Fourier Transform Summary

	$f(x)$	$\widehat{f}(k)$
Definition and inversion		
	$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\widehat{f}(k)dk$	$\int_{-\infty}^{\infty} e^{-ikx}f(x)dx$
Examples		
1	$\dfrac{1}{1+x^2}$	$\pi e^{-\lvert k\rvert}$
2	$e^{-\lvert x\rvert}$	$\dfrac{2}{1+k^2}$
3	$\begin{cases}1 & \lvert x\rvert<1\\ 1/2 & x=0\\ 0 & \lvert x\rvert\geq 1\end{cases}$	$2\,\dfrac{\sin(k)}{k}$
4	$\dfrac{\sin(x)}{x}$	$\pi\begin{cases}1 & \lvert k\rvert<1\\ 1/2 & k=0\\ 0 & \lvert k\rvert>1\end{cases}$
5	$e^{-x^2/(2\sigma^2)}$	$\sqrt{2\pi}\,\sigma e^{-\sigma^2k^2/2}$
Properties		
0	$c_1f_1(x)+c_2f_2(x)$	$c_1\widehat{f}_1(k)+c_2\widehat{f}_2(k)$
1	$f(x+a)$	$e^{iak}\widehat{f}(k)$
2	$e^{iax}f(x)$	$\widehat{f}(k-a)$
3	$f(ax)$	$a^{-1}\widehat{f}(k/a)$
4	$f'(x)$	$ik\widehat{f}(k)$
5	$ixf(x)$	$-\widehat{f}'(k)$
6	$f*g(x)$	$\widehat{f}(k)\widehat{g}(k)$
7	$f(x)g(x)$	$(2\pi)^{-1}\widehat{f}*\widehat{g}(k)$

Note about the examples 3 and 4: The value of a Fourier or inverse Fourier transform is insensitive to changes of the input function at a single point. In property 6: $f*g(x)=\int_{-\infty}^{\infty} f(x-y)g(y)dy$

Laplace Transform Summary

	$y(t)$	$Y(s)$
Definition and inversion		
	$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{st}Y(s)ds$	$\int_0^{\infty} e^{-st}y(t)dt$
Examples		
1	e^{-at}	$\dfrac{1}{s+a}$
2	$\sin(\omega t)$	$\dfrac{\omega}{s^2+\omega^2}$
3	$\cos(\omega t)$	$\dfrac{s}{s^2+\omega^2}$
Properties		
0	$c_1y_1(t)+c_2y_2(t)$	$c_1Y_1(s)+c_2Y_2(s)$
1	$y'(t)$	$sY(s)-y(0)$
2	$ty(t)$	$-Y'(s)$
3	$e^{at}y(t)$	$Y(s-a)$
4	$u(t-a)y(t-a)$	$e^{-ac}Y(s)$
5	$y_1*y_2(t)$	$Y_1(s)Y_2(s)$

In property 4: $u(x)=\begin{cases}0 & x<0\\ 1 & x\geq 0\end{cases}$. In property 5: $y_1*y_2(t)=\int_0^t y_1(t-\tau)y_2(\tau)d\tau$

University of British Columbia
Math 301, Section 201

Final exam

Date: April 18, 2012
Time: 8:30 - 11:00am

Name (print):
Student ID Number:
Signature:

Instructor: Richard Froese

Instructions:

1. No notes, books or calculators are allowed. A summary sheet with properties of Fourier and Laplace transforms is provided.
2. Read the questions carefully and make sure you provide all the information that is asked for in the question.
3. Show all your work. Answers without any explanation or without the correct accompanying work could receive no credit, even if they are correct.
4. Answer the questions in the space provided. Continue on the back of the page if necessary.

Question	Mark	Maximum
1		14
2		14
3		15
4		14
5		15
6		14
7		14
Total		100

[14] 1. Evaluate

$$f(x) = \int_{-\infty}^{\infty} \frac{e^{ikx}}{(k^2+1)^2} dk. \qquad (\text{for } x \in \mathbb{R}).$$

For $x \geq 0$ close the contour in the upper half plane. There is a poles of order 2 at $k = i$. Thus

$$f(x) = 2\pi i \operatorname{Res}\left[\frac{e^{ikx}}{(k-i)^2(k+i)^2} ; i\right]$$

$$= 2\pi i \lim_{k \to i} \frac{d}{dk} (k-i)^2 \frac{e^{ikx}}{(k-i)^2(k+i)^2}$$

$$= 2\pi i \lim_{k \to i} \left[\frac{ix e^{ikx}(k+i)^2 - 2(k+i)e^{ikx}}{(k+i)^4}\right]$$

$$= 2\pi i \left[\frac{ix e^{-x}(2i)^2 - 2(2i)e^{-x}}{(2i)^4}\right] = \frac{2\pi i\, e^{-x}(-4ix - 4i)}{16}$$

$$= \frac{\pi}{2} e^{-x}(x+1)$$

For $x \leq 0$ we can notice $f(x) = \overline{f(-x)} = \overline{\frac{\pi}{2} e^{x}(-x+1)}$

