

University of British Columbia
Math 301, Section 201

Midterm 1

Date: February 13, 2012

Time: 11:00 - 11:50pm

Name (print):

Student ID Number:

Signature:

Instructor: Richard Froese

Instructions:

1. No notes, books or calculators are allowed.
2. Read the questions carefully and make sure you provide all the information that is asked for in the question.
3. Show all your work. Answers without any explanation or without the correct accompanying work could receive no credit, even if they are correct.
4. Answer the questions in the space provided. Continue on the back of the page if necessary.

Question	Mark	Maximum
1		12
2		12
3		16
Total		40

[4]

1. (a) Evaluate $I = \int_0^{\infty} \frac{1}{(x^2+1)^2} dx$.

$$I = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2} dx \quad \text{by symmetry}$$

$$= \frac{1}{2} 2\pi i \operatorname{Res} \left[\frac{1}{(z^2+1)^2}; i \right]$$

$$= \pi i \lim_{z \rightarrow i} \frac{d}{dz} (z-i)^2 \frac{1}{(z-i)^2 (z+i)^2}$$

$$= \pi i \lim_{z \rightarrow i} (-2)(z+i)^{-3} = \frac{-2\pi i}{(2i)^3} = \frac{\pi}{4}$$

[4]

(b) Evaluate $I = \int_0^{2\pi} \frac{1}{2\cos(\theta)+3} d\theta$.

$$I = -i \int_0^{2\pi} \frac{ie^{i\theta} d\theta}{(e^{i\theta} + e^{-i\theta} + 3)e^{i\theta}} = -i \oint_{|z|=1} \frac{dz}{z^2 + 3z + 1}$$

Roots are at $z_{\pm} = \frac{-3 \pm \sqrt{5}}{2}$. Since $z_+ z_- = 1$, one is inside

and one is outside: z_+ is closer to 0, hence inside.

$$\operatorname{Res} \left[\frac{1}{z^2 + 3z + 1}, z_+ \right] = \operatorname{Res} \left[\frac{1}{(z-z_+)(z-z_-)}; z_+ \right] = \frac{1}{z_+ - z_-}$$

$$= \frac{1}{\sqrt{5}}. \quad \text{Thus } I = 2\pi i (-i) \frac{1}{\sqrt{5}} = \frac{2\pi}{\sqrt{5}}$$

[4]

- (c) Evaluate $I = \oint_{|z|=1} z^n e^{1/z} dz$ for all $n \in \mathbb{Z}$, where the contour is traversed in the counter-clockwise direction.

There is an essential singularity at $z=0$, we want the residue.

$$\begin{aligned} z^n e^{1/z} &= z^n \left(1 + \frac{1}{z} + \frac{1}{2z^2} + \dots + \frac{1}{k!} z^{-k} + \dots \right) \\ &= z^n + z^{n-1} + \frac{1}{2} z^{n-2} + \dots + \frac{1}{k!} z^{n-k} \end{aligned}$$

$$I = 2\pi i \operatorname{Res} [z^n e^{1/z}; 0] = 2\pi i \begin{cases} 0 & \text{if } n < -1 \\ \frac{1}{(n+1)!} & \text{if } n \geq -1 \end{cases}$$

$$\text{Since } n-k = -1 \Leftrightarrow k = n+1$$

[4]

2. (a) Identify the branch points of the multivalued function $\log((z+1)/(z-1))$. Don't forget to check $z = \infty$.

As a multivalued function $\log((z+1)/(z-1)) = \log(z+1) - \log(z-1)$.

So possible branch points are $z = -1, +1, \infty$.

Circle about	Change in $\log(z+1)$	Change in $-\log(z-1)$	total
-1	$2\pi i$	0	$2\pi i$
1	0	$-2\pi i$	$-2\pi i$
∞	$2\pi i$	$-2\pi i$	0

So the branch pts are ± 1 (and not ∞).

