Math 152: Linear Systems – Winter 2003

Section 2: Systems of Linear Equations and Gaussian Elimination

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Systems of linear equations

So far, we have seen systems of linear equations as the equations that describe points, lines and planes. However, linear systems of equations show up in many other ways in engineering problems. We will look at examples involving computing the equilibrium configuration of a system of weights and springs and of a LCR circuit. Other examples would be the calculation of equilibrium temperature distributions or electric fields. Such examples often involve the discretization of a continuous function. In other words, a continuous function like the temperature distribution in a body (which has a value for each of the infinitely many points in the body) will be replaced by a list of temperatures at a large but finite number n of closely spaced points. This gives rise to a system of linear equations in n unknowns, where n can be in the tens of thousands, or higher. Therefore, we want to develop a technique to solve systems of linear equations in n unknowns when n is large.

The most general form of a linear system of equations is

$$\begin{array}{ccccccc}
a_{1,1}x_1 & + & a_{1,2}x_2 & +\cdots & + & a_{1,n}x_n & = & c_1 \\
a_{2,1}x_1 & + & a_{2,2}x_2 & +\cdots & + & a_{2,n}x_n & = & c_2 \\
\vdots & \vdots & & & & \vdots & & \vdots \\
a_{m,1}x_1 & + & a_{m,2}x_2 & +\cdots & + & a_{m,n}x_n & = & c_m
\end{array}$$

Here the numbers $a_{i,j}$ and c_j are known, and the goal is to find all values of $x_1, \ldots, x_n$ that satisfy all the equations.

Lets start with some examples. Consider the system of equations

$$\begin{array}{ccccccc}
x_1 & + & x_2 & + & x_3 & = & 6 \\
x_1 & - & x_2 & + & x_3 & = & 0 \\
2x_1 & + & x_2 & - & 8x_3 & = & -11
\end{array}$$

One could try to proceed as follows. Solve the first equations for, say, x_3. This gives

$$x_3 = 6 - x_1 - x_2.$$

Now substitute this value for x_3 into the second and third equations. This gives

$$\begin{array}{ccccccc}
x_1 & - & x_2 & + & (6 - x_1 - x_2) & = & 0 \\
2x_1 & + & x_2 & - & 8(6 - x_1 - x_2) & = & -11
\end{array}$$

or

$$-2x_2 = -6$$
$$10x_1 + 9x_2 = 37$$

Now solve the first of these equations for x_2 and substitute into the last equation. This gives $x_2 = 3$ and $x_1 = 1$. Finally we can go back and calculate $x_3 = 6 - 1 - 3 = 2$.

Although this procedure works fine for $n = 2$ or even $n = 3$, it rapidly becomes unwieldy for larger values of n. We will now introduce a technique called Gaussian elimination that works well for large n and can be easily implemented on a computer.

We have already observed that there may be many systems of equations with the same solution. When there are only two unknowns, this amounts to saying that different pairs of lines may intersect in the same point. Gaussian elimnation is based on the following idea. We introduce three elementary row operations. These operations change the system of the equations into another system with exactly the same the set of solutions. We then apply these elementary row operations in a systematic way to change the system of equations into a system that is easily solved.

Elementary row operations

The first elementary row operation is

1. *Multiplication of a row by a non-zero number*

For example, if we multilply the first equation in the system above by 3, we end up with

$$\begin{array}{rcrcrcr} 3x_1 & + & 3x_2 & + & 3x_3 & = & 18 \\ x_1 & - & x_2 & + & x_3 & = & 0 \\ 2x_1 & + & x_2 & - & 8x_3 & = & -11 \end{array}$$

This new system of equations has exactly the same solutions as the original system, because we can undo the elementary row operation simply by dividing the first equation by 3. Thus the values x_1, x_2, x_3 solve this system if and only if they solve the original system. (Notice that this would not be true if we multiplied by zero. In that case we could not undo the operation, and the new system of equations could well have more solutions than the original system.)

The second elementary row operation is

2. *Adding a multiple of one row to another row*

For example, if added 2 times the first row to the second row in our example we would obtain the system

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 6 \\ 3x_1 & + & x_2 & + & 3x_3 & = & 12 \\ 2x_1 & + & x_2 & - & 8x_3 & = & -11 \end{array}$$

Again, the new system of equations has exactly the same solutions as the original system, since we could undo this elementary row operation by subtracting 2 times the first row from the second row.

The third and final elementary row operation is

3. *Interchanging two rows*

For example, if we swapped the first and second equations in our original system we would end up with

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & x_3 & = & 0 \\ x_1 & + & x_2 & + & x_3 & = & 6 \\ 2x_1 & + & x_2 & - & 8x_3 & = & -11 \end{array}$$

This obviously doesn't change the solutions of the system.

Problem 2.1: Start with the system

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 6 \\ x_1 & - & x_2 & + & x_3 & = & 0 \\ 2x_1 & + & x_2 & - & 8x_3 & = & -11 \end{array}$$

and perform the following sequence of row operations:

1. Subtract the first row from the second row
2. Subtract twice the first row from the third row
3. Multiply the second row by $-1/2$
4. Add the second row to the third row
5. Multiply the third row by $-1/10$

Solve the resulting system of equations by starting with the third equation, then the second and then the first.

Gaussian Elimination

To save unnecessary writing, we now set up an streamlined notation for systems of linear equations. Notice that the only thing that distinguished one system of equations from another are the coefficients. So, as shorthand, we can write the system of equations

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 3 \\ x_1 & - & x_2 & + & x_3 & = & 3 \\ 2x_1 & + & x_2 & - & 8x_3 & = & -4 \end{array}$$

as

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 3 \\ 1 & -1 & 1 & 3 \\ 2 & 1 & -8 & -4 \end{array}\right]$$

This is called an augmented matrix. "Augmented" refers to the column to the right of the line that contains the information about the right side of each equation.

Recall that we want to use a sequence of elementary row operations to turn an arbitary system of equations into an easily solved system of equations (with exactly the same solutions). What equations are easily solved? Well, the easiest possible equations to solve are ones of the form

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -4 \end{array}\right]$$

If we translate from the shorthand back to the equations they represent, the first row says $x_1 = 3$, the second row says $x_2 = 3$ and the third row says $x_3 = -4$. In other words, we can just read off the values of x_1, x_2 and x_3 in the rightmost column. The equations are already solved, and there is nothing left to do!

Slightly more work, but still easy to do, are upper triangular systems. These are systems where all the entries below the diagonal are equal to zero, as in

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 3 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & -4 & -8 \end{array}\right]$$

The reason these are easy to solve is that the equation represented by the jth row only involves the variables $x_j, x_{j+1}, \ldots, x_n$. So if we start with the last equation (in the example $-8x_3 = -4$), we can solve it immediately for x_n (in the example $x_3 = 2$). Now we move up one equation. This equation only involves x_{n-1} and x_n, and we already know x_n. So we can solve it for x_{n-1} (in the example $-x_2 + x_3 = 3$ so $-x_2 + 2 = 3$ so $x_2 = -1$). We can continue in this way until all the x_n's have been found. (In the example there is one more equation $x_1 + x_2 + x_3 = 3$ or $x_1 - 1 + 2 = 3$ or $x_1 = 2$.)