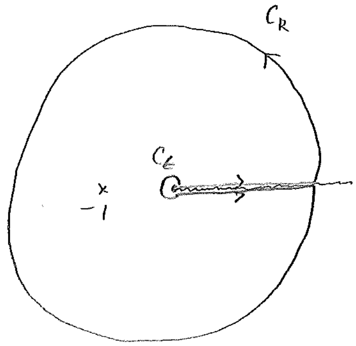
$$= \frac{\pi}{2} e^{x}(-x+1) = \frac{\pi}{2} e^{-|x|}(|x|+1).$$

So $f(x) = \frac{\pi}{2} e^{-|x|}(|x|+1)$ for all $x \in \mathbb{R}$.

[14] 2. Evaluate

$$I = \int_0^\infty \frac{x^{\alpha-1}}{1+x} dx, \quad 0 < \alpha < 1.$$

explaining all your steps.



Choose the branch cut on the positive real axis.

We have

$$\int_\epsilon^R \frac{x^{\alpha-1}}{1+x} dx - \int_\epsilon^R \frac{e^{2\pi i(\alpha-1)} x^{\alpha-1}}{1+x} dx + \int_{C_\epsilon} + \int_{C_R} =$$

$$2\pi i \operatorname{Res}\left[\frac{z^{\alpha-1}}{1+z}, -1\right] = 2\pi i \, e^{\pi i(\alpha-1)}$$

$$\int_{C_\epsilon} \frac{z^{\alpha-1}}{1+z} dz \le \frac{\epsilon^{\alpha-1}}{1-\epsilon} \cdot 2\pi\epsilon \sim \epsilon^{\alpha} \to 0 \text{ as } \epsilon \downarrow 0$$

$$\int_{C_R} \frac{z^{\alpha-1}}{1+z} \le \frac{R^{\alpha-1}}{R-1} \cdot 2\pi R \sim R^{\alpha-1} \to 0 \text{ as } R \uparrow \infty$$

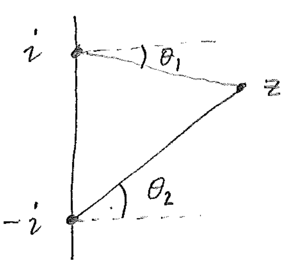
So $\left(1 - e^{2\pi i(\alpha-1)}\right) I = 2\pi i \, e^{i\pi(\alpha-1)}$

$$I = \frac{2\pi i}{e^{-i\pi(\alpha-1)} - e^{i\pi(\alpha-1)}} = \frac{\pi}{\sin(\pi(1-\alpha))}$$

[5] 2. (a) What are the branch points of the multivalued function $f(z) = (z^2+1)^{1/2}$? Is infinity a branch point?

The branch points are $+i$ and $-i$. Infinity is not a branch point since we can write $f(z) = z\left(1+\frac{1}{z^2}\right)^{\frac{1}{2}}$ (as a multivalued fcn). This returns to the same value when z goes around a large circle.

[5] (b) Using the range of angles method, define a branch of $f(z)$ that is continous and positive for $z \in \mathbb{R}$.



Let $\theta_1(z)$ $\theta_2(z)$ be the angles shown.

where $\theta_1 \in \left(-\frac{3\pi}{2}, \frac{\pi}{2}\right]$, $\theta_2 \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Then $f(z) = \sqrt{|z^2+1|}\, e^{i(\theta_1+\theta_2)/2}$ is the required branch.

[5] (c) Using the range of angles method, define a branch of $f(z)$ that is analytic outside the unit circle and negative for large positive $z \in \mathbb{R}$.

Change the ranges to $\theta_1 \in \left(-\frac{3\pi}{2}, \frac{\pi}{2}\right]$, $\theta_2 \in \left[\frac{\pi}{2}, \frac{5\pi}{2}\right)$

For z on the positive real axis we have $\theta_1 = -\theta_2 + 2\pi K$ for some k.

