Math 152: Linear Systems – Winter 2005

Section 4: Eigenvalues and Eigenvectors

Eigenvalues and eigenvectors

Let A be an $n \times n$ matrix. A number λ and a vector $\mathbf{x}$ are called an eigenvalue eigenvector pair if

(1) $\mathbf{x} \neq \mathbf{0}$

(2) $A\mathbf{x} = \lambda\mathbf{x}$

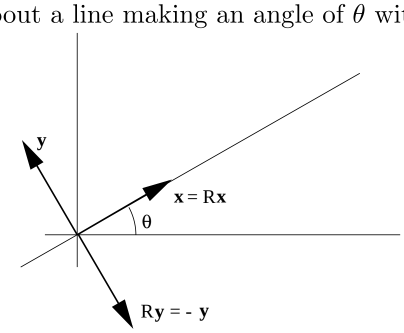
In other words, the action of A on the vector $\mathbf{x}$ is to stretch or shrink it by an amount λ without changing its direction.

We say λ is an eigenvalues of A if there exists a vector $\mathbf{x}$ so that λ and $\mathbf{x}$ are an eigenvalue eigenvector pair.

Notice that we do not allow $\mathbf{x} = \mathbf{0}$. If we did, any number λ would be an eigenvalue. However we do allow $\lambda = 0$. Saying that 0 is and eigenvalue of A means that there is a non-zero solution $\mathbf{x}$ (the eigenvector) of $A\mathbf{x} = 0\mathbf{x} = \mathbf{0}$. So we see that 0 is an eigenvalue of A precisely when A is not invertible.

Lets look at some examples.

Consider the matrix of reflection about a line making an angle of θ with the x axis.



Let $\mathbf{x}$ be any vector that lies along the line. Then the reflection doesn't affect $\mathbf{x}$. This means that $R\mathbf{x} = \mathbf{x}$. In other words, $\mathbf{x}$ is an eigenvector with eigenvalue 1

On the other hand, suppose that $\mathbf{y}$ is a vector at right angles to the line. Then the reflection flips $\mathbf{y}$ into minus itself. So $R\mathbf{y} = -\mathbf{y}$. In other words, $\mathbf{y}$ is an eigenvector with eigenvalue -1

If we take any other vector and reflect it, we don't end up with a vector that lies on the same line as the original vector. Thus there are no further eigenvectors or eigenvalues.

An important point to notice is that the eigenvector is not uniquely determined. The vector $\mathbf{x}$ could be *any* vector along the line, and $\mathbf{y}$ could be *any* vector orthogonal to the line. In fact, if we go back to the original definition of eigenvalue and eigenvector we can see that if λ and $\mathbf{x}$ are an eigenvalue eigenvector pair, then so are λ and $s\mathbf{x}$ for any non-zero number s, since (1) $s\mathbf{x} \neq \mathbf{0}$ and (2) $As\mathbf{x} = sA\mathbf{x} = s\lambda\mathbf{x} = \lambda s\mathbf{x}$. So the important thing about an eigenvector is its direction, not its length.

However, there is no such ambiguity in the definition of the eigenvalue. The reflection matrix has exactly two eigenvalues 1 and -1.

In some sense, the reflection matrix R illustrates the most satisfactory situation. R is a 2×2 matrix with two distinct eigenvalues. The corresponding eigenvectors $\mathbf{x}$ and $\mathbf{y}$ are linearly independent (in fact they are orthogonal) and form a basis for two dimensional space.

It will be important in applications to determine whether or not there exists a basis of eigenvectors of a given matrix. In this example, $\mathbf{x}$ and $\mathbf{y}$ are a basis of eigenvectors of R.

As our next example, consider the identity matrix I. Since the identity matrix doesn't change a vector, we have $I\mathbf{x} = \mathbf{x}$ for any vector $\mathbf{x}$. Thus any vector $\mathbf{x}$ is an eigenvector of I with eigenvalue 1.

This example shows that a given eigenvalue may have many eigenvectors associated with it. However, in this example, there still exists a basis of eigenvectors: any basis at all is a basis of eigenvectors.

Next we will consider the rotation matrix ... and run into trouble. Suppose R is the matrix of rotation by $\pi/4$ (ie $45°$). Then $R\mathbf{x}$ is never in the same direction as $\mathbf{x}$, since R changes the direction of $\mathbf{x}$ by $\pi/4$. So R has no eigenvalues and no eigenvectors.

This unfortunate state of affairs will cause us to make a considerable detour into the theory of complex numbers. It turns out that if we work with complex numbers rather than real numbers, then the rotation matrix has eigenvalues too.

Problem 4.1: Show that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are eigenvectors for the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. What are the corresponding eigenvalues?

Problem 4.2: Suppose P is a projection matrix. What are the eigenvalues and eigenvectors of P?

Computing the eigenvalues and eigenvectors

We now consider the problem of finding all eigenvalue eigenvector pairs for a given $n \times n$ matrix A.

To start, suppose someone tells you that a particular value λ is an eigenvalue of A. How can you find the corresponding eigenvector $\mathbf{x}$? This amounts to solving the equation $A\mathbf{x} = \lambda\mathbf{x}$ for $\mathbf{x}$. This can be rewritten

$$(A - \lambda I)\mathbf{x} = \mathbf{0},$$

where I denotes the identity matrix. In other words $\mathbf{x}$ is a non-zero solution to a homogeneous equation. It can be found by Gaussian elimination.

For example, suppose you know that 4 is an eigenvalue of

$$\begin{bmatrix} 3 & -6 & -7 \\ 1 & 8 & 5 \\ -1 & -2 & 1 \end{bmatrix}.$$

To find the corresponding eigenvector, we must solve

$$\left(\begin{bmatrix} 3 & -6 & -7 \\ 1 & 8 & 5 \\ -1 & -2 & 1 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This can be written

$$\begin{bmatrix} -1 & -6 & -7 \\ 1 & 4 & 5 \\ -1 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

To solve this we reduce the matrix. This yields

$$\begin{bmatrix} -1 & -6 & -7 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

The fact that the rank of this matrix is less than 3 confirms that 4 is indeed an eigenvalue. If the rank of the matrix were 3 then the only solution to the equation would be $\mathbf{0}$ which is not a valid eigenvector.

Taking $x_3 = s$ as a parameter, we find that $x_2 = -s$ and $x_1 = -s$. Thus

$$\mathbf{x} = s \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

is an eigenvector (for any non-zero choice of s). In particular, we could take $s = -1$. Then

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Now that we have a method for finding the eigenvectors once we know the eigenvalues, the natural question is: Is there a way to determine the eigenvalues without knowing the eigenvectors? This is where determinants come in. The number λ is an eigenvector if there is some non-zero solution $\mathbf{x}$ to the equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$. In other words, λ is an eigenvector if the matrix $(A - \lambda I)$ is not invertible. This happens precisely when $\det(A - \lambda I) = 0$.

This gives us a method for finding the eigenvalues. Compute $\det(A - \lambda I)$. This will be a polynomial in λ. The eigenvalues will be exactly the values of λ that make this polynomial zero, i.e., the roots of the polynomial.

So here is the algorithm for finding the eigenvalues and eigenvectors.

(1) Compute $\det(A - \lambda I)$ and find the values of λ for which it is zero. These are the eigenvalues.

(2) For each eigenvalue, find the non-zero solutions to $(A - \lambda I)\mathbf{x} = \mathbf{0}$. These are the eigenvectors.

I should mention that this is actually only a practical way to find eigenvalues when the matrix is small. Finding eigenvalues of large matrices is an important problem and many efficient methods have been developed for use on computers.

Example 1

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

First we compute

$$\begin{aligned} \det(A - \lambda I) &= (2 - \lambda)(2 - \lambda) - 1 \\ &= \lambda^2 - 4\lambda + 3 \end{aligned}$$

We can find the roots of this polynomial using the quadratic formula or by factoring it by inspection. We get

$$\lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3)$$

So the eigenvalues are 1 and 3.