Problem 2.2: Show that the lower triangular system of equations represented by

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 3 \\ 1 & -1 & 0 & 3 \\ 2 & 1 & -8 & -4 \end{array}\right]$$

is also easily solved, by easily solving it! It's just a matter of convention whether we aim for upper triangular or lower triangular systems in the elimination procedure.

In practise (i.e., in a typical computer program) the procedure that is actually used is to apply a sequence of row operations to turn the system of equations into an upper triangular system. Then the equations are solved one by one, starting at the bottom and working up. This is the most efficient way to solve a system of equations. However, its sometimes convenient to apply row operations to bring the equation into the "completely solved" form. Then, you can just read off the solution from the last column.

Let us now do a bunch of examples to illustrate this procedure. I'll cook them up so that everything that possibly can go wrong, does go wrong. (I'll also cook them up so that the numbers come out looking nice. This will definitely not be the case in an example coming up in a real application!). Here is a shorthand for indicating which elementary row operation was done. The notation 3(1) means the first row was multiplied by the non-zero number 3. The notation $(2) - 4(5)$ means that 4 times the fifth row was subtracted from the second row. Finally, $(2) \leftrightarrow (3)$ means that the second and third row were interchanged.

Lets start with

$$\left[\begin{array}{cccc|c} 1 & 2 & -2 & -7 & -29 \\ 1 & 2 & -1 & -5 & -18 \\ 0 & 3 & 0 & -3 & -6 \\ -1 & 4 & 1 & 1 & 14 \end{array}\right].$$

We are trying to put this matrix in upper triangular form. So we start by trying to produce zero entries in the first column under the top entry. We can do this by adding multiples of the first row to the other rows. So, the first move is to subtract the first row from the second row. The result is

$$\left[\begin{array}{cccc|c} 1 & 2 & -2 & -7 & -29 \\ 0 & 0 & 1 & 2 & 11 \\ 0 & 3 & 0 & -3 & -6 \\ -1 & 4 & 1 & 1 & 14 \end{array}\right] \begin{array}{l} \\ (2)-(1) \\ \\ \\ \end{array}$$

The third row already has a zero in the first column, so there is nothing to do here. To put a zero in the fourth row we add the first row to the last row.

$$\left[\begin{array}{cccc|c} 1 & 2 & -2 & -7 & -29 \\ 0 & 0 & 1 & 2 & 11 \\ 0 & 3 & 0 & -3 & -6 \\ 0 & 6 & -1 & -6 & -15 \end{array}\right] \begin{array}{l} \\ \\ \\ (4)+(1) \end{array}$$

Now we shift our attention to the second column. We want to produce zeros below the diagonal. If we attempt to do this by adding multiples of the first row to other rows, we will destroy the zeros that we have already produced. So we try to use the second row. This is where we run into the first glitch. Since the diagonal entry in the second row is zero, adding a multiple of this row to the others won't have any effect on the numbers in the second column that we are trying to change. To rememdy this we simply swap the second and third columns.

$$\left[\begin{array}{cccc|c} 1 & 2 & -2 & -7 & -29 \\ 0 & 3 & 0 & -3 & -6 \\ 0 & 0 & 1 & 2 & 11 \\ 0 & 6 & -1 & -6 & -15 \end{array}\right] \begin{array}{l} \\ (2)\leftrightarrow(3) \\ (2)\leftrightarrow(3) \\ \\ \end{array}$$

Now we can complete the job on the second column by subtracting 2 times the second row from the last row.

$$\left[\begin{array}{cccc|c} 1 & 2 & -2 & -7 & -29 \\ 0 & 3 & 0 & -3 & -6 \\ 0 & 0 & 1 & 2 & 11 \\ 0 & 0 & -1 & 0 & -3 \end{array}\right] \begin{array}{l} \\ \\ \\ (4)-2(2) \end{array}$$

Now we shift our attention to the third column. To produce a zero in the entry below the diagonal we must add the third row to the last row.

$$\begin{array}{cl}\left[\begin{array}{cccc|c} 1 & 2 & -2 & -7 & -29 \\ 0 & 3 & 0 & -3 & -6 \\ 0 & 0 & 1 & 2 & 11 \\ 0 & 0 & 0 & 2 & 8 \end{array}\right] & \begin{array}{l} \\ \\ \\ (4)+(3) \end{array}\end{array}$$

The matrix is now in upper triangular form. Lets find the solution. This last row is shorthand for the equation $2x_4 = 8$. So $x_4 = 2$. The third row now gives $x_3 + 2(4) = 11$, so $x_3 = 3$. The second row gives $3x_2 - 3(4) = -6$ so $x_2 = 2$. Finally the first row gives $x_1 + 2(2) - 2(3) - 7(4) = -29$ so $x_1 = 1$.

There is really no need to do anything more, but lets continue with elementary row operations to put the equations into the "completely solved" form, just to see how this goes. First we divide the second row by 3.

$$\begin{array}{cl}\left[\begin{array}{cccc|c} 1 & 2 & -2 & -7 & -29 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 2 & 11 \\ 0 & 0 & 0 & 2 & 8 \end{array}\right] & \begin{array}{l} \\ (1/3)(2) \\ \\ \\ \end{array}\end{array}$$

Now we subtract twice the second row from the first row.

$$\begin{array}{cl}\left[\begin{array}{cccc|c} 1 & 0 & -2 & -5 & -25 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 2 & 11 \\ 0 & 0 & 0 & 2 & 8 \end{array}\right] & \begin{array}{l} (1)-2(2) \\ \\ \\ \\ \end{array}\end{array}$$

Now add twice the third row to the first row. Then divide the last row by 2.

$$\begin{array}{cl}\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -3 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 2 & 11 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right] & \begin{array}{l} (1)+2(3) \\ \\ \\ (1/2)(4) \end{array}\end{array}$$

Finally, we add various multiples of the last row to the previous rows.

$$\begin{array}{cl}\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right] & \begin{array}{l} (1)+(4) \\ (2)+(4) \\ (3)-2(4) \\ \\ \end{array}\end{array}$$

We now can read the solution off from the last column.

