University of British Columbia
Math 301, Section 201

Midterm 2

Date: March 19, 2012
Time: 11:00 - 11:50pm

Name (print): Solutions
Student ID Number:
Signature:

Instructor: Richard Froese

Instructions:

1. No notes, books or calculators are allowed.
2. Read the questions carefully and make sure you provide all the information that is asked for in the question.
3. Show all your work. Answers without any explanation or without the correct accompanying work could receive no credit, even if they are correct.
4. Answer the questions in the space provided. Continue on the back of the page if necessary.

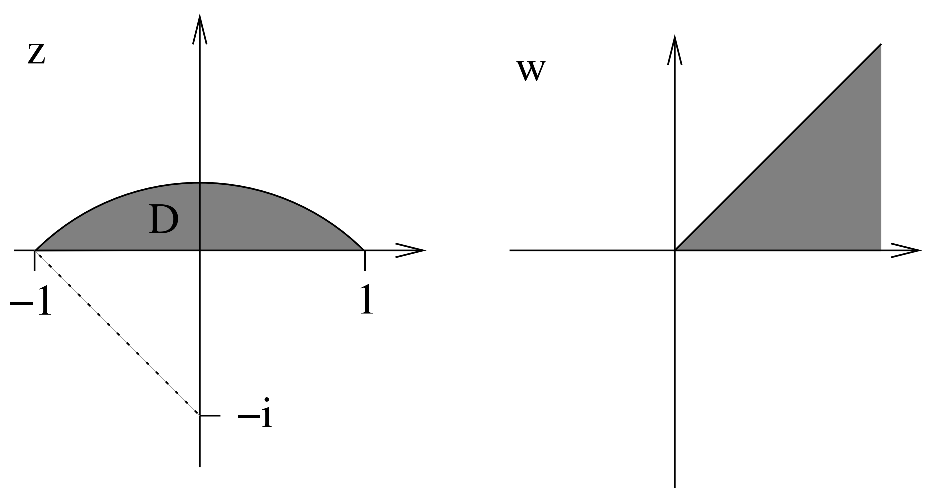
Question	Mark	Maximum
1		15
2		8
3		9
4		8
Total		40

[5] 1. (a) Write down the fractional linear transformation $f(z)$ satisfying $f(-1) = 0$, $f(0) = 1$ and $f(1) = \infty$.

Solution:

$$f(z) = -\frac{z+1}{z-1}$$

Let D be the region depicted below. (D is bounded below by the segment $[-1, 1]$ on the real line and above by an arc of the circle centred at $-i$ going through -1 and 1.)



[3] (b) What is the image of the top (circular) part of the boundary of D under $w = f(z)$? Give a complete explanation of your answer.

Solution: The image is the diagonal ray $\{z = x + iy : x = y, x \geq 0\}$. Explanation: The segment $[-1, 1]$ maps to $[0, \infty)$ since $-1 \to 0$, $1 \to \infty$ and 0 maps to a real value between 0 and ∞. The arc segment since it passes through -1 and 1 must also map to a ray from 0 to ∞. To decide which ray, either (1) use conformality: the angle of $\pi/4$ at -1 is preserved on the image at 0. or (2) compute the image of one other point on the arc, e.g., $f((\sqrt{2}-1)i) = (1/\sqrt{2} + i/\sqrt{2})$. Then the ray must pass through this point.

[2] (c) Draw the image of the region D the map $w = f(z)$ on the diagram above.

[5] (d) Solve Laplace's equation $\Delta\varphi(x,y) = 0$ for $z = x + iy \in D$ with boundary conditions $\varphi(x,0) = 1$ for $-1 \le x \le 1$ and $\varphi(x,y) = 0$ for $x + iy$ on the circular portion of the boundary.

Solution: First solve the problem on the target region as $\Phi(w) = A\mathrm{Arg}(w)+B$. Matching the boundary conditions gives $A = -4/\pi$ and $B = 1$. So we can write the solution as

$$\phi(z) - -4/\pi\mathrm{Arg}(f(z)) + 1$$

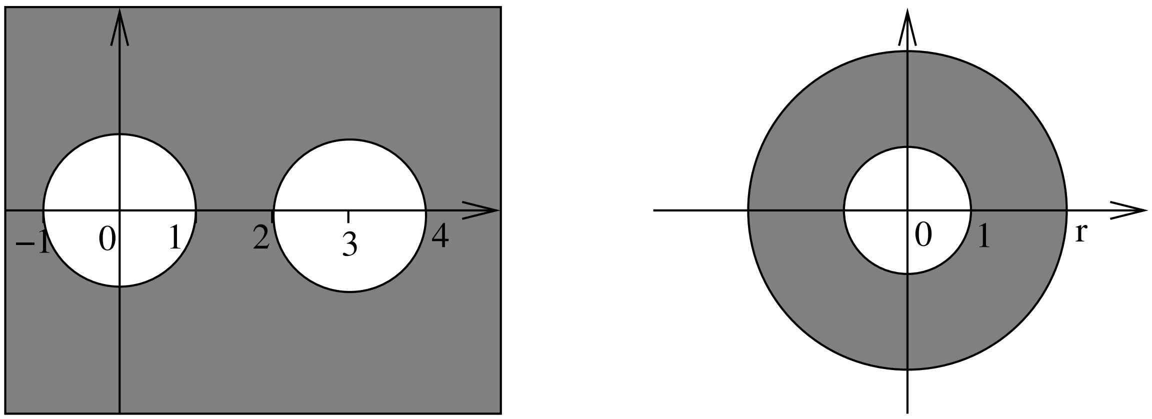
This would suffice as an answer, but we can also do it more explicitly by using $\mathrm{Arg}(u + iv) = \cot^{-1}(u/v)$ and computing