Now we find the eigenvector for $\lambda = 1$. We must solve $(A - I)\mathbf{x} = \mathbf{0}$. The matrix for this homogeneous system of equations is

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Reducing this matrix yields

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

so an eigenvector is

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Next we find the eigenvector for $\lambda = 3$. We must solve $(A - 3I)\mathbf{x} = \mathbf{0}$. The matrix for this homogeneous system of equations is

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

Reducing this matrix yields

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

so an eigenvector is

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Example 2 Let us find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 3 & -6 & -7 \\ 1 & 8 & 5 \\ -1 & -2 & 1 \end{bmatrix}.$$

First we compute

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 3-\lambda & -6 & -7 \\ 1 & 8-\lambda & 5 \\ -1 & -2 & 1-\lambda \end{bmatrix} \\ &= (3-\lambda)((8-\lambda)(1-\lambda)+10) + 6((1-\lambda)+10) - 7(-2+(8-\lambda)) \\ &= -\lambda^3 + 12\lambda^2 - 44\lambda + 48 \end{aligned}$$

Its not always easy to find the zeros of a polynomial of degree 3. However, if we already know one solution, we can find the other two. Sometimes, one can find one solution by guessing. In this case we already know that 4 is a solution (since this is the same matrix that appeared in the example in the last section). We can check this:

$$-64 + 12 \times 16 - 44 \times 4 - 48 = 0$$

This means that $\lambda^3 + 12\lambda^2 - 44\lambda + 48$ can be factored as $-\lambda^3 + 12\lambda^2 - 44\lambda + 48 = (\lambda - 4)q(\lambda)$, where $q(\lambda)$ is a second degree polynomial. To find $q(\lambda)$ we can use long division of polynomials.

$$\begin{array}{rrrrr} & -\lambda^2 & +8\lambda & -12 & \\ \lambda - 4 \;\Big) & -\lambda^3 & +12\lambda^2 & -44\lambda & +48 \\ & -\lambda^3 & +4\lambda^2 & & \\ & & 8\lambda^2 & -44\lambda & \\ & & 8\lambda^2 & -32\lambda & \\ & & & -12\lambda & +48 \\ & & & -12\lambda & +48 \end{array}$$

This yields $q(\lambda) = -\lambda^2 + 8\lambda - 12$ This can be factored using the quadratic formula (or by inspection) as $q(\lambda) = -(\lambda - 2)(\lambda - 6)$ So we conclude

$$-\lambda^3 + 12\lambda^2 - 44\lambda + 48 = -(\lambda - 4)(\lambda - 2)(\lambda - 6)$$

and the eigenvalues are 2, 4 and 6.

Now we find the eigenvector for $\lambda = 2$. We must solve $(A - 2I)\mathbf{x} = \mathbf{0}$. The matrix for this homogeneous system of equations is

$$\begin{bmatrix} 1 & -6 & -7 \\ 1 & 6 & 5 \\ -1 & -2 & -1 \end{bmatrix}$$

Reducing this matrix yields

$$\begin{bmatrix} 1 & -6 & -7 \\ 0 & -8 & -8 \\ 0 & 0 & 0 \end{bmatrix}$$

so an eigenvector is

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Next we find the eigenvector for $\lambda = 4$. We must solve $(A - 4I)\mathbf{x} = \mathbf{0}$. The matrix for this homogeneous system of equations is

$$\begin{bmatrix} -1 & -6 & -7 \\ 1 & 4 & 5 \\ -1 & -2 & -3 \end{bmatrix}$$

Reducing this matrix yields

$$\begin{bmatrix} -1 & -6 & -7 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

so an eigenvector is

$$\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Finally we find the eigenvector for $\lambda = 6$. We must solve $(A - 4I)\mathbf{x} = \mathbf{0}$. The matrix for this homogeneous system of equations is

$$\begin{bmatrix} -3 & -6 & -7 \\ 1 & 2 & 5 \\ -1 & -2 & -5 \end{bmatrix}$$

Reducing this matrix yields

$$\begin{bmatrix} -3 & -6 & -7 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

so an eigenvector is

$$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Example 3

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & -1 & 1-\lambda \end{bmatrix}.$$

In this case it makes sense to expand along the last column. This yields

$$\det(A - \lambda I) = 0 - 0 + (1-\lambda)(1-\lambda)(2-\lambda) = (1-\lambda)^2(2-\lambda)$$

This is already factored, so the zeros are $\lambda = 1$ and $\lambda = 2$. Notice that the factor $(1 - \lambda)$ occurs occurs to the second power. In this situation there are fewer distinct eigenvalues than we expect.

Lets compute the eigenvectors.

To find the eigenvectors for $\lambda = 1$ we must solve the homogeneous equation with matrix $A - I$, i.e.,

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

This reduces to

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and we find that there are two parameters in the solution. The set of solutions in parametric form is

$$s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We can find two linearly independent solutions by setting $s = 1$, $t = 0$ and $s = 0$, $t = 1$. This gives

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

To find the eigenvectors for $\lambda = 2$ we must solve the homogeneous equation with matrix $A - 2I$, i.e.,

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}$$

This reduces to

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

and we find that the set of solutions in parametric form is

$$s \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Setting $s = 1$ gives the eigenvector

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

In this 3×3 example, even though there are only two distinct eigenvalues, 1 and 2, there are still three independent eigenvectors (i.e., a basis), because the eigenvalue 1 has two independent eigenvectors associated to it.

Example 4

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

Here

$$\det(A - \lambda I) = (\lambda - 2)^2$$

so there is only one eigenvalues $\lambda = 2$.

To find the eigenvectors, we must solve the homogeneous system with matrix

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The solutions are

$$s \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

so there is only one independent eigenvector.

So here is a matrix that does not have a basis of eigenvectors. Fortunately, matrices like this, that have too few eigenvectors, are rare. But they do occur!

Problem 4.3: Find the eigenvalues and eigenvectors for

$$a) \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} \quad b) \begin{bmatrix} -2 & -8 \\ 4 & 10 \end{bmatrix} \quad c) \begin{bmatrix} 29 & -10 \\ 105 & -36 \end{bmatrix} \quad d) \begin{bmatrix} -9 & -14 \\ 7 & 12 \end{bmatrix}$$

Problem 4.4: Find the eigenvalues and eigenvectors for

$$a) \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 2 \\ 2 & 0 & 2 \end{bmatrix} \quad b) \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix} \quad c) \begin{bmatrix} 7 & -9 & -15 \\ 0 & 4 & 0 \\ 3 & -9 & -11 \end{bmatrix} \quad d) \begin{bmatrix} 31 & -100 & 70 \\ 18 & -59 & 42 \\ 12 & -40 & 29 \end{bmatrix}$$

Complex numbers

Complex numbers can be thought of as points on the xy plane. The point $[x, y]$, thought of as a complex number, is written $x + iy$ (or $x + jy$ if you are an electrical engineer).

If $z = x + iy$ then x is called the real part of z and y is called the imaginary part of z.

Complex numbers are added just as if they were vectors in two dimensions. If $z = x + iy$ and $w = s + it$, then

$$z + w = (x + iy) + (s + it) = (x + s) + i(y + t)$$

To multiply two complex numbers, just remember that $i^2 = -1$. So if $z = x + iy$ and $w = s + it$, then

$$zw = (x + iy)(s + it) = xs + i^2ytr + iys + ixt = (xs - yt) + i(xt + ys)$$

The modulus of a complex number, denoted $|z|$ is simply the length of the corresponding vector in two dimensions. If $z = x + iy$

$$|z| = |x + iy| = \sqrt{x^2 + y^2}$$

An important property is

$$|zw| = |z||w|$$

The complex conjugate of a complex number z, denoted $\bar{z}$, is the reflection of z across the x axis. Thus $\overline{x + iy} = x - iy$. Thus complex conjugate is obtained by changing all the i's to $-i$'s. We have

$$\overline{zw} = \bar{z}\bar{w}$$

and

$$z\bar{z} = |z|^2$$

This last equality is useful for simplifying fractions of complex numbers by turnint the denominator into a real number, since

$$\frac{z}{w} = \frac{z\bar{w}}{|w|^2}$$

For example, to simplify $(1 + i)/(1 - i)$ we can write

$$\frac{1+i}{1-i} = \frac{(1+i)^2}{(1-i)(1+i)} = \frac{1-1+2i}{(2} = i$$

A complex number z is real (i.e. the y part in $x + iy$ is zero) whenever $\bar{z} = z$. We also have the following formulas for the real and imaginary part. If $z = x + iy$ then $x = (z + \bar{z})/2$ and $y = (z - \bar{z})/(2i)$

Complex numbers are indispensible in many practical calculations. We will discuss complex exponentials when we talk about differential equations. The reason why we are interested in them now is the following fact:

If we use complex numbers, every polynomial can be completely factored.

In other words given a polynomial $\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$, there exist (possibly complex) numbers $r_1, r_2, \ldots, r_n$ such that

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = (\lambda - r_1)(\lambda - r_2)\cdots(\lambda - r_n)$$

The numbers r_1 are the values of λ for which the polynomial is zero.

So for example the polynomial λ^2+1 has no real roots, since there is no real number λ for which it is zero. However there are two complex roots, $\pm i$ and

$$\lambda^2+1=(\lambda+i)(\lambda-i)$$

Of course, actually finding the roots of a high degree polynomial is difficult. Here are some points to keep in mind.