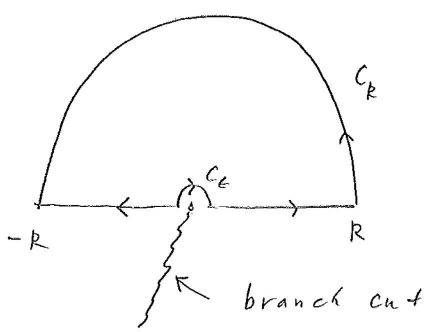
Determine k: if $\theta_2 \in \left[\frac{\pi}{2}, \frac{5\pi}{2}\right)$ then for z on the positive real axis $\theta_2 \in \left(2\pi, \frac{5\pi}{2}\right]$, $-\theta_2 \in \left[-\frac{5\pi}{2}, -2\pi\right)$, $-\theta_2 + 2\pi \in \left[-\frac{\pi}{2}, 0\right)$ which is in the range of θ_1, so $k=1$. So for z on pos. real axis

$f(z) = \sqrt{|z^2+1|}\, e^{i(\theta_1+\theta_2)/2} = \sqrt{|z^2+1|}\, e^{i\pi} = -\sqrt{|z^2+1|}$

3. In this question we evaluate

$$I = \int_0^\infty \frac{\sqrt{x}}{x^2+1}\,dx.$$

[3] (a) Draw a diagram that shows the contour you are using, and the branch cut for $\sqrt{z}$.



Contour is $[0,R] + C_R - [0,-R] + C_\epsilon$

[4] (b) What does the Cauchy residue theorem say for your contour? Compute the residues that appear.

Let $f(z) = \dfrac{\sqrt{z}}{z^2+1}$. Then

$$\int_{[\epsilon,R]} f\,dz + \int_{C_R} f\,dz - \int_{[-\epsilon,-R]} f\,dz + \int_{C_\epsilon} f\,dz = 2\pi i \operatorname{Res}[f(z); i]$$

$$= \frac{2\pi i \sqrt{i}}{2i} = \pi e^{i\pi/4}$$

To evaluate $\int_{[-\epsilon,-R]} f\,dz$ let $z(r) = -r$ $\quad dz = -dr$

$$\int_{[-\epsilon,-R]} f(z)\,dz = -\int_\epsilon^R f(-r)\,dr = -\int_\epsilon^R \frac{i\sqrt{r}}{r^2+1}\,dr$$

[4] (c) Which integrals (over portions of your contour) tend to zero in the limit? Give estimates that show this.

$$\left|\int_{C_R} f\,dz\right| \leq \max_{|z|=R}\left|\frac{\sqrt{z}}{z^2+1}\right| \cdot \pi R \leq \frac{R^{\frac{1}{2}}}{R^2-1}\pi R \to 0 \quad \text{as } R\to\infty$$

$$\left|\int_{C_\epsilon} f\,dz\right| \leq \max_{|z|=\epsilon}\left|\frac{\sqrt{z}}{z^2+1}\right| \cdot \pi\epsilon \leq \frac{\epsilon^{\frac{1}{2}}}{1-\epsilon^2}\pi\epsilon \to 0 \quad \text{as } \epsilon\to 0$$

[4] (d) Evaluate I.

$$\lim_{\epsilon\downarrow 0}\lim_{R\downarrow 0}\int_{[\epsilon,R]} f\,dz = I$$

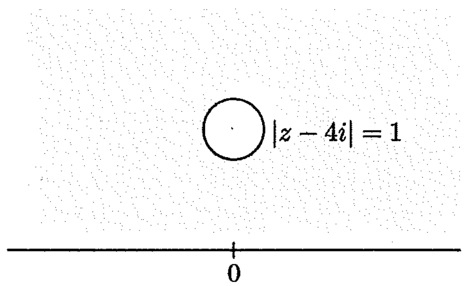
$$\lim_{\epsilon\downarrow 0}\lim_{R\downarrow 0}\int_{[-\epsilon,-R]} f\,dz = -i\int_\epsilon^R \frac{\sqrt{x}}{x^2+1}\,dx \to -i\,I$$

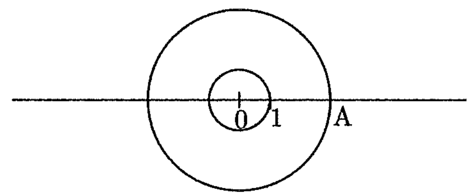
So $(1+i)I = \pi e^{i\pi/4}$,

$$I = \frac{\pi e^{i\pi/4}}{1+e^{i\pi/2}} = \frac{\pi}{2\cos(\pi/4)} = \frac{\sqrt{2}\,\pi}{2}$$

[14]

4. Find a fractional linear transformation that maps the shaded region $\{z : \text{Im}z \geq 0, |z-4i| \geq 1\}$ to the annulus $\{z : 1 \leq |z| \leq A\}$ with the real line mapping to the unit circle. What is the outer radius A? (Recall that α and α^* are symmetric with respect to a circle centred at a with radius R if $\alpha^* = a + R^2/(\overline{\alpha} - \overline{a})$.)