$$\begin{aligned} f(x+iy) &= -\frac{(x+iy+1)}{(x+iy-1)}\frac{(x-iy-1)}{(x-iy-1)} \\ &= \frac{1-x^2-y^2}{(x-1)^2+y^2} + i\frac{2y}{(x-1)^2+y^2} \\ &= u(x,y) + iv(x,y) \end{aligned}$$

so that

$$\phi(x,y) = -\frac{4}{\pi}\cot^{-1}\left(\frac{1-x^2-y^2}{2y}\right) + 1$$

[8] 2. Find a conformal map that maps the region outside the circles $\{|z| \leq 1\}$ and $\{|z-3| \leq 1\}$ (depicted on the left) to the annulus $1 < |z| < r$ (depicted on the right). What is the value of r?



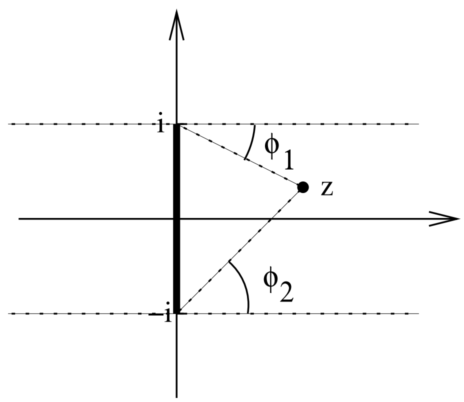
Solution: The points a and b with which both circles are symmetric satisfy $ab = 1$ and $(3-a)(3-b) = 1$. This gives $9 - 3a - 3b + ab = 1$, $3 - a - 1/a = 0$, $a^2 - 3a + 1 = 0$. Thus $a = (3-\sqrt{5})/2$ and $b = (3+\sqrt{5})/2$ are the desired points. Now we use the fractional linear transformation mapping $a \to 0$, $1 \to 1$ and $b \to \infty$. This is

$$f(z) = \frac{(z-a)}{(z-b)} \frac{(1-b)}{(1-a)}.$$

The point r is the image of 2, so

$$r = f(2) = \frac{(2-a)}{(2-b)} \frac{(1-b)}{(1-a)}$$

3. Consider the complex velocity potential $\Omega(z) = v_0(z^2+1)^{1/2}$, $v_0 > 0$, where the branch is chosen so that $\Omega(z)$ has a branch cut on the imaginary axis between $-i$ and i. Concretely, $\Omega(z) = v_0|z-i|^{1/2}|z+i|^{1/2}\exp((\phi_1+\phi_2)/2)$ in terms of the angles ϕ_1 and ϕ_2 depicted below, with $\phi_1, \phi_2 \in [-\pi/2, 3\pi/2]$.



[3]

(a) How does the complex fluid velocity behave as $|z| \to \infty$?

Solution: $\Omega'(z) = v_0 z(z^2+1)^{-1/2} = v_0(1+z^{-2})^{-1/2} \sim v_0$ as $|z| \to \infty$. So the complex fluid velocity behaves like $\overline{\Omega}'(z) \sim \overline{v}_0 = v_0$.

[3]

(b) Is the set $\mathbb{R}\backslash\{0\} = \{z = x+iy : y = 0, x \neq 0\}$ a streamline for this flow? Give a reason.

Solution: Yes because the if $z = x \in \mathbb{R}$ then $z^2+1 = x^2+1$ is a positive real number and $(x^2+1)^{1/2}$ is real (no matter what branch you choose). So the imaginary part of $\Omega(x)$ is zero, hence constant.

You could also figure out what happens to the angles ϕ_1 and ϕ_2, as in the next part.

[3]

(c) Show that this potential represents idealized inviscid fluid flow around a thin plate positioned on the branch cut.

Solution: We have to show that $\mathrm{Im}(\Omega(z))$ is constant on the left and right sides of the plate. We have $\mathrm{Im}(\Omega(z)) = v_0|z-i|^{1/2}|z+i|^{1/2}\sin((\phi_1+\phi_2)/2)$

On the right side of the plate, $\phi_1 = -\pi/2$ and $\phi_2 = \pi/2$ so $\sin((\phi_1+\phi_2)/2) = \sin(0) = 0$. Thus $\mathrm{Im}(\Omega(z)) = 0$ on the right side of the plate.

On the left side of the plate $\phi_1 = 3\pi/2$ and $\phi_2 = \pi/2$ so $\sin((\phi_1+\phi_2)/2) = \sin(\pi) = 0$. Thus $\mathrm{Im}(\Omega(z)) = 0$ on the right side of the plate too.

[8] 4. How many zeros does $p(z) = z^5 + z^3 + z + 1$ have in the right half plane?

Solution: We must compute the change in the argument of $p(iy)$ as y travels from $+\infty$ to $-\infty$ on the imaginary axis. We have $p(iy) = iy^5 - iy^3 + iy + 1$ so $\mathrm{Re}(p(iy)) = 1$ and $\mathrm{Im}(p(iy)) = y^5 - y^3 + y$. Since the real part never vanishes, there is no winding and the argument changes from $\pi/2$ to $-\pi/2$. So $\Delta(\arg(p)) = -\pi$ and therefore $N_+(p) = (-\pi + 5\pi)/(2\pi) = 2$.