1,4]$ and $[4,3,10]$

The dot product

The dot product of two vectors is defined in both two and three dimensions (actually in any dimension). The result is a number. Two main uses of the dot product are testing for orthogonality and computing projections.

The dot product of $\mathbf{a}=[a_1,a_2]$ and $\mathbf{b}=[b_1,b_2]$ is given by

$$\mathbf{a}\cdot\mathbf{b}=a_1b_1+a_2b_2.$$

Similarly, the dot product of $\mathbf{a}=[a_1,a_2,a_3]$ and $\mathbf{b}=[b_1,b_2,b_3]$ is given by

$$\mathbf{a}\cdot\mathbf{b}=a_1b_1+a_2b_2+a_3b_3.$$

The properties of the dot product are as follows:

0. If $\mathbf{a}$ and $\mathbf{b}$ are vectors, then $\mathbf{a}\cdot\mathbf{b}$ is a number.
1. $\mathbf{a}\cdot\mathbf{a}=\|a\|^2$.
2. $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$.
3. $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$.
4. $s(\mathbf{a}\cdot\mathbf{b})=(s\mathbf{a})\cdot\mathbf{b}$.
5. $\mathbf{0}\cdot\mathbf{a}=0$.
6. $\mathbf{a}\cdot\mathbf{b}=\|\mathbf{a}\|\|\mathbf{b}\|\cos(\theta)$, where θ is angle between $\mathbf{a}$ and $\mathbf{b}$.
7. $\mathbf{a}\cdot\mathbf{b}=0 \iff \mathbf{a}=\mathbf{0}$ or $\mathbf{b}=\mathbf{0}$ or $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (i.e,. perpendicular).

Properties 0 to 5 are easy consequences of the definitions. For example, to verify property 5 we write

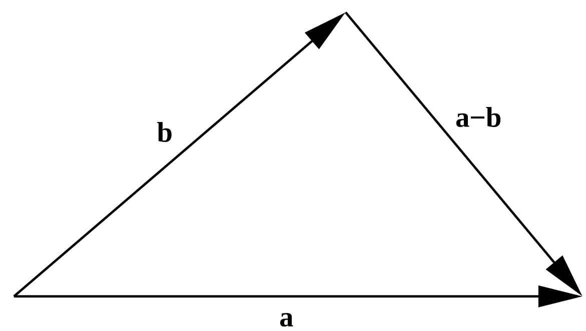
$$\mathbf{0}\cdot\mathbf{a}=[0,0,0]\cdot[a_1,a_2,a_3]=0a_1+0a_2+0a_3=0.$$

Property 6 is the most important property and is often taken as the definition. Notice that our definition is given in terms of the components of the vectors, which depend on how we chose the co-ordinate axes. It is not at all clear that if we change the co-ordinate axis, and hence the co-ordinates of the vectors, that we will get the same answer for the dot product. However, property 6 says that the dot product only depends on the lengths of the vectors and the angle between them. These quantities are independent of how co-ordinate axes are chosen, and hence so is the dot product.

To show that property 6 holds we compute $\|\mathbf{a}-\mathbf{b}\|^2$ in two different ways. First of all, using properties 1 to 5, we have

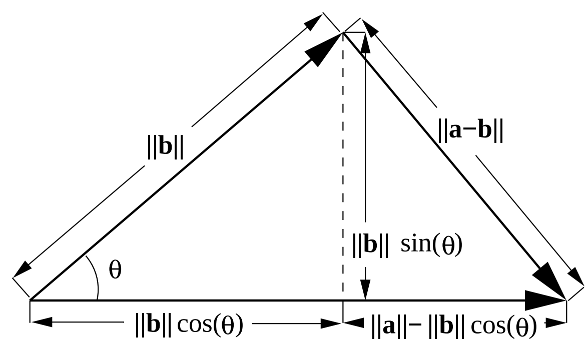
$$\begin{aligned}\|\mathbf{a}-\mathbf{b}\|^2 &= (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})\\ &= \mathbf{a}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{b}-\mathbf{b}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}\\ &= \|\mathbf{a}\|^2+\|\mathbf{b}\|^2-2\mathbf{a}\cdot\mathbf{b}\end{aligned}$$

(Which properties were used in each step?) Next we compute $\|\mathbf{a}-\mathbf{b}\|$ from the following diagram.



The vectors $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{a}-\mathbf{b}$

We mark the lengths of each of the line segments:



Lengths of segments

Using Pythagoros' theorem for the right angled triangle on the right of this diagram, we see that

$$\|\mathbf{a}-\mathbf{b}\|^2 = (\|\mathbf{a}\| - \|\mathbf{b}\|\cos(\theta))^2 + \|\mathbf{b}\|^2\sin^2(\theta).$$

Thus, using $\cos^2(\theta)+\sin^2(\theta)=1$,

$$\begin{aligned}\|\mathbf{a}-\mathbf{b}\|^2 &= \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2\cos^2(\theta) - 2\|\mathbf{a}\|\|\mathbf{b}\|\cos(\theta) + \|\mathbf{b}\|^2\sin^2(\theta)\\ &= \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\|\mathbf{b}\|\cos(\theta)\end{aligned}$$

Now we equate the two expressions for $\|\mathbf{a}-\mathbf{b}\|^2$. This gives

$$\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\|\mathbf{b}\|\cos(\theta)$$

Subtracting $\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2$ from both sides and dividing by -2 now yields

$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos(\theta).$$

This proves property 6.

Property 7 now follows directly from 6. If $\mathbf{a}\cdot\mathbf{b}=0$ then $\|\mathbf{a}\|\|\mathbf{b}\|\cos(\theta)=0$ so either $\|\mathbf{a}\|=0$, in which case $\mathbf{a}=\mathbf{0}$, or $\|\mathbf{b}\|=0$, in which case $\mathbf{b}=\mathbf{0}$, or $\cos(\theta)=0$, which implies that $\theta=\pi/2$ (since θ lies between 0 and π). This implies $\mathbf{a}$ and $\mathbf{b}$ are orthogonal.

Property 6 can be used to compute the angle between two vectors. For example, what is the angle between the vectors whose tails lie at the centre of a cube and whose heads lie on adjacent vertices? To compute this take a cube of side length 2 and centre it at the origin, so that the vertices lie at the points $[\pm 1, \pm 1, \pm 1]$. Then we must find the angle between $\mathbf{a} = [1, 1, 1]$ and $\mathbf{b} = [-1, 1, 1]$. Since

$$\mathbf{a} \cdot \mathbf{b} = -1 + 1 + 1 = 1 = \|\mathbf{a}\|\|\mathbf{b}\| \cos(\theta) = \sqrt{3}\sqrt{3}\cos(\theta)$$

we obtain

$$\theta = \arccos(1/3) \sim 1.231 \quad (\sim 70.528^\circ)$$

Problem 1.7: Compute the dot product of the vectors $\mathbf{a}$ and $\mathbf{b}$ and find the angle between them.

(i) $\mathbf{a} = [1, 2]$, $\mathbf{b} = [-2, 3]$

(ii) $\mathbf{a} = [-1, 2]$, $\mathbf{b} = [1, 1]$

(iii) $\mathbf{a} = [1, 1]$, $\mathbf{b} = [2, 2]$

(iv) $\mathbf{a} = [1, 2, 1]$, $\mathbf{b} = [-1, 1, 1]$

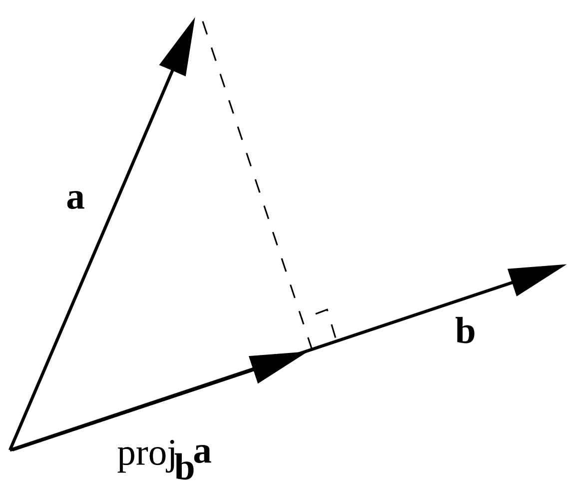
(v) $\mathbf{a} = [-1, 2, 3]$, $\mathbf{b} = [3, 0, 1]$

Problem 1.8: For which value of s is the vector $[1, 2, s]$ orthogonal to $[-1, 1, 1]$?