Find α, α^* that are symmetric w.r.t the circle and the real axis. We need $\alpha^* = \overline{\alpha}$ and $\alpha^* = 4i + \dfrac{1}{\overline{\alpha} + 4i}$. So

$$\overline{\alpha} = 4i + \frac{1}{\overline{\alpha} + 4i}, \quad (\overline{\alpha} - 4i)(\overline{\alpha} + 4i) = 1 \quad \overline{\alpha}^2 + 16 = 1$$

$$\overline{\alpha}^2 = -15 \qquad \overline{\alpha} = \pm i\sqrt{15}.$$

Map $-i\sqrt{15} \to 0$ $\quad 0 \to 1$ $\quad i\sqrt{15} \to \infty$

$$f(z) = \frac{z + i\sqrt{15}}{z - i\sqrt{15}}(-1) = \frac{-z - i\sqrt{15}}{z - i\sqrt{15}}$$

When $z = 3i$ $\quad f(z) = \dfrac{-3i - i\sqrt{15}}{3i - i\sqrt{15}} = \dfrac{\sqrt{15} + 3}{\sqrt{15} - 3} = A$

5. Consider the complex velocity potentials for ideal inviscid flow given by $\Omega_1(z) = iV_0(z - 1/z)$ and $\Omega_2(z) = i\gamma \log(z)$. Here $V_0 > 0$ and $\gamma > 0$.

[4] (a) Show that $\Omega_1(z)$ describes a flow around the unit circle $\{z : |z| = 1\}$.

We must show $\operatorname{Im} \Omega_1(e^{i\theta}) = \text{const.}$

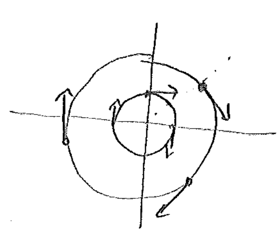
$\Omega_1(e^{i\theta}) = iV_0(e^{i\theta} - e^{-i\theta}) = -2V_0 \sin(\theta)$ so

$\operatorname{Im}(\Omega_1(e^{i\theta})) = 0.$

[4] (b) What are the streamlines for the flow described by $\Omega_2(z)$? Plot several of these, and indicate with arrows the direction of the flow.

$\Omega_2(z) = i\gamma \log(z) = i\gamma \log|z| + i\gamma i \arg(z)$

$\operatorname{Im} \Omega_2 = \gamma \log|z| = c$ when $|z| = \text{const}$. So streamlines are circles. The complex velocity is $\overline{\Omega_2'(z)} = \frac{-i\gamma}{\bar{z}}$, so if $z = Re^{i\theta}$ $\overline{\Omega_2'(z)} = \frac{-ie^{i\theta}}{R}$



[3]

(c) Explain why $\Omega(z) = \Omega_1(z)+\Omega_2(z)$ also describes a flow around the unit circle $\{z : |z| = 1\}$. What is the asympotic velocity of this flow as $|z| \to \infty$?

Both $\mathrm{Im}\,\Omega_1$ and $\mathrm{Im}\,\Omega_2$ are const on the unit circle so $\mathrm{Im}(\Omega_1+\Omega_2) = \text{const}$ there too

$$\lim_{|z|\to\infty} \overline{\Omega'(z)} = -i V_0$$

[4]

(d) Where are the stagnation point for the flow described by $\Omega(z)$? For what values of $\gamma > 0$ are the stagnation points on the boundary of the the unit circle?

Need $iV_0\left(1+\frac{1}{z^2}\right) + \frac{i\gamma}{z} = 0$ $\quad z^2 + \frac{\gamma}{V_0} z + 1 = 0$

$z = -\frac{\gamma}{2V_0} \pm \sqrt{\frac{\gamma^2}{4V_0^2} - 1}$. If $\frac{\gamma}{2V_0} < 1$ this lies on the unit circle.

