University of British Columbia
Math 301, Section 201

Midterm 1

Date: February 13, 2013
Time: 11:00 - 11:50pm

Name (print):
Student ID Number:
Signature:

Instructor: Richard Froese

Instructions:

1. No notes, books or calculators are allowed.
2. Read the questions carefully and make sure you provide all the information that is asked for in the question.
3. Show all your work. Answers without any explanation or without the correct accompanying work could receive no credit, even if they are correct.
4. Answer the questions in the space provided. Continue on the back of the page if necessary.

Question	Mark	Maximum
1		13
2		12
3		8
4		7
Total		40

[2] 1. (a) Find and classify the singularities of $f(z) = \dfrac{\cot(\pi z)}{z^2}$

$$\frac{\cot(\pi z)}{z^2} = \frac{\cos(\pi z)}{z^2 \sin(\pi z)}$$

- simple poles at $z = n \in \mathbb{Z}$, $z \neq 0$
- pole of order 3 at $z = 0$

[3] (b) Calculate the residue of $f(z) = \dfrac{\cot(\pi z)}{z^2}$ at each of its singularities. (Hint: Sometimes the easiest way to find the residue is to compute the Laurent series directly by manipulating series.)

For $n \in \mathbb{Z}$, $n \neq 0$, $\text{Res}\left[\dfrac{\cos(\pi z)}{z^2 \sin(\pi z)}; n\right] = \dfrac{\cos(\pi z)}{n^2 \pi \cos(\pi z)} = \dfrac{1}{\pi n^2}$

For $n = 0$, compute Laurent expansion:

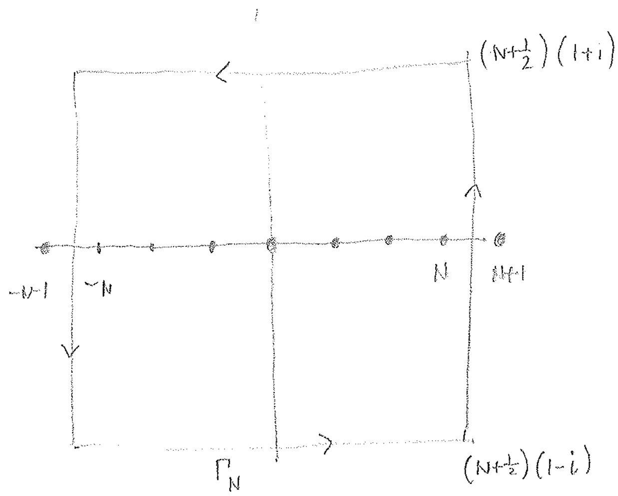
$$\frac{\cos(\pi z)}{z^2 \sin(\pi z)} = \frac{1 - \frac{\pi^2 z^2}{2} + O(z^4)}{z^2\left(\pi z - \frac{1}{6}\pi^3 z^3 + O(z^5)\right)} = \frac{\text{same}}{\pi z^3\left(1 - \frac{1}{6}\pi^2 z^2 + O(z^4)\right)}$$

$$= \frac{\left(1 - \frac{\pi^2 z^2}{2} + O(z^4)\right)\left(1 + \frac{1}{6}\pi^2 z^2 + O(z^4)\right)}{\pi z^3}$$

$$= \frac{1}{\pi z^3}\left(1 - \frac{1}{3}\pi^2 z^2 + O(z^4)\right) = \frac{1}{\pi z^3} - \frac{\pi}{3}\left(\frac{1}{z}\right) + O(z)$$

Thus $\text{Res}\left[\dfrac{\cos(\pi z)}{z^2 \sin(\pi z)}, 0\right] = -\dfrac{\pi}{3}$

[4] (c) The sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$ can be evaluated by integrating $f(z) = \frac{\cot(\pi z)}{z^2}$ over a suitable contour Γ_N and taking $N \to \infty$. Draw Γ_N and mark the singularites on your diagram. What does the Cauchy residue theorem say when applied to Γ_N?



$$\int_{\Gamma_N} f(z)\,dz = 2\pi i\left(-\frac{\pi}{3} + \sum_{\substack{n=-N \\ n\neq 0}}^{N} \frac{1}{\pi n^2}\right)$$

[4] (d) State what estimates are required to perform the evaluation of $\sum_{n=1}^{\infty} \frac{1}{n^2}$. (You need not prove them.) What is the value of the infinite sum?

Need $\left|\int_{\Gamma_N} f(z)\,dz\right| \to 0$ as $N \to \infty$ then

$$-\frac{\pi}{3} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n^2} = 0 \quad \text{or}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

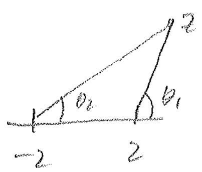
2. When we use the range of angles method to construct a branch of $f(z) = (4 - z^2)^{-1/2}$, we write $(z-2) = |z-2|\,e^{i\theta_1}$, $(z+2) = |z+2|\,e^{i\theta_2}$, and use an expression for $f(z)$ in terms of these quantites. Branches can then be specified by choosing ranges for the angles θ_1 and θ_2.

[4] (a) Write down the expression for $f(z)$

$$f(z) = (-1)(z-2)^{-\frac{1}{2}}(z+2)^{-\frac{1}{2}} = |z-2|^{-\frac{1}{2}}|z+2|^{-\frac{1}{2}} e^{i\frac{\pi-\theta_1-\theta_2}{2}}$$

$$= \frac{1}{\sqrt{|z^2-4|}} e^{i\frac{\pi-\theta_1-\theta_2}{2}} \quad \left[\frac{1}{\sqrt{z^2-4}} e^{i\frac{-\pi-\theta_1-\theta_2}{2}} \text{ is also ok but will result in different ranges of angles} \right]$$

[4] (b) What range of angles results in a branch cut on the interval $[-2, 2]$ and positive values of $f(z)$ on the top lip of the cut? Does this branch have a residue at infinity? If so, compute it.