Problem 1.9: Does the triangle with vertices $[1, 2, 3]$, $[4, 0, 5]$ and $[3, 4, 6]$ have a right angle?

Projections

Suppose $\mathbf{a}$ and $\mathbf{b}$ are two vectors. The projection of $\mathbf{a}$ in the direction of $\mathbf{b}$, denoted $\mathrm{proj}_{\mathbf{b}}\mathbf{a}$, is the vector in the direction of $\mathbf{b}$ whose length is determined by drawing a line perpendicular to $\mathbf{b}$ that goes through $\mathbf{a}$. In other words, the length of $\mathrm{proj}_{\mathbf{b}}\mathbf{a}$ is the component of $\mathbf{a}$ in the direction of $\mathbf{b}$.



Projection

To compute $\text{proj}_{\mathbf{b}}\mathbf{a}$, we first note that it is a multiple of $\mathbf{b}$. Thus $\text{proj}_{\mathbf{b}}\mathbf{a} = s\mathbf{b}$ for some number s. To compute s, we use the fact that the vector $\text{proj}_{\mathbf{b}}\mathbf{a} - \mathbf{a}$ (along the dotted line in the diagram) is orthogonal to $\mathbf{b}$. Thus $(\text{proj}_{\mathbf{b}}\mathbf{a} - \mathbf{a}) \cdot \mathbf{b} = 0$, or $(s\mathbf{b} - \mathbf{a}) \cdot \mathbf{b} = 0$, or $s = \mathbf{a} \cdot \mathbf{b}/\mathbf{b} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b}/\|\mathbf{b}\|^2$. Thus

$$\text{proj}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b}.$$

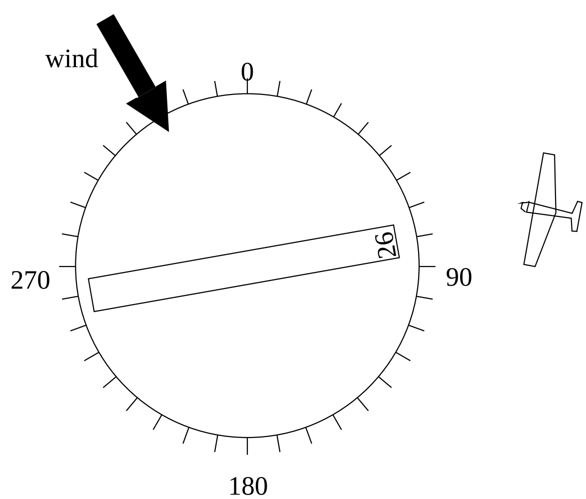
If $\mathbf{b}$ is a unit vector (i.e., $\|\mathbf{b}\| = 1$) this expression is even simpler. In this case

$$\text{proj}_{\mathbf{b}}\mathbf{a} = (\mathbf{a} \cdot \mathbf{b})\,\mathbf{b}.$$

Projections are useful for computing the components of a vector in various directions. An easy example is given be the co-ordinates of a vector. These are simply the components of a vector in the direction of the standard basis vectors. So in two dimensions

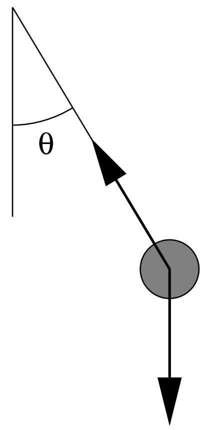
$$\begin{aligned} a_1 &= \mathbf{a} \cdot \mathbf{i} = [a_1, a_2] \cdot [1, 0] \\ a_2 &= \mathbf{a} \cdot \mathbf{j} = [a_1, a_2] \cdot [0, 1] \end{aligned}$$

Problem 1.10: An airplane with an approach speed of 70 knots is on approach to runway 26 (i.e., pointing in the direction of 260 degrees). If the wind is from 330 degrees at 10 knots, what heading should the pilot maintain to stay lined up with the runway? What is the groundspeed of the airplane?



Runway

Problem 1.11: Suppose the angle of the pendulum shaft makes an angle of θ with the vertical direction (see diagram). The force of gravity has magnitude (length) equal to mg and points downwards. The force along the shaft of the pendulum acts to keep the shaft rigid, i.e., the component of the total force along the shaft is zero. Write down the co-ordinates of the two forces and the total force using two different sets of co-ordinate axes — one horizontal and vertical, and one parallel to and orthogonal to the shaft of the pendulum.



Pendulum

The determinant in two and three dimensions

The determinant is a number that is associated with a square matrix, that is, a square array of numbers. In two dimensions it is defined by

$$\det\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = a_1b_2 - a_2b_1.$$

The definition in three dimensions is

$$\begin{aligned}\det\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} &= a_1\det\begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix} - a_2\det\begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix} + a_3\det\begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix} \\ &= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1a_2 - a_3b_2c_1\end{aligned}$$

We want to determine the relationship between the determinant and the vectors $\mathbf{a} = [a_1, a_2]$ and $\mathbf{b} = [b_1, b_2]$ (in two dimensions) and $\mathbf{a} = [a_1, a_2, a_3]$, $\mathbf{b} = [b_1, b_2, b_3]$ and $\mathbf{c} = [c_1, c_2, c_3]$ (in three dimensions). We will do the two dimensional case now, but postpone the three dimensional case until after we have discussed the cross product.

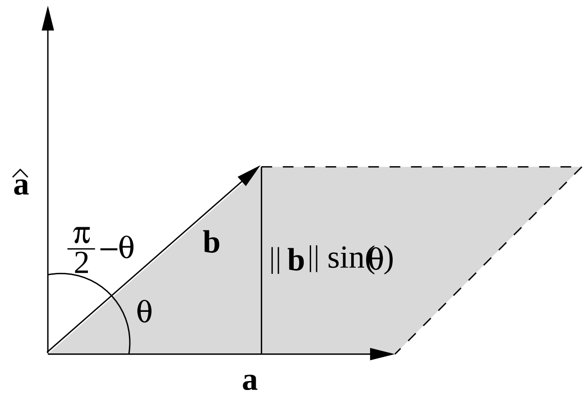
So let $\mathbf{a} = [a_1, a_2]$ and $\mathbf{b} = [b_1, b_2]$ be two vectors in the plane. Define

$$\hat{\mathbf{a}} = [-a_2, a_1].$$

Notice that $\hat{\mathbf{a}}$ has the same length as $\mathbf{a}$, and is perpendicular to $\mathbf{a}$, since

$$\hat{\mathbf{a}} \cdot \mathbf{a} = -a_2a_1 + a_1a_2 = 0.$$

There are exactly two vectors with these properties. The vector $\hat{\mathbf{a}}$ is the one that is obtained from $\mathbf{a}$ by a counterclockwise rotation of $\pi/2$ (i.e., 90°). To see this, notice that if $\mathbf{a}$ lies in the first quadrant (that is, $a_1 > 0$ and $a_2 > 0$) then $\hat{\mathbf{a}}$ lies in the second quadrant, and so on. Later in the course we will study rotations and this will be a special case.