You can always find the roots of a quadratic polynomial using the quadratic formula. In other words the roots of $a\lambda^2+b\lambda+c$ are

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

If the quantity inside the square root is negative, then the roots are complex. So, for example the roots of $\lambda^2+\lambda+1$ are

$$\frac{-1\pm\sqrt{1^2-4}}{2}=\frac{-1\pm\sqrt{-3}}{2}=\frac{-1\pm\sqrt{-1}\sqrt{3}}{2}=\frac{-1}{2}\pm i\frac{\sqrt{3}}{2}$$

Problem 4.5: Show that $|zw|=|z||w|$ for complex numbers z and w.

Problem 4.6: Show that $\overline{zw}=\bar{z}\bar{w}$ for complex numbers z and w.

Problem 4.7: Show that $z\bar{z}=|z|^2$ for every complex numbers z.

Example 5

Lets consider the matrix of rotation by $\pi/2$. This is the matrix

$$A=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}.$$

We compute

$$\det(A-\lambda I)=\det\begin{bmatrix}-\lambda & 1\\ -1 & -\lambda\end{bmatrix}=\lambda^2+1$$

The roots are $\pm i$ so the eigenvalues are i and $-i$.

Now we compute the eigenvector corresponding to the eigenvalue i. We must solve the homogeneous equation with matrix

$$\begin{bmatrix}-i & 1\\ -1 & -i\end{bmatrix}$$

Notice that we will have to do complex arithmetic to achieve this, since the matrix now has complex entries. To reduce this matrix we have to add i times the first row to the second row. This gives

$$\begin{bmatrix}-i & 1\\ -1+-i^2 & -i+i\end{bmatrix}=\begin{bmatrix}-i & 1\\ 0 & 0\end{bmatrix}$$

So if we let the $x_2=s$, then $-ix_1+s=0$, or $x_1=-is$. So the solution is

$$s\begin{bmatrix}-i\\ 1\end{bmatrix}$$

and we may choose $s = 1$. Lets check that this is really an eigenvector:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix} = i \begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

To find the other eigenvector we can use a trick. Suppose that the original matrix A has only real entries. This will always be the case in our examples. Suppose that A has a complex eigenvalue eigenvector pair λ and $\mathbf{x}$. Then $A\mathbf{x} = \lambda\mathbf{x}$. Taking the complex conjugate of this equation, we obtain $\bar{A}\bar{\mathbf{x}} = \bar{\lambda}\bar{\mathbf{x}}$. (Here conjugating a matrix or a vector just means conjugating each entry). Now, since A has real entries, $\bar{A} = A$. Hence $A\bar{\mathbf{x}} = \bar{\lambda}\bar{\mathbf{x}}$. In other words $\bar{\lambda}$ is an eigenvalue with eigenvector $\bar{\mathbf{x}}$.

In the present example, we already know that $\bar{i} = -i$ is an eigenvalue. But now we don't have to compute the eigenvector that goes along with it. It is simply the conjugate of the one we already computed. So the eigenvector corresponding to $-i$ is

$$\begin{bmatrix} i \\ 1 \end{bmatrix}$$

Eigenvalues and eigenvectors: summary

The eigenvalues of A are the zeros or roots of the polynomial $\det(A - \lambda I)$. If we use complex numbers then $\det(A - \lambda I)$ can be completely factored, i.e.,

$$\det(A - \lambda I) = \pm(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$$

Finding the roots may be difficult. However for 2×2 matrices we may use the quadratic formula.

If all the roots are distinct (i.e., $\lambda_i \neq \lambda_j$ for $i \neq j$) then the corresponding eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are linearly independent (I didn't show you why this is true, so I'm just asking you to believe it!) and therefore form a basis.

If there are repeated roots, then there are fewer than n distinct eigenvalues. In this situation, it might happen that there are not enough eigenvectors to form a basis. However it also might happen that more than one eigenvector associated to a given eigenvalue, so that in the end there are enough eigenvectors to form a basis. Unfortunately the only way we have to find out is to try to compute them all.

Complex exponential

We begin by considering the differential equation

$$y'(t) = y(t)$$

In other words we are looking for a function whose derivative is equal to the function. The exponential is such a function, so $y(t) = e^t$ is a solution to this differential equation. So is $y(t) = Ce^t$, where C is a constant.

Now consider the equation

$$y'(t) = ay(t)$$

where i a is real number. Then, using the chain rule, we see that $y(t) = Ce^{at}$ is a solution for any choice of constant C.

Notice that the constant C is $x(0)$, the value of the solution at time zero. If we insist that the solution at time $t = 0$ take on a particular value

$$x(0) = y_0$$

Then this forces the constant C to be y_0

Now consider the equation

$$y'(t) = iy(t)$$

A solution to this equation is given by

$$y(t) = \cos(t) + i\sin(t)$$

To check this, just differentiate.

$$\begin{aligned} y'(t) &= \cos'(t) + i\sin'(t) \\ &= -\sin(t) + i\cos(t) \\ &= i(\cos(t) + i\sin(t)) \\ &= iy(t) \end{aligned}$$

So it is natural to define the exponential, e^{it}, of a purely imaginary number it to be

$$e^{it} = \cos(t) + i\sin(t)$$

The complex exponential satisfies the familiar rule $e^{i(s+t)} = e^{is}e^{it}$ since by the addition formulas for sine and cosine

$$\begin{aligned} e^{i(s+t)} &= \cos(s+t) + i\sin(s+t) \\ &= \cos(s)\cos(t) - \sin(s)\sin(t) + i(\sin(s)\cos(t) + \cos(s)\sin(t)) \\ &= (\cos(s) + i\sin(s))(\cos(t) + i\sin(t)) \\ &= e^{is}e^{it} \end{aligned}$$

Now it easy to check that solutions to

$$y'(t) = iby(t)$$

are given by $y(t) = Ce^{ibt}$, where C is an arbitrary constant. Since we are dealing with complex numbers, we allow C to be complex too.

The exponential of a number that has both a real and imaginary part is defined in the natural way.

$$e^{a+ib} = e^a e^{ib} = e^a(\cos(b) + i\sin(b))$$

and it is easy to check that the solution to the differential equation

$$y'(t) = \lambda y(t) = (a + ib)y(t)$$

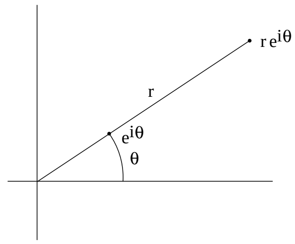
is given by $y(t) = Ce^{\lambda t} = Ce^{(a+ib)t}$. As before, if we insist that the solution at time $t = 0$ take on a particular value

$$x(0) = y_0,$$

then this forces the constant C to be y_0.

Polar representation of a complex number

Notice that the number $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ lies on the unit circle on the complex plane, at the point making an angle of θ (radians) with the x axis. If we multiply $e^{i\theta}$ by a real number r, then we obtain the complex number whose polar co-ordinates are r and θ.



Notice that r is exactly the modulus of the complex number $re^{i\theta}$. The angle θ is called the argument. This representation of complex numbers makes the definition of complex multiplication more transparent. We have

$$r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

In other words, when we multiply two complex numbers, the moduli get multiplied and the arguments get added.

Systems of linear differential equations

Consider the system of differential equations

$$\begin{array}{lll} y_1'(t) & = a_{1,1}y_1(t) & +a_{1,2}y_2(t) \\ y_2'(t) & = a_{2,1}y_1(t) & +a_{2,2}y_2(t) \end{array}$$

This system of equations describes a situation where we have two quantities y_1 and y_2, where the rate of change of each one of the quantities depends on the values of both.

We can rewrite this as a matrix equation. Let $\mathbf{y}(t)$ be the vector

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}.$$

and define the derivative of a vector to be the vector of derivatives, i.e.,

$$\mathbf{y}'(t) = \begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix}.$$

Define A to be the matrix

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}.$$

Then this system of equations can be rewritten

$$\mathbf{y}'(t) = A\mathbf{y}$$

A general system of linear equations has this form, except $\mathbf{y}(t)$ is an n-dimensional vector and A is an $n \times n$ matrix.

How can we find solutions to such a system of equations? Taking a hint from the scalar case, we can try to find solutions of the form

$$\mathbf{y}(t) = e^{\lambda t}\mathbf{x}$$

where $\mathbf{x}$ is a fixed vector (not depending on t). With this definition

$$\mathbf{y}'(t) = \lambda e^{\lambda t}\mathbf{x}$$

so that $\mathbf{y}' = A\mathbf{y}$ whenever

$$\lambda e^{\lambda t}\mathbf{x} = Ae^{\lambda t}\mathbf{x} = e^{\lambda t}A\mathbf{x}$$

Dividing by $e^{\lambda t}$, this condition becomes

$$\lambda\mathbf{x} = A\mathbf{x}.$$

In other words, $\mathbf{y}(t) = e^{\lambda t}\mathbf{x}$ is a solution exactly whenever λ and $\mathbf{x}$ are an eigenvalue eigenvector pair for A.