In the previous example there was a unique solution to the system of equations. We already know, from the geometrical meaning of the equations, that sometimes there will be lots of solutions depending on a parameter. This is expected to happen when there are fewer equations than unknowns (e.g., the intersections of two planes in three dimensional space is usually a line) but will also occur in certain degenerate cases when the number of equations is equal to or more than the number of unknown (e.g., three, or even four, planes may intersect in a line too). What happens in the procedure of row reductions when there are parameters in the solution? Lets look at another example.

$$\left[\begin{array}{cccc|c} 1 & 3 & 2 & -2 & -1 \\ 1 & 3 & 4 & -2 & 3 \\ -2 & -6 & -4 & 5 & 5 \\ -1 & -3 & 2 & 1 & 6 \end{array}\right]$$

We begin, as before, by trying to produce zeros in the first column under the diagonal entry. This procedure yields

$$\begin{array}{cl}\left[\begin{array}{cccc|c} 1 & 3 & 2 & -2 & -1 \\ 0 & 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 4 & -1 & 5 \end{array}\right] & \begin{array}{l} \\ (2)-(1) \\ (3)+2(1) \\ (4)+(1) \end{array}\end{array}$$

As in the previous example, there is now a zero sitting in the diagonal spot in the second column. Last time, we swapped rows at this point to put a non-zero entry in this place. But now, all the other entries below this one are zero too! So there is nothing we can swap in to save the situation. (Clearly, swapping the first row down is not a good idea, since that would destroy the zero in the first column.) So we just have to admit defeat, and shift our attention one column to the right. We subtract twice the second row from the fourth row.

$$\left[\begin{array}{cccc|c} 1 & 3 & 2 & -2 & -1 \\ 0 & 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -1 & -3 \end{array}\right] \begin{array}{l} \\ (2)-(1) \\ \\ (4)-2(2) \end{array}$$

Now we complete the job by adding the third row to the last row.

$$\left[\begin{array}{cccc|c} 1 & 3 & 2 & -2 & -1 \\ 0 & 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ (2)-(1) \\ \\ (4)+(3) \end{array}$$

What the solutions? The third equation says $x_4 = 3$ and the second equation says $x_3 = 2$. There is nothing new here. However the first equation introduces not just one, but two new variables x_1 and x_2. It reads $x_1 + 3x_2 + 2(2) - 2(3) = -1$, or, $x_1 + 3x_2 = 1$ Clearly, there are infinitely many values of x_1 and x_2 that satisfy this equation. In fact, if we fix x_2 to be any arbitrary value $x_2 = s$, and then set $x_1 = 1 - 3s$, x_1 and x_2 will be solutions. So for any choice of s

$$x_1 = 1 - 3s, x_2 = s, x_3 = 2, x_4 = 3$$

is a solution. There are infinitely many solutions depending on a parameter s. We could also write this as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix} + s \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

and recognize the solutions as a line in four dimensional space passing through $[1, 0, 2, 3]$ in the direction $[-3, 1, 0, 0]$.

Problem 2.3: The following equations have already been put in upper triangular form. In each case there are infinitely many solutions, depending on one or more parameters. Write down the general expression for the solution in terms of parameters.

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 4 \end{array}\right]$$

There is one situation left to consider, namely when there are no solutions at all. Geometrically, this happens, for example, when we are trying to find the intersection of two parallel planes. Lets look at an example.

$$\left[\begin{array}{cc|c} 1 & 3 & 1 \\ 1 & 4 & 2 \\ -1 & -3 & 0 \\ 2 & 6 & 4 \end{array}\right]$$

We begin in the usual way.

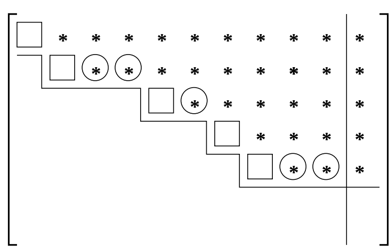
$$\begin{array}{cl} \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{array}\right] & \begin{array}{l} \\ (2)-(1) \\ (3)+(1) \\ (4)-2(1) \end{array} \end{array}$$

There is nothing left to do in the second column, so we shift our attention to the third column and subtract twice the third row from the fourth row.

$$\begin{array}{cl} \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right] & \begin{array}{l} \\ \\ \\ (4)-2(3) \end{array} \end{array}$$

Now we are done. If we write down the equation corresponding to the third row, we get $0x_1 + 0x_2 = 1$, or $0 = 1$. Clearly there is no choice of x_1 or x_2 that makes this true. Similarly, the third equation is $0 = 1$. So this is a system of equations with no solutions.

Lets summarize what we have done in this section. Every system of equations can be brought into upper triangular form using a sequence of elementary row transformations. The resulting upper triangular matrix will look something like



In this diagram, all the entries below the staircase line are zero. The boxes represent non-zero entries. The stars represent arbitary entries, that may or may not be zero. Each circled star corresponds to a parameter that must be introduced.

If we want to put this example in completely reduced form, we use elementary row operations to zero out the entries lying about the boxes too. Then we multiply each row by a number so that the corner entries (in the boxes) become 1. The the completely reduced form for the example above would look like this. (The official name of this form is the reduced row echelon form.)

$$\left[\begin{array}{cccccccccc|c}
\boxed{1} & 0 & * & * & 0 & * & 0 & 0 & * & * & * \\
 & \boxed{1} & * & * & 0 & * & 0 & 0 & * & * & * \\
 & & & & \boxed{1} & * & 0 & 0 & * & * & * \\
 & & & & & & \boxed{1} & 0 & * & * & * \\
 & & & & & & & \boxed{1} & * & * & * \\
 & & & & & & & & & & \\
 & & & & & & & & & &
\end{array}\right]$$

If the bottom of the matrix has a row that is all zeroes, except for the augmented entry, then the system of equations has no solutions. This is because the bottom row stands for an equation of the form $0 = \square$ with $\square \neq 0$. Here is a typical example.

$$\left[\begin{array}{cccccccccc|c}
\square & * & * & * & * & * & * & * & * & * & * \\
 & \square & * & * & * & * & * & * & * & * & * \\
 & & & & \square & * & * & * & * & * & * \\
 & & & & & & \square & * & * & * & * \\
 & & & & & & & \square & * & * & * \\
 & & & & & & & & & & \square \\
 & & & & & & & & & &
\end{array}\right]$$

If all the steps on the staircase in the non-augmented part of the matrix have size one, then there are no parameters to introduce, and the solution is unique. Notice that in this case there are the same number of equations as variables. Here is a typical example.

$$\begin{bmatrix}
\square & * & * & * & * & * & * \\
 & \square & * & * & * & * & * \\
 & & \square & * & * & * & * \\
 & & & \square & * & * & * \\
 & & & & \square & * & * \\
 & & & & & \square & * \\
 & & & & & & \square
\end{bmatrix}$$

Finally, we introduce some terminology. The *rank* of a matrix is the number of non-zero rows in the matrix obtained after reducing it to the upper triangular form described above. In other words the rank is the number of boxes in the diagrams above. We can now rephrase the different possibilities in terms of rank. If the rank of the augmented matrix is greater than the rank of the unaugmented matrix (i.e., the matrix without the last column) then there are no solutions. If the rank of the matrix is equal to the number of unknowns then the solution is unique. If the rank r of the matrix is equal to the rank of the unaugmented matrix, but less than the number n of unknowns, then there are $n - r$ parameters in the solution.