The vector $\hat{\mathbf{a}}$

Notice now that the determinant can be written as a dot product.

$$\hat{\mathbf{a}} \cdot \mathbf{b} = -a_2 b_1 + a_1 b_2 = \det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$$

We want to use the geometric formula for the dot product of $\hat{\mathbf{a}}$ and $\mathbf{b}$. Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$ and $\pi/2 - \theta$ be the angle between $\hat{\mathbf{a}}$ and $\mathbf{b}$, as shown in the diagram.

Using the geometric meaning of the dot product, we obtain

$$\begin{aligned} \det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} &= \hat{\mathbf{a}} \cdot \mathbf{b} \\ &= \|\hat{\mathbf{a}}\| \|\mathbf{b}\| \cos(\pi/2 - \theta) \\ &= \|\mathbf{a}\| \|\mathbf{b}\| \sin(\theta) \end{aligned}$$

We need to be a bit careful here. When we were discussing the dot product, we always assumed that the angle between two vectors was in the range 0 to π. In fact, the geometric formula for the dot product is not sensitive to how we measure the angle. Suppose that instead of θ in the range 0 to π we use $\theta_1 = -\theta$ (measuring the angle "backwards") or $\theta_2 = 2\pi - \theta$ (measuring the angle going the long way around the circle). Since $\cos(\theta) = \cos(-\theta) = \cos(2\pi - \theta)$ we have

$$\mathbf{c} \cdot \mathbf{d} = \|\mathbf{c}\| \|\mathbf{d}\| \cos(\theta) = \|\mathbf{c}\| \|\mathbf{d}\| \cos(\theta_1) = \|\mathbf{c}\| \|\mathbf{d}\| \cos(\theta_2).$$

In other words, the geometric formula for the dot product still is true.

In the diagram above, we want to let the angle θ between $\mathbf{a}$ and $\mathbf{b}$ range between $-\pi$ and π. In this case the angle $\pi/2 - \theta$ between $\hat{\mathbf{a}}$ and $\mathbf{b}$ is sometimes not in the range between 0 or 2π. But if this happens, then it is still the angle between $\hat{\mathbf{a}}$ and $\mathbf{b}$, just "backwards" or "the long way around." Thus the geometric formula above still is correct.

Values of θ between 0 and π correspond to the situation where the direction of $\mathbf{b}$ is obtained from the direction of $\mathbf{a}$ by a counterclockwise rotation of less than π. This is the case in the diagram. On the other hand, θ between $-\pi$ and 0 corresponds to the case where a clockwise rotation of less than π is needed to get from the direction of $\mathbf{a}$ to the direction of $\mathbf{b}$.

The quantity $\sin(\theta)$ can be positive or negative, depending on the orientations of $\mathbf{a}$ and $\mathbf{b}$, but in any case the positive quantity $\|\mathbf{b}\| |\sin(\theta)|$ is the height of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$ if we take $\mathbf{a}$ to be the base. In this case, the length of the base is $\|\mathbf{a}\|$. Recall that the area of a parallelogram is the length of the base times the height. Thus

$$\left| \det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \right| = \text{Area of parallelogram spanned by } \mathbf{a} \text{ and } \mathbf{b}$$

The determinant is positive if $\sin(\theta)$ is positive, that is, if θ is positive. This is the case if the direction of $\mathbf{b}$ is obtained by a counterclockwise rotation of half a circle or less from the direction of $\mathbf{a}$. Otherwise the determinant is negative.

Notice that the determinant whose rows are the components of two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ is zero exactly when the vectors $\mathbf{a}$ and $\mathbf{b}$ are pointing in the same direction, or in the opposite direction, that is, if one is obtained from the other by scalar multiplication. The sign of the determinant gives information about their relative orientation.

The cross product

Unlike the dot product, the cross product is only defined for vectors in three dimensions. And unlike the dot product, the cross product of two vectors is another vector, not a number. If $\mathbf{a} = [a_1, a_2, a_3]$ and $\mathbf{b} = [b_1, b_2, b_3]$, then $\mathbf{a} \times \mathbf{b}$ is a vector given by

$$\mathbf{a} \times \mathbf{b} = [a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1].$$

An easy way to remember this is to write down a 3×3 matrix whose first row contains the unit basis vectors and whose second and third rows contain the components of $\mathbf{a}$ and $\mathbf{b}$. Then the cross product is obtained by following the usual rules for computing a 3×3 determinant.

$$\begin{aligned} \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} &= \mathbf{i} \det \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} - \mathbf{j} \det \begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} + \mathbf{k} \det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \\ &= [a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1] \end{aligned}$$

The geometric meaning of the cross product is given the following three properties.

1. $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$ and to $\mathbf{b}$

2. $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \sin(\theta)$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. In this formula, θ lies between 0 and π, so that $\sin(\theta)$ is positive. This is the same as saying that the length of $\mathbf{a} \times \mathbf{b}$ is the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.

3. The vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$ obey the right hand rule

This geometric description of the cross product shows that the definition of the cross product is independent of how we choose our co-ordinate axes.

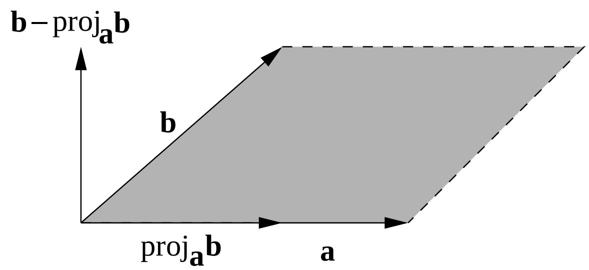
To verify 1, we compute the dot products $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})$ and $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b})$ and verify that they are zero. (This is one of the problems below.)

To verify 2 we must show that the length of $\mathbf{a} \times \mathbf{b}$ is the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$, since the quantity $\|\mathbf{a}\|\|\mathbf{b}\| \sin(\theta)$ is precisely this area.

Since both the length and the area are positive quantities , it is enough to compare their squares. We have

$$\|\mathbf{a} \times \mathbf{b}\|^2 = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \tag{1.1}$$

On the other hand, the area A of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$ is length $\|\mathbf{a}\|$ times the height. This height is the length of the vector $\mathbf{b} - \text{proj}_{\mathbf{a}}\mathbf{b} = \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{a}/\|\mathbf{a}\|^2$



The parallelogram spanned by $\|\mathbf{a}\|$ and $\|\mathbf{b}\|$

Using these facts, we arrive at the following formula for the square of the area of the parallelogram.

$$\begin{aligned} A^2 &= \|\mathbf{a}\|^2 \|\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{a}/\|\mathbf{a}\|^2\|^2 \\ &= \|\mathbf{a}\|^2 \left(\|\mathbf{b}\|^2 + (\mathbf{a}\cdot\mathbf{b})^2\|\mathbf{a}\|^2/\|\mathbf{a}\|^4 - 2(\mathbf{a}\cdot\mathbf{b})^2/\|\mathbf{a}\|^2\right) \\ &= \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a}\cdot\mathbf{b})^2 \\ &= (a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2) - (a_1b_1+a_2b_2+a_3b_3)^2 \end{aligned} \tag{1.2}$$

Expanding the expressions in (1.1) and (1.2) reveals that they are equal.

Notice that there are exactly two vectors satisfying properties 1 and 2, that is, perpendicular to the plane spanned by $\mathbf{a}$ and $\mathbf{b}$ and of a given length. The cross product of $\mathbf{a}$ and $\mathbf{b}$ is the one that satisfies the right hand rule. We say that vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ (the order is important) satisfy the right hand rule if you can point the index finger of your right hand in the direction of $\mathbf{a}$ and the middle finger in the direction of $\mathbf{b}$ and the thumb in the direction of $\mathbf{c}$. Try to convince yourself that if $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ (in that order) satisfy the right hand rule, then so do $\mathbf{b}$, $\mathbf{c}$, $\mathbf{a}$ and $\mathbf{c}$, $\mathbf{a}$, $\mathbf{b}$.

Here are some properties of the cross product that are useful in doing computations. The first two are maybe not what you expect.