So we can find as many solutions as we have eigenvalue eigenvector pairs.

To proceed we first notice that if $\mathbf{y}_1(t)$ and $\mathbf{y}_2(t)$ are two solutions to the equation $\mathbf{y}' = A\mathbf{y}$, then a linear combination $c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t)$ is also a solution, since

$$\begin{aligned}\frac{d}{dt}\Big(c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t)\Big) &= c_1\mathbf{y}_1'(t) + c_2\mathbf{y}_2'(t)\\ &= c_1A\mathbf{y}_1(t) + c_2A\mathbf{y}_2(t)\\ &= A\Big(c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t)\Big)\end{aligned}$$

Notice that we are assuming that the constants c_1 and c_2 do not depend on t.

Similarly, if $\mathbf{y}_1(t)$, $\mathbf{y}_2(t)$, …, $\mathbf{y}_n(t)$ are n solutions then $c_1\mathbf{y}_1(t)+c_2\mathbf{y}_2(t)+\cdots+c_n\mathbf{y}_n(t)$ is a solution for any choice of constants $c_1, c_2, \ldots, c_n$.

Now suppose that A is an $n \times n$ matrix. Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_k$ are its eigenvalues with eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$. Then we have that for any choice of constants $c_1, c_2, \ldots, c_k$,

$$\mathbf{y}(t) = c_1e^{\lambda_1 t}\mathbf{x}_1 + c_2e^{\lambda_2 t}\mathbf{x}_2 + \cdots + c_ke^{\lambda_k t}\mathbf{x}_k \tag{4.1}$$

is a solution.

Have we found all solutions? In other words, could there be a solution of the equation that is not this form, or is every solution of the form (4.1) for some choice of $c_1, c_2, \ldots, c_k$?

There is a theorem in differential equations that says that given an initial condition $\mathbf{x}_0$ then there is one and only one solution of $\mathbf{y}' = A\mathbf{y}$ satisfying $\mathbf{y}(0) = \mathbf{x}_0$.

So our theoretical question above is equivalent to the following quite practical question. Given an initial vector $\mathbf{x}_0$, does there exist a solution $\mathbf{y}(t)$ of the form (4.1) whose value at zero is the given initial condition, i.e., $\mathbf{y}(0) = \mathbf{x}_0$.

This will be true if, given any vector $\mathbf{x}_0$, one can find $c_1, c_2, \ldots, c_k$ so that

$$\mathbf{y}(0) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k = \mathbf{x}_0$$

This is exactly the condition that the eigenvectors form a basis.

It turns out that in the "bad" cases where there are not enough eigenvectors of A to form a basis, there are solutions that don't have the form (4.1).

Now suppose that there are n eigenvectors that do form a basis. How can we actually find the numbers $c_1, c_2, \ldots, c_n$ such that

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_n = \mathbf{x}_0?$$

Just notice that this is a system linear equations

$$\left[\mathbf{x}_1\middle|\mathbf{x}_2\middle|\cdots\middle|\mathbf{x}_n\right]\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \mathbf{x}_0$$

so you know what to do.

Example 1

Lets find the general solution to the system of equations

$$\begin{aligned} x_1'(t) &= 2x_1(t) + x_2(t) \\ x_2'(t) &= x_1(t) + 2x_2(t) \end{aligned}$$

This is equivalent to the matrix equation

$$\mathbf{y}'(t) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\mathbf{y}(t)$$

The matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has eigenvector and eigenvalues $\lambda_1 = 1$, $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\lambda_2 = 3$, $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ form a basis, so the general solution is

$$\mathbf{y}(t) = c_1e^{\lambda_1 t}\mathbf{x}_1 + c_2e^{\lambda_2 t}\mathbf{x}_2 = c_1e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Now lets find the solution satisfying the initial condition

$$\mathbf{y}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

We have to find constants c_1 and c_2 so that

$$c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

This is the same as solving

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The solution is

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix}$$

Example 2

Now lets do an example where the eigenvalues are complex. Consider the equation

$$\mathbf{y}'(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\mathbf{y}(t)$$

The matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has eigenvector and eigenvalues $\lambda_1 = i$, $\mathbf{x}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$ and complex conjugate $\lambda_2 = -i$, $\mathbf{x}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$. The eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ form a basis, so the general solution is

$$\mathbf{y}(t) = c_1e^{\lambda_1 t}\mathbf{x}_1 + c_2e^{\lambda_2 t}\mathbf{x}_2 = c_1e^{it}\begin{bmatrix} -i \\ 1 \end{bmatrix} + c_2e^{-it}\begin{bmatrix} i \\ 1 \end{bmatrix}$$

In most applications, the solutions that we are interested in are real. The solution above looks decidedly complex! Remember, however, that the constants c_1 and c_2 can be complex too. Perhaps for special choices of c_1 and c_2 the solution will turn out to be real.

This is always true when the original matrix is real. In this case the complex eigenvalues and eigenvectors occur in conjugate pairs. So if

$$\mathbf{y}_1(t) = e^{\lambda t}\mathbf{x}$$

is a solution, then so is

$$\bar{\mathbf{y}}_1(t) = e^{\bar{\lambda} t}\bar{\mathbf{x}}$$

So if we choose $c_1 = a/2$ and $c_2 = a/2$ for a real number a, then

$$\begin{aligned} c_1 e^{\lambda t}\mathbf{x} + c_2 e^{\bar{\lambda} t}\bar{\mathbf{x}} &= a/2(e^{\lambda t}\mathbf{x} + e^{\bar{\lambda} t}\bar{\mathbf{x}}) \\ &= a\,\mathrm{Re}(e^{\lambda t}\mathbf{x}) \end{aligned}$$

(here Re stands for the real part. We used that for a complex number z, $z + \bar{z} = 2\,\mathrm{Re}\,z$). Similarly, if we choose $c_1 = a/2i$ and $c_2 = -a/2i$, then

$$\begin{aligned} c_1 e^{\lambda t}\mathbf{x} + c_2 e^{\bar{\lambda} t}\bar{\mathbf{x}} &= a/2i(e^{\lambda t}\mathbf{x} - e^{\bar{\lambda} t}\bar{\mathbf{x}}) \\ &= a\,\mathrm{Im}(e^{\lambda t}\mathbf{x}) \end{aligned}$$

The upshot is that the real and imaginary parts of a solution are also solutions. Its sometimes easier to just start with one the complex solutions and find its real and imaginary parts. This gives us two reals solutions to work with. Notice that it doesn't matter which one of the complex solutions we pick. Because they are conjugate, their real parts are the same, and their imaginary parts differ by only a minus sign. In the example we have

$$\begin{aligned} \mathbf{y}_1(t) &= e^{it}\begin{bmatrix} -i \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -ie^{it} \\ e^{it} \end{bmatrix} \\ &= \begin{bmatrix} -i(\cos(t) + i\sin(t)) \\ \cos(t) + i\sin(t) \end{bmatrix} \\ &= \begin{bmatrix} -i\cos(t) + \sin(t) \\ \cos(t) + i\sin(t) \end{bmatrix} \\ &= \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} + i\begin{bmatrix} -\cos(t) \\ +\sin(t) \end{bmatrix} \end{aligned}$$

The real part and imaginary part are

$$\begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$$

and

$$\begin{bmatrix} -\cos(t) \\ \sin(t) \end{bmatrix}$$

One can check directly that these are solutions to the original equation. The general solution can also be written

$$a_1\begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} + a_2\begin{bmatrix} -\cos(t) \\ \sin(t) \end{bmatrix}$$

The advantage of this way of writing the solution is that if we choose a_1 and a_2 to be real the solution is real too.

Now suppose we want to satisfy an initial condition. Lets find the solution $\mathbf{y}(t)$ of the equation that satisfies

$$\mathbf{y}(0) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

There are two ways to proceed. Either we use the complex form of the general solution. Then we must find c_1 and c_2 such that

$$c_1 \begin{bmatrix} -i \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

This amounts to solving

$$\begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

The solution is

$$\begin{aligned} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -2 \end{bmatrix} \\ &= \frac{1}{-2i} \begin{bmatrix} 1 & -i \\ -1 & -i \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} i + 1/2 \\ i - 1/2 \end{bmatrix} \end{aligned}$$

So $c_1 = i + 1/2$ and $c_2 = i - 1/2$. If we plug these into the expression for the general solution we get the right answer. However there is still a fair amount of complex arithmetic needed to show explicitly that the solution is real.