Problem 2.4: Solve the following system of equations.

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & 3x_3 & = & 2 \\ 2x_1 & - & 3x_2 & + & 2x_3 & = & 2 \\ 3x_1 & + & 2x_2 & - & 4x_3 & = & 9 \end{array}$$

Problem 2.5: Solve the following system of equations.

$$\begin{array}{rcrcrcr} 2x_1 & + & x_2 & - & 1x_3 & = & 6 \\ x_1 & - & 2x_2 & - & 2x_3 & = & 1 \\ -x_1 & + & 12x_2 & + & 8x_3 & = & 7 \end{array}$$

Problem 2.6: Solve the following system of equations.

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 4x_3 & = & 1 \\ x_1 & + & x_2 & + & 3x_3 & = & 2 \\ 2x_1 & + & 5x_2 & + & 9x_3 & = & 1 \end{array}$$

Problem 2.7: Solve the following system of equations.

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 4x_3 & = & 1 \\ x_1 & + & x_2 & + & 3x_3 & = & 2 \\ 2x_1 & + & 5x_2 & + & 9x_3 & = & 3 \end{array}$$

Problem 2.8: Solve the following system of equations.

$$\begin{array}{rcrcrcrcr} 3x_1 & + & x_2 & - & x_3 & + & 2x_4 & = & 7 \\ 2x_1 & - & 2x_2 & + & 5x_3 & - & 7x_4 & = & 1 \\ -4x_1 & - & 4x_2 & + & 7x_3 & - & 11x_4 & = & -13 \end{array}$$

Problem 2.9: For what values of a, b, c, d, α and β does the system of equations

$$\begin{array}{rcrcl} ax_1 & + & bx_2 & = & \alpha \\ cx_1 & + & dx_2 & = & \beta \end{array}$$

have a unique solution?

Using MATLAB for row reductions

The MATLAB program that you are using in the labs has a built in command called `rref` that reduces a matrix to reduced row echelon form. Lets try it on the example in the previous section. First we define the initial matrix `A`. Remember that the last column of this matrix is the augmented part.

```
A = [1 2 -2 -7 -29; 1 2 -1 -5 -18; 0 3 0 -3 -6; -1 4 1 1 14]
```

To find the reduced row echelon form, simply type

```
>> rref(A)

ans =

    1 0 0 0 1
```

```
    0 1 0 0 2
    0 0 1 0 3
    0 0 0 1 4
```

This gives the final answer instantly, but not the intermediate steps. We can compute these as well. In MATLAB `A(1,1:end)` is the first row of the matrix `A`. So to do the first two steps of the row reduction we could type

```
>> A(2,1:end)=A(2,1:end)-A(1,1:end)
A =
     1 2 -2 -7 -29
     0 0  1  2  11
     0 3  0 -3 -6
    -1 4  1  1  14
>> A(4,1:end)=A(4,1:end)+A(1,1:end)
A =
    1 2 -2 -7 -29
    0 0  1  2  11
    0 3  0 -3 -6
    0 6 -1 -6 -15
```

Swapping the second and third row could be done as follows.

```
>> temprow=A(2,1:end);
>> A(2,1:end)=A(3,1:end);
>> A(3,1:end)=temprow
A =
    1 2 -2 -7 -29
    0 3  0 -3 -6
    0 0  1  2  11
    0 6 -1 -6 -15
```

Finally, if we wanted to multiply the second row by 1/3 we could do this as follows.

```
>> A(2,1:end)=(1/3)*A(2,1:end)
A =
1 2 -2 -7 -29
0 1  0 -1 -2
0 0  1  2  11
0 6 -1 -6 -15
```

Homogeneous equations and the structure of solutions

If the coefficients on the right sides of a system of equations are all zero, the system is said to be *homogeneous*. In other words, a homogeneous system is a system of equations of the form

$$\begin{array}{ccccccc} b_{1,1}x_1 & + & b_{1,2}x_2 & +\cdots & + & b_{1,n}x_n & = & 0 \\ b_{2,1}x_1 & + & b_{2,2}x_2 & +\cdots & + & b_{2,n}x_n & = & 0 \\ \vdots & \vdots & & & & \vdots & & \vdots \\ b_{m,1}x_1 & + & b_{m,2}x_2 & +\cdots & + & b_{m,n}x_n & = & 0 \end{array}$$

Given a system of equations, the *associated homogeneous system* is the homogeneous system of equations you get by setting all the right sides to zero.

Geometrically, homogeneous systems describe points, lines and planes that pass through the origin. In fact $\mathbf{x} = \mathbf{0}$, i.e., $x_1 = 0, x_2 = 0, \ldots, x_n = 0$ is always a solution to a homogeneous system of equations.

When are there **other (nonzero)** solutions to the above homogeneous system? We have n unknowns and m equations. When we perform the Gaussian reduction, the right-hand sides of the equations will stay zero so the augmented matrix will generally have the form

$$\left[\begin{array}{cccccccc|c} 1 & * & * & * & \cdots & \cdots & \cdots & * & 0 \\ 0 & 1 & * & * & \cdots & \cdots & \cdots & * & 0 \\ 0 & 0 & 0 & 1 & * & \cdots & \cdots & * & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 1 & * & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The last several lines may be identically zero. In the last section we saw that there are solutions depending on parameters if the number of variables is greater than the rank of the matrix. Thus, **if n (the number of unknowns) is bigger than the number of non-zero lines in the above row-reduced matrix, then there exists a non-zero solution.** Otherwise only a trivial solution $x_1 = 0, x_2 = 0, \ldots, x_n = 0$ is present. We illustrate the idea with examples.

1. Consider a homogeneous system

$$\begin{array}{ccccccc} 3x_1 & + & 6x_2 & + & x_3 & = & 0 \\ 6x_1 & + & 2x_2 & + & 2x_3 & = & 0 \\ x_1 & + & x_2 & + & 3x_3 & = & 0 \end{array}$$

The augmented matrix can be reduced by row operations to the form (check!)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right],$$

which implies $x_1 = x_2 = x_3 = 0$. And, in agreement with our above statement, the number of variables (3) is not more than the number of non-zero rows (also 3).

2. Another homogeneous system:

$$\begin{array}{rcrcrcl} -x_1 & + & 2x_2 & + & 4x_3 & = & 0 \\ 2x_1 & - & 4x_2 & - & 8x_3 & = & 0\,. \\ x_1 & - & x_2 & + & 3x_3 & = & 0 \end{array}$$

Its augmented matrix

$$\left[\begin{array}{rrr|r} -1 & 2 & 4 & 0 \\ 2 & -4 & -8 & 0 \\ 1 & -1 & 3 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} -1 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 7 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 10 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

and the number of nonzero rows is 2, which is less than the number of unknowns, 3. Hence by the above statement there must be a nonzero solution. We find $x_1 = -10x_3$,$x_2 = -7x_3$, with no requirement on x_3. Hence x_3 is any number t, and we obtain infinitely many nonzero solutions

$$x_1 = -10t, x_2 = -7t, x_3 = t, \qquad t \in (-\infty, \infty),$$

one for each value of t.