1. $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$
2. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\cdot\mathbf{a})\mathbf{b} - (\mathbf{b}\cdot\mathbf{a})\mathbf{c}$.
3. $s(\mathbf{a}\times\mathbf{b}) = (s\mathbf{a})\times\mathbf{b} = \mathbf{a}\times(s\mathbf{b})$.
4. $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$.
5. $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$.

Problem 1.12: Compute $[1,2,3]\times[4,5,6]$

Problem 1.13: Verify that $\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) = 0$ and $\mathbf{b}\cdot(\mathbf{a}\times\mathbf{b}) = 0$.

Problem 1.14: Explain why $\|\mathbf{a}\|\|\mathbf{b}\|\sin(\theta)$ is the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$. Here θ is the angle between the two vectors.

Problem 1.15: Find examples to show that in general $\mathbf{a}\times\mathbf{b} \neq \mathbf{b}\times\mathbf{a}$ and $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \neq (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$.

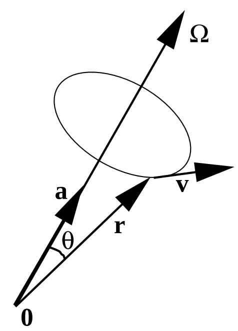
Problem 1.16: Show that $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$.

Problem 1.17: Derive an expression for $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})$ that involves dot products but not cross products.

Problem 1.18: What is the analog of the cross product in two dimensions? How about four dimensions?

The cross product and rotational motion

Consider a rigid body rotating about an axis given by the unit vector $\mathbf{a}$ at a rate of Ω radians per second. Let $\mathbf{r}$ be the position vector of a point on the body.



Rotational motion

What is the velocity of the point? The point travels on a circle with radius $\|\mathbf{r}\|\sin(\theta)$, where θ is the angle that $\mathbf{r}$ makes with the axis. Therefore, in one second, the point travels a distance of $\Omega\|\mathbf{r}\|\sin(\theta)$. Thus

(i) the magnitude of the velocity is $\|\mathbf{v}\| = \Omega\|\mathbf{r}\|\sin(\theta)$.

Now notice that

(ii) $\mathbf{v}$ is orthogonal to the plane spanned by $\mathbf{a}$ and $\mathbf{r}$.

Finally notice that

(iii) $\Omega\mathbf{a}$, $\mathbf{r}$ and $\mathbf{v}$ obey the right hand rule, for the direction of rotation shown in the diagram (that is, counterclockwise when viewed from above).

The facts (i), (ii) and (iii) imply that $\mathbf{v}$ is exactly the cross product of $\Omega\mathbf{a}$ and $\mathbf{r}$. It is customary to let $\boldsymbol{\Omega}$ denote the vector $\Omega\mathbf{a}$. Then

$$\mathbf{v} = \boldsymbol{\Omega} \times \mathbf{r}.$$

If the direction of rotation is reversed (that is, clockwise when viewed from above) then $\mathbf{v}$ points in the opposite direction, so

$$\mathbf{v} = -\boldsymbol{\Omega} \times \mathbf{r}.$$

Problem 1.19: A body rotates at an angular velocity of 10 rad/wec about the axis through the points $[1, 1, -1]$ and $[2, -3, 1]$. Find the velocity of the point $[1, 2, 3]$ on the body.

Problem 1.20: Imagine a plate that lies in the xy–plane and is rotating about the z–axis. Let P ge a point that is painted on this plane. Denote by r the distance from P to the origin, by $\theta(t)$ the angle at time t between the line from the origin to P and the x–axis and by $[x(t), y(t)]$ the co-ordinates of P at the time t. Find $x(t)$ and $y(t)$ in terms of $\theta(t)$. Compute the velocity of P in two ways: 1. by differentiating $[x(t), y(t)]$ and 2. by computing $\boldsymbol{\Omega} \times \mathbf{r}$.

The triple product and the determinant in three dimensions

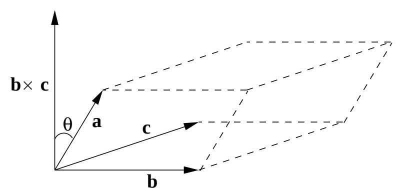
The cross product is defined so that the dot product of $\mathbf{a}$ with $\mathbf{b} \times \mathbf{c}$ is a determinant:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

This determinant is called the triple product of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Using this fact we can show that the absolute value of the triple product is the volume of the parallelepiped spanned by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Here is a diagram of the parallelepiped.



The triple product

The absolute value of the triple product is

$$|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = \|\mathbf{a}\| \cos(\theta) \|\mathbf{b} \times \mathbf{c}\|.$$

Here θ is the angle between $\mathbf{a}$ and $\mathbf{b} \times \mathbf{c}$. The quantity $\|\mathbf{a}\| \cos(\theta)$ is the height of parallelepiped and $\|\mathbf{b} \times \mathbf{c}\|$ is the area of the base. The product of these is the volume of the parallopiped, as claimed. Thus

$$\left| \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \right| = \text{Volume of the parallelepiped spanned by } \mathbf{a}, \mathbf{b} \text{ and } \mathbf{c}$$

The sign of the triple product is postitive if θ is between zero and $\pi/2$ and negative if θ lies between $\pi/2$ and π. This is the case if $\mathbf{a}$ is on the same side of the plane spanned by $\mathbf{b}$ and $\mathbf{c}$ as $\mathbf{b} \times \mathbf{c}$. This holds if the vectors $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{a}$ (in that order) satisfy the right hand rule. Equivalently $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ (in that order) sastify the right hand rule.

Mathematically, it is more satisfactory to define the right hand rule using the determinant. That is, we say that vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ satisfy the right hand rule if the determinant $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is positive.

Describing linear sets

The following sections we will consider points, lines, planes and space in two and three dimensions. Each of these sets have two complementary (or dual) descriptions. One is is called the parametric form and the other the equation form. Roughly speaking, the parametric form specifies the set using vectors that are parallel to the set, while the equation form uses vectors that are orthogonal to the set. Using the parametric description, it is easy to write down explicitly all the elements of the set, but difficult to check whether a given point lies in the set. Using the equation description its the other way around: if someone gives you a point, it is easy to check whether it lies in the set, but it is difficult to write down explicitly even a single member of the set.

One way of thinking about solving a system of linear equations is simply going from one description to the other. This will (hopefully) become clear later on.

These sections will always follow the same pattern. We will consider the parametric and equation descriptions, first in the special case when the set passes through the origin. Then we consider the general case.

Recall that when we say "the point $\mathbf{x}$" this means "the point at the head of the vector $\mathbf{x}$ whose tail is at the origin."

In these sections I have used the notation $[x_1, x_2]$ instead of $[x, y]$ and $[x_1, x_2, x_3]$ instead of $[x, y, z]$ for typical points in two and three dimensions.

Lines in two dimensions: Parametric form

First we consider lines passing through the origin. Let $\mathbf{a} = [a_1, a_2]$ be vector in the direction of the line. Then all the points $\mathbf{x}$ on the line are of the form

$$\mathbf{x} = s\mathbf{a}$$

for some number s. The number s is called a parameter. Every value of s corresponds to exactly one point (namely $s\mathbf{a}$) on the line.

Now we consider the general case. Let $\mathbf{q}$ be a point on the line and $\mathbf{a}$ lie in the direction of the line. Then the points on the line can be thought of as the points on the line through the origin in the direction of $\mathbf{a}$ shifted or translated by $\mathbf{q}$. Then a point $\mathbf{x}$ lies on the line through $\mathbf{q}$ in the direction of $\mathbf{a}$ exactly when

$$\mathbf{x} = \mathbf{q} + s\mathbf{a}$$

for some value of s.