Its easier to start with the real solutions. In this approach we must find a_1 and a_2 so that

$$a_1 \begin{bmatrix} \sin(0) \\ \cos(0) \end{bmatrix} + a_2 \begin{bmatrix} -\cos(0) \\ \sin(0) \end{bmatrix} = a_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

Thus $a_1 = a_2 = -2$ so the solution is

$$-2 \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} + -2 \begin{bmatrix} -\cos(t) \\ \sin(t) \end{bmatrix} = \begin{bmatrix} -2\sin(t) + 2\cos(t) \\ -2\cos(t) - 2\sin(t) \end{bmatrix}$$

Example 3 Now lets do an example where the eigenvalues are complex, and have both a real and imaginary part. Lets solve

$$\mathbf{y}'(t) = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{y}(t)$$

with initial condition

$$\mathbf{y}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The first step is to find the eigenvalues and eigenvectors. I'll omit the computations. The result is $\lambda_1 = -1+i$ with eigenvector $\mathbf{x}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$ and the complex conjugates $\lambda_2 = -1 - i$ with eigenvector $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$. Thus a solution is

$$\mathbf{y}_1(t) = e^{(-1+i)t} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

To find real solutions we calculate the real and imaginary parts of this.

$$\begin{aligned} \mathbf{y}_1(t) &= \begin{bmatrix} e^{(-1+i)t} \\ ie^{(-1+i)t} \end{bmatrix} \\ &= \begin{bmatrix} e^{-t}e^{it} \\ ie^{-t}e^{it} \end{bmatrix} \\ &= \begin{bmatrix} e^{-t}(\cos(t) + i\sin(t)) \\ ie^{-t}(\cos(t) + i\sin(t)) \end{bmatrix} \\ &= \begin{bmatrix} e^{-t}\cos(t) \\ -e^{-t}\sin(t) \end{bmatrix} + i \begin{bmatrix} e^{-t}\sin(t) \\ e^{-t}\cos(t) \end{bmatrix} \end{aligned}$$

So the general solution can be written

$$a_1 \begin{bmatrix} e^{-t}\cos(t) \\ -e^{-t}\sin(t) \end{bmatrix} + a_2 \begin{bmatrix} e^{-t}\sin(t) \\ e^{-t}\cos(t) \end{bmatrix}$$

To satisfy the initial condition, we need

$$a_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so that $a_1 = 1$ and $a_2 = 1$. Thus the solution is

$$\mathbf{y}(t) = e^{-t} \begin{bmatrix} \cos(t) + \sin(t) \\ -\sin(t) + \cos(t) \end{bmatrix}$$

Problem 4.8: Find the general solution to $\mathbf{y}' = A\mathbf{y}$ when $A = \begin{bmatrix} -2 & -8 \\ 4 & 10 \end{bmatrix}$. (Hint: This matrix appeared in the problems of last chapter). Find the solution satisfying the initial condition $\mathbf{y}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Problem 4.9: Find the general solution to $\mathbf{y}' = A\mathbf{y}$ when $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$. Find both the complex form and the real form. Find the solution satisfying the initial condition $\mathbf{y}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Problem 4.10: Find the general solution to $\mathbf{y}' = A\mathbf{y}$ when $A = \begin{bmatrix} 6 & 0 & 13 \\ 5 & 1 & 13 \\ -2 & 0 & -4 \end{bmatrix}$. Find both the complex form and the real form. Find the solution satisfying the initial condition $\mathbf{y}(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Problem 4.11: Is it true that every 3×3 matrix with real entries always has at least one real eigenvalue? Why?

Diagonalization

Diagonal matrices (that is, matrices that have zero entries except on the diagonal) are extremely easy to work with. For a start, the eigenvalues of a diagonal matrix are exactly the diagonal entries. If

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

then $\det(D - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda)$ which is zero precisely when λ equals one of $\lambda_1, \lambda_2, \ldots, \lambda_n$. The corresponding eigenvectors are just the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$.

It is also easy to compute powers of a diagonal matrix. We simply obtain

$$D^k = \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix}$$

This formula makes it easy to compute the matrix exponential of D. Recall that the matrix e^{tD} is defined to be the matrix power series

$$e^{tD} = I + tD + \frac{t^2}{2}D^2 + \frac{t^3}{3!}D^3 + \cdots$$

Using the formula above we find that

$$\begin{aligned} e^{tD} &= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} + \begin{bmatrix} t\lambda_1 & 0 & \cdots & 0 \\ 0 & t\lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & t\lambda_n \end{bmatrix} + \begin{bmatrix} \frac{t^2\lambda_1^2}{2} & 0 & \cdots & 0 \\ 0 & \frac{t^2\lambda_2^2}{2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \frac{t^2\lambda_n^2}{2} \end{bmatrix} + \cdots \\ &= \begin{bmatrix} 1 + t\lambda_1 + \frac{t^2\lambda_1^2}{2} + \cdots & 0 & \cdots & 0 \\ 0 & 1 + t\lambda_2 + \frac{t^2\lambda_2^2}{2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 + t\lambda_n + \frac{t^2\lambda_n^2}{2} + \cdots \end{bmatrix} \\ &= \begin{bmatrix} e^{t\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{t\lambda_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & e^{t\lambda_n} & \end{bmatrix} \end{aligned}$$

Things are not quite so simple for an arbitrary $n \times n$ matrix A. However, **if A has a basis of eigenvectors** then it turns out that there exists an invertible matrix B such that $AB = DB$, where D is the diagonal matrix whose diagonal elements are the eigenvalues of A. Multiplying by B^{-1} from either the left or right gives

$$A = BDB^{-1}, \quad D = B^{-1}AB$$

In fact, B is simply the matrix whose columns are the eigenvectors of A. In other words, if $\mathbf{x}_1$, $\mathbf{x}_2$, …, $\mathbf{x}_n$ are the eigenvectors for A then $B = \left[\mathbf{x}_1 \middle| \mathbf{x}_2 \middle| \cdots \middle| \mathbf{x}_n\right]$. To see this notice that

$$\begin{aligned} AB &= A\left[\mathbf{x}_1 \middle| \mathbf{x}_2 \middle| \cdots \middle| \mathbf{x}_n\right] \\ &= \left[A\mathbf{x}_1 \middle| A\mathbf{x}_2 \middle| \cdots \middle| A\mathbf{x}_n\right] \\ &= \left[\lambda_1\mathbf{x}_1 \middle| \lambda_2\mathbf{x}_2 \middle| \cdots \middle| \lambda_n\mathbf{x}_n\right] \\ &= \left[\mathbf{x}_1 \middle| \mathbf{x}_2 \middle| \cdots \middle| \mathbf{x}_n\right] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \\ &= BD \end{aligned}$$

The assumption that A has a basis of eigenvectors implies that the matrix B is invertible.

Using the representation $A = BDB^{-1}$ it is easy to calculate powers of A. We have

$$A^2 = BDB^{-1}BDB^{-1} = BDIDB^{-1} = BD^2B^{-1}$$

and similarly

$$A^k = BD^kB^{-1}$$

Therefore we can now also sum the power series for the exponential and obtain

$$e^{tA} = Be^{tD}B^{-1}$$

Computing high powers of a matrix

Recall that when we were discussing the random walk problem, we ended up with the problem of computing the limit for large n of $P^n\mathbf{x}_0$ where P is the matrix of transition probabilities.

We can now solve this problem using diagonalization. Lets do an example. Suppose that

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{2}{3} & \frac{3}{4} \end{bmatrix}$$

We wish to compute P^n for large n.

We begin by diagonalizing P. This involves finding the eigenvalues and eigenvectors. I won't give the details of this computation. The results are $\lambda_1 = 1$, $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 8/3 \end{bmatrix}$ and $\lambda_2 = 1/12$, $\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. So

$$P = \begin{bmatrix} 1 & -1 \\ \frac{8}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{12} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ \frac{8}{3} & 1 \end{bmatrix}^{-1}$$

and

$$P^n = \begin{bmatrix} 1 & -1 \\ \frac{8}{3} & 1 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & (\frac{1}{12})^n \end{bmatrix} \begin{bmatrix} 1 & -1 \\ \frac{8}{3} & 1 \end{bmatrix}^{-1}$$

But $1^n = 1$ for all n and $\left(\frac{1}{12}\right)^n \to 0$ as $n \to \infty$, since $\frac{1}{12} < 1$. So

$$\begin{aligned} \lim_{n\to\infty} P^n &= \begin{bmatrix} 1 & -1 \\ \frac{8}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ \frac{8}{3} & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & -1 \\ \frac{8}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{3}{11} & \frac{3}{11} \\ \frac{-8}{11} & \frac{3}{11} \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{11} & \frac{3}{11} \\ \frac{8}{11} & \frac{8}{11} \end{bmatrix} \end{aligned}$$

Another formula for the determinant

If A has a basis of eigenvectors, then we can get another formula for the determinant. Using the multiplicative property of the determinant, we have

$$\det(A) = \det(BDB^{-1}) = \det(B)\det(D)\det(B)^{-1} = \det(D).$$

But $\det(D)$ is just the product of the diagonal elements, i.e., the eigenvalues. Thus the determinant of A is the product of its eigenvalues:

$$\det(A) = \lambda_1\lambda_2\cdots\lambda_n.$$

Actually, this is true even if A doesn't have a basis of eigenvectors and isn't diagonalizable.