In a similar manner, if for some homogeneous system with 4 variables the augmented matrix has only 2 nonzero rows, then the general solution has 4-2=2 free (undefined) variables on which the other two depend.

Properties of solutions of homogeneous systems.

1. A homogeneous system has either one zero-solution ($x_1 = ... = x_n = 0$) or infinitely-many solutions that depend on parameters.

2. If $(x_1, ..., x_n)$ and $(y_1, ..., y_n)$ are solutions to a given homogeneous system, $(x_1 + y_1, ..., x_n + y_n)$ is also a solution. (Solutions are additive.)

3. If $(x_1, ..., x_n)$ is a solution to a given homogeneous system, $(ax_1, ..., ax_n)$ is also a solution, for any number a. (Solutions are scalable.)

the first statement follows from our previous discussion; the other two are easy to verify, using the initial homogeneous system.

Connection of solutions to homogeneous and inhomogeneous systems.

The importance of homogeneous equations comes from the following fact. If $\mathbf{x} = [x_1, x_2, \ldots, x_n]$ and $\mathbf{y} = [y_1, y_2, \ldots, y_n]$ are two solutions to a (not neccesarily homogeneous) system of equations,

$$\begin{array}{ccccccccc} b_{1,1}x_1 & + & b_{1,2}x_2 & +\cdots & + & b_{1,n}x_n & = & c_1 \\ b_{2,1}x_1 & + & b_{2,2}x_2 & +\cdots & + & b_{2,n}x_n & = & c_2 \\ \vdots & \vdots & & & & \vdots & & \vdots \\ b_{m,1}x_1 & + & b_{m,2}x_2 & +\cdots & + & b_{m,n}x_n & = & c_m \end{array}$$

then the difference $\mathbf{x} - \mathbf{y} = [x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n]$ solves the associated homogeneous system. This is a simple calculation

$$\begin{array}{cccccccccc} b_{1,1}(x_1 - y_1) & + & b_{1,2}(x_2 - y_2) & +\cdots & + & b_{1,n}(x_n - y_n) & = & (c_1 - c_1) & = & 0 \\ b_{2,1}(x_1 - y_1) & + & b_{2,2}(x_2 - y_2) & +\cdots & + & b_{2,n}(x_n - y_n) & = & (c_2 - c_2) & = & 0 \\ \vdots & \vdots & & & & \vdots & & \vdots & & \\ b_{m,1}(x_1 - y_1) & + & b_{m,2}(x_2 - y_2) & +\cdots & + & b_{m,n}(x_n - y_n) & = & (c_m - c_m) & = & 0 \end{array}$$

To see the implications of this lets suppose that $\mathbf{x} = \mathbf{q}$ is any particular solution to a (non-homogeneous) system of equations. Then if $\mathbf{y}$ is any other solution $\mathbf{y} - \mathbf{x} = \mathbf{z}$ is a solution of the corresponding homogenous system. So $\mathbf{y} = \mathbf{q} + \mathbf{z}$. In other words any solution can be written as $\mathbf{q}+$ some solution of the corresponding homogenous system. Going the other way, if $\mathbf{z}$ is any solution of the corresponding homogenous system, then $\mathbf{q}+\mathbf{z}$ solves the original system. This can be seen by plugging $\mathbf{q}+\mathbf{z}$ into the equation. So structure of the set of solutions is

$$\mathbf{x} = \mathbf{q} + (\textsf{\textbf{solution to homogeneous system}})$$

As you run through all solutions to the homogenous system on the right, $\mathbf{x}$ runs through all solution of the original system. Notice that it doesn't matter which $\mathbf{q}$ you choose as the starting point. This is completely analogous to the parametric form for a line, where the base point can be any point on the line.

If we have applied the process of Gaussian elimination to the orginal system, and concluded that the general solution has parameters, we will end up with a general solution of the form

$$\mathbf{q} + s_1\mathbf{a}_1 + \cdots + s_n\mathbf{a}_n.$$

Notice that $\mathbf{q}$ is a particular solution (corresponding to all parameters equal to zero) and $s_1\mathbf{a}_1 + \cdots + s_n\mathbf{a}_n$ is the general solution to the corresponding homogeneous system.

These considerations have practical importance if you have to solve a bunch of systems, all with the same coefficients on the left side, but with different coefficients on the right. In this situation, you could first find the general solution to the corresponding homogeneous system of equations. Then to find the general solution to one of the systems, you would only need to find a single particular solution, and then add the general solution to the homogeneous system to obtain all solutions. The only trouble with this is that it might not really be any easier to find a single particular solution than it is to find all solutions.

Problem 2.10: Find the general solution of the system of equations

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ -1 & -1 & 1 & 2 & 1 \\ 3 & 3 & -1 & -2 & 1 \end{array}\right]$$

In the form $\mathbf{x} = \mathbf{q} + s_1\mathbf{a}_1 + s_2\mathbf{a}_2$. Verify that $\mathbf{a}_1$ and $\mathbf{a}_2$ solve the corresponding homogeneous equation.

Problem 2.11: Consider the system of equations

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ -1 & -1 & 1 & 2 & -1 \\ 3 & 3 & -1 & -2 & 9 \end{array}\right]$$

Verify that $[4, 0, 3, 0]$ is a solution and write down the general solution.

Some geometric applications

Now we will apply Gaussian elimination to some of the geometry problems we studied in the first part of this course.

Lets start with the question of linear independence. Recall that a collection of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is called linearly dependent if we can find some non-zero coefficients $c_1, c_2, \ldots, c_n$ such that

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n = \mathbf{0}$$

This is actually a homogeneous system of linear equations for the numbers $c_1, \ldots, c_n$. If $c_1 = c_2 = \cdots = c_n = 0$ is the only solution, then the vectors are linearly indpendent. Otherwise, they are linearly dependent. To decide, we must set up the matrix for the system of equations and perform a row reduction to decide if there is a unique solution or not. In setting up the equations, it is convenient to treat the $\mathbf{x}_i$'s as row vectors.

For example, lets decide if

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

are linearly independent. The equation $c1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = \mathbf{0}$ can be written

$$\begin{array}{cccc} c_1 & +c_2 & +c_3 & = 0 \\ 2c_1 & +c_2 & +2c_3 & = 0 \\ 0c_1 & +c_2 & +c_3 & = 0 \end{array}$$

The matrix for this system of equations is

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

Since this is a homogeneous system, we don't have to write the augmented part of the matrix. Performing a row reduction yields

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the number of non-zero rows is the same as the number of variables (three) there are no non-zero solutions. Therefore the vectors are linearly independent.