[14] 6. Solve

$$u_t(x,t) = u_{xx}(x,t) + u(x,t). \quad -\infty < x < \infty, \quad t > 0$$
$$u(x,0) = e^{-x^2}$$

Take Fourier transform:

$$\hat{u}_t(k,t) = -k^2\hat{u}(k,t) + \hat{u}(k,t) = (1-k^2)\hat{u}(k,t)$$

$$\hat{u}(k,0) = \sqrt{\pi}\, e^{-k^2/4}$$

$$\hat{u}(k,t) = C(k)\, e^{(1-k^2)t}$$

$$\hat{u}(k,0) = C(k) = \sqrt{\pi}\, e^{-k^2/4}$$

$$\hat{u}(k,t) = \sqrt{\pi}\, e^{-k^2/4}\, e^{(1-k^2)t} = \sqrt{\pi}\, e^{t}\, e^{-k^2(\frac{1}{4}+t)}$$

$$= \sqrt{\pi}\, e^{t}\, e^{-k^2(\frac{1}{2}+2t)/2}$$

$$= \frac{\sqrt{2\pi}}{\sqrt{2}}\, e^{t}\, \frac{\sqrt{\frac{1}{2}+2t}}{\sqrt{\frac{1}{2}+2t}}\, e^{-k^2(\frac{1}{2}+2t)/2} \qquad \text{let } \sigma = \sqrt{\tfrac{1}{2}+2t}$$

$$= \frac{1}{\sqrt{1+4t}}\, e^{t}\, \sigma\, e^{-k^2\sigma^2/2}$$

$$u(x,t) = \frac{e^{t}}{\sqrt{1+4t}}\, e^{-x^2/(1+4t)}$$

[7] 7. (a) What is the Laplace transform $Y(s)$ of the solution $y(t)$ to the initial value problem

$$y'''(t) - 6y''(t) + 11y'(t) - 6y(t) = e^{-2t}, \quad y(0) = y'(0) = 0, \quad y''(0) = 1?$$

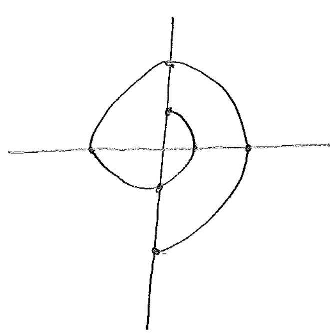
$$s^3 Y(s) - s^2 y(0) - s y'(0) - y''(0) - 6s^2 Y(s) + 6 s y(0) + 6 y'(0) + 11 s Y(s) - 11 y(0)$$
$$- 6 Y(s) = \frac{1}{s+2}$$
$$(s^3 - 6s^2 + 11s - 6) Y(s) = 1 + \frac{1}{s+2}, \quad Y(s) = \frac{s+3}{(s^3-6s^2+11s-6)(s+2)}$$

[7] (b) Use the Nyquist criterion to determine whether the solution $y(t)$ is bounded as t tends to infinity.

The poles of Y are at $s = -2$ and at the zeros of $p(s) = s^3 - 6s^2 + 11s - 6$. Use Nyquist to determine if $p(s)$ has poles in the right half plane.

$$p(iy) = -iy^3 + 6y^2 + i\,11y - 6 \quad \text{so}$$
$$Re(p(iy)) = 6y^2 - 6, \quad Im(p(iy)) = -y^3 + 11y$$
$$Re\, p = 0 \text{ when } y = \pm 1, \quad Im\, p = 0 \text{ when } y = 0, \pm\sqrt{11}$$

y	Re	Im
∞	$\sim 6y^2$	$\sim -y^3$
$\sqrt{11}$	>0	0
1	0	>0
0	<0	0
-1	0	<0
$-\sqrt{11}$	>0	0
$-\infty$		$+$



$$N_+(p) = \frac{1}{2\pi}(3\pi + 3\pi) = 3$$

All the zeros of p are in the RHP $\longrightarrow$ Soln $y(t)$ is not bounded.

(in fact $p(s) = (s-1)(s-2)(s-3)$!)

[14] 1. Evaluate

$$I = \int_0^\infty \frac{\cos(\pi x)}{1-4x^2}dx,$$

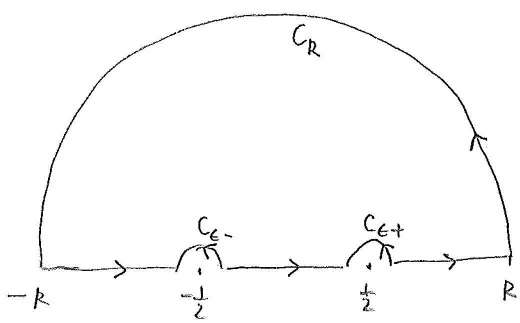
explaining all your steps.

Denominator vanishes at $x = \pm\frac{1}{2}$, but so does $\cos(\pi x)$ so the singularity is removable at $x = \pm\frac{1}{2}$.