Lines in two dimensions: Equation form

First we consider lines passing through the origin. Let $\mathbf{b} = [b_1, b_2]$ be orthogonal to the direction of line. The a point $\mathbf{x}$ is on the line exactly when $\mathbf{x} \cdot \mathbf{b} = 0$. This can be written

$$x_1 b_1 + x_2 b_2 = 0.$$

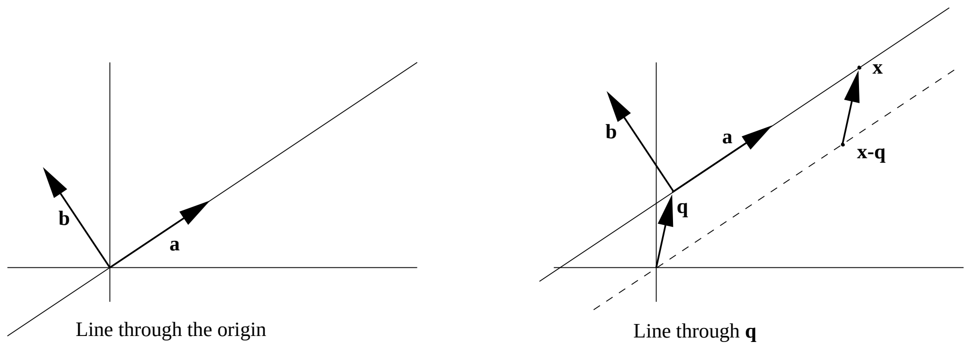
Now we consider the general case. Let $\mathbf{q}$ be a point on the line and $\mathbf{b} = [b_1, b_2]$ be orthogonal to the direction of line. A point $\mathbf{x}$ lies on the line through $\mathbf{q}$ in the direction of $\mathbf{a}$ exactly when $\mathbf{x} - \mathbf{q}$ lies on the line through the origin in the direction of $\mathbf{a}$. Thus $(\mathbf{x} - \mathbf{q}) \cdot \mathbf{b} = 0$. This can be written

$$(x_1 - q_1)b_1 + (x_2 - q_2)b_2 = 0$$

or

$$x_1 b_1 + x_2 b_2 = c,$$

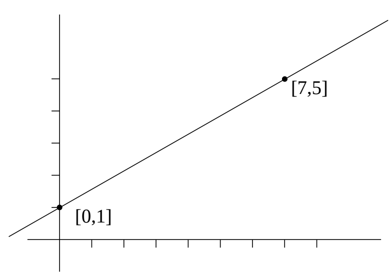
where $c = \mathbf{q} \cdot \mathbf{b}$.



A line in two dimensions

Problem 1.21: A line orthogonal to $\mathbf{b}$ can be described as the set of all points $\mathbf{x}$ whose projections onto $\mathbf{b}$ all have the same value. Using the formula for projections, show that this leads to the equation description of the line.

Problem 1.22: Find both the parametric form and equation form for the line in the diagram. Write down five points on the line (notice that the parametric form is more useful for this). Check whether the point $[\frac{1012}{3}, \frac{1069}{21}]$ is on the line (notice that the equation form is more useful for this.)



Problem 1.23: Find the equation form for the line $[1,1] + s[-1,2]$.

Problem 1.24: Find the parametric form for the line $x_1 - 3x_2 = 5$

Problem 1.25: Use a projection to find the distane from the point $[-2,3]$ to the line $3x_1 - 4x_2 = -4$

Problem 1.26: Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be the verices of a triangle. By definition, the median of a triangle is a straight line that passes through a vertex of the traingle and through the midpoint of the opposite side.

(i) Find the parametric form of the equation for each median. (ii) Do all the medians meet at a common point? If so, which point?

Lines in three dimensions: Parametric form

The parametric form of a line in three (or higher) dimensions looks just the same as in two dimensions. The points $\mathbf{x}$ on the obtained by starting at some point $\mathbf{q}$ on the line and then adding all multiples of all the multiples of a vector $\mathbf{a}$ pointing in the direction of the line. So

$$\mathbf{x} = \mathbf{q} + s\mathbf{a}$$

The only difference is that now $\mathbf{q}$ and $\mathbf{a}$ are vectors in three dimensions.

Lines in three dimensions: Equation form

We begin with lines through the origin. A line through the origin can be described as all vectors orthogonal to a plane. Choose two vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ lying in the plane that are not collinear (or zero). Then a vector is orthogonal to the plane if and only if it is orthogonal to both $\mathbf{b}_1$ and $\mathbf{b}_2$. Therefore the line consists of all points $\mathbf{x}$ such that $\mathbf{x}\cdot\mathbf{b}_1 = 0$ and $\mathbf{x}\cdot\mathbf{b}_2 = 0$. If $\mathbf{b}_1 = [b_{1,1}, b_{1,2}, b_{1,3}]$ and $\mathbf{b}_2 = [b_{2,1}, b_{2,2}, b_{2,3}]$ then these equations can be written

$$\begin{array}{ccccccc} b_{1,1}x_1 & + & b_{1,2}x_2 & + & b_{1,3}x_3 & = & 0 \\ b_{2,1}x_1 & + & b_{2,2}x_2 & + & b_{2,3}x_3 & = & 0 \end{array}$$

Notice that there are many possible choices for the vectors $\mathbf{b}_1$ and $\mathbf{b}_2$. The method of Gaussian elimination, studied later in this course, is a method of replacing the vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ with equivalent vectors in such a way that the equations become easier to solve.

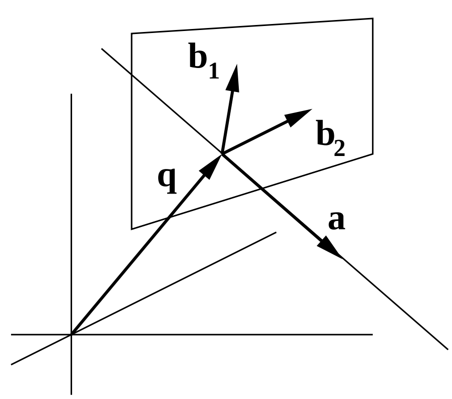
Now consider a line passing through the point $\mathbf{q}$ and orthogonal to the directions $\mathbf{b}_1$ and $\mathbf{b}_2$. A point $\mathbf{x}$ lies on this line precisely when $\mathbf{x}-\mathbf{q}$ lies on the line through the origin that is orthogonal to $\mathbf{b}_1$ and $\mathbf{b}_2$. Thus $(\mathbf{x}-\mathbf{q})\cdot\mathbf{b}_1 = 0$ and $(\mathbf{x}-\mathbf{q})\cdot\mathbf{b}_2 = 0$. This can be written

$$\begin{array}{ccccccc} b_{1,1}(x_1-q_1) & + & b_{1,2}(x_2-q_2) & + & b_{1,3}(x_3-q_3) & = & 0 \\ b_{2,1}(x_1-q_1) & + & b_{2,2}(x_2-q_2) & + & b_{2,3}(x_3-q_3) & = & 0, \end{array}$$

or

$$\begin{array}{ccccccc} b_{1,1}x_1 & + & b_{1,2}x_2 & + & b_{1,3}x_3 & = & c_1 \\ b_{2,1}x_1 & + & b_{2,2}x_2 & + & b_{2,3}x_3 & = & c_2 \end{array}$$

where $c_1 = \mathbf{q}\cdot\mathbf{b}_1$ and $c_2 = \mathbf{q}\cdot\mathbf{b}_2$



A line in three dimensions

Planes in three dimensions: Parametric form

We begin with planes through the origin. Since a plane is a two dimensional object, we will need two parameters to describe points on the plane. Let $\mathbf{a}_1$ and $\mathbf{a}_2$ be non-collinear vectors in the direction of the plane. Then every point on the plane can be reached by adding some multiple of $\mathbf{a}_1$ to some other multiple of $\mathbf{a}_2$. In other words, points $\mathbf{x}$ on the plane are all points of the form

$$\mathbf{x} = s\mathbf{a}_1 + t\mathbf{a}_2$$

for some values of s and t.