This same row reduction also shows that any vector $\mathbf{y}$ in $\mathbb{R}^3$ can be written as a linear combination of $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$. Writing $\mathbf{y}$ as a linear combination of $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ means finding coefficients c_1, c_2 and c_3 such that $c1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = \mathbf{y}$. This is a (non-homogeneous) system of linear equations with augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & y_1 \\ 2 & 1 & 2 & y_2 \\ 0 & 1 & 1 & y_3 \end{array}\right]$$

Using the same Gaussian elimination steps as above, this matrix reduces to

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$$

where the $*$'s are some numbers. This system has a (unique) solution.

Here is another geometric example. Do the planes whose equations are given by $x_1 + x_2 + x_3 = 1$, $2x_1 + x_2 + 2x_1 = 1$ and $x_2 = 1$ intersect in a single point? To anwer this, we note that the intersection of the three planes is given by the set of points that satisfy all three equations. In other words they satisfy the system of equations whose augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

A row reduction yields

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus solutions are given by

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

This is the parametric equation of a line. Thus the three planes intersect in a line, not a point.

Problem 2.12: Are the following vectors are linearly dependent or independent?

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Can every vector in $\mathbb{R}^4$ be written as a linear combination of these vectors? How about the vector the

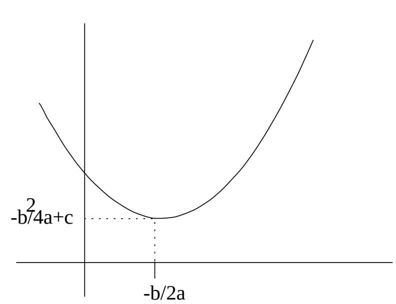
$$\mathbf{y}_1 = \begin{bmatrix} 2 \\ 4 \\ -3 \\ 4 \end{bmatrix}?$$

Minimizing quadratic functions

Let begin by recalling how we would find the minimum of a quadratic function in one variable, namely a parabola given by $f(x) = ax^2 + bx + c$. We simply find the value of x for which the derivative is zero, that is, we solve $f'(x) = 0$. Notice that since f is quadratic, this is a linear equation

$$2ax + b = 0$$

which is easily solved for $x = -b/2a$ (provided $a \neq 0$). So the minimum value is $f(-b/2a) = -b^2/(4a) + c$.



Of course, if a is negative, then the parabola points downwards, and we have found the minimum value, not the maximum value.

A quadratic function of two variables x_1 and x_2 is a function of the form

$$f(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2 + dx_1 + ex_2 + f.$$

(The 2 in front of b is just for convenience.) For what values of x_1 and x_2 is $f(x_1, x_2)$ the smallest? Just like with the parabola in one variable, there may be no such values. It could be that f has a maximum instead, or that f has what is called a saddle point. However if f does have a minimum, the procedure described below is guaranteed to find it. (If f has a maximum or saddle point, the procedure will find these points instead.)

The idea behind finding the minimum is simple. Suppose that x_1 and x_2 are the values for which $f(x_1, x_2)$ is smallest. Then the function $g(s) = f(x_1 + s, x_2)$ must have a minimum at $s = 0$. So $g'(0) = 0$. But

$$\begin{aligned} g'(s) &= \frac{d}{ds} f(x_1 + s, x_2) \\ &= \frac{d}{ds} a(x_1 + s)^2 + 2b(x_1 + s)x_2 + cx_2^2 + d(x_1 + s) + ex_2 + f \\ &= 2a(x_1 + s) + 2bx_2 + d \end{aligned}$$

so that the condition is

$$g'(0) = 2ax_1 + 2bx_2 + d = 0.$$

Notice that this expression can be obtained by holding x_2 fixed and differentiating with respect to x_1. It is called the partial derivative of f with respect to x_1 and is denoted $\frac{\partial f}{\partial x_1}$.

The same argument can be applied to $h(s) = f(x_1, x_2 + s)$ (or $\frac{\partial f}{\partial x_2}$.) This yields

$$h'(0) = \frac{\partial f(x_1, x_2)}{\partial x_2} = 2bx_1 + 2cx_2 + e = 0.$$

Therefore we conclude that the pair of values x_1 and x_2 at which f achieves its minimum satisfy the system of linear equations

$$\begin{array}{rrl} 2ax_1 & +2bx_2 & = -d \\ 2bx_1 & +2cx_2 & = -e \end{array}$$

This is a 2 by 2 system with augmented matrix

$$\left[\begin{array}{cc|c} 2a & 2b & -d \\ 2b & 2c & -e \end{array}\right]$$

This is easily generalized to n variables. In this case the quadratic function is given by

$$f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} x_i x_j + \sum_{i=1}^{n} b_i x_i + c$$

To see this is the same, lets expand out the first term when $n = 2$. Then

$$\begin{aligned} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} x_i x_j &= a_{1,1}x_1x_1 + a_{1,2}x_1x_2 + a_{2,1}x_2x_1 + a_{2,2}x_2x_2 \\ &= a_{1,1}x_1^2 + (a_{1,2} + a_{2,1})x_1x_2 + a_{2,2}x_2^2 \end{aligned}$$

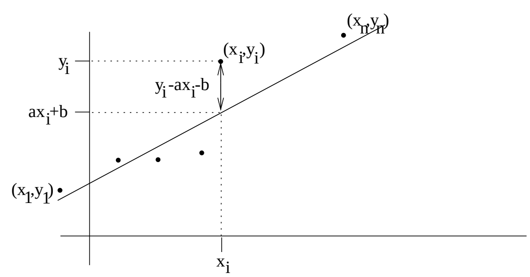
So this is just the same as before with $a_{1,1} = a$, $a_{1,2} + a_{2,1} = 2b$ and $a_{2,2} = c$. Notice that we might as well assume that $a_{i,j} = a_{j,i}$, since replacing both $a_{i,j}$ and $a_{j,i}$ with $(a_{1,2} + a_{2,1})/2$ doesn't change f.

If this function f has a minimum we can find it by generalizing the procedure above. In other words we try to find values of $x_1, \ldots, x_n$ for which $\partial f/\partial x_1 = \partial f/\partial x_2 = \cdots = \partial f/\partial x_n = 0$. This leads to a system of n linear equations whose associated augmented matrix is

$$\left[\begin{array}{cccc|c} 2a_{1,1} & 2a_{1,2} & \ldots & 2a_{1,n} & -b_1 \\ 2a_{2,1} & 2a_{2,2} & \ldots & 2a_{2,n} & -b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 2a_{n,1} & 2a_{n,2} & \ldots & 2a_{n,n} & -b_n \end{array}\right]$$

1. Least squares fit

As a first application lets consider the problem of finding the "best" straight line going through a collection of data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. (Careful! the x_i's are not the unknowns in this problem, but rather the known fixed data points, together with the y_i's.) Which straight line fits best? There is no one answer. One can measure how good the fit of a straight line is in various ways. However the following way of measuring the fit results in a problem that is easy to solve.