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos(\pi x)}{1-4x^2}\,dx$$

$$= \frac{1}{2}\,\text{P.V.}\int_{-\infty}^{\infty} \frac{\cos(\pi x)}{1-4x^2}\,dx = \lim_{R\to\infty}\lim_{\epsilon\to 0} \underbrace{\frac{1}{2}\int_{-R}^{-\frac{1}{2}-\epsilon} + \frac{1}{2}\int_{-\frac{1}{2}+\epsilon}^{\frac{1}{2}-\epsilon} + \frac{1}{2}\int_{\frac{1}{2}+\epsilon}^{R}}_{I_{R,\epsilon}}$$

$$= \text{Re}\left[\frac{1}{2}\,\text{P.V.}\int_{-\infty}^{\infty} \frac{e^{i\pi x}}{1-4x^2}\,dx\right]$$



$$I_{R,\epsilon} - \int_{C_{\epsilon-}} - \int_{C_{\epsilon+}} + \int_{C_R} = 0$$

$$\lim_{\epsilon\downarrow 0}\int_{C_{\epsilon-}} = \pi i\,\text{Res}\left[\frac{e^{i\pi z}}{1-4z^2}; -\frac{1}{2}\right] = \frac{i\pi\, e^{-i\pi/2}}{4}$$

$$\lim_{\epsilon\downarrow 0}\int_{C_{\epsilon+}} = i\pi\,\text{Res}\left[\frac{e^{i\pi z}}{1-4z^2}; \frac{1}{2}\right] = \frac{i\pi\, e^{i\pi/2}}{-4}$$

$$\lim_{R\to\infty}\int_{C_R} = 0 \text{ (Jordan's lemma)}$$

So $I = \frac{1}{2}\left(\frac{i\pi e^{-i\pi/2}}{4} - \frac{i\pi e^{i\pi/2}}{4}\right)$

$$= \frac{\pi}{4}$$

3. Let $g(z)$ be the branch of $(z(z-1))^{1/2}$ defined by $g(z) = \exp((\mathrm{Log}(z) + \mathrm{Log}(z-1))/2)$, where Log denotes the principal branch of the logarithm.

[4] (a) Compute $\lim_{z\to -1} g(z)$ when z approaches -1 from the upper half plane, and from the lower half plane. Is $g(z)$ continuous at -1?

$g(z) = e^{\frac{1}{2}(\mathrm{Log}(z) + \mathrm{Log}(z-1))}$, when $z \to -1$ from UHP,

then $\mathrm{Log}(z) \to i\pi$ and $\mathrm{Log}(z-1) \to \ln 2 + i\pi$ so

$g(z) \to e^{\frac{1}{2}(i\pi + i\pi + \ln 2)} = -\sqrt{2}$. When $z \to -1$ from LHP

then $\mathrm{Log}(z) \to -i\pi$ and $\mathrm{Log}(z-1) \to \ln 2 - i\pi$ so

$g(z) \to e^{\frac{1}{2}(-i\pi - i\pi + \log(2))} = -\sqrt{2}$

Yes $g(z)$ is cts at -1.

[3] (b) Where are the branch cuts for $g(z)$?

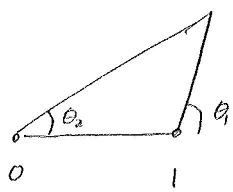
Branch cut is $[0,1]$. On $(-\infty, 0]$ the branch cuts cancel.

[4] (c) Show how to construct $g(z)$ using the 'range of angles' method.

To use the range of angles method we write

$g(z) = |z(z-1)|^{\frac{1}{2}} e^{i(\theta_1+\theta_2)/2}$ with $\theta_1 \in (-\pi, \pi]$ and $\theta_2 \in (-\pi, \pi]$.

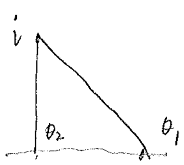
This has the same branch cut on $[0,1]$ since the double branch cut on $(-\infty, 0]$ cancels.



Also when $x > 1$ then $g(x)$ is positive for both the original defn and the range of angles defn (where $\theta_1 = \theta_2 = 0$).

Same branch cut + same values at $x > 0$ $\Rightarrow$ same branch.

[4] (d) What is the value of $g(i)$ and $g(-i)$?