If the plane passes through some point $\mathbf{q}$ in the directions of $\mathbf{a}_1$ and $\mathbf{a}_2$, then we simply shift all the points on the parallel plane through the origin by $\mathbf{q}$. So $\mathbf{x}$ lies on the plane if

$$\mathbf{x} = \mathbf{q} + s\mathbf{a}_1 + t\mathbf{a}_2$$

for some values of s and t.

Planes in three dimensions: Equation form

A plane through the origin can be described as all vectors orthogonal to a given vector $\mathbf{b}$. (In this situation, if $\mathbf{b}$ has unit length it is called the normal vector to the plane and is often denoted $\mathbf{n}$.) Therefore $\mathbf{x}$ lies on the plane whenever $\mathbf{x} \cdot \mathbf{b} = 0$, or

$$b_1x_1 \quad + \quad b_2x_2 \quad + \quad b_3x_3 \quad = \quad 0.$$

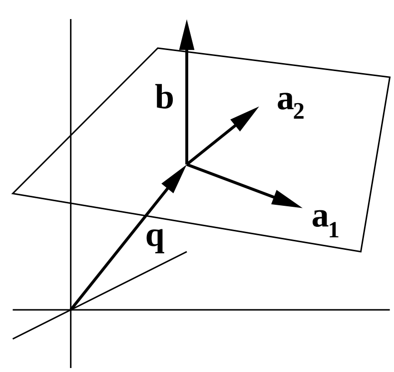
If a plane with normal vector $\mathbf{b}$ is translated so that it passes through the point $\mathbf{q}$, then $\mathbf{x}$ lies on the plane whenever $\mathbf{x}-\mathbf{q}$ lies on the parallel plane through the origin. Thus $\mathbf{x}$ lies on the plane whenever $(\mathbf{x}-\mathbf{q})\cdot\mathbf{b} = 0$. Equivalently

$$b_1(x_1 - q_1) \quad + \quad b_2(x_2 - q_2) \quad + \quad b_3(x_3 - q_3) \quad = \quad 0,$$

or

$$b_1x_1 \quad + \quad b_2x_2 \quad + \quad b_3x_3 \quad = \quad c,$$

where $\mathbf{c} = \mathbf{q} \cdot \mathbf{b}$.



A plane in three dimensions

Problem 1.27: Find the equation of the plane containing the points $[1, 0, 1]$, $[1, 1, 0]$ and $[0, 1, 1]$.

Problem 1.28: Find the equation of the sphere which has the two planes $x_1+x_2+x_3 = 3$ and $x_1+x_2+x_3 = 9$ as tangent planes if the center of the sphere is on the planes $2x_1 - x_2 = 0$, $3x_1 - x_3 = 0$.

Problem 1.29: Find the equation of the plane that passes through the point $[-2, 0, 1]$ and through the line of intesection of $2x_1 + 3x_2 - x_3 = 0$, $x_2 - 4x_2 + 2x_3 = -5$.

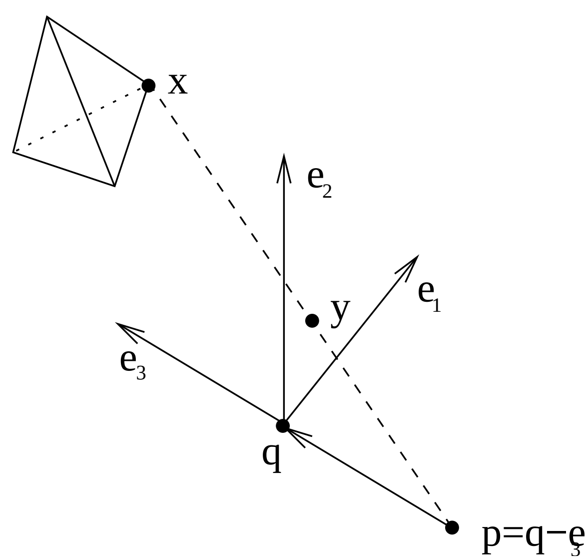
Problem 1.30: What's wrong with the question "Find the equation for the plane containing $[1, 2, 3]$, $[2, 3, 4]$ and $[3, 4, 5]$."?

Problem 1.31: Find the distance from the point $\mathbf{p}$ to the plane $\mathbf{b} \cdot \mathbf{x} = c$.

Problem 1.32: Find the equation fo the line through $[2, -1, -1]$ and parallel to each of the two planes $x_1+x_2 = 0$ and $x_1-x_2+2x_3 = 0$. Express the equation fo the line in both parametric and equation form.

Application: 3-D graphics

How can we represent a three dimensional object on piece of paper or computer screen? Imagine the object in space and, a certain distance away, a point $\mathbf{p}$ representing the eye of the observer. Between the observer and the object is a plane called the view plane. The position of the origin of this plane is described by a point $\mathbf{q}$, and its orientation is given by three orthogonal unit vectors of length 1, denoted $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. (These are not the same as the standard basis vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ in this problem.) As usual, only the heads of the vectors (points) $\mathbf{p}$, $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{q}$ are shown on the diagram. (The origin, where the tails of these vectors lie, is not depicted at all.) We will assume that the view plane is a distance one from the observer in the direction $\mathbf{e}_3$. Thus, $\mathbf{e}_3$ can be thought of as the direction that the observer is looking. Think of light rays leaving the object at point $\mathbf{x}$ and travelling to the observer's eye at $\mathbf{p}$. At some point $\mathbf{y}$ this line intersects the view plane. All the vectors $\mathbf{y}$ on the view plane that correspond to some vector $\mathbf{x}$ on our object will furnish the two dimensional representation of the object.



Perspective

How do we determine the point $\mathbf{y}$? The parametric form of points on the plane is $\mathbf{q} + s_1\mathbf{e}_1 + s_2\mathbf{e}_2$. So we must have that $\mathbf{y} = \mathbf{q} + s_1\mathbf{e}_1 + s_2\mathbf{e}_2$ for some values of s_1 and s_2. We also know that the vector $\mathbf{x} - \mathbf{p}$ is in the same direction as $\mathbf{y}-\mathbf{p}$. Therefore they must be multiples, i.e., $\mathbf{y}-\mathbf{p}=\lambda(\mathbf{x}-\mathbf{p})$ for some number λ. Substituting in our expression for $\mathbf{y}$ yields

$$\mathbf{q}+s_1\mathbf{e}_1+s_2\mathbf{e}_2-\mathbf{p}=\lambda(\mathbf{x}-\mathbf{p}).$$

Since $\mathbf{q}-\mathbf{p}=\mathbf{e}_3$ this gives

$$\mathbf{e}_3+s_1\mathbf{e}_1+s_2\mathbf{e}_2=\lambda(\mathbf{x}-\mathbf{p}).$$

Let us take the dot product of both sides of this equation with the unit vectors $\mathbf{e}_3$, $\mathbf{e}_1$ and $\mathbf{e}_2$. We can use the fact that $\mathbf{e}_i\cdot\mathbf{e}_j$ is zero if $i\neq j$ and 1 if $i=j$. Start with $\mathbf{e}_3$. This gives

$$\mathbf{e}_3\cdot\mathbf{e}_3=1=\lambda\mathbf{e}_3\cdot(\mathbf{x}-\mathbf{p}).$$

This determines λ.

$$\lambda=\frac{1}{\mathbf{e}_3\cdot(\mathbf{x}-\mathbf{p})}.$$

Now take the dot product with $\mathbf{e}_1$. This gives

$$s_1=\lambda\mathbf{e}_1\cdot(\mathbf{x}-\mathbf{p})=\frac{\mathbf{e}_1\cdot(\mathbf{x}-\mathbf{p})}{\mathbf{e}_3\cdot(\mathbf{x}-\mathbf{p})}$$

Similarly, taking the dot product with $\mathbf{e}_2$ leads to

$$s_2=\frac{\mathbf{e}_2\cdot(\mathbf{x}-\mathbf{p})}{\mathbf{e}_3\cdot(\mathbf{x}-\mathbf{p})}$$

To plot the image of an object, we now simply plot the co-ordinates s_1 and s_2 corresponding to all the points on the object on the s_1–s_2 plane.