Each line is given by an equation $y = ax + b$. So the variables in this problem are a and b. We want to find the values of a and b that give the best fitting line. The vertical distance between the point (x_i, y_i) and the line is given by $|y_i - ax_i - b|$. We will take as a measure of the fit, the square of this quantity, added up over all the data points. So

$$\begin{aligned} f(a,b) &= \sum_i (y_i - ax_i - b)^2 \\ &= \sum_i \left(y_i^2 + x_i^2 a^2 + b^2 - 2x_i y_i a - 2y_i b + 2x_i ab\right) \\ &= \left(\sum_i x_i^2\right) a^2 + 2\left(\sum_i x_i\right) ab + nb^2 - 2\left(\sum_i x_i y_i\right) a - 2\left(\sum_i y_i\right) b + \left(\sum_i y_i^2\right) \end{aligned}$$

Here we used that $(\sum_i 1) = n$, the number of points. Therefore the linear equations we must solve for a and b are

$$\left[\begin{array}{cc|c} 2\left(\sum_i x_i^2\right) & 2\left(\sum_i x_i\right) & 2\left(\sum_i x_i y_i\right) \\ 2\left(\sum_i x_i\right) & 2n & 2\left(\sum_i y_i\right) \end{array}\right]$$

We could solve these equations numerically in each particular case, but since its just a 2 by 2 system we can also write down the answer explicitly. In fact, the solution to

$$\left[\begin{array}{cc|c} A & B & E \\ C & D & F \end{array}\right]$$

is

$$\begin{bmatrix} \dfrac{DE-BF}{AD-BC} \\ \dfrac{AF-CE}{AD-BC} \end{bmatrix},$$

provided $AD - BC \neq 0$, as you may check directly, or derive using a sequence of row transformations. So in this case

$$a = \frac{n(\sum x_i y_i) - (\sum x_i)(\sum y_i)}{n(\sum x_i^2) - (\sum x_i)^2}$$
$$b = \frac{(\sum x_i^2)(\sum y_i) - (\sum x_i)(\sum x_i y_i)}{n(\sum x_i^2) - (\sum x_i)^2}$$

Lets do an example. Suppose we want to find the best straight line through the points $(1, 1)$, $(2, 2)$, $(2, 3)$, $(3, 3)$ and $(3, 4)$. Calculate

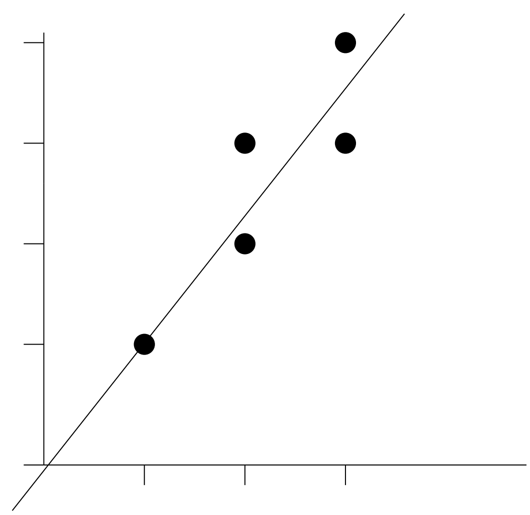
$$\begin{array}{lll} \sum 1 & = n & = 5 \\ \sum x_i & = 1+2+2+3+3 & = 11 \\ \sum y_i & = 1+2+3+3+4 & = 13 \\ \sum x_i^2 & = 1+4+4+9+9 & = 27 \\ \sum y_i^2 & = 1+4+9+9+16 & = 39 \\ \sum x_i y_i & = 1+4+6+9+12 & = 32 \end{array}$$

so

$$a = (5 \cdot 32 - 11 \cdot 13)/(5 \cdot 27 - 11^2) = 17/14 = 1.214\ldots$$

and

$$b = (27 \cdot 13 - 11 \cdot 32)/(5 \cdot 27 - 11^2) = -1/14 = -0.0714\ldots$$



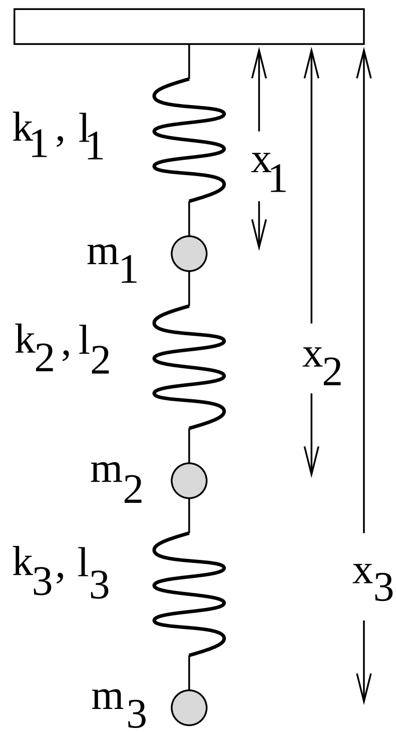
Problem 2.13: Find the "best" straight line going through the points $(1, 1)$, $(2, 1)$, $(2, 3)$, $(3, 4)$, $(3, 5)$ and $(4, 4)$.

Problem 2.14: Consider the problem of finding the parabola $y = ax^2 + bx + c$ that best fits the n data points $(x_1, y_1) \ldots (x_n, y_n)$. Derive the system of three linear equations which determine a, b and c. (You need not solve solve them!)

2. Equilibrium configuration of hanging weights and springs

Consider the problem of n vertically hanging weight connected by springs. What is the equilibrium configuration? We can solve this problem by calculating the total potential energy of the system. The equilibrium configuration minimizes the total potential energy.

Here is the diagram of the setup. Our goal is to compute the numbers x_1, …, x_n. In the diagram $n = 3$.



There are two sources of potential energy. One is the potential energy stored in the spring. This is equal to $ks^2/2$, where k is the spring constant that measures the stiffness of the spring, and s is the amount that the spring has been streched from its natural length. In our problem, suppose that the spring constant of the ith spring is k_i and its natural length is l_i. Then the potential energy stored in the ith spring is $k_i(x_i - x_{i-1} - l_i)^2/2$. To make this formula work out correctly for the first spring we set $x_0 = 0$.

The other source of potential energy is gravity. The gravitational potential energy of the ith weight is $-m_i g x_i$. The reason for the minus sign is that we are measuring distances downward.