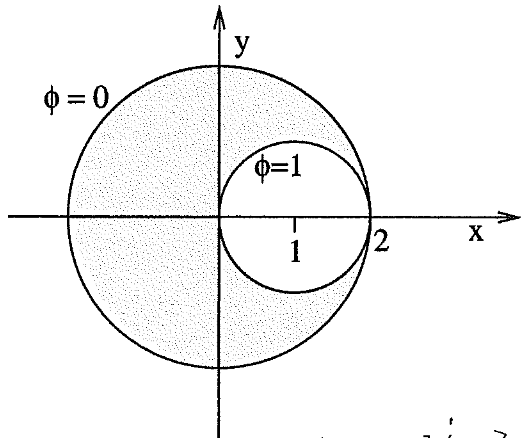
When $z = i$ $\theta_1 = \frac{3\pi}{4}$ $\theta_2 = \frac{\pi}{2}$ $g(z) = |i(i-1)|^{\frac{1}{2}} e^{i\frac{5\pi}{8}}$
$= 2^{\frac{1}{4}} e^{i5\pi/8}$ When $z = -i$ $\theta_1 = -\frac{3\pi}{4}$ $\theta_2 = -\frac{\pi}{2}$ so
$g(z) = 2^{\frac{1}{4}} e^{-i5\pi/8}$

Compare with the original defn: When $z = i$ $\mathrm{Log}(z) = \frac{i\pi}{2}$
$\mathrm{Log}(i-1) = \ln\sqrt{2} + i\frac{3\pi}{4}$ $\quad e^{\frac{1}{2}\left(\frac{i\pi}{2} + \ln\sqrt{2} + i\frac{3\pi}{4}\right)} = 2^{\frac{1}{4}} e^{i\frac{5\pi}{8}}$

$z = -i$ similar.

[14]

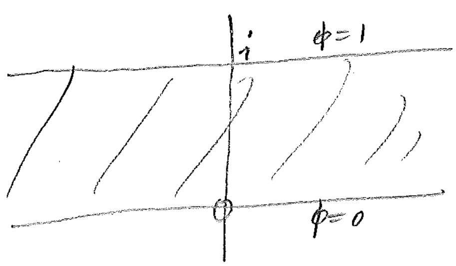
4. Solve Laplaces equation $\Delta\phi = 0$ in the region between the circles $\{z : |z| = 2\}$ and $\{z : |z-1| = 1\}$ with boundary condition $\phi = 1$ on the inner circle and $\phi = 0$ on the outer circle.



Find the FLT that maps $2 \to \infty$ $2i \to 0$ and $-2 \to 1$

$$f(z) = \frac{z-2i}{z-2}\,\frac{-4}{-2-2i} = \frac{(z-2i)(1-i)}{z-2}$$

$$f(1+i) = \frac{(1-i)(1-i)}{-1+i} = i-1$$ so region maps to (check eg $f(-1)$).

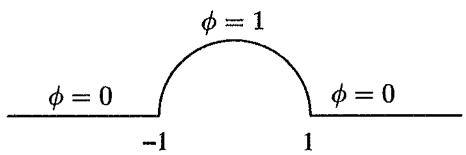


$$\Phi(u,v) = v$$

Write $f(x+iy) = u(x,y) + i\,v(x,y)$, $\quad v(x,y) = \dfrac{4-x^2-y^2}{x^2+y^2-4x+4}$

is the solution

5. Consider the following region with boundary data.



[7]

(a) Show that the Joukowski map $J(z) = (1/2)(z + 1/z)$ maps the shaded region onto the upper half plane.

Check on the boundary:

When $z = x \in (-\infty, -1]$ $\quad J(x) = \frac{1}{2}\left(x + \frac{1}{x}\right) \in (-\infty, -1]$

When $z = e^{i\theta}$, $\theta \in [0, \pi]$ $\quad J(e^{i\theta}) = \cos(\theta) \in [-1, 1]$.

When $z = x \in [1, \infty)$ $\quad J(x) = \frac{1}{2}\left(x + \frac{1}{x}\right) \in [1, \infty)$

So the boundary maps to the boundary of the upper half plane. To check that the map is to the upper (not lower) half plane, check orientation, or an interior pt. eg.

$2i \longmapsto \frac{1}{2}\left(2i + \frac{1}{2i}\right) = \frac{1}{2}\left(2i - \frac{1}{2}i\right) = \frac{1}{2}\frac{3}{2}i = \frac{3i}{4} \in$ U.H.P.

[7] (b) Solve Laplaces equation $\Delta\phi = 0$ in the shaded region, with the indicated boundary conditions.