$\theta_1 \in [0, 2\pi]$ $\theta_2 \in [0, 2\pi]$. Then the cuts cancel on $[2, \infty)$ and we are left with a cut on $[-2, 2]$.

When $z = x$ on top lip then $\theta_1 = \pi$, $\theta_2 = 0$ so

$$f(x) = \frac{1}{\sqrt{|x^2-4|}} e^{i\frac{\pi-\pi}{2}} = \frac{1}{\sqrt{4-x^2}} > 0$$

Since f is analytic near ∞, residue at ∞ exists

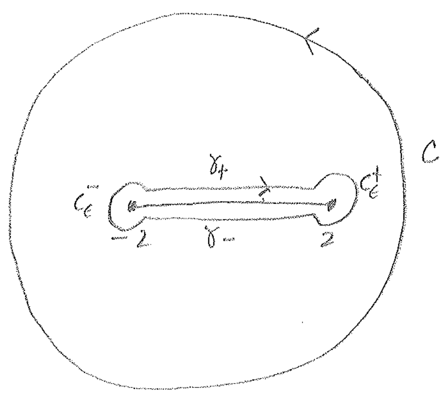
Then, since $\lim_{|z|\to\infty} f(z) = 0$, $Res[f, \infty] = \lim_{|z|\to\infty} z f(z)$

Compute this as z goes along pos. imag. axis

Then $\theta_1 \to \frac{\pi}{2}$ $\theta_2 \to \frac{\pi}{2}$ so

$$Res[f, \infty] = \lim_{\lambda\to\infty} i\lambda \frac{1}{\sqrt{\lambda^2+4}} e^{i\frac{\pi - \pi/2 - \pi/2}{2}} = i$$

[4] (c) Evaluate the integral $I = \int_{-2}^{2} \frac{1}{\sqrt{4-x^2}} dx$ by integrating one of the branches from the previous parts around a suitable contour and taking a limit. Draw the contour and indicate which parts of the integral vanish in the limit. You need not prove the needed estimates.



Integrals over C_ϵ^+ and C_ϵ^- vanish.

CRT then $\Rightarrow$

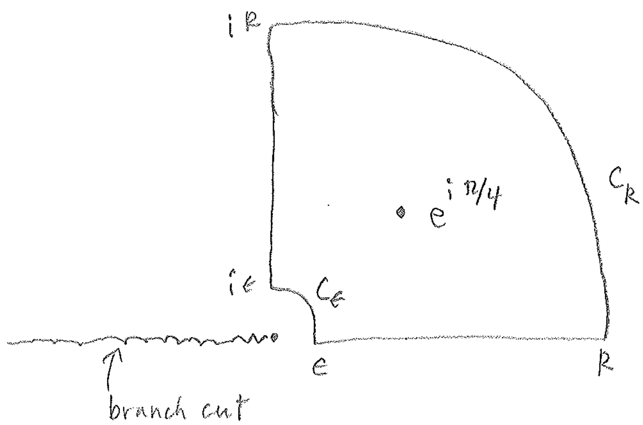
$$2I + 2\pi i \operatorname{Res}[f, \infty] = 0$$

$$2I + 2\pi i i = 0$$

$$I = \pi$$

[3]

3. (a) Draw the contour and the branch cut you would use to evaluate $I = \int_0^\infty \frac{x^\alpha}{1+x^4}dx$. Where are the singularites enclosed by the contour located?



[5]

(b) The procedure for evaluating I relies on the integral over some portions of the contour tending to zero in the limit. Provide the needed estimates and give range of α for which each estimate will work.

$$\left| \int_{C_R} \frac{z^\alpha}{1+z^4}\, dz \right| \le \max_{|z|=R} \frac{|z|^\alpha}{|1+z^4|} \cdot \frac{\pi}{2} \cdot R$$

$$\le \frac{R^\alpha}{R^4-1} \frac{\pi}{2} R \to 0 \text{ if } \alpha < 3$$

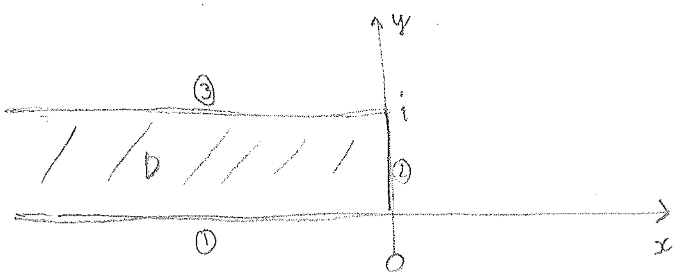
$$\left| \int_{C_\epsilon} \frac{z^\alpha}{1+z^4}\, dz \right| \le \max_{|z|=\epsilon} \frac{|z|^\alpha}{|1+z^4|} \cdot \frac{\pi}{2} \epsilon$$

$$\le \frac{\epsilon^\alpha}{1-\epsilon^4} \frac{\pi}{2} \epsilon \to 0 \quad \text{if } \alpha > -1$$

4. Let D be the half strip

$$D = \{x + iy : x \le 0, 0 \le y \le 1\}$$

(a) What is the image of D under $f(z) = z^2$?



Check the boundary:

① $(-\infty, 0]$ maps to the positive real axis $[0, \infty)$

② The segment $[0, i]$ maps to $[-1, 0]$

③ If $z = x + i$ for $x \le 0$, then $z^2 = x^2 - 1 + 2ix = u + iv$ where

$v \le 0$ and $u = -1 + \left(\frac{v}{2}\right)^2$ — a parabola passing through -1