Thus the total potential energy in the system for n weights is the function

$$f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} \frac{k_i}{2}(x_i - x_{i-1} - l_i)^2 - m_i g x_i.$$

When $n = 3$ this becomes

$$f(x_1, x_2, x_3) = \frac{k_1}{2}(x_1 - l_1)^2 + \frac{k_2}{2}(x_2 - x_1 - l_2)^2 + \frac{k_3}{2}(x_3 - x_2 - l_3)^2 - m_1 g x_1 - m_2 g x_2 - m_3 g x_3$$

This is a quadratic function, so we know how to find the minimum. The equations are obtained by taking partial derivatives: To get the first equation we hold x_2 and x_3 fixed and differentiate with respect to x_1

and so on. Thus the equations are

$$\begin{aligned} k_1(x_1 - l_1) - k_2(x_2 - x_1 - l_2) - m_1 g &= 0 \\ k_2(x_2 - x_1 - l_2) - k_3(x_3 - x_2 - l_3) - m_2 g &= 0 \\ k_3(x_3 - x_2 - l_3) - m_3 g &= 0 \end{aligned}$$

The augmented matrix for this system is

$$\left[\begin{array}{ccc|c} k_1 + k_2 & -k_2 & 0 & m_1 g + k_1 l_1 - k_2 l_2 \\ -k_2 & k_2 + k_3 & -k_3 & m_2 g + k_2 l_2 - k_3 l_3 \\ 0 & -k_3 & k_3 & m_3 g + k_3 l_3 \end{array}\right]$$

Suppose that the spring constants are $k_1 = 1$, $k_2 = 2$ and $k_3 = 1$. The masses are all equal to 1, $g = 10$ and the natural length of the springs is 1 for all springs (in appropriate units). Then to find the equilibrium configuration we must solve

$$\left[\begin{array}{ccc|c} 3 & -2 & 0 & 9 \\ -2 & 3 & -1 & 11 \\ 0 & -1 & 1 & 11 \end{array}\right]$$

Gaussian elimination gives

$$\left[\begin{array}{ccc|c} 3 & -2 & 0 & 9 \\ 0 & -1 & 1 & 11 \\ 0 & 0 & 2 & 106 \end{array}\right]$$

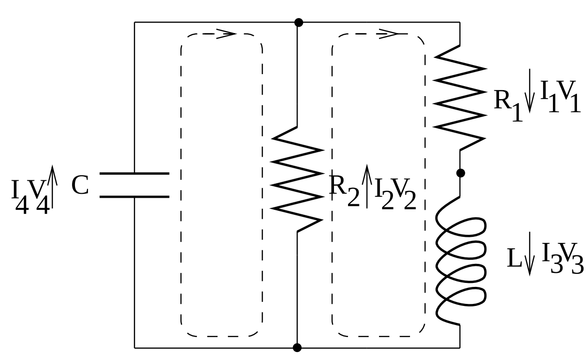
which can be solved to give $x_1 = 31$, $x_2 = 42$, $x_3 = 53$.

Problem 2.15: Write down the augmented matrix for a system of n weights and springs.

Problem 2.16: Write down the system of equations you would have to solve if there are 5 identical springs with $k_i = 1$ and $l_i = 1$ and five weights with $m_1 = 1$, $m_2 = 2$, $m_3 = 3$, $m_4 = 4$, and $m_5 = 5$.

Choosing basic variables in a circuit

So far our examples have always involved equations with a unique solution. Here is an application using equations that don't have a unique solution. Consider the following circuit.



We won't be able to solve this circuit until we a studied differential equations in the last part of this course. However we can make some progress using what we know already.

There are three types of components: resistors, inductors (coils) and capacitors. Associated with each component is the current I flowing through that component, and the voltage drop V across that component. If there are n different components in a circuit, then there are $2n$ variables (currents and voltages) to determine. In the circuit above there are 8.

Of course, these variables are not all independent. They satisfy two types of linear relations: algebraic and differential. We won't touch the differential relations for now, but we can consider the algebraic relations.

The first algebraic relation relates the current and voltage drop across a resistor. If R is the resistance and I and V are the current and voltage drop respectively, then $V = IR$. In our example, this gives two equations

$$\begin{aligned} V_1 &= I_1 R_1 \\ V_2 &= I_2 R_2 \end{aligned}$$

The other two algebraic relations are Kirchoff's laws. The first of these states that the total voltage drop across any loop in the circuit is zero. For the two loops in the example circuit, this gives the equations

$$\begin{aligned} V_4 - V_2 &= 0 \\ V_1 + V_3 + V_2 &= 0 \end{aligned}$$

Notice we have to take the direction of the arrows into account. The second Kirchoff law states that current cannot accumulate at a node. At each node, the current flowing in must equal the current flowing out. In the example circuit there are three nodes, giving the equations.

$$\begin{aligned} I_4 + I_2 - I_1 &= 0 \\ I_1 - I_3 &= 0 \\ I_3 - I_2 - I_4 &= 0 \end{aligned}$$

We now want to pick a few of the variables, and solve for all the rest in terms of these. In a small circuit like the example, this can be done "by hand." For example, its pretty obvious that $I_1 = I_3$ and $V_2 = V_4$ so one could eliminate two variables right off the bat. However, it is also useful to have a systematic way of doing it, that will work for any circuit (but probably will require a computer for anything but the simplest circuit).

As a rule of thumb, you can pick the voltages across the capacitor and the currents across the inductors as basic variables and solve for the rest in terms of these. In other words, we want I_3 and V_4 to be *parameters* when we solve the system of equations. To accomplish this we will choose the order of the variables with I_3 and V_4 at the end of the list. With this in mind we choose the order $I_1, I_2, I_4, V_1, V_2, V_3, I_3, V_4$. Then the equations become

$$\begin{array}{cccccccccl}
R_1 I_1 & & & -V_1 & & & & & = 0 \\
& R_2 I_2 & & & -V_2 & & & & = 0 \\
& & & & -V_2 & & & +V_4 & = 0 \\
& & & V_1 & +V_2 & +V_3 & & & = 0 \\
-I_1 & +I_2 & +I_4 & & & & & & = 0 \\
I_1 & & & & & & -I_3 & & = 0 \\
& -I_2 & -I_4 & & & & +I_3 & & = 0
\end{array}$$

The matrix for this system is (since it is a homogeneous system of equations, we don't have to bother writing the augmented part)

$$\begin{bmatrix} R_1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Here is the reduced form of this matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{R_2} \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & \frac{1}{R_2} \\ 0 & 0 & 0 & 1 & 0 & 0 & -R_1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & R_1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus

$$\begin{aligned} I_1 &= I_3 \\ I_2 &= \frac{1}{R_2}V_4 \\ I_4 &= I_3 - \frac{1}{R_2}V_4 \\ V_1 &= R_1 I_3 \\ V_2 &= V_4 \\ V_3 &= -R_1 I_3 - V_4 \end{aligned}$$

So we have succeeded in expressing all the variables in terms of I_3 and V_4. We therefore need only determine these to solve the circuit completely.

Problem 2.17: If a circuit contains only resistors, then we can solve it completely using the ideas of this section. Write down the linear equations satisfied by the currents in the following circuit. In this diagram, the component on the far left is a voltage source (battery). The voltage across the voltage source is always E.