As an example, lets take $\mathbf{p}=[11,0,0]$, $\mathbf{q}=[10,0,0]$, $\mathbf{e}_1=[0,1,0]$, $\mathbf{e}_2=[0,0,1]$ and $\mathbf{e}_3=[-1,0,0]$.

What is the image of the point $\mathbf{x}=[1,1,1]$? We compute $\mathbf{x}-\mathbf{p}=[-10,1,1]$ so that

$$\begin{aligned}\mathbf{e}_1\cdot(\mathbf{x}-\mathbf{p})&=1\\ \mathbf{e}_2\cdot(\mathbf{x}-\mathbf{p})&=1\\ \mathbf{e}_3\cdot(\mathbf{x}-\mathbf{p})&=10\end{aligned}$$

So $s_1=s_2=1/10$.

Now let us compute the image of a line segment between $[1,1,1]$ and $[2,0,1]$. These are all points of the form $\mathbf{x}=[1,1,1]+t([2,0,1]-[1,1,1])=[1+t,1-t,1]$ as t varies between 0 and 1. This time we have $\mathbf{x}-\mathbf{p}=[1+t-11,1-t,1]$ so that

$$\begin{aligned}\mathbf{e}_1\cdot(\mathbf{x}-\mathbf{p})&=1-t\\ \mathbf{e}_2\cdot(\mathbf{x}-\mathbf{p})&=1\\ \mathbf{e}_3\cdot(\mathbf{x}-\mathbf{p})&=10-t\end{aligned}$$

Thus $s_1=(1-t)/(10-t)$ and $s_2=1/(10-t)$. Even though it is not immediately obvious, the points $[s_1,s_2]$, as t varies, all lie on a line segment. In fact

$$s_1+9s_2=(1-t)/(10-t)+9/(10-t)=(10-t)/(10-t)=1.$$

This shows that the points $[s_1,s_2]$ lie on a line perpendicular to $[1,9]$.

In fact, it is possible to show that any line segment in space maps to a line segment on the $s1$–$s2$ plane. Thus, to plot the image of an object consisting of staight line segments (such as the tetrahedron in the picture) it is only neccesary to plot the vertices and then join them by straight lines.

Problem 1.33: What are the s_1 and s_2 co=ordinates of the point $\mathbf{x}=[1,2,3]$, if $\mathbf{p}$, $\mathbf{q}$ are as above, $\mathbf{e}_1=[0,1/\sqrt{2},1/\sqrt{2}]$ and $\mathbf{e}_2=[0,-1/\sqrt{2},1/\sqrt{2}]$.

Problem 1.34: Plot the image on the s_1–s_2 plane of the tetrahedron whose vertices are located at $[0,0,0]$, $[0,1,0]$, $[0,1/2,\sqrt{3}/2]$ and $[\sqrt{3}/6,\sqrt{6}/3,1/2]$ (Use the same values as before: $\mathbf{p}=[-10,0,0]$, $\mathbf{q}=[10,0,0]$, $\mathbf{e}_1=[0,1,0]$, $\mathbf{e}_2=[0,0,1]$ and $\mathbf{e}_3=[-1,0,0]$.)

Problem 1.35: Suppose that that points $\mathbf{x}$ lie on the line $\mathbf{x}=\mathbf{x}_0+t\mathbf{v}$. Show that correponding planar points $[s_1,s_2]$ also lie on a line. (Hint: show that there are numbers a, b, c that do not depend on t, so that $as_1+bs_2=c$ for every t.)

Problem 1.36: Consider a different drawing procedure where the point $\mathbf{x}$ maps to the point on the view plane given by the intersection of the plane with the line through $\mathbf{x}$ parallel to $\mathbf{e}_3$. Find a formula for the s_1 and s_2 co-ordinates of $\mathbf{x}$.

Description of points and the geometry of solutions to systems of equations

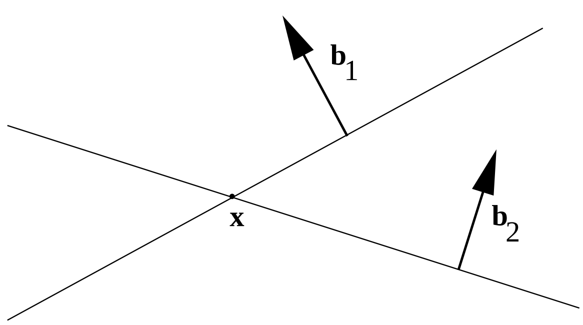
So far we have considered the parametric and equation descriptions of lines and planes in two and three dimensions. We can also try to describe points in the same way. This will help you get a geometric picture of what it means to solve a system of equations.

The "parametric" description of a point doesn't have any parameters! It simply is the name of the point $\mathbf{x}=\mathbf{q}$.

In two dimensions the equation form for describing a point will look like

$$\begin{array}{ccccc} b_{1,1}x_1 & + & b_{1,2}x_2 & = & c_1 \\ b_{2,1}x_1 & + & b_{2,2}x_2 & = & c_2 \end{array}$$

where the vectors $\mathbf{b}_1=[b_{1,1},b_{1,2}]$ and $\mathbf{b}_2=[b_{2,1},b_{2,2}]$ are not colinear. Each one of this equations describes a line. The point $\mathbf{x}=[x_1,x_2]$ will satisfy both equations if it lies on both lines, i.e., on the intersection. Since the vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ are not co-linear, the lines are not parallel, so the intersection is a single point.



Intersections

In three dimensions the equation form for describing a point will look like

$$\begin{array}{ccccccc} b_{1,1}x_1 & + & b_{1,2}x_2 & + & b_{1,3}x_3 & = & c_1 \\ b_{2,1}x_1 & + & b_{2,2}x_2 & + & b_{2,3}x_3 & = & c_2 \\ b_{3,1}x_1 & + & b_{3,2}x_2 & + & b_{3,3}x_3 & = & c_3 \end{array}$$

$\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$ don't all lie on the same plane. This can be interpreted as the intersection of three planes in a single point.

Notice that going from the equation description of a point to the parametric description just means finding the solution of the system of equations. If, in two dimensions, the vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ are not colinear , or, in three dimensions, $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$ don't all lie on the same plane, then the system of equations has a unique solution.

Now suppose that you are handed an arbitrary system of equations

$$\begin{array}{ccccccc} b_{1,1}x_1 & + & b_{1,2}x_2 & + & b_{1,3}x_3 & = & c_1 \\ b_{2,1}x_1 & + & b_{2,2}x_2 & + & b_{2,3}x_3 & = & c_2 \\ b_{3,1}x_1 & + & b_{3,2}x_2 & + & b_{3,3}x_3 & = & c_3 \end{array}$$

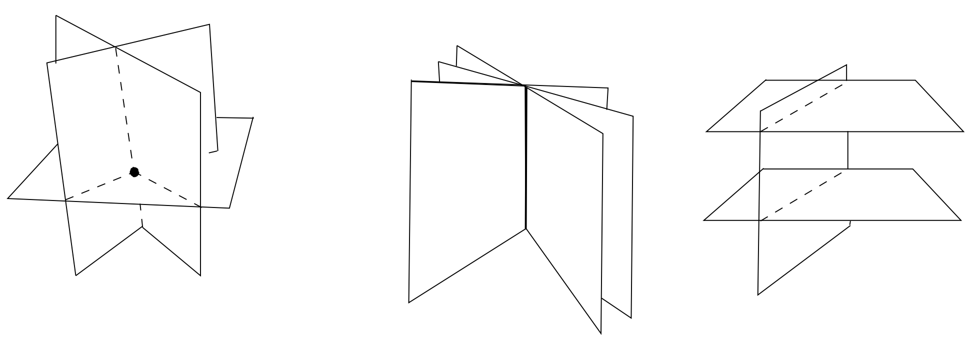
What does the set of solutions $\mathbf{x} = [x_1, x_2, x_3]$ look like? As we just have seen, if $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$ don't all lie on the same plane, there is a unique solution given as the intersection of three planes. Recall that the determinant can be used to test whether the vectors $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$ lie on the same plane. So a unique solution exists to the equation precisely when

$$\det \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{bmatrix} \neq 0$$

What happens when the determinant is zero and three vectors $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$ do lie on the same plane? Then it could be that the three planes intersect in a line. In this case every point on that line is a solution of the system of equations, and the solution set has a parametric description of the form $\mathbf{x} = \mathbf{q} + s\mathbf{a}$.