[4]

- (b) If a single valued branch of the function above is defined as $\text{Log}((z+1)/(z-1))$ (using the principal branch of the logarithm) where are the branch cuts?

Branch cut occurs when $\frac{z+1}{z-1} = -p \quad p \in [0, \infty)$

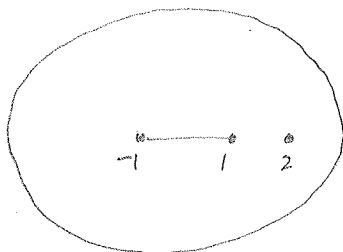
Same as $z+1 = -zp+p \quad , \quad z = \frac{p-1}{p+1}$. This

ranges from -1 to 1 as p ranges from 0 to ∞

Branch cut = $[-1, 1]$.

[4]

- (c) Evaluate $I = \oint_{|z|=3} (z-2)^{-1} \text{Log}((z+1)/(z-1)) dz$ where the contour is traversed in the counterclockwise direction.



There are no singularities outside $|z|=3$ so we need the residue at ∞ . Note that

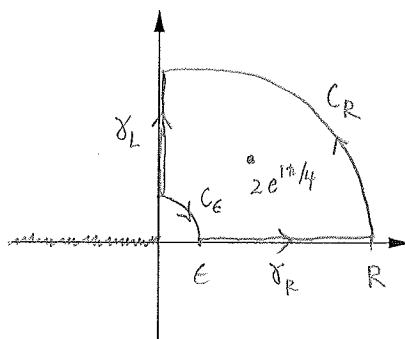
$$\lim_{z \rightarrow \infty} (z-2)^{-1} \text{Log}((z+1)/(z-1)) = 0 \quad \text{so}$$

$$\begin{aligned} \text{Res} \left[(z-2)^{-1} \text{Log} \left(\frac{z+1}{z-1} \right); \infty \right] &= \lim_{z \rightarrow \infty} z (z-2)^{-1} \text{Log} \left(\frac{z+1}{z-1} \right) = \text{Log}(1) \\ &= 0 \quad , \quad \text{so } I = 0 \text{ too.} \end{aligned}$$

3. Suppose that you wish to calculate $I = \int_0^\infty \frac{x^\alpha}{x^4 + 16} dx$ by integrating $f(z) = \frac{z^\alpha}{z^4 + 16}$ around a suitable contour and taking limits.

[2]

- (a) Sketch the contour you would use to evaluate this integral.



[3]

- (b) Specify the branch of z^α that you would use by describing by the range of angles method how you would compute your branch of z^α for every $z \in \mathbb{C}$. Draw the branch cut on your sketch.

Principal branch would work. To define this branch of z^α by range of angles method write $z = r e^{i\theta}$ with $\theta \in (-\pi, \pi]$. Then $z^\alpha = r^\alpha e^{i\alpha\theta}$. Branch cut is on negative real axis

[2]

(c) Are there other branches of z^α that would work in your integral? If so, describe them.

Any branch whose branch cut is a ray from 0 to ∞ that does not intersect the contour would work. In addition we want z^α to be real for $z \in [0, \infty)$ to make $\int_{\gamma_R}$ converge to I , although this is not strictly necessary.

[2]

(d) Find and classify the isolated singular points inside your contour. Draw them onto your sketch. Calculate the residue at these points.

Singularities occur when $z^4 = -16$. Write $z = re^{i\theta}$. Then $r^4 e^{i4\theta} = 16 e^{i(\pi + 2\pi k)}$ so $r = 2$ and $\theta = \pi/4 + \pi k/2$, $k \in \mathbb{Z}$. The only singularity inside is $2e^{i\pi/4}$. This is a simple pole, so

$$\begin{aligned} \text{Res} \left[\frac{z^\alpha}{z^4 + 16} ; 2e^{i\pi/4} \right] &= \frac{(2e^{i\pi/4})^\alpha}{4(2e^{i\pi/4})^3} \\ &= \frac{2^\alpha e^{i\alpha\pi/4}}{2^5 e^{i3\pi/4}} = 2^{\alpha-5} e^{\frac{i\pi}{4}(\alpha-3)} \end{aligned}$$

[3]

- (e) For each piece of the contour where you want the integral to vanish in the limit, indicate the range of α where this happens, and provide an explicit estimate to prove it.