The matrix exponential and differential equations

The matrix exponential e^{tA} can be used to solve the differential equation

$$\mathbf{y}'(t) = A\mathbf{y}(t)$$

with initial condition

$$\mathbf{y}(0) = \mathbf{x}_0$$

To see this notice that e^{tA} satisfies the differential equation $\frac{d}{dt}e^{tA} = Ae^{tA}$. This follows from the power series representation

$$e^{tA} = I + tA + \frac{t^2}{2}A^2 + \frac{t^3}{3!}A^3 + \cdots$$

since

$$\begin{aligned}\frac{d}{dt}e^{tA} &= A + \frac{2t}{2}A^2 + \frac{3t^2}{3!}A^3 + \cdots \\ &= A + tA^2 + \frac{t^2}{2!}A^3 + \cdots \\ &= A(I + tA + \frac{t^2}{2}A^2 + \cdots) \\ &= Ae^{tA}\end{aligned}$$

Also

$$e^{0A} = I$$

These two facts imply that $\mathbf{y}(t) = e^{tA}\mathbf{x}_0$ is the solution to our differential equation and initial condition, since $\mathbf{y}'(t) = \frac{d}{dt}e^{tA}\mathbf{x}_0 = Ae^{tA}\mathbf{x}_0 = A\mathbf{y}(t)$ and $\mathbf{y}(0) = e^{0A}\mathbf{x}_0 = I\mathbf{x}_0 = \mathbf{x}_0$.

The matrix exponential is a nice theoretical construction. However, to actually compute the matrix exponential using diagonalization involves just the same ingredients—computing the eigenvalues and vectors—as our original solution. In fact it is more work.

However, there is one situation where the matrix exponential gives us something new. This is the situation where A does not have a basis of eigenvectors. The power series definition of the matrix exponential still makes sense, and can compute it in certain special cases. Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. This matrix does not have a basis of eigenvectors. So it cannot be diagonalized. However, in a homework problem, you showed that $e^{tA} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$. Thus the solution to

$$\mathbf{y}'(t) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\mathbf{y}(t)$$

with initial condition

$$\mathbf{y}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

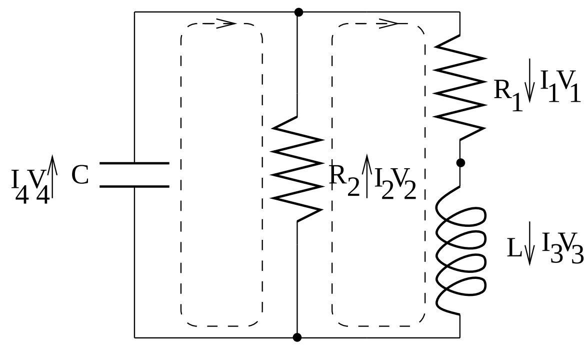
is

$$\mathbf{y}(t) = e^{tA}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2e^t + te^t \\ e^t \end{bmatrix}$$

Notice that this solution involves a power of t in addition to exponentials.

LCR circuits

We now return to the circuit that we discussed previously.



Recall that we chose as basic variables I_3 and V_4 and solved for all the other variables in terms of these. The result was

$$
\begin{aligned}
I_1 &= I_3 \\
I_2 &= \frac{1}{R_2} V_4 \\
I_4 &= I_3 - \frac{1}{R_2} V_4 \\
V_1 &= R_1 I_3 \\
V_2 &= V_4 \\
V_3 &= -R_1 I_3 - V_4
\end{aligned}
$$

Now we can complete the job and determine I_3 and V_4. We have to take into account now that the currents and voltages are functions of time. The relations between currents and voltages across capacitors and inductors involves the time derivatives.

If I and V are the current flowing through and the voltage across a capacitor with capacitance C, then

$$\frac{dV}{dt} = \frac{1}{C} I$$

If I and V are the current flowing through and the voltage across an inductor with inductance L, then

$$\frac{dI}{dt} = \frac{1}{L} V$$

Notice that we have chosen as basic the variables that get differentiated.

Using these relations for I_3 and V_4 yields

$$
\begin{aligned}
\frac{dI_3}{dt} &= \frac{1}{L} V_3 \\
\frac{dV_4}{dt} &= \frac{1}{C} I_4
\end{aligned}
$$

Now we reexpress everything in terms of I_3 and V_4 using the equations we derived previously.

$$
\begin{aligned}
\frac{dI_3}{dt} &= \frac{1}{L}(-R_1 I_3 - V_4) = \frac{-R_1}{L} I_3 - \frac{1}{L} V_4 \\
\frac{dV_4}{dt} &= \frac{1}{C}(I_3 - \frac{1}{R_2} V_4) = \frac{1}{C} I_3 - \frac{1}{R_2 C} V_4
\end{aligned}
$$

This can be written as

$$\begin{bmatrix} I_3 \\ V_4 \end{bmatrix}' = \begin{bmatrix} \frac{-R_1}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{R_2 C} \end{bmatrix} \begin{bmatrix} I_3 \\ V_4 \end{bmatrix}$$

Lets try to determine for what values of R_1, L, C and R_2 the circuit exhibits oscillations. Recall that the solution will involve sines and cosines whenever the matrix has complex eigenvalues.

The polynomial $\det(A - \lambda I) = \lambda^2 + b\lambda + c$, where

$$b = \frac{R_1}{L} + \frac{1}{R_2 C}$$

and

$$c = \frac{R_1}{R_2 LC} + \frac{1}{LC}.$$

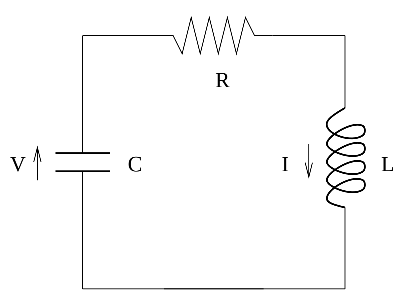
The eigenvalues are the roots of this polynomial, given by $(-b \pm \sqrt{b^2 - 4c})/2$. These will be complex if $b^2 < 4c$, i.e., if

$$\left(\frac{R_1}{L} + \frac{1}{R_2 C}\right)^2 < 4\left(\frac{R_1}{R_2 LC} + \frac{1}{LC}\right)$$

Notice that this can be achieved by decreasing R_1 and increasing R_2

Problem 4.12: In the circuit above, suppose that $R_1 = R_2 = 1$ ohm, $C = 1$ farad and $L = 1$ henry. If the intial current across the inductor is $I_3(0) = 1$ ampere and initial voltage across the capacitor is $V_4(0) = 1$ volt, find $I_3(t)$ and $V_4(t)$ for all later times. What is $V_1(t)$?

Problem 4.13: Consider the circuit with diagram



Write down the system of equations satisfied by I and V. For what values of L, C and R does the circuit exhibit oscillations? Suppose that $R = 1$ ohm, $C = 1$ farad and $L = 1$ henry. If the intial current across the inductor is $I(0) = 1$ ampere and initial voltage across the capacitor is $V(0) = 1$ volt, find $I(t)$ and $V(t)$ for all later times.

Converting higher order equations into first order systems

So far we have only considered first order differential equations. In other words, the equations have only involved first derivatives $y'(t)$ and not higher derivatives like $y''(t)$. However higher order equations, especially second order equations, occur often in practical problems. In this section we will show that a higher order linear differential equation can be converted into an equivalent first order system.