First solve the problem in the U.H.P. ($w = u + iv$' plane)

$\Phi = 0$ $\quad -1 \quad$ $\Phi = 1$ $\quad 1 \quad$ $\Phi = 0$

$$\Phi(w) = A\,\mathrm{Arg}(w+1) + B\,\mathrm{Arg}(w-1) + C$$

Boundary conds $\rightarrow$ $C = 0$, $\pi B = 1$, $\pi A + \pi B = 0$

$B = \frac{1}{\pi}$, $A = -\frac{1}{\pi}$

$$\Phi(u,v) = -\frac{1}{\pi}\cot^{-1}\left(\frac{u+1}{v}\right) + \frac{1}{\pi}\cot^{-1}\left(\frac{u-1}{v}\right).$$

Write $J(x+iy) = \frac{1}{2}\left(x + iy + \frac{x - iy}{x^2+y^2}\right)$

$$= \underbrace{\frac{1}{2}\left(x + \frac{x}{x^2+y^2}\right)}_{u(x,y)} + i\,\underbrace{\left(\frac{1}{2}\left(y - \frac{y}{x^2+y^2}\right)\right)}_{v(x,y)}.$$

$$\phi(x,y) = \Phi\left(u(x,y), v(x,y)\right).$$

[7]

6. (a) What is the Fourier transform of $x^2e^{-x^2/2}$?

$$\mathcal{F}(e^{-x^2/2}) = \sqrt{2\pi}\, e^{-k^2/2} \quad (\text{from table with } \sigma = 1).$$

$$\mathcal{F}(x^2 e^{-x^2/2}) = -\frac{d^2}{dk^2}\sqrt{2\pi}\, e^{-k^2/2}$$

$$= -\sqrt{2\pi}\,\frac{d}{dk} - k\, e^{-k^2/2}$$

$$= \sqrt{2\pi}\left(e^{-k^2/2} - k^2 e^{-k^2/2}\right).$$

[7]

(b) Use the Fourier transform to compute $\int_{-\infty}^{\infty} e^{-|x-y|}e^{-|y|}dy$. (Hint: you can use the result from an earlier problem for the last step.)

$\mathcal{F}[e^{-|x|}] = \frac{2}{1+k^2}$ so $\mathcal{F}[e^{-|x|} * e^{-|x|}]$

$= \frac{2}{1+k^2} \cdot \frac{2}{1+k^2} = \frac{4}{(1+k^2)^2}$. Thus $e^{-|x|} * e^{-|x|} =$

$$\int_{-\infty}^{\infty} e^{-|x-y|}e^{-|y|}\,dy = \mathcal{F}^{-1}\left[\frac{4}{(1+k^2)^2}\right] = \frac{4}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\frac{1}{(1+k^2)^2}\,dk$$

$$= \frac{2}{\pi}\frac{\pi}{2}\, e^{-|x|}(|x|+1) = e^{-|x|}(|x|+1)$$

[7] 7. (a) Compute the Laplace transform $Y(s)$ of the solution $y(t)$ to

$$y''''(t) + y'(t) - y(t) = \cos(t),$$
$$y'''(0) = 1, \quad y''(0) = y'(0) = y(0) = 0,$$

$$s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) + s Y(s) - y(0) - Y(s) = \frac{s}{s^2+1}$$

$$(s^4 + s - 1)\, Y(s) - 1 = \frac{s}{s^2+1}$$

$$Y(s) = \frac{1}{s^4+s-1}\left(\frac{s}{s^2+1} + 1\right)$$

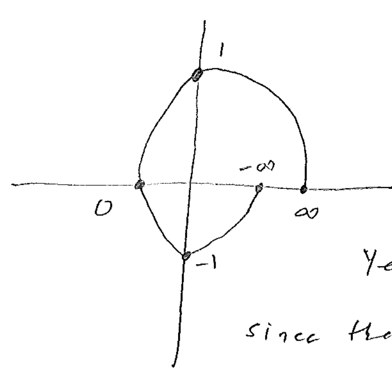
[7] (b) Use the Nyquist criterion to decide whether the solution $y(t)$ grows exponentially as $t \to \infty$ or not.

Need to check whether $p(s) = s^4+s-1$ has any zeros in RHP.

$p(iy) = y^4 + iy - 1 \qquad Re(p(iy)) = y^4 - 1 \qquad Im(p(iy)) = y.$

The real sol'ns to $Re(p(iy)) = 0$ are $y = \pm 1$

y	$Re\,p$	$Im\,p$
∞	$\sim y^4$	$\sim y$
1	0	1
0	-1	0
-1	0	-1
$-\infty$	$\sim y^4$	$\sim y$



$\Delta \arg p = 2\pi$

$N_+(p) = \frac{1}{2\pi}(2\pi + 4\pi)$

$= 3$

Yes, solns grow exponentially since there are 3 zeros in RHP.