It could also be that all three planes are the same, in which case the solution set is the plane. In this case the solution set has a parametric description of the form $\mathbf{x} = \mathbf{q} + s_1\mathbf{a}_1 + s_2\mathbf{a}_2$

Another possibility is that two of the planes could be parallel with no intersection. In this case there are no solutions at all! Some of these possibilities are illustrated below:

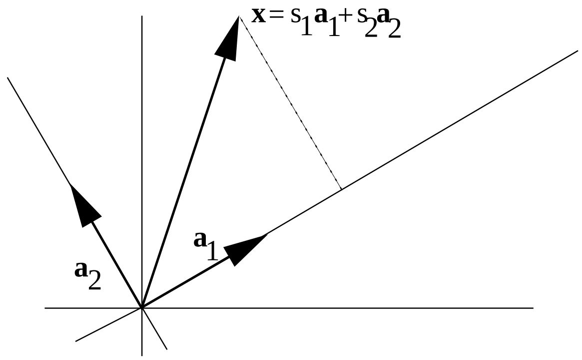


Planes intersecting

Describing the whole plane in two dimensions and all of space in three dimensions

If the set we are trying to describe is the whole plane in two dimensions or all of space in three dimensions, then we don't need any equations, since there are no restrictions on the points. However it does make sense to think about the parametric form.

Lets start with two dimensions. If we pick any two vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ that don't lie on the same line, then any vector $\mathbf{x} = [x_1, x_2]$ in the plane can be written as $s_1\mathbf{a}_1 + s_2\mathbf{a}_2$. Notice that every choice of s_1 and s_2 corresponds to exactly one vector $\mathbf{x}$. In this situation we could use the parameters s_1 and s_2 as co-ordinates instead of x_1 and x_2. In fact if $\mathbf{a}_1$ and $\mathbf{a}_2$ are unit vectors orthogonal to each other, this just amounts to changing the co-ordinate axes to lie along $\mathbf{a}_1$ and $\mathbf{a}_2$. (The new co-ordinates $[s_1, s_2]$ are then just what we were calling $[x_1', x_2']$ before.) (In fact, even if the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are not unit vectors orthogonal to each other, we can still think of them of lying along new co-ordinate axes. However, now the axes have been stretched and sheared instead of just rotated, and need not lie at right angles any more.)



A basis

The situation in three dimensions is similar. Now we must pick three vectors $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ that don't lie on the same plane. Then every vector $\mathbf{x}$ has a unique representation $\mathbf{x} = s_1\mathbf{a}_1 + s_2\mathbf{a}_2 + s_3\mathbf{a}_3$. Again, we could use s_1, s_2 and s_3 as co-ordinates in place of x_1, x_2 and x_3. Again, if $\mathbf{a}_1$, $\mathbf{a}_2$ and $3\mathbf{a}_3$ are orthogonal with unit length, then this amounts to choosing new (orthogonal) co-ordinate axes.

Linear dependence and independence

The condition in two dimensions that two vectors are not co-linear, and the condition in three dimensions that three vectors do not lie on the same plane has now come up several times — in ensuring that a system of equations has a unique solutions and in ensuring that every vector can be written in a unique way in parametric form using those vectors. This condition can be tested by computing a determinant.

We will now give this condition a name and define the analogous condition in any number of dimensions.

First, some terminology. If $\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_n$ are a collection of vectors then a vector of the form

$$s_1\mathbf{a}_1 + s_2\mathbf{a}_2 + \cdots s_n\mathbf{a}_n$$

for some choice of numbers $s_1, \ldots s_n$ is called a *linear combination* of $\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_n$.

Now, the definition. A collection of vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_n$ is called *linearly dependent* if some linear combination of them equals zero, i.e.,

$$s_1\mathbf{a}_1 + s_2\mathbf{a}_2 + \cdots s_n\mathbf{a}_n = \mathbf{0}$$

for $s_1, \ldots s_n$ not all zero. A collection of vectors is said to be *linearly independent* if it is not linearly dependent. In other words, the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_n$ are linearly independent if the only way a linear combination of them $s_1\mathbf{a}_1 + s_2\mathbf{a}_2 + \cdots s_n\mathbf{a}_n$ can equal zero is for $s_1 = s_2 = \cdots = s_n = 0$.

What does linear dependence mean in three dimensions? Suppose that $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ are linearly dependent. Then there are some numbers s_1, s_2 and s_3, not all zero, such that

$$s_1\mathbf{a}_1 + s_2\mathbf{a}_2 + s_3\mathbf{a}_3 = \mathbf{0}.$$

Suppose that s_1 is one of the non-zero numbers. Then we can divide by $-s_1$ and find that

$$-\mathbf{a}_1 + s_2'\mathbf{a}_2 + s_3'\mathbf{a}_3 = \mathbf{0}$$

for $s_2' = -s_2/s_1$ and $s_3' = -s_3/s_1$. Thus

$$\mathbf{a}_1 = s_2'\mathbf{a}_2 + s_3'\mathbf{a}_3,$$

or $\mathbf{a}_1$ is a linear combination of $\mathbf{a}_2$ and $\mathbf{a}_3$. But this implies that $\mathbf{a}_1$ lies on the plane spanned by $\mathbf{a}_2$ and $\mathbf{a}_3$, i.e., the vectors all lie on the same plane. If s_1 happens to be zero we can repeat the same argument with one of the s_i's which is not zero. Thus linear dependence implies that all three vectors lie on the same plane. Conversely, if all three vectors lie on the same plane, then we can write one vector as a linear combination of the other two, $\mathbf{a}_1 = s_2'\mathbf{a}_2 + s_3'\mathbf{a}_3$ which implies $-\mathbf{a}_1 + s_2'\mathbf{a}_2 + s_3'\mathbf{a}_3 = \mathbf{0}$ which says that the vectors are linearly dependent.

So in three dimensions, linear dependence means the vectors lie on the same plane.

Similarly, in two dimensions, linear dependence means the vectors are co-linear.

One final piece of terminology. A collection of n linearly independent vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_n$ in n dimensional space is called a *basis*.

If $\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_n$ is a basis, then every vector $\mathbf{x}$ can be written in a unique way as a linear combination

$$\mathbf{x} = s_1\mathbf{a}_1 + s_2\mathbf{a}_2 + \cdots s_n\mathbf{a}_n$$

Problem 1.37: Is the collection of vectors $\mathbf{a}_1 = [1, 1]$, $\mathbf{a}_2 = [1, 0]$ a basis for two dimensional space? If so, express the vector $\mathbf{x} = [0, 1]$ as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$

Problem 1.38: Is it possible for four vectors to be linearly independent in three dimensional space?

Problem 1.39: Suppose that $\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_n$ is a basis. Show that if some vector $\mathbf{x}$ has representation $\mathbf{x} = s_1\mathbf{a}_1 + s_2\mathbf{a}_2 + \cdots s_n\mathbf{a}_n$ and $\mathbf{x} = t_1\mathbf{a}_1 + t_2\mathbf{a}_2 + \cdots t_n\mathbf{a}_n$, then $s_1 = t_1$, $s_2 = t_2$,. . .,$s_n = t_n$. (Hint: subtract the two expressions for $\mathbf{x}$ and use the fact that the basis vectors are linearly independent.)