There are 2 such contours: C_R and C_ϵ .

$$\begin{aligned}
 \left| \int_{C_R} \frac{z^\alpha}{z^4 + 16} dz \right| &\leq \max_{z \in C_R} \frac{|z^\alpha|}{|z^4 - 16|} \cdot \text{length}(C_R) \\
 &\leq \max_{|z|=R} \frac{|z|^\alpha}{|z|^4 - 16} \cdot \frac{\pi}{2} R \quad (\text{if } |z|^4 > 16) \\
 &= \frac{\pi}{2} \frac{R^{\alpha+1}}{R^4 - 16} \sim R^{\alpha-3} \quad \text{as } R \rightarrow \infty
 \end{aligned}$$

this $\rightarrow 0$ as $R \rightarrow \infty$ if $\alpha < 3$.

$$\begin{aligned}
 \left| \int_{C_\epsilon} \frac{z^\alpha}{z^4 + 16} dz \right| &\leq \max_{|z|=\epsilon} \frac{|z|^\alpha}{16 - |z|^4} \cdot \frac{\pi}{2} \epsilon \quad (\text{if } 16 > |z|^4) \\
 &= \frac{\pi}{2} \frac{\epsilon^{\alpha+1}}{16 - \epsilon^4} \sim \epsilon^{\alpha+1} \quad \text{as } \epsilon \rightarrow 0
 \end{aligned}$$

this $\rightarrow 0$ as $\epsilon \rightarrow 0$ if $\alpha > -1$.

[3]

(f) Calculate the integral over the remaining pieces of your contour, and use the result to find I . For what range of α is your formula valid.

$$\int_{\gamma_R} \frac{z^\alpha}{z^4 + 16} dz \rightarrow I \quad \text{as } R \rightarrow \infty, \epsilon \rightarrow 0.$$

On γ_L parametrize with $z = e^{in/2} \cdot r$, $dz = e^{in/2} dr$ $r \in [0, \infty)$

Then $z^\alpha = e^{i\alpha n/2} r^\alpha$ so

$$\int_{\gamma_L} \frac{z^\alpha}{z^4 + 16} dz \rightarrow e^{i\frac{\alpha n}{2}} e^{i\frac{n}{2}} I = e^{i\frac{n}{2}(\alpha+1)} I$$

$$\text{Now CRT gives } (1 - e^{i\frac{n}{2}(\alpha+1)}) I = 2\pi i \left(2^{\alpha-5} e^{i\frac{n}{4}(\alpha-3)} \right)$$

$$I = \frac{1}{1 - e^{i\frac{n}{2}(\alpha+1)}} \cdot \pi i 2^{\alpha-4} e^{i\frac{n}{4}(\alpha-3)}$$

$$= \frac{\pi 2^{\alpha-4} e^{i\frac{n}{4}(\alpha-1)}}{1 - e^{i\frac{n}{2}(\alpha+1)}}$$

[1]

(g) Write I in a pretty form involving only real quantities.

$$\text{since } -e^{i\frac{n}{2}(\alpha+1)} = e^{i\frac{n}{2}(\alpha-1)},$$

$$I = \frac{\pi 2^{\alpha-4} e^{i\frac{n}{4}(\alpha-1)}}{1 + e^{i\frac{n}{2}(\alpha-1)}} = \frac{\pi 2^{\alpha-4}}{e^{-i\frac{n}{4}(\alpha-1)} + e^{i\frac{n}{4}(\alpha-1)}} = \frac{\pi 2^{\alpha-5}}{\cos\left(\frac{n}{4}(\alpha-1)\right)}$$