Suppose we want to solve the equation

$$y''(t) + ay'(t) + by(t) = 0$$

with initial conditions

$$\begin{aligned} y(0) &= y_0 \\ y'(0) &= y_0' \end{aligned}$$

Define the new functions $z_1(t)$ and $z_2(t)$ by

$$\begin{aligned} z_1(t) &= y(t) \\ z_2(t) &= y'(t) \end{aligned}$$

Then

$$\begin{aligned} z_1'(t) &= y'(t) & &= z_2(t) & & \\ z_2'(t) &= y''(t) & &= -ay'(t) - by(t) & &= -az_2(t) - bz_1(t) \end{aligned}$$

and

$$\begin{aligned} z_1(0) &= y_0 \\ z_2(0) &= y_0' \end{aligned}$$

In other words the vector $\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$ satisfies the equation

$$\frac{d}{dt}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$$

with initial condition

$$\begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}.$$

Here is an example. Suppose we want to solve the second order equation

$$y'' + 4y' + y = 0$$

with initial conditons

$$y(0) = 1, y'(0) = 0$$

If we let $z_1(t) = y(t)$ and $z_2(t) = y'(t)$ then

$$\frac{d}{dt}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$$

with initial condition

$$\begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

To solve this we first find the eigenvalues and eigenvectors. They are $\lambda_1 = -2+\sqrt{3}$, $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -2+\sqrt{3} \end{bmatrix}$ and $\lambda_1 = -2-\sqrt{3}$, $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -2-\sqrt{3} \end{bmatrix}$ So the general solution is

$$c_1 e^{(-2+\sqrt{3})t} \begin{bmatrix} 1 \\ -2+\sqrt{3} \end{bmatrix} + c_2 e^{(-2-\sqrt{3})t} \begin{bmatrix} 1 \\ -2-\sqrt{3} \end{bmatrix}$$

To satisfy the initial condition, we need

$$c_1 \begin{bmatrix} 1 \\ -2+\sqrt{3} \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2-\sqrt{3} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The solution is

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/3 + 1/2 \\ -\sqrt{3}/3 + 1/2 \end{bmatrix}$$

Thus

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = (\sqrt{3}/3 + 1/2) e^{(-2+\sqrt{3})t} \begin{bmatrix} 1 \\ -2+\sqrt{3} \end{bmatrix} + (-\sqrt{3}/3 + 1/2) e^{(-2-\sqrt{3})t} \begin{bmatrix} 1 \\ -2-\sqrt{3} \end{bmatrix}$$

and so

$$y(t) = z_1(t) = (\sqrt{3}/3 + 1/2) e^{(-2+\sqrt{3})t} + (-\sqrt{3}/3 + 1/2) e^{(-2-\sqrt{3})t}$$

Actually, to solve the equation

$$y''(t) + a y'(t) + b y(t) = 0$$

its not really neccesary to turn it into a first order system. We can simply try to find solutions of the form $y(t) = e^{\lambda t}$. If we plug this into the equation we get $(\lambda^2 + a\lambda + b)e^{\lambda t}$ which is zero if λ is a root of $\lambda^2 + a\lambda + b$. This polynomial has two roots, which yields two solutions.

Still, the procedure of turning a higher order equation into a first order system is important. This is because on a computer it is much easier to solve a first order system than a high order equation. If the coefficients a and b are functions of t, then exact solutions (like exponentials) usually can't be found. However, one can still turn the equation into a first order system $\mathbf{y}'(t) = A(t)\mathbf{y}(t)$ where the matrix now depends on t and solve this on a computer.

Problem 4.14: Consider the second order equation

$$y'' - 5y' + 6y = 0$$

with initial conditions $y(0) = 1$, $y'(0) = 0$. Solve this by turning it into a 2×2 system. Then solve it directly by using trial solutions $e^{\lambda t}$.

Problem 4.15: Consider the second order equation

$$y'' + y' + y = 0$$

with initial conditions $y(0) = 1$, $y'(0) = 0$. Solve this by turning it into a 2×2 system. Then solve it directly by using trial solutions $e^{\lambda t}$.

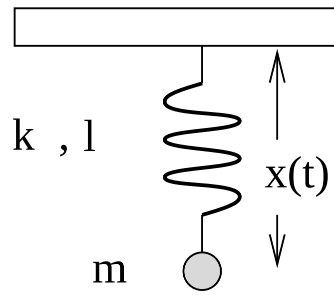
Problem 4.16: How can you turn a third order equation

$$y''' + ay'' + by' + cy = 0$$

into an equivalent 3×3 system of equations?

Springs and weights

To begin, lets consider the situation where we have a single weight hanging on a spring.



We want to determine how the weight moves in time. To do this we calculate the forces acting on the weight and use Newton's law of motion $F = ma$.

One force acting on the weight are the force of gravity. This acts in the positive x direction (i.e., downward) and has magnitude mg. The other force is due to the spring. It's magnitude is $k(x - l)$ in the negative x direction. The acceleration is the second derivative $x''(t)$. Thus the total force is $F = mg - k(x(t) - l)$ and $ma = mx''(t)$ Newton's law reads

$$mx''(t) = mg - k(x - l) = -kx + mg + lk$$

This is not quite in the form we can handle, due to the term $mg + lk$ on the right. What we must do is first find the equilibrium solution. In a previous lecture we found the equilibrium position by minimizing the potential energy. There is another, equivalent, way. That is to find the value of x for which the total force is zero. In other words

$$-kx_{\text{eq}} + mg + lk = 0$$

or

$$x_{\text{eq}} = (mg + lk)/k$$

Notice that the total force can be written

$$-kx + mg + lk = -k(x - x_{\text{eq}})$$

Now let $y(t) = x(t) - x_{\text{eq}}$ be the displacement from the equilibrium point. Notice that $y'(t) = x'(t)$ and $y''(t) = x''(t)$, since x_{eq} is a constant. So the equation for $y(t)$ is

$$my''(t) = -ky(t)$$

or

$$y''(t) + \frac{k}{m}y(t) = 0$$

We could turn this into a first order system. However, it is easier to try solutions of the form $e^{\lambda t}$. Substituting this into the equation yields

$$(\lambda^2 + k/m)e^{\lambda t} = 0$$

so we require that $\lambda^2 + k/m = 0$, or $\lambda = \pm i\sqrt{k/m}$. Thus, solutions are $e^{i\sqrt{k/m}t}$ and $e^{-i\sqrt{k/m}t}$. To obtain real solutions, we can take the real and imaginary parts. This gives as solutions $\sin(\sqrt{k/m}t)$ and $\cos(\sqrt{k/m}t)$, and the general solution is

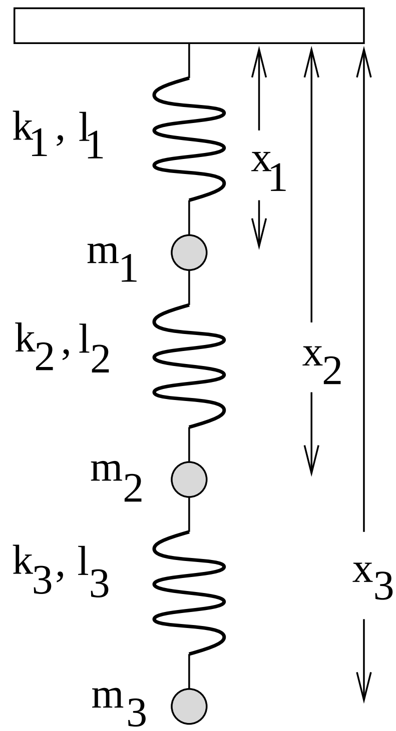
$$c_1 \sin(\sqrt{k/m}t) + c_2 \cos(\sqrt{k/m}t)$$

We can make the equation a little more interesting by adding friction. A frictional force is proportional to the velocity and acts in the direction opposite to the motion. Thus the equation with friction reads

$$y''(t) + \beta y'(t) + \frac{k}{m} y(t) = 0$$

This can be solved by turning it into a first order system, or directly, using trial solution of the form $e^{\lambda t}$ as above.

Now we turn the problem with several weights and springs. In this problem matrices play an essential role.



We begin by computing the forces acting on each weight. Let us start with the first weight. Gravity is pulling down, and the springs above and below are pushing or pulling with a force proportional to their extensions. Thus the total force on the first weight is $m_1 g - k_1(x_1 - l_1) + k_2(x_2 - x_1 - l_2)$. To get the signs right on the spring forces, think of what happens when one of the x_i's gets large. For example, when x_1 gets large, the first spring streches and pulls up, so the sign of the force should be negative for large x_1. So Newton's equation for the first weight is

$$m_1 x_1''(t) = m_1 g - k_1(x_1 - l_1) + k_2(x_2 - x_1 - l_2) = -(k_1 + k_2)x_1 + k_2 x_2 + m_1 g + k_1 l_1 - k_2 l_2$$

or

$$x_1''(t) = -\frac{k_1 + k_2}{m_1} x_1 + \frac{k_2}{m_1} x_2 + g + \frac{k_1 l_1 - k_2 l_2}{m_1}$$

Similary the equations for the second and third weights are

$$\begin{aligned} x_2''(t) &= \frac{k_2}{m_2} x_1 - \frac{k_2 + k_3}{m_2} x_2 + \frac{k_3}{m_2} x_3 + g + \frac{k_2 l_2 - k_3 l_3}{m_2} \\ x_3''(t) &= \frac{k_3}{m_3} x_2 - \frac{k_3}{m_3} x_3 + g + \frac{k_3 l_3}{m_3} \end{aligned}$$

Thus can be written as a second order matrix equation

$$\mathbf{x}''(t) = K\mathbf{x}(t) + \mathbf{b}$$

where

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix},$$

$$K = \begin{bmatrix} -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & 0 \\ \frac{k_2}{m_2} & -\frac{k_2+k_3}{m_2} & \frac{k_3}{m_2} \\ 0 & \frac{k_3}{m_3} & -\frac{k_3}{m_3} \end{bmatrix}$$

and

$$\mathbf{b} = \begin{bmatrix} g + \frac{k_1 l_1 - k_2 l_2}{m_1} \\ g + \frac{k_2 l_2 - k_3 l_3}{m_2} \\ g + \frac{k_3 l_3}{m_3} \end{bmatrix}.$$

With this notation, the equilibrium solution is the value of $\mathbf{x}$ that makes all the forces zero. That is,

$$K\mathbf{x}_{\mathrm{eq}} + \mathbf{b} = \mathbf{0}$$

or,

$$\mathbf{x}_{\mathrm{eq}} = -K^{-1}\mathbf{b}$$

As in the case of one weight the force side of the equation can now be written as

$$K\mathbf{x} + b = K(\mathbf{x} + K^{-1}\mathbf{b}) = K(\mathbf{x} - \mathbf{x}_{\mathrm{eq}})$$

so if we define

$$\mathbf{y}(t) = \mathbf{x}(t) - \mathbf{x}_{\mathrm{eq}},$$

the equation for $\mathbf{y}(t)$ is

$$\mathbf{y}''(t) = K\mathbf{y}(t)$$

To solve this second order 3×3 system, we could turn it in to a first order 6×6 system. However, just as in the case of a single higher order equation we can proceed directly. We try trial solutions of the form $e^{\kappa t}\mathbf{y}$. Substituting this into the equation, we see that the equation is satisfied if

$$\kappa^2\mathbf{y} = K\mathbf{y}$$

in other words, κ^2 is an eigenvalue of K with eigenvector $\mathbf{y}$, or κ is one of the two square roots of and eigenvalue.

So, if K has eigenvalues λ_1, λ_2 and λ_3 with eigenvectors $\mathbf{y}_1$, $\mathbf{y}_2$ and $\mathbf{y}_3$, then six solutions of the equation are given by $e^{\sqrt{\lambda_1}t}\mathbf{y}_1$, $e^{-\sqrt{\lambda_1}t}\mathbf{y}_1$, $e^{\sqrt{\lambda_2}t}\mathbf{y}_2$, $e^{-\sqrt{\lambda_2}t}\mathbf{y}_2$, $e^{\sqrt{\lambda_3}t}\mathbf{y}_3$ and $e^{-\sqrt{\lambda_3}t}\mathbf{y}_3$. If some of the λ_i's are negative, then these solutions are complex exponentials, and we may take their real and imaginary parts to get real solutions. The general solution is a linear combination of these, and the coefficients in the linear combination may be chosen to satisfy an initial condtion.

To make this clear we will do an example. Suppose that all the masses m_i, lengths l_i and spring constants k_i are equal to 1. Then

$$K = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

Suppose that the intial postion of the weights is $x_1 = 30$, $x_2 = 60$ and $x_3 = 70$, and that the initial velocities are $x_1' = 1$ and $x_2' = x_3' = 0$. We will determine the positions of the weights for all subsequent times.

The numbers in this problem don't turn out particularly nicely, so I'll just give them to 3 significant figures.

The first step is to find the eigenvalues and eigenvectors of K. They are given by

$$\lambda_1 = -0.198 \quad \lambda_2 = -1.55 \quad \lambda_3 = -3.25$$

$$\mathbf{x}_1 = \begin{bmatrix} 0.445 \\ 0.802 \\ 1.00 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} -1.25 \\ -0.555 \\ 1.00 \end{bmatrix} \quad \mathbf{x}_3 = \begin{bmatrix} 1.80 \\ -2.25 \\ 1.00 \end{bmatrix}$$

Let $\mu_1 = \sqrt{0.198} = 0.445$, $\mu_2 = \sqrt{1.55} = 1.25$ and $\mu_3 = \sqrt{3.25} = 1.80$ Then if $\mathbf{y}(t) = \mathbf{x}(t) - \mathbf{x}_{\mathrm{eq}}$, then general solution for $\mathbf{y}(t)$ is

$$\mathbf{y}(t) = (c_1 e^{i\mu_1 t} + d_1 e^{-i\mu_1 t})\mathbf{x}_1 + (c_2 e^{i\mu_2 t} + d_2 e^{-i\mu_2 t})\mathbf{x}_2 + (c_3 e^{i\mu_3 t} + d_3 e^{-i\mu_3 t})\mathbf{x}_3$$

where $c_1, d_1, c_2, d_2, c_3, d_3$ are arbitrary constants. Taking real and imaginary parts, we can also write the general solution as

$$\mathbf{y}(t) = (a_1 \cos(\mu_1 t) + b_1 \sin(\mu_1 t))\mathbf{x}_1 + (a_2 \cos(\mu_2 t) + b_2 \sin(\mu_2 t))\mathbf{x}_2 + (a_3 \cos(\mu_3 t) + b_3 \sin(\mu_3 t))\mathbf{x}_3$$

where $a_1, b_1, a_2, b_2, a_3, b_3$ are arbitrary constants. Notice that we can find the general solution for $\mathbf{y}(t) = \mathbf{x}(t) - \mathbf{x}_{\mathrm{eq}}$ without knowing $\mathbf{x}_{\mathrm{eq}}$. However, since the initial conditions were given in terms of $\mathbf{x}$ and not $\mathbf{y}$, we now have to find $\mathbf{x}_{\mathrm{eq}}$ to be able to convert intital conditions for $\mathbf{x}$ to initial conditions for $\mathbf{y}$. If we work in units where $g = 10$ then

$$\mathbf{b} = \begin{bmatrix} 10 \\ 10 \\ 11 \end{bmatrix}$$

and

$$\mathbf{x}_{\mathrm{eq}} = -K^{-1}\mathbf{b} = \begin{bmatrix} 31 \\ 52 \\ 63 \end{bmatrix}$$

so

$$\mathbf{y}(0) = \mathbf{x}(0) - \mathbf{x}_{\mathrm{eq}} = \begin{bmatrix} 30 \\ 60 \\ 70 \end{bmatrix} - \begin{bmatrix} 31 \\ 52 \\ 63 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \\ 7 \end{bmatrix}$$

Also

$$\mathbf{y}'(0) = \mathbf{x}'(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

So to satisfy the first initial condition, since $\cos(0) = 1$ and $\sin(0) = 0$, we need that

$$\mathbf{y}(0) = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + a_3\mathbf{x}_3 = \begin{bmatrix} -1 \\ 8 \\ 7 \end{bmatrix}.$$

Explicitly, we need to solve

$$[\mathbf{x}_1|\mathbf{x}_2|\mathbf{x}_3] \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \\ 7 \end{bmatrix},$$

or

$$\begin{bmatrix} 0.445 & -1.25 & 1.80 \\ 0.802 & -0.555 & -2.25 \\ 1.00 & 1.00 & 1.00 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \\ 7 \end{bmatrix}$$

This is not a pretty sight, but I can punch the numbers into my computer and find that

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 7.04 \\ 1.33 \\ -1.37 \end{bmatrix}$$

To satisfy the second initial conditon, we differentiate the expression for the general solution of $\mathbf{y}(t)$ and set $t = 0$. This gives

$$\mu_1 b_1 \mathbf{x}_1 + \mu_2 b_2 \mathbf{x}_2 + \mu_3 b_3 \mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Solving this numerically gives

$$\begin{bmatrix} \mu_1 b_1 \\ \mu_2 b_2 \\ \mu_3 b_3 \end{bmatrix} = \begin{bmatrix} 0.242 \\ -0.435 \\ 0.194 \end{bmatrix}$$

Finally, we divide by the $\mu_i's$ to give

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0.543 \\ -0.348 \\ 1.80 \end{bmatrix}$$

Now we have completely determined all the constants, so the solution is complete.

Problem 4.17: Suppose K is a 3×3 matrix with eigenvalues and eigenvectors given by

$$\lambda_1 = -1 \quad \lambda_2 = -4 \quad \lambda_3 = -9$$

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Find the solution of

$$\mathbf{y}''(t) = K\mathbf{y}(t)$$

satisfying

$$\mathbf{y}(0) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\mathbf{y}'(0) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Problem 4.18: Consider a system of two hanging weights and springs. Suppose that all the masses, spring constants and spring lengths are equal to one, and that $g = 10$. Find the positions $x_1(t)$ and $x_2(t)$ for all times if $x_1(0) = 20$, $x_2(0) = 30$, $x_1'(0) = 1$, $x_2'(0) = 